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# Some notes concerning Riemannian metrics of Cheeger Gromoll type

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ABSTRACT

Let $(M, g)$ be an $n$-dimensional Riemannian manifold and $TM$ its tangent bundle. The purpose of the present paper is three-fold. Firstly, to study paraholomorphy property of two Riemannian metrics $g_a$ and $g_{a,b}$ of Cheeger Gromoll type depending on one parameter and two parameters by using compatible paracomplex structures $J_a$ and $J_{a,b}$ on the tangent bundle $TM$. Secondly, to classify Killing vector fields on the tangent bundle $TM$ equipped with the Riemannian metric $g_{a,b}$. Finally, to give a detailed description of geodesics on the tangent bundle $TM$ with respect to the Riemannian metric $g_{a,b}$.

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## 1. Introduction

The research in the topic of differential geometry of tangent bundles over Riemannian manifolds begun with Sasaki, who constructed, in the original paper [1] of 1958, a Riemannian metric ${}^Sg$ on the tangent bundle $TM$ of a Riemannian manifold $(M, g)$, which depends closely on the base metric $g$. Although the Sasaki metric is naturally defined, it has been shown in many papers that the Sasaki metric presents a kind of rigidity. In [2], Kowalski proved that if the Sasaki metric ${}^Sg$ is locally symmetric, then the base metric $g$ is flat and hence ${}^Sg$ is also flat. In [3], Musso and Tricerri have demonstrated an extreme rigidity of ${}^Sg$ in the following sense: if $(TM, {}^Sg)$ is of constant scalar curvature, then $(M, g)$ is flat. Inspired by a paper of Cheeger and Gromoll, they also defined a new $g$-natural metric ${}^{CG}g$ on the tangent bundle $TM$, which they called the Cheeger Gromoll metric [4]. Sekizawa (see [5]) computed geometric objects related to this metric. Later, Gudmundson and Kappos, in [6,7], have completed these results and shown that the scalar curvature of the Cheeger Gromoll metric is never constant if the metric on the base manifold has constant sectional curvature. Furthermore, Abbassi and Sarih have proved that the tangent bundle $TM$ with the Cheeger Gromoll metric is never a space of constant sectional curvature (see [8]). In [9], the first author and his collaborators studied the paraholomorphy property of the Sasaki and Cheeger Gromoll metrics by using compatible paracomplex stuctures on the tangent bundle and showed that the Cheeger Gromoll metric is never paraholomorphic with respect to the compatible paracomplex structure.

A more general metric is given by Anastasiei in [10] which generalizes both of the two metrics mentioned above in the following sense: it preserves the orthogonality of the two distributions, on the horizontal distribution it is the same as on the base manifold, and finally the Sasaki and the Cheeger Gromoll metric can be obtained as particular cases of this metric. A compatible almost complex structure is also introduced and the tangent bundle $TM$ becomes a locally conformal almost Kählerian manifold. In [11], Munteanu studied another Riemannian metric on the tangent bundle $TM$ of a Riemannian manifold $M$ which generalizes the Sasaki metric and Cheeger Gromoll metric and a compatible almost complex structure which confers a structure of locally conformal almost K ählerian manifold to $TM$ together with the metric. He found conditions under which the tangent bundle $TM$ is almost Kählerian, locally conformal Kählerian or Kählerian

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when the tangent bundle $TM$ has constant sectional curvature or constant scalar curvature. On the other hand, Oproiu and his collaborators constructed natural metrics on the tangent bundles of Riemannian manifolds which possess interesting geometric properties [12–15]. All the preceding metrics belong to the wide class of the so-called $g$-natural metrics on the tangent bundle, initially classified by Kowalski and Sekizawa [16] and fully characterized by Abbassi and Sarih [17–19] (see also [20] for other presentation of the basic result from [16] and for more details about the concept of naturality).

