J. Math. Anal. Appl. 393 (2012) 25–32

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

The winding number of $Pf+1$ for polynomials P and meromorphic extendibility of f

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ARTICLE INFO

Article history:
Received 8 February 2012
Available online 13 April 2012
Submitted by D. Khavinson

Keywords:
Meromorphic extensions
Argument principle

ABSTRACT

Let Δ be the open unit disc in $\mathbb{C}$. The paper deals with the following conjecture: if f is a continuous function on $b\Delta$ such that the change of argument of $Pf+1$ around $b\Delta$ is nonnegative for every polynomial P such that $Pf+1$ has no zero on $b\Delta$ then f extends holomorphically through Δ. We prove a related result on meromorphic extendibility for smooth functions with finitely many zeros of finite order, which, in particular, implies that the conjecture holds for real analytic functions.

1. Introduction

Let Δ be the open unit disc in $\mathbb{C}$. Given a continuous function φ on $b\Delta$ with no zero on $b\Delta$ we denote by $W(\varphi)$ the winding number of φ around 0, so $2\pi W(\varphi)$ is the change of argument of $\varphi(z)$ as z runs around $b\Delta$ in the positive direction. If a function g is holomorphic on Δ then we denote by $Z(g)$ the number of zeros of g counting multiplicity. We denote by $A(\Delta)$ the disc algebra, that is the algebra of all continuous functions on $\overline{\Delta}$ which are holomorphic on Δ. It is known that one can characterize holomorphic extendibility in terms of the argument principle:

Theorem 1.1 ([1–3]). *A continuous function f on $b\Delta$ extends holomorphically through Δ if and only if $W(f+Q)\geq 0$ for every polynomial Q such that $f+Q\neq 0$ on $b\Delta$.*

Note that the "only if" part is a consequence of the argument principle.

One can view $f+Q$ above as $Pf+Q$ with $P\equiv 1$. We believe that an analogous theorem holds for $Q\equiv 1$:

Conjecture 1.2. *Let f be a continuous function on $b\Delta$ such that*

$$W(Pf+1)\geq 0 \tag{1.1}$$

whenever P is a polynomial such that $Pf+1\neq 0$ on $b\Delta$. Then f extends holomorphically through Δ.

The present note is the result of an unsuccessful attempt to prove this conjecture. In the paper we prove the conjecture for sufficiently smooth functions with finitely many zeros of finite order. In particular, the conjecture holds for real analytic functions.

2. Functions with no zeros

Suppose that the function f has no zero. In this case (1.1) implies that $W(f)\geq 0$. Indeed, there is an $\varepsilon>0$ such that $W(f)=W(f+\eta)$ for all $\eta,\ |\eta|<\varepsilon$. Choosing for P a constant $c,\ |c|>1/\varepsilon$, (1.1) implies that $W(f)\geq 0$. Note that (1.1) implies that

$$W(f)+W\left(P+\frac{1}{f}\right)=W(Pf+1)\geq 0$$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2012.03.061

so

$$W(P+1/f) \geq -W(f)$$

for every polynomial P such that $P+1/f \neq 0$ on $b\Delta$. If $W(f)=0$ then Theorem 1.1 implies that $1/f$ extends holomorphically through Δ and since $W(f)=0$ the argument principle shows that this holomorphic extension has no zero on Δ which gives:

Proposition 2.1. *Let f be a continuous function on $b\Delta$ which has no zero and which satisfies $W(f)=0$. If $W(Pf+1)\geq 0$ whenever P is a polynomial such that $Pf+1\neq 0$ on $b\Delta$ then f extends holomorphically through Δ.*

Now, let $W(f)=N>0$. Then $W(Pf+1)=W(f)+W(P+1/f)\geq 0$, so $W(1/f+P)\geq -N$ for every polynomial P such that $1/f+P\neq 0$ on $b\Delta$. A recent theorem of Raghupathi and Yattselev [4, Theorem 2] applies to show that if f is α-Hölder continuous with $\alpha>1/2$ then $1/f$ has a meromorphic extension through Δ which has at most N poles, counting multiplicity. So in this case there are a function H in the disc algebra and a polynomial R of degree not exceeding N, with all zeros contained in Δ, such that

$$\frac{1}{f(z)}=\frac{H(z)}{R(z)} \quad (z\in b\Delta).$$

Since $W(1/f)=-N$ and since $\deg R\leq N$ it follows by the argument principle that R has exactly N zeros in Δ, counting multiplicity, and H has no zero on $\overline{\Delta}$. It follows that $f=R/H$ extends holomorphically through Δ. This proves:

Proposition 2.2. *Let f be an α-Hölder continuous function on $b\Delta$ with $\alpha>1/2$ which has no zero on $b\Delta$. If $W(Pf+1)\geq 0$ whenever P is a polynomial such that $Pf+1\neq 0$ on $b\Delta$ then f extends holomorphically through Δ.*

3. Functions with finitely many zeros

The reasoning in Section 2 is no longer possible if f has zeros on $b\Delta$. Suppose that f has the form

$$f(z)=(z-b_1)^{m_1}(z-b_2)^{m_2}\cdots(z-b_n)^{m_n}g(z) \quad (z\in b\Delta),$$

where $b_i\in b\Delta,\ 1\leq i\leq n$, $b_i\neq b_j$ $(i\neq j)$ and where g is a continuous function with no zeros. (In particular, this holds when f is real analytic). Assume that $W(g)=N$. Then (1.1) implies that

$$W(1/g+(z-b_1)^{m_1}\cdots(z-b_n)^{m_n}P)\geq -N.$$

If $m_1=\cdots=m_n=0$ as in the preceding section then, if $N\geq 0$, [4, Theorem 2] implies that $1/g$ has a meromorphic extension through Δ with at most N poles. So the relevant question now is whether the same is true in general:

Question 3.1. Let $b_1,b_2,\ldots,b_n\in b\Delta,\ b_i\neq b_j$ if $i\neq j$, and let $m_1,\ldots,m_n\in\mathbb{N}$. Let $\mathcal{P}$ be the family of all polynomials Q of the form

$$Q(z)=(z-b_1)^{m_1}\cdots(z-b_n)^{m_n}p(z),$$

where p is a polynomial, and let $J\in\mathbb{N}\cup\{0\}$. Suppose that f is a continuous function on $b\Delta$ such that

$$f(b_j)\neq 0 \quad (1\leq j\leq n) \tag{3.1}$$

and such that $W(f+Q)\geq -J$ for each $Q\in\mathcal{P}$ such that $f+Q\neq 0$ on $b\Delta$. Must f extend meromorphically through Δ with the extension having at most J poles, counting multiplicity?

Note that one has to assume (3.1) since otherwise $W(f+Q)$ is undefined for every $Q\in\mathcal{P}$. If $m_1=\cdots=m_n=0$ and $J=0$ then the positive answer is provided by Theorem 1.1. If $m_1=\cdots=m_n=0$ and $J\geq 1$ and if f is α-Hölder continuous with $\alpha>1/2$ then the answer is positive by [4, Theorem 2] which was proved by using the theorem on rigid interpolaton:

Theorem 3.2 *([4, Theorem 5]). Suppose that g is a holomorphic function on Δ and let $N\in\mathbb{N}$. Suppose that for every nonnegative integer n and for every polynomial p of degree not exceeding n we have*

$$Z(z^n g+p)\leq N+n. \tag{3.2}$$

Then g is a quotient of polynomials of degree not exceeding N.

Note that if g is a quotient of polynomials of degree not exceeding N then (3.2) holds for every polynomial p of degree not exceeding n. In the present paper we use Theorem 3.2 as an essential tool.

4. On Question 3.1

Theorem 4.1. *Let $b_j \in b\Delta,\ 1 \le j \le n,\ b_i \ne b_j\ (i \ne j)$, and let $m_j \in \mathbb{N},\ 1 \le j \le n$. Let $\mathcal{P}$ be the family of all polynomials Q of the form*

$$Q(z) = (z-b_1)^{m_1}(z-b_2)^{m_2}\cdots(z-b_n)^{m_n}p(z),$$

where p is a polynomial. Let $N = m_1 + m_2 + \cdots + m_n$. Suppose that $f \in \mathcal{C}^{N+1}(b\Delta)$ is such that $f(b_j) \ne 0\ (1 \le j \le n)$, let J be a nonnegative integer and assume that

$$W(f+Q) \ge -J \quad \text{for every } Q \in \mathcal{P} \text{ such that } f+Q \ne 0 \text{ on } b\Delta.$$

Then f has a meromorphic extension through Δ having at most J poles in Δ, counting multiplicity.

