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Equilibria of nonconvex valued maps under constraints ☆

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ABSTRACT

In the paper the notion of n-tangency for set-valued maps defined on a subset of a Banach space is considered. The existence of equilibria of upper semicontinuous map being n-tangent to a sleek retract with the nontrivial Euler characteristic is established.

0. Introduction

Let M be a *compact* subset of a Banach space E and $\varphi : M \multimap E$ be an *upper semicontinuous* set-valued map with *compact* values. In the paper we ask about the existence of *equilibria* of φ, i.e. $x_0 \in M$ such that $0 \in \varphi(x_0)$.

A classical result due to Browder and Fan [4,9] says that if M is convex, $\varphi : M \multimap E$ has convex values and is *inward* in the sense that

$$\varphi(x) \cap T_M(x) \neq \emptyset \quad \text{for each } x \in M, \tag{1}$$

where $T_M(x) = \mathrm{cl}(\bigcup_{h>0} h(M - x))$, then φ admits an equilibrium. Observe that $T_M(x)$ is a tangent cone to M at x and therefore the inwardness condition (1) can be interpreted as a tangency condition.

This result has been generalized many times, e.g. [6–8]. In [2], Ben-El-Mechaiekh and Kryszewski relaxed the convexity of M and obtained a similar result. Namely, if M is $\mathcal{L}$*-retract* with the nontrivial Euler characteristic ($\chi(M) \neq 0$), φ is as above but tangent with respect to the Clarke cone, i.e.

$$\varphi(x) \cap C_M(x) \neq \emptyset \quad \text{for each } x \in M, \tag{2}$$

where $C_M(x)$ stands for the Clarke cone tangent to M at x, then an equilibrium still exists.

A natural problem concerning the relaxation of convexity of values of φ arises. As shown in [2], if M is as above and φ has acyclic (e.g. contractible or cell-like) values and satisfies the strong tangency condition

$$\varphi(x) \subset C_M(x) \quad \text{for each } x \in M, \tag{3}$$

then there equilibria exist.

The following conjecture was posed in [2]:

(*C*) *If M is an $\mathcal{L}$-retract such that $\chi(M) \neq 0$, φ has acyclic values and condition* (2) *is satisfied, then there exists an equilibrium of* φ.

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We show that the very conjecture is false.

Example 0.1. Let $M = [0,1] \times [0,1]$, $E = \mathbb{R}^2$ and $\varphi : M \multimap E$ be a map defined by:

$$\varphi(x, y) := \begin{cases} \operatorname{conv}(\{(-1,0),(0,1)\}) \cup \operatorname{conv}(\{(0,1),(1,0)\}), & \text{if } (x,y) \in \{1\} \times [0,1], \\ \{(1,0)\}, & \text{if } (x,y) \in [0,1) \times [0,1]. \end{cases}$$

Since M is convex, then for any $x \in M$, $C_M(x) = T_M(x)$ and $\varphi(x) \cap C_M(x)$ is nonempty and convex. Then condition (2) is satisfied and φ is upper semicontinuous with contractible, hence acyclic, values. Moreover M is compact and convex, thus M is an $\mathcal{L}$-retract and $\chi(M) = 1$ (see Section 1). However, it is clear that $0 \notin \varphi(x)$ for each $x \in M$.

Hence, it appears that the pointwise tangency condition (2) together with the acyclicity (or even contractibility) of values of φ are too weak for the existence of equilibria. In order to obtain a positive answer it seems that one needs to study the local behavior of φ with respect to M in terms of homotopical triviality. We provide a class of the so-called n-tangent set-valued maps with not necessarily convex values (see Definition 2.1) and show that for that class the problem of existence of equilibria has a solution (see Theorem 2.3, Corollary 2.7).

1. Preliminaries

We consider set-valued maps $\varphi : X \multimap Y$, where X and Y are metric spaces, that assign to each $x \in X$, a nonempty subset $\varphi(x)$ of Y. By the *graph* of φ we mean the set $\operatorname{Gr}(\varphi) := \{(x,y) \in X \times Y \mid y \in \varphi(x)\}$. We say that a set-valued map φ is *lower semicontinuous* if for any open set $U \subset Y$, the preimage $\varphi^{-1}(U) := \{x \in X\colon\ \varphi(x) \cap U \neq \emptyset\}$ is open; φ is *upper semicontinuous* if for any open set $U \subset Y$, the small preimage $\varphi^{+1}(U) := \{x \in X\colon\ \varphi(x) \subset U\}$ is open; φ is *continuous* if it is upper and lower semicontinuous simultaneously. By a *selection* of φ we mean a continuous map $f : X \to Y$ such that $f(x) \in \varphi(x)$ for any $x \in X$.