The work is organized as follows: In Section 2, some introductory materials concerning with the tangent bundle $TM$ over an $n$-dimensional Riemannian manifold $M$ are collected. In Section 3, we first introduce paraholomorphic Norden manifolds (or para-Kähler-Norden manifolds) and then investigate the paraholomorphy property of two Cheeger Gromoll type Riemannian metrics $g_a$ and $g_{a,b}$ by using compatible paracomplex structures $J_a$ and $J_{a,b}$ on the tangent bundle $TM$, respectively. In Section 4, the adapted frame which allows the tensor calculus to be efficiently done is inserted in the tangent bundle $TM$. Killing vector fields on $(TM, g_{a,b})$ are classified; that is, general forms of all Killing vector fields on $(TM, g_{a,b})$ are found. Also, it is shown that if $(TM, g_{a,b})$ is the tangent bundle with the Cheeger Gromoll type Riemannian metric $g_{a,b}$ of a Riemannian, compact and orientable manifold $(M, g)$ with vanishing first and second Betti numbers, then the Lie algebras of Killing vector fields on $(M, g)$ and on $(TM, g_{a,b})$ are isomorphic. In Section 5, we study relations between geodesics on the base manifold $(M, g)$ and geodesics on the tangent bundle $(TM, g_{a,b})$ by means of the adapted frame.

## 2. Preliminaries

Let $M$ be an $n$-dimensional Riemannian manifold. Throughout this paper, all manifolds, tensor fields and connections are always assumed to be differentiable of class $C^\infty$. Also, we denote by $\Im^p_q(M)$ the set of all tensor fields of type $(p, q)$ on $M$.

*Basic formulas on tangent bundles*: Let $TM$ be the tangent bundle over an $n$-dimensional Riemannian manifold $M$, and $\pi$ the natural projection $\pi : TM \to M$. Let the manifold $M$ be covered by a system of coordinate neighborhoods $(U, x^i)$, where $(x^i)$, $i = 1, \ldots, n$ is a local coordinate system defined in the neighborhood $U$. Let $(y^i)$ be the Cartesian coordinates in each tangent space $T_PM$ at $P \in M$ with respect to the natural base $\left\{ \left.\frac{\partial}{\partial x^i}\right|_P \right\}$, $P$ being an arbitrary point in $U$ whose coordinates are $(x^i)$. Then we can introduce local coordinates $(x^i, y^i)$ on open set $\pi^{-1}(U) \subset TM$. We call them induced coordinates on $\pi^{-1}(U)$ from $(U, x^i)$. The projection $\pi$ is represented by $(x^i, y^i) \to (x^i)$. The indices $I, J, \ldots$ run from 1 to 2 $n$, the indices $\bar{i}, \bar{j}, \ldots$ run from $n + 1$ to $2n$. Summation over repeated indices is always implied.

Let $X = X^i \frac{\partial}{\partial x^i}$ be the local expression in $U$ of a vector field $X$ on $M$. Then the vertical lift ${}^VX$, the horizontal lift ${}^HX$ and the complete lift ${}^CX$ of $X$ are given, with respect to the induced coordinates, by

$$ {}^VX = X^i \partial_{\bar{i}}, \tag{2.1} $$

$$ {}^HX = X^i \partial_i - y^j \Gamma^i_{jk} X^k \partial_{\bar{i}}, \tag{2.2} $$

and

$$ {}^CX = X^i \partial_i + y^s \partial_s X^i \partial_{\bar{i}}, \tag{2.3} $$

where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \frac{\partial}{\partial y^i}$ and $\Gamma^i_{jk}$ are the coefficients of the Levi-Civita connection $\nabla$ of $g$.

In particular, we have the vertical spray ${}^Vu$ and the horizontal spray ${}^Hu$ on $TM$ defined by

$$ {}^Vu = y^{i\,V}(\partial_i) = y^i \partial_{\bar{i}}, \qquad {}^Hu = y^{i\,H}(\partial_i) = y^i \delta_i, \tag{2.4} $$

where $\delta_i = \partial_i - y^j \Gamma^s_{ji} \partial_{\bar{s}}$. ${}^Vu$ is also called the canonical or Liouville vector field on $TM$.

Now, let $r$ be the norm of a vector $u \in TM$. Then, for any smooth function $f$ of R to R, we have

$$ {}^HX\left(f(r^2)\right) = 0 \tag{2.5} $$

$$ {}^VX\left(f(r^2)\right) = 2f'(r^2) g(X, u) \tag{2.6} $$

and in particular, we get

$$ {}^HX(r^2) = 0. \tag{2.7} $$

$$ {}^VX(r^2) = 2g(X, u). \tag{2.8} $$

Let $X$, $Y$ and $Z$ be any vector fields on $M$, then we have

$$ {}^HX(g(Y, u)) = g\left((\nabla_X Y), u\right), \tag{2.9} $$

$$ {}^VX(g(Y, u)) = g\left(X, Y\right), \tag{2.10} $$

$$ {}^HX\left({}^V(g(Y, Z))\right) = X\left(g(Y, Z)\right) \tag{2.11} $$

$$ {}^VX\left({}^V(g(Y, Z))\right) = 0\ [19]. \tag{2.12} $$

Suppose that a tensor field $S \in \Im_q^p(M)$, $q > 1$, is given. We then define a tensor field $\gamma S \in \Im_{q-1}^p(TM)$ on $\pi^{-1}(U)$ by