We shall rewrite the condition in Theorem 4.1 in a slightly different form. Let $a_j \in b\Delta,\ 1 \le j \le N$, and let $\mathcal{S}$ be the family of all polynomials Q of the form

$$Q(z) = (z-a_1)\cdots(z-a_N)p(z),$$

where p is a polynomial. Note that we do not require that $a_i \ne a_j$ if $i \ne j$. To prove Theorem 4.1 it will be enough to prove the following:

$$\left.\begin{array}{l}\text{Suppose that } f \text{ is a function of class } \mathcal{C}^{N+1} \text{ on } b\Delta \text{ such that } f(a_j) \ne 0,\\ 1 \le j \le N, \text{ and such that } W(f+Q) \ge -J \text{ whenever } Q \in \mathcal{S} \text{ and } f+Q \ne 0\\ \text{on } b\Delta. \text{ Then } f \text{ has a meromorphic extension through } \Delta \text{ having at most } J\\ \text{poles in } \Delta \text{ counting multiplicity.}\end{array}\right\} \tag{4.1}$$

Proposition 4.2. *A continuous function f on $b\Delta$ satisfies*

$$W(f + (z-a_1)\cdots(z-a_N)g) \ge -J \tag{4.2}$$

for every polynomial g such that

$$f + (z-a_1)\cdots(z-a_N)g \ne 0 \quad \text{on } b\Delta \tag{4.3}$$

if and only if f satisfies (4.2) for every function $g \in A(\Delta)$ which satisfies (4.3).

Proof. If for some $g \in A(\Delta)$ satisfying (4.3) we have $W(f+(z-a_1)\cdots(z-a_n)g) \le -J-1$ then this holds for all sufficiently small perturbations of g. In particular, it holds for some polynomial sufficiently close to g, contradicting (4.2), completing the proof. □

Lemma 4.3. *Let $I \subset b\Delta$ be an arc centered at a and let $f \in \mathcal{C}^{n+1}(I)$. There are a polynomial p of degree not exceeding $n-1$ and a function $h \in \mathcal{C}^1(I)$ such that $f(z) = p(z) + (z-a)^n h(z)\ (z \in I)$.*

Proof. Write $a = e^{it_0}$ and let J be a segment on $\mathbb{R}$ centered at t_0. Then

$$f(e^{it}) = f(e^{it_0}) + c_1(t-t_0) + \cdots + c_{n-1}(t-t_0)^{n-1} + (t-t_0)^n g(t) \quad (t \in J), \tag{4.4}$$

where $g \in \mathcal{C}^1(J)$. For all t close to $t_0,\ t \ne t_0$, we have

$$\frac{t-t_0}{e^{it}-e^{it_0}} = a_0 + a_1(e^{it}-e^{it_0}) + a_2(e^{it}-e^{it_0})^2 + \cdots, \tag{4.5}$$

where the series converges for t near t_0. Now, by (4.4) and (4.5),

$$f(e^{it}) = f(e^{it_0}) + \sum_{k=1}^{n-1} c_k \left[\sum_{j=0}^{\infty} a_j (e^{it}-e^{it_0})^j\right]^k (e^{it}-e^{it_0})^k + \left[\sum_{j=0}^{\infty} a_j(e^{it}-e^{it_0})^j\right]^n (e^{it}-e^{it_0})^n g(t).$$

Computing the powers and rearranging we get

$$f(e^{it}) = f(e^{it_0}) + b_1(e^{it}-e^{it_0}) + \cdots + b_{n-1}(e^{it}-e^{it_0})^{n-1} + (e^{it}-e^{it_0})^n\big[g(t)+w(t)\big],$$

where, as a sum of a convergent power series, w is real analytic and so $g + w = h$ is of class $\mathcal{C}^1$. The proof is complete. □