If $A \subset B$, then $A \hookrightarrow B$ is *homotopy n-trivial* provided that for any $0 \leqslant k \leqslant n$, every continuous map $f_0 : S^k \to A$ admits a continuous extension $f : D^{k+1} \to B$, i.e. $f(x) = f_0(x)$ for any $x \in S^k$, where S^k and D^{k+1} stand for a unit sphere and a closed ball in $\mathbb{R}^{k+1}$. A map φ has *acyclic* values if $\check{H}^q(\varphi(x)) \approx \check{H}^q(pt)$ for any $q \in \mathbb{Z}$ and $x \in X$, where $\check{H}$ denotes the Čech cohomology functor and pt is a one point space. In particular, if for any $x \in X$, $\varphi(x)$ is convex, contractible, cell-like, then for any $n = 0,1,2,\ldots$, $\varphi(x) \in UV^n$,[1] and hence φ has acyclic values [10,3].

It is well known that approximation methods are helpful in the study of fixed points or equilibria of set-valued maps. However in the context of our problem we would like to look for a graph approximation $f : M \to E$ of φ satisfying the additional tangency condition: $f(x) \in C_M(x)$ for any $x \in M$. In [11, Th. 2.1] we have obtained a useful result in this direction. Below we recall an appropriate version of this result convenient for our purposes (comp. [11, Cor. 2.2, Rem. 2.3], [5,12]).

Theorem 1.1. *Let $n \geqslant 0$, X be a metric space, E be a Banach space, $\varphi : X \multimap E$ be upper semicontinuous with compact values, $C : X \multimap E$ be lower semicontinuous with closed and convex values. Then for any open neighborhood $\mathcal{U}$ of $\operatorname{Gr}(\varphi)$, there is a continuous selection $f : X \to E$ of C such that $\operatorname{Gr}(f) \subset \mathcal{U}$ provided that $\dim(X) \leqslant n+1$[2] and the following conditions hold*:

- (T) *for any $x \in X$, $\varphi(x) \cap C(x) \neq \emptyset$,*
- (C_n) *for any $x \in X$, for any open neighborhood U of $\varphi(x)$, there are an open neighborhood $V \subset U$ of $\varphi(x)$ and an open neighborhood W of x such that for any $y \in W$ the inclusion $V \cap C(y) \hookrightarrow U \cap C(y)$ is homotopy n-trivial.*

If condition (T) holds, then (C_n) is satisfied provided that φ has convex values. Moreover, if the strong tangency condition is satisfied, i.e. for any $x \in X$, $\varphi(x) \subset C(x)$, then (C_n) is equivalent to the condition: $\varphi(x) \in UV^n$ for any $x \in X$ (see [11, Lem. 2.13]).

In what follows we recall notions of tangent cones in a Banach space. Given a closed subset M of a Banach space E, for any $x \in M$, by

$$C_M(x) := \left\{ v \;\middle|\; \limsup_{t \to 0^+,\, x' \to_M x} \frac{d(x' + tv, M)}{t} = 0 \right\},$$

we denote the *Clarke tangent cone* to M at $x \in M$. It is well known that $C_M(x)$ is closed and convex and if M is convex, then $C_M(x) = T_M(x)$ (see [1]).

By $T_M^B(x)$ we denote the *Bouligand tangent cone* to M at x, i.e.

$$T_M^B(x) := \left\{ v \;\middle|\; \liminf_{t \to 0^+} \frac{d(x + tv, M)}{t} = 0 \right\}.$$

[1] Recall that for a subset A of metric space X, $A \in UV^n$ if for any open neighborhood U of A there is an open neighborhood $V \subset U$ of A such that the inclusion $V \hookrightarrow U$ is homotopy n-trivial.

[2] $\dim(X)$ denotes the covering dimension of the metric space X.

In general $C_M(x) \subset T_M^B(x)$ for any $x \in M$. We say that M is *sleek* if the set-valued map $M \ni x \mapsto T_M^B(x) \subset E$ is lower semicontinuous. Then $T_M^B(x) = C_M(x)$ for any $x \in M$, and hence the set-valued map $M \ni x \mapsto C_M(x) \subset E$ is lower semicontinuous, too [1].