$$\gamma S = (y^{\bar{l}} S_{li_2\ldots i_q}^{j_1\ldots j_p})\partial_{\overline{j_1}} \otimes \cdots \otimes \partial_{\overline{j_p}} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_q}$$

with respect to the induced coordinates $(x^i, y^i)$ [21, p. 12]. The tensor field $\gamma S$ defined on each $\pi^{-1}(U)$ determines global tensor field on $TM$. We easily see that $\gamma P$ has components, with respect to the induced coordinates $(x^i, y^i)$,

$$(\gamma P) = \begin{pmatrix} 0 \\ y^i P_j^i \end{pmatrix}$$

for any $P \in \Im_1^1(M)$ and $(\gamma P)({}^V f) = 0, f \in \Im_0^0(M)$, i.e. $\gamma P$ is a vertical vector field on $TM$.

Explicit expressions for the Lie bracket [, ] of the tangent bundle $TM$ are given by Dombrowski in [22]. The bracket operation of vertical and horizontal vector fields is given by the formulas

$$\begin{cases} \left[{}^H X, {}^H Y\right] = {}^H [X, Y] - {}^V (R(X, Y)u) \\ \left[{}^H X, {}^V Y\right] = {}^V (\nabla_X Y) \\ \left[{}^V X, {}^V Y\right] = 0 \end{cases} \tag{2.13}$$

for all vector fields $X$ and $Y$ on $M$, where $R$ is the Riemannian curvature of $g$ defined by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

## 3. Almost paracomplex structures with Norden metrics

An almost paracomplex manifold is an almost product manifold $(M_{2k}, \varphi)$, $\varphi^2 = id$, $\varphi \neq \pm id$ such that the two eigenbundles $T^+M_{2k}$ and $T^-M_{2k}$ associated with the two eigenvalues $+1$ and $-1$ of $\varphi$, respectively, have the same rank. Note that the dimension of an almost paracomplex manifold is necessarily even. This structure is said to be integrable if the matrix $\varphi = (\varphi_j^i)$ is reduced to the constant form in a certain holonomic natural frame in a neighborhood $U_x$ of every point $x \in M_{2k}$. On the other hand, an almost paracomplex structure is integrable if and only if one can introduce a torsion-free linear connection such that $\nabla\varphi = 0$. A paracomplex manifold is an almost paracomplex manifold $(M_{2k}, \varphi)$ such that the $G$-structure defined by the affinor field $\varphi$ is integrable. Also it can give another equivalent-definition of paracomplex manifold in terms of local homeomorphisms in the space $R^k(j) = \left\{(X^1, \ldots, X^k)/X^i \in R(j),\ i = 1, \ldots, k\right\}$ and paraholomorphic changes of charts in a way similar to [23] (see also [24]), i.e. a manifold $M_{2k}$ with an integrable paracomplex structure $\varphi$ is a real realization of the paraholomorphic manifold $M_k(R(j))$ over the algebra $R(j)$.

A tensor field $\omega$ of type $(0, q)$ is called a pure tensor field with respect to $\varphi$ if

$$\omega(\varphi X_1, X_2, \ldots, X_q) = \omega(X_1, \varphi X_2, \ldots, X_q) = \cdots = \omega(X_1, X_2, \ldots, \varphi X_q)$$

for any $X_1, \ldots, X_q \in \Im_0^1(M_{2k})$. The real model of a paracomplex tensor field $\overset{*}{\omega}$ on $M_k(R(j))$ is a $(0, q)$-tensor field on $M_{2k}$, being pure with respect to $\varphi$. Pure tensors have been studied by many authors (see, e.g., [9,15,24–32]). Consider an operator $\Phi_\varphi : \Im_q^0(M_{2k}) \to \Im_{q+1}^0(M_{2k})$ applied to the pure tensor field $\omega$ by (see [32])