Lemma 4.4. *Let f be a function of class $\mathcal{C}^{N+1}$ on $b\Delta$ and let $b_j \in b\Delta,\ 1 \le j \le n,\ b_i \ne b_j\ (j \ne i)$. Let $m_1, \ldots, m_n$ be positive integers such that $m_1 + \cdots m_n = N$ and let*

$$\begin{aligned}
&a_k = b_1 \quad (1 \le k \le m_1)\\
&a_k = b_2 \quad (m_1 + 1 \le k \le m_1 + m_2)\\
&\cdots\\
&a_k = b_n \quad (m_1 + \cdots m_{n-1} + 1 \le k \le m_1 + \cdots m_n.)
\end{aligned}$$

There are constants $A_k,\ 0 \le k \le N-1$, *and a function* g *of class* $\mathcal{C}^1$ *on* $b\Delta$ *such that*

$$f(z) = A_0 + A_1(z-a_1) + A_2(z-a_1)(z-a_2) + \cdots + A_{N-1}(z-a_1)\cdots(z-a_{N-1}) + g(z)(z-a_1)\cdots(z-a_N).$$

Proof. By Lemma 4.3 we have

$$f(z) = B_{10} + B_{11}(z-b_1) + \cdots + B_{1,m_1-1}(z-b_1)^{m_1-1} + g_1(z)(z-b_1)^{m_1} \quad (z \in b\Delta),$$

where g_1 is of class $\mathcal{C}^1$ on $b\Delta$ and of class $\mathcal{C}^{N+1}$ on $b\Delta \setminus \{b_1\}$. We repeat the procedure to write

$$g_1(z) = B_{20} + B_{21}(z-b_2) + \cdots + B_{2,m_2-1}(z-b_2)^{m_2-1} + g_2(z)(z-b_2)^{m_2} \quad (z \in b\Delta),$$

where the function g_2 is of class $\mathcal{C}^1$ on $b\Delta$ and of class $\mathcal{C}^{N+1}$ on $b\Delta \setminus \{b_1, b_2\}$. Repeating this procedure we get the functions $g_2, g_3, \cdots, g_n$, all of class $\mathcal{C}^1$ on $b\Delta$, such that

$$g_{n-1}(z) = B_{n0} + B_{n1}(z-b_n) + \cdots + B_{n,m_n-1}(z-b_n)^{m_n-1} + g_n(z)(z-b_n)^{m_n}.$$

Putting $g = g_n$ and substituting the expression for g_{n-1} into the expression for g_{n-2} and so on we get the result with

$$\begin{aligned}
&A_0 = B_{10}, \ldots, A_{m_1-1} = B_{1,m_1-1},\\
&A_{m_1} = B_{20},\ A_{m_1+1} = B_{21}, \ldots, A_{m_2-1} = B_{2,m_2-1},\\
&\cdots
\end{aligned}$$

which completes the proof. $\square$

5. Proof of Theorem 4.1

As already mentioned, we have to prove (4.1). So suppose that $f \in \mathcal{C}^{N+1}(b\Delta)$ satisfies $f(a_j) \ne 0$ $(1 \le j \le N)$ and satisfies (4.2) for every polynomial g satisfying (4.3). By Proposition 4.2, f satisfies (4.2) for every g in the disc algebra that satisfies (4.3). By Lemma 4.4 there are numbers $A_0, A_1, \ldots, A_{N-1}$ and a function g of class $\mathcal{C}^1$ on $b\Delta$ such that if

$$D(z) = A_0 + A_1(z-a_1) + \cdots + A_{N-1}(z-a_1)\cdots(z-a_{N-1})$$

then

$$f(z) = D(z) + (z-a_1)\cdots(z-a_N)g(z) \quad (z \in b\Delta)$$

which implies that

$$W\big(D + (z-a_1)\cdots(z-a_N)g + (z-a_1)\cdots(z-a_N)P\big) \ge -J$$

for every P in the disc algebra such that the expression in parentheses is different from 0 on $b\Delta$. Since g is of class $\mathcal{C}^1$ we can write

$$g(z) = F(z) + \overline{G(z)} \quad (z \in b\Delta),$$

where F and G are in the disc algebra with boundary values of class H^α for every $\alpha < 1$ which implies that

$$W\big(D + (z-a_1)\cdots(z-a_N)(F + \overline{G} + P)\big) \ge -J$$

for every P in the disc algebra such that the expression in parentheses is different from 0 on $b\Delta$, so

$$W\big(D + (z-a_1)\cdots(z-a_N)(\overline{G} + P)\big) \ge -J \tag{5.1}$$

whenever P in the disc algebra is such that

$$D + (z-a_1)\cdots(z-a_N)(\overline{G} + P) \ne 0 \quad \text{on } b\Delta. \tag{5.2}$$