We say that M is an *$\mathcal{L}$-retract* [2] if there are an open neighborhood Ω of M in E, $L \geqslant 1$ and a retraction $r : \Omega \to M$ such that

$$\|r(x) - x\| \leqslant L \cdot d(x, M)$$

for any $x \in \Omega$. It is well known that any closed and convex subset of Banach space is sleek (see [1]) and by [2] is $\mathcal{L}$-retract. Moreover, any compact C^1-manifold in Euclidean space is a sleek $\mathcal{L}$-retract and Clarke tangent cones coincide with the tangent spaces of the manifold.

2. Equilibria of *n*-tangent set-valued maps

Let M be a closed subset M of a Banach space E. Now we introduce a class of n-tangent set-valued maps.

Definition 2.1. We say that a map $\varphi : M \multimap E$ with compact values is *n-tangent*, if tangency condition (2) holds and

> for any $x \in M$, for any open neighborhood U of $\varphi(x)$, there are an open neighborhood $V \subset U$ of $\varphi(x)$ and an open neighborhood W of x such that for any $y \in W$ the inclusion $V \cap C_M(y) \hookrightarrow U \cap C_M(y)$ is homotopy n-trivial. (4)

Let $C : M \multimap E$ be given by the formula: $C(x) := C_M(x)$ for any $x \in M$. If $\varphi : M \multimap E$ is n-tangent, then condition (4) coincides with condition (C_n).

Example 2.2. (1) If M is sleek, $\varphi : M \multimap E$ has compact and convex values, $\varphi(x) \cap C_M(x) \neq \emptyset$ for any $x \in M$, then φ is n-tangent for any $n \geqslant 0$. Indeed, given $x \in M$, for any open neighborhood U of $\varphi(x)$, there is an open and convex neighborhood $V \subset U$ of $\varphi(x)$. The set $W := C^{-1}(V) = \{y \in M \mid V \cap C_M(y) \neq \emptyset\}$ is open, since C is lower semicontinuous. For any $y \in W$ the set $V \cap C_M(y)$ is nonempty and convex, and then condition (4) is satisfied.

(2) Let M be sleek, $\varphi : M \multimap E$ be a map with compact values such that $\varphi(x) \subset C_M(x)$ for any $x \in M$. If for any $x \in M$, $\varphi(x) \in UV^n$, then φ is n-tangent. In particular, if φ has contractible or cell-like values, then φ is n-tangent.

Observe that the map φ given in Example 0.1 is not n-tangent for any $n \geqslant 0$, since any open neighborhood U of $\varphi(1,1)$ contains an open neighborhood V of $\varphi(1,1)$ and there is $t_0 \in (0,1)$ such that for any $t \in (t_0, 1)$, the set $V \cap C_M(t,1)$ has two path-connected components.

Theorem 2.3. *If an $\mathcal{L}$-retract M is compact and sleek, $\chi(M) \neq 0$, $\dim(M) \leqslant n+1$, and $\varphi : M \multimap E$ is n-tangent upper semicontinuous set-valued map, then there exists $x_0 \in M$ such that $0 \in \varphi(x_0)$.*

Proof. In view of Theorem 1.1, for any $\varepsilon > 0$, there is an approximation $f_\varepsilon : M \to E$ such that

$$\operatorname{Gr}(f_\varepsilon) \subset B\big(\operatorname{Gr}(\varphi), \varepsilon\big)$$

and $f_\varepsilon(x) \in C_M(x)$ for any $x \in M$. Hence, by the above mentioned Ben-El-Mechaiekh and Kryszewski result, there is $x_\varepsilon \in M$ such that $0 = f_\varepsilon(x_\varepsilon)$. $\varphi(M)$ is compact since M is compact and φ is upper semicontinuous with compact values. Then it is easy to check that φ has an equilibrium. $\square$

In general, condition (4) seems to be difficult to verify. Therefore we provide a more natural class of maps (see Example 2.5 and Corollary 2.7).

Definition 2.4. We say that a map $\varphi : M \multimap E$ with compact values is *locally uniformly n-tangent*, if tangency condition (2) holds and

> for any $x \in M$, for any $\varepsilon > 0$ there are $0 < \delta < \varepsilon$ and an open neighborhood W of x such that for any $y \in W$ the inclusion $B(\varphi(y), \delta) \cap C_M(y) \hookrightarrow B(\varphi(y), \varepsilon) \cap C_M(y)$ is homotopy n-trivial. (5)

The following example justifies the relevance of the class.