$$\begin{aligned}(\Phi_\varphi \omega)(X, Y_1, Y_2, \ldots, Y_q) = {} & (\varphi X)(\omega(Y_1, Y_2, \ldots, Y_q)) - X(\omega(\varphi Y_1, Y_2, \ldots, Y_q)) \\ & + \omega((L_{Y_1}\varphi)X, Y_2, \ldots, Y_q) + \cdots + \omega(Y_1, Y_2, \ldots, (L_{Y_q}\varphi)X),\end{aligned}$$

where $L_Y$ denotes the Lie differentiation with respect to $Y$. Let $\varphi$ be a (an almost) paracomplex structure on $M_{2k}$ and $\Phi_\varphi \omega = 0$, the (almost) paracomplex tensor field $\overset{*}{\omega}$ on $M_k(R(j))$ is said to be (almost) paraholomorphic (see [25,32,33]). Thus a (an almost) paraholomorphic tensor field $\overset{*}{\omega}$ on $M_k(R(j))$ is realized on $M_{2k}$ in the form of a pure tensor field $\omega$, such that

$$(\Phi_\varphi \omega)(X, Y_1, Y_2, \ldots, Y_q) = 0$$

for any $X, Y_1, \ldots, Y_q \in \Im_0^1(M_{2k})$. Therefore, the tensor field $\omega$ on $M_{2k}$ is also called a (an almost) paraholomorphic tensor field.

An almost paracomplex Norden manifold $(M_{2k}, \varphi, g)$ is defined to be a real differentiable manifold $M_{2k}$ endowed with an almost paracomplex structure $\varphi$ and a Riemannian metric $g$ satisfying the Nordenian property (or purity condition)

$$g(\varphi X, Y) = g(X, \varphi Y)$$

for any $X, Y \in \Im_0^1(M_{2k})$. Manifolds of this kind are referred to as anti-Hermitian and $B$-manifolds (see [15,24,28,31]). If $\varphi$ is integrable, we say that $(M_{2k}, \varphi, g)$ is a paracomplex Norden manifold. A paracomplex Norden manifold $(M_{2k}, \varphi, g)$ is a realization of the paraholomorphic manifold $(M_k(R(j)), \overset{*}{g})$, where $\overset{*}{g} = (\overset{*}{g}_{uv})$, $u, v = 1, \ldots, k$ is a paracomplex metric tensor field on $M_k(R(j))$.

In a paracomplex Norden manifold, a paracomplex Norden metric $g$ is called paraholomorphic if

$$(\Phi_\varphi g)(X, Y, Z) = 0 \tag{3.1}$$

for any $X, Y, Z \in \Im_0^1(M_{2k})$. The paracomplex Norden manifold with paraholomorphic Norden metric $(M_{2k}, \varphi, g)$ is called a paraholomorphic Norden manifold.

In [28], Salimov and his collaborators have proved that for an almost paracomplex manifold with Norden metric $g$, the condition $\Phi_{\varphi}g = 0$ is equivalent to $\nabla\varphi = 0$, where $\nabla$ is the Levi-Civita connection of $g$ (for complex version see [26]). By virtue of this point of view, paraholomorphic Norden manifolds are similar to Kähler manifolds. Therefore, there exist a one-to-one correspondence between para-Kähler–Norden manifolds and paracomplex Norden manifolds with a paraholomorphic metric. Recall that in such a manifold, the Riemannian curvature tensor is pure and paraholomorphic, also the curvature scalar is a locally paraholomorphic function (see [28]).

Let $(M, g)$ be an $n$-dimensional Riemannian manifold and denote by $r$, the norm, a vector $u = (y^i)$, i.e. $r^2 = g_{ji}y^jy^i$. The Cheeger Gromoll metric ${}^{CG}g$ on the tangent bundle $TM$ is given by

$$\begin{aligned}
&{}^{CG}g({}^HX, {}^HY) = {}^V(g(X, Y)),\\
&{}^{CG}g({}^HX, {}^VY) = 0,\\
&{}^{CG}g({}^VX, {}^VY) = \frac{1}{\alpha}\left[{}^V(g(X, Y)) + g(X, u)g(Y, u)\right]
\end{aligned}$$

for all vector fields $X,\ Y \in \Im^1_0(M)$, where ${}^V(g(X, Y)) = (g(X, Y)) \circ \pi$ and $\alpha = 1 + r^2$.