By our assumption we have $f(a_j) \ne 0$ $(1 \le j \le N)$ which implies that

$$D(a_j) \ne 0 \quad (1 \le j \le N). \tag{5.3}$$

Recall that (5.1) holds whenever P is in the disc algebra and is such that (5.2) holds. In particular, (5.1) holds whenever P is a polynomial satisfying (5.2). Conjugating (5.1) we get

$$W\big(\overline{D} + (\overline{z} - \overline{a_1})\cdots(\overline{z} - \overline{a_N})(G + \overline{P})\big) \le J$$

which, multiplying the expression in parentheses with z^N, gives

$$W\big(z^N\overline{D} + (1-\overline{a_1}z)\cdots(1-\overline{a_N}z)(G+\overline{P})\big) \le N+J$$

for every polynomial P such that the expression in parentheses does not vanish on $b\Delta$. In particular, if A is the polynomial such that

$$A(z) = z^N\overline{D(z)} \quad (z\in b\Delta) \quad \text{and if} \quad B(z) = (1-\overline{a_1}z)\cdots(1-\overline{a_N}z)$$

then the degree of A does not exceed N and we have

$$W(A+BG+B\overline{P}) \le N+J$$

for every polynomial P such that $A+BG+B\overline{P}\neq 0$ on $b\Delta$. On $b\Delta$ we have $\overline{P(z)} = p(z)/z^m$ where $m\in\mathbb{N}\cup\{0\}$ and where p is a polynomial of degree not exceeding m, so it follows that

$$W(z^m(A+BG)+Bp_m) \le N+J+m$$

whenever $m\in\mathbb{N}\cup\{0\}$ and p_m is a polynomial of degree not exceeding m such that

$$z^m(A+BG)+Bp_m \neq 0 \quad \text{on } b\Delta. \tag{5.4}$$

The argument principle implies that

$$Z\left(z^m\left(\frac{A}{B}+G\right)+p_m\right) \le N+J+m \tag{5.5}$$

for every $m\in\mathbb{N}\cup\{0\}$ and for every polynomial p_m of degree not exceeding m such that (5.4) holds. Now, by (5.3) we have $A(a_j)\neq 0$ $(1\le j\le N)$. Since $B(a_j)=0$ $(1\le j\le N)$, (5.4) holds if and only if

$$z^m\left(\frac{A(z)}{B(z)}+G(z)\right)+p_m(z)\neq 0 \quad (z\in b\Delta,\ z\neq a_j,\ 1\le j\le N). \tag{5.6}$$

It follows that (5.5) holds for every polynomial p_m of degree not exceeding m, without condition (5.6). Indeed if

$$Z\left(z^m\left(\frac{A}{B}+G\right)+q_m\right) \ge N+J+m+1 \tag{5.7}$$

for some $m\in\mathbb{N}\cup\{0\}$ and for some polynomial q_m of degree not exceeding m then, by the argument principle, the same holds for q_m replaced by $q_m+\eta$ for all sufficiently small η. However, since G is α-Hölder smooth with $\alpha > 1/2$, the same holds for the function $z\mapsto z^m\big(A(z)/B(z)+G(z)\big)+q_m(z)$ which implies that the set

$$\left\{z^m\left(\frac{A(z)}{B(z)}+G(z)\right)+q_m(z)\colon z\in b\Delta,\ z\neq a_j,\ 1\le j\le N\right\}$$

has planar measure zero, so there are arbitrarily small η such that (5.7) holds for q_m replaced by $p_m=q_m+\eta$ where p_m satisfies (5.6). This proves that (5.5) holds for any $m\in\mathbb{N}\cup\{0\}$ and for any polynomial p_m of degree not exceeding m. Theorem 3.2 now applies to show that there are polynomials P, Q of degree not exceeding $N+J$ without common factors such that