Example 2.5. Let $\varphi : M \multimap E$ have compact values and for any $x \in M$, $\varphi(x) \cap C_M(x) \neq \emptyset$. If there is $\varepsilon_0 > 0$ such that $B(\varphi(x), \varepsilon) \cap C_M(x)$ is contractible for any $x \in M$ and $0 < \varepsilon \leqslant \varepsilon_0$, then φ is locally uniformly n-tangent. In particular, if for any $x \in M$ and $\varepsilon > 0$, $B(\varphi(x), \varepsilon) \cap C_M(x)$ is convex, then φ is locally uniformly n-tangent.

Proposition 2.6. *If $\varphi : M \multimap E$ is locally uniformly n-tangent continuous set-valued map, then φ is n-tangent.*

Proof. Let $x \in M$ and let U be an open neighborhood of $\varphi(x)$. Then $B(\varphi(x), \varepsilon) \subset U$ for some $\varepsilon > 0$. Since the map φ is upper semicontinuous, there is an open neighborhood W_1 of x such that

$$B\big(\varphi(y), \varepsilon/2\big) \subset B\big(\varphi(x), \varepsilon\big)$$

for any $y \in W_1$. By (5), we find $\delta > 0$ and an open neighborhood W_2 of x such that for any $y \in W_2$ and for any $0 \leqslant k \leqslant n$, every map $f_0 : S^k \to B(\varphi(y), \delta) \cap C_M(y)$ admits an extension $f : D^{k+1} \to B(\varphi(y), \varepsilon/2) \cap C_M(y)$. The set-valued map φ is lower semicontinuous with compact values, then φ is H-lower semicontinuous, i.e. for any $x \in X$ and $\varepsilon' > 0$, there is $\delta' > 0$ such that $\sup_{a \in \varphi(x)} d(a, \varphi(y)) < \varepsilon'$ for any $y \in M$ such that $d(x, y) < \delta'$. Hence, there is an open neighborhood W_3 of x such that

$$B\big(\varphi(x), \delta/2\big) \subset B\big(\varphi(y), \delta\big)$$

for any $y \in W_3$. Let $W := W_1 \cap W_2 \cap W_3$ and $V := B(\varphi(x), \delta/2)$. Then it is easy to check that for any $y \in W$ and for any $0 \leqslant k \leqslant n$, every map $f_0 : S^k \to V \cap C_M(y)$ admits an extension $f : D^{k+1} \to U \cap C_M(y)$. $\square$

Corollary 2.7. *If an $\mathcal{L}$-retract M is compact and sleek, $\chi(M) \neq 0$, $\dim(M) \leqslant n+1$, and $\varphi : M \multimap E$ is locally uniformly n-tangent continuous set-valued map, then there exists $x_0 \in M$ such that $0 \in \varphi(x_0)$.*

Proof. The conclusion follows from Proposition 2.6 and Theorem 2.3. $\square$

Note that in Corollary 2.7 the assumption on the continuity of φ cannot be weakened by the upper semicontinuity. Indeed, the map φ, given in Example 0.1, satisfies the following condition: for any $x \in M$ and $\varepsilon > 0$, $B(\varphi(x), \varepsilon) \cap C_M(x)$ is convex. Hence condition (5) is satisfied. However φ has no equilibria.

Observe that, if condition (5) is satisfied and φ has compact values, then $\varphi(x) \cap C_M(x) \in UV^n$ in $C_M(x)$, hence $\varphi(x) \cap C_M(x) \in UV^n$ in E (see [3, Lem. 2]). Therefore if φ is locally uniformly n-tangent, then

$$\text{for any } x \in M, \quad \varphi(x) \cap C_M(x) \in UV^n. \tag{6}$$

In Corollary 2.7 condition (5) cannot be relaxed by (6).

Example 2.8. Let $M = [0,1] \times [0,1]$, $E = \mathbb{R}^2$, and let $\varphi : M \multimap E$ be defined as follows:

$$\varphi(x, y) := \begin{cases} \operatorname{conv}(\{(1,0), (1-2x, 2x)\}), & \text{if } x \in [0, 1/2], \\ \operatorname{conv}(\{(1,0), (0,1)\}) \cup \operatorname{conv}(\{(0,1), (-2x+1, 2-2x)\}), & \text{if } x \in (1/2, 1]. \end{cases}$$

Then φ is continuous, $\varphi(x)$, $\varphi(x) \cap C_M(x)$ are contractible, hence $\varphi(x) \in UV^n$ and $\varphi(x) \cap C_M(x) \in UV^n$ for any $x \in M$. Moreover, M is compact and convex. Thus M is a sleek $\mathcal{L}$-retract and $\chi(M) = 1$. However $0 \notin \varphi(x)$ for any $x \in M$.

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