In [9], we defined an almost paracomplex structure $J_{CG}$ on $TM$ by the formulas

$$\begin{cases}
J_{CG}({}^HX) = \sqrt{\alpha}\ {}^VX - \dfrac{1}{1+\sqrt{\alpha}}g(X, u)\ {}^Vu,\\
J_{CG}({}^VX) = \dfrac{1}{\sqrt{\alpha}}\ {}^HX + \dfrac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})}g(X, u)\ {}^Hu,
\end{cases}$$

and we have proved that the almost paracomplex Norden manifold $\left(TM, J_{CG}, {}^{CG}g\right)$ is never a para-Kähler-Norden manifold (or a paraholomorphic Norden manifold).

Consider a Riemannian metric $g_a$ of Cheeger Gromoll type defined by the following formulas (see also, [10,34])

$$\begin{aligned}
&g_a({}^HX, {}^HY) = {}^V(g(X, Y)),\\
&g_a({}^HX, {}^VY) = 0,\\
&g_a({}^VX, {}^VY) = a(r^2)\left[{}^V(g(X, Y)) + g(X, u)g(Y, u)\right]
\end{aligned} \tag{3.2}$$

for all vector fields $X,\ Y \in \Im^1_0(M)$, where $a : [0, \infty) \longrightarrow (0, \infty)$. This metric is a generalization of the Cheeger Gromoll metric.

We define an almost paracomplex structure $J_a$ on $TM$ by

$$\begin{cases}
J_a({}^HX) = \dfrac{1}{\sqrt{a}}\left({}^VX - \dfrac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})}g(X, u)\ {}^Vu\right),\\
J_a({}^VX) = \sqrt{a}\left({}^HX + \dfrac{1}{1+\sqrt{\alpha}}g(X, u)\ {}^Hu\right),
\end{cases} \tag{3.3}$$

for all $X,\ Y \in \Im^1_0(M)$. Note that $J_a({}^Hu) = \frac{1}{\sqrt{\alpha \cdot a}}{}^Vu$ and $J_a({}^Vu) = \sqrt{\alpha \cdot a}\ {}^Hu$. By using (3.2) and (3.3), one can easily check that the Riemannian metric $g_a$ is pure with respect to the almost paracomplex structure $J_a$. Hence we state the following theorem.

**Theorem 1.** *Let $(M, g)$ be a Riemannian manifold and TM be its tangent bundle equipped with the Riemannian metric $g_a$ defined by (3.2) and the paracomplex structure $J_a$ defined by (3.3). The triple $(TM, J_a, g_a)$ is an almost paracomplex Norden manifold.*

We now are interested in the paraholomorphy property of the Riemannian metric $g_a$ with respect to the almost paracomplex structure $J_a$. Therefore, we shall need the following proposition.

**Proposition 1** (*[34]*). *Let $(M, g)$ be a Riemannian manifold and equip its tangent bundle TM with the Riemannian metric $g_a$. Then the corresponding Levi-Civita connection $\nabla^a$ satisfies the following*

$$\begin{cases}
\nabla^a_{{}^HX}{}^HY = {}^H(\nabla_XY) - \dfrac{1}{2}{}^V(R(X, Y)u),\\
\nabla^a_{{}^HX}{}^VY = {}^V(\nabla_XY) + \dfrac{a}{2}{}^H(R(u, Y)X),\\
\nabla^a_{{}^VX}{}^HY = \dfrac{a}{2}{}^H(R(u, X)Y),\\
\nabla^a_{{}^VX}{}^VY = L\left\{g(X, u){}^VY + g(Y, u){}^VX\right\} + \dfrac{1-L}{\alpha}g(X, Y){}^Vu - \dfrac{1}{\alpha}g(X, u)g(Y, u){}^Vu,
\end{cases} \tag{3.4}$$