$$\frac{A(z)}{B(z)}+G(z) = \frac{P(z)}{Q(z)} \quad (z\in b\Delta).$$

Recall that

$$B(z) = \frac{1}{a_1a_2\cdots a_n}(a_1-z)(a_2-z)\cdots(a_N-z)$$

and that $A(a_j)\neq 0$ $(1\le j\le N)$. We have $G=P/Q-A/B$. The function G is continuous on $\overline{\Delta}$, so if a factor $(\alpha-z)^k$ occurs in B then Q has to be divisible by $(\alpha-z)^k$. In fact, if Q contained only ℓ factors $(\alpha-z)$ with $\ell<k$ then we would get

$$(z-\alpha)^\ell G = \frac{P}{Q_1} - \frac{A}{(z-\alpha)^{k-\ell}B_1},$$

where Q_1 is a polynomial, $Q_1(\alpha)\neq 0$, and where B_1 is a polynomial. Since the left side is continuous at α, since $Q_1(\alpha)\neq 0$ and since $A(\alpha)\neq 0$, this is not possible. Thus $Q=RB$ where R is a polynomial of degree not exceeding J. It follows that

$$G = \frac{P}{Q}-\frac{A}{B} = \frac{P-RA}{RB}.$$

Since G is continuous on $\overline{\Delta}$ it follows that $P-RA$ must be divisible by B. Since $\deg A\le N$, $\deg R\le J$, $\deg P\le N+J$ it follows that $P-RA$ is a polynomial of degree not exceeding $N+J$. Thus, on $b\Delta$, G is a quotient of two polynomials of degree not exceeding J, which implies that the same holds for $\overline{G}$. Thus, $g=F+\overline{G}$ extends meromorphically through Δ with the number of poles not exceeding J and so the same holds for

$$f = A_0 + A_1(z-a_1)+\cdots+A_{N-1}(z-a_1)\cdots(z-a_{N-1})+(z-a_1)\cdots(z-a_N)g.$$

The proof is complete. $\square$

Remark. Another look at the proof of Theorem 4.1 shows that it is enough to assume that $f \in \mathcal{C}^{L+1}(b\Delta)$ where $L = \max\{m_1, m_2, \ldots, m_n\}$.

6. On Conjecture 1.2

We prove somewhat more general result than Conjecture 1.2 for sufficiently smooth functions f with finitely many zeros on $b\Delta$ of finite order:

Theorem 6.1. *Suppose that f is of class $\mathcal{C}^\infty$ on $b\Delta$ with at most finitely many zeros of finite order and let $J \in \mathbb{N} \cup \{0\}$. Then*

$$W(Pf + 1) \geq -J$$

for each polynomial P such that $Pf + 1 \neq 0$ on $b\Delta$ if and only if f extends meromorphically through Δ with the extension having at most J poles, counting multiplicity. In particular, f extends holomorphically through Δ if and only if $W(Pf + 1) \geq 0$ for every polynomial P such that $Pf + 1 \neq 0$ on $b\Delta$.

Corollary 6.2. *A real analytic function f on $b\Delta$ extends meromorphically through Δ with at most J poles in Δ, counting multiplicity, if and only if $W(Pf + 1) \geq -J$ for each polynomial P such that $Pf + 1 \neq 0$ on $b\Delta$. In particular, f extends holomorphically through Δ if and only if $W(Pf + 1) \geq 0$ for each polynomial P such that $Pf + 1 \neq 0$ on $b\Delta$.*

Proof of Theorem 6.1. Let J be a nonnegative integer and let f be a smooth function on $b\Delta$ that satisfies

$$W(Pf + 1) \geq -J \tag{6.1}$$

for all polynomials P such that $Pf + 1 \neq 0$ on $b\Delta$. This happens if and only if (6.1) holds for all functions P in the disc algebra such that $Pf + 1 \neq 0$ on $b\Delta$. Indeed, if we have $W(Pf + 1) \leq -J - 1$ for some P_0 in the disc algebra then the same holds for all P in the disc algebra sufficiently close to P_0. In particular, it holds for some polynomial P. We assume that f has at most finitely many zeros (of finite order) on $b\Delta$, so $f = \Pi g$ where $\Pi(z) = (z - a_1)(z - a_2)\cdots(z - a_n)$ and g is a smooth function on $b\Delta$ without zeros. Now (6.1) becomes

$$W(P\Pi g + 1) \geq -J$$

which gives

$$W\left(P\Pi + \frac{1}{g}\right) \geq -J - W(g).$$

Suppose first that $W(g) = N \geq -J$ and so $N + J \geq 0$; hence

$$W\left(\frac{1}{g} + \Pi P\right) \geq -(N + J)$$

for every polynomial P such that $1/g + \Pi P \neq 0$ on $b\Delta$. By Theorem 4.1,

$$\frac{1}{g(z)} = \frac{H(z)}{Q(z)} \quad (z \in b\Delta),$$

where H is in the disc algebra and Q is a polynomial with at most $N + J$ zeros on Δ. The argument principle now shows that $N = W(g) = W(Q) - W(H) = Z(Q) - Z(H) \leq N + J - Z(H)$ which implies that $Z(H) \leq J$ which shows that g, and consequently $f = \Pi g$, has a meromorphic extension through Δ with at most J poles, counting multiplicity.

We now complete the proof by showing that $N + J < 0$ is impossible. Assume that $-(N + J) \geq 1$. Since g is smooth and $W(g) = N < 0$ one can write

$$g(z) = F(z)\overline{G(z)}z^N \quad (z \in b\Delta),$$

where F and G are in the disc algebra with no zeros on $\overline{\Delta}$ and with smooth boundary values. We get

$$W\left(P\Pi F\overline{G}z^N + 1\right) \geq -J$$

whenever P in the disc algebra is such that the expression in parentheses is different from zero on $b\Delta$. Since F has no zero on $\overline{\Delta}$ this happens if and only if

$$W\left(P\Pi\overline{G} + z^{-N}\right) \geq -J - N \geq 1$$

and, since G has no zero on $\overline{\Delta}$, we have

$$W\left(\frac{z^{-N}}{\overline{G}} + P\Pi\right) \geq 1$$

whenever P in the disc algebra is such that

$$\frac{z^{-N}}{\overline{G}} + P\Pi \neq 0 \quad \text{on } b\Delta.$$

By Theorem 4.1 it follows that $z^{-N}/\overline{G}$ extends holomorphically through Δ which is possible if and only if

$$\frac{z^{-N}}{\overline{G(z)}} = Q(z) \quad (z \in b\Delta)$$

where Q is a polynomial of degree not exceeding $-N$. In particular,

$$W(Q + P\Pi) \geq 1 \tag{6.2}$$

for all functions P in the disc algebra such that $Q + P\Pi \neq 0$ on $b\Delta$. However, one can choose P in the disc algebra such that $Q + P\Pi = e^{\Psi}$ with Ψ entire. To do this one has to choose Ψ in such a way that $(e^{\Psi} - Q)/\Pi$ is holomorphic. This is easy to do; see [5]. The argument principle now shows that with this P, we have $W(F + P\Pi) = 0$, so (6.2) fails. This shows that $N + J < 0$ is impossible and completes the proof of Theorem 6.1. □

7. Generalizations of Theorem 4.1

Suppose that f is a continuous function and R is a fixed polynomial. Suppose that f is a continuous function on $b\Delta$ that does not vanish at any zero of R contained in $b\Delta$, such that

$$W(f + Rp) \geq 0$$

for every polynomial p such that $f + Rp \neq 0$ on $b\Delta$. Must f extend holomorphically through Δ? We know from Section 4 that the answer is positive provided that all zeros of R are on $b\Delta$ and provided that f is sufficiently smooth.

For general R the answer is negative. To see this, let $f(z) = z/(z - 1/2)$ $(z \in b\Delta)$. If p is a polynomial such that $f + zp \neq 0$ on $b\Delta$ then the argument principle implies that

$$\begin{aligned} W(f + zp) &= W\left(\frac{z}{z - 1/2} + zp\right) = W\left(\frac{z}{z - 1/2}\,(1 + (z - 1/2)p)\right) \\ &= W\left(\frac{z}{z - 1/2}\right) + W\big(1 + (z - 1/2)p\big) \geq 0 \end{aligned}$$

yet f does not extend holomorphically through Δ. We now show that for sufficiently smooth functions the answer to the question above is positive provided that R has no zero in Δ.

Let Π_1 be a product of $N \geq 0$ factors of the form $z - a$, $a \in \Delta$, let Π_2 be a finite product of factors of the form $z - a$, $a \in b\Delta$, and let Π_3 be a finite product of factors of the form $z - a$, $a \in \mathbb{C} \setminus \overline{\Delta}$. Let $\Pi = \Pi_1\Pi_2\Pi_3$, let J be a nonnegative integer and suppose that f is a smooth function on $b\Delta$ such that f does not vanish at any zero of Π_2 and such that

$$W(f + \Pi p) \geq -J \tag{7.1}$$

whenever p is a polynomial such that $f + \Pi p \neq 0$ on $b\Delta$. We know that this happens if and only if (7.1) holds for each p in the disc algebra such that $f + \Pi p \neq 0$ on $b\Delta$. Now, since the zeros of Π_3 are in $\mathbb{C} \setminus \overline{\Delta}$ it follows that p is in the disc algebra if and only $\Pi_3 p$ is in the disc algebra. It follows that (7.1) holds for every p in the disc algebra such that $f + \Pi p \neq 0$ on $b\Delta$ if and only if

$$W(f + \Pi_1\Pi_2 p) \geq -J \tag{7.2}$$

for each p in the disc algebra such that $f + \Pi_1\Pi_2 p \neq 0$ on $b\Delta$. Now, (7.2) implies that

$$W\left(\frac{f}{\Pi_1} + \Pi_2 p\right) \geq -J - N$$

whenever p is a polynomial such that $f/\Pi_1 + \Pi_2 p \neq 0$ on $b\Delta$. If f is sufficiently smooth then Theorem 4.1 implies that f/Π_1 has a meromorphic extension through Δ which has at most $J + N$ poles in Δ, counting multiplicity. This proves:

Theorem 7.1. *Let $\Pi = \Pi_1\Pi_2\Pi_3$ where Π_1 is a product of N factors of the form $z - a$, $a \in \Delta$, where Π_2 is a finite product of factors of the form $z - a$, $a \in b\Delta$, and where Π_3 is a finite product of factors of the form $z - a$, $a \in \mathbb{C} \setminus \overline{\Delta}$. Assume that $f \in \mathcal{C}^{\infty}(b\Delta)$ vanishes at no zero of Π_2 and assume that J is a nonnegative integer. Then f satisfies*

$$W(f + \Pi p) \geq -J$$

for every polynomial p such that $f + \Pi p \neq 0$ on $b\Delta$ if and only if f/Π_1 has a meromorphic extension through Δ which has at most $N + J$ poles in Δ, counting multiplicity.

8. Remarks

If D is a bounded domain in $\mathbb{C}$, we denote by $A(D)$ the algebra of all continuous functions on $\overline{D}$ which extend holomorphically through D. Theorem 1.1 has been generalized to:

Theorem 8.1 ([6]). *Let $D \subset \mathbb{C}$ be a bounded domain whose boundary consists of a finite number of pairwise disjoint simple closed curves. Let J be a nonnegative integer. Then $W(Pf + Q) \geq -J$ for each P, Q in $A(D)$ such that $Pf + Q \neq 0$ on bD if and only if f has a meromorphic extension through D with at most J poles counting multiplicity.*

If $J = 0$, that is, if we speak of holomorphic extendibility, then one can take $P \equiv 1$ [2]. It remains an open question whether one can take $P \equiv 1$ in general. Ragupathi and Yattselev [4] made progress by proving that one can take $P \equiv 1$ in the case where $D = \Delta$ and f is α-Hölder continuous with $\alpha > 1/2$. Conjecture 1.2 deals with the open question of whether one can take $Q \equiv 1$ in Theorem 8.1. We conclude by mentioning a related result which holds for all continuous functions:

Theorem 8.2. *Let f be a continuous function on $b\Delta$ and assume that*

$$W\big(P(f + c) + 1\big) \geq 0$$

whenever c is a constant and P is a polynomial such that $P(f + c) + 1 \neq 0$ on $b\Delta$. Then f extends holomorphically through Δ.

Proof. Observe that by choosing c large enough, $W(f + c) = 0$ and so, by Proposition 2.1, $f + c$ extends holomorphically through Δ and so does f. □

Acknowledgment

This paper was supported in part by the Ministry of Higher Education, Science and Technology of Slovenia through the research program Analysis and Geometry, Contract No. P1-02091 (B).

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