*where $L = \frac{a'(r^2)}{2a(r^2)}$ and $\alpha = 1 + r^2$.*

The covariant derivative

$$(\nabla^a_{\widetilde{X}} J_a)\widetilde{Y} = \nabla^a_{\widetilde{X}}(J_a\widetilde{Y}) - J_a(\nabla^a_{\widetilde{X}}\widetilde{Y})$$

of the almost paracomplex structure $J_a$ computed taking into account (3.4) is

$$\begin{cases} \text{(i) } (\nabla^a_{{}^HX} J_a)^H Y = \dfrac{\sqrt{a}}{2}{}^H(R(X,Y)u + R(u,Y)X), \\ \text{(ii) } (\nabla^a_{{}^HX} J_a)^V Y = -\dfrac{\sqrt{a}}{2}{}^V(R(X,Y)u + R(u,Y)X) + \dfrac{\sqrt{a}}{2(1+\sqrt{\alpha})}g(Y,u)^V(R(u,X)u) \\ \quad + \dfrac{1}{\sqrt{a}\sqrt{\alpha}(1+\sqrt{\alpha})}g(R(u,Y)X,u)^V u, \\ \text{(iii) } (\nabla^a_{{}^VX} J_a)^H Y = \dfrac{-L}{\sqrt{a}}g(X,u)^V Y + \dfrac{L}{\sqrt{a}\sqrt{\alpha}}g(Y,u)^V X \\ \quad + \dfrac{1+2\sqrt{\alpha} - L(1+\sqrt{\alpha})}{\sqrt{a}\alpha(1+\sqrt{\alpha})}g(X,Y)^V u + g(X,u)g(Y,u)^V u \\ \quad - \dfrac{\sqrt{a}}{2}{}^V(R(u,X)Y) + \dfrac{\sqrt{a}}{2\sqrt{\alpha}(1+\sqrt{\alpha})}g(R(u,X)Y,u)^V u, \\ \text{(iv) } (\nabla^a_{{}^VX} J_a)^V Y = \dfrac{a'}{2\sqrt{a}}g(X,u)^H Y - \dfrac{a'}{2\sqrt{a}}g(Y,u)^H X \\ \quad + \dfrac{a\sqrt{a}}{2}{}^H(R(u,X)Y) + \dfrac{a\sqrt{a}}{2(1+\sqrt{\alpha})}{}^H(R(u,X)u) \\ \quad + \dfrac{a'(1+\sqrt{\alpha}) - 2a}{2\sqrt{a}\sqrt{\alpha}(1+\sqrt{\alpha})}g(X,Y)^H u + \dfrac{a'(1+\sqrt{\alpha}) - 2a\sqrt{\alpha}}{2\sqrt{a}(1+\sqrt{\alpha})^2}g(X,u)g(Y,u)^H u. \end{cases} \tag{3.5}$$

From (3.5), we certainly say that $\nabla^a J_a \neq 0$. In this case $\Phi_{J_a} g_a \neq 0$. Hence we have the theorem below.

**Theorem 2.** *Let $(M, g)$ be a Riemannian manifold and let $TM$ be its tangent bundle with the Riemannian metric $g_a$ and the paracomplex structure $J_a$. The Riemannian metric $g_a$ is never paraholomorphic with respect to the paracomplex structure $J_a$, i.e. the triple $(TM, J_a, g_a)$ is never a para-Kähler-Norden manifold.*

Let us consider a more general metric of Cheeger Gromoll type, which is a family of Riemannian metrics depending on two parameters. This metric is defined in [11] by the following formulas

$$\begin{aligned} &g_{a,b}({}^HX, {}^HY) = {}^V(g(X,Y)), \\ &g_{a,b}({}^HX, {}^VY) = 0, \\ &g_{a,b}({}^VX, {}^VY) = a(r^2)^V(g(X,Y)) + b(r^2)g(X,u)g(Y,u) \end{aligned} \tag{3.6}$$

for all vector field $X,\ Y \in \Im^1_0(M)$, where $a, b : [0,\infty) \longrightarrow (0,\infty)$, $a > 0$ and also called it $g_{a,b}$. The Sasaki metric and the Cheeger Gromoll metric are particular cases of this metric. Really, for $a = 1$ and $b = 0$, the Sasaki metric is obtained, while the Cheeger Gromoll metric for $a = b = \frac{1}{\alpha}$.

An almost paracomplex structure on $TM$, for which the Riemannian metric $g_{a,b}$ is pure with respect to the structure, is defined by

$$\begin{cases} J_{a,b}({}^HX) = \dfrac{1}{\sqrt{a}}{}^VX - \dfrac{1}{\alpha - 1}\left(\dfrac{1}{\sqrt{a}} + \dfrac{1}{\sqrt{a + b(\alpha-1)}}\right)g(X,u)^V u, \\ J_{a,b}({}^VX) = \sqrt{a}^H X - \dfrac{1}{\alpha-1}(\sqrt{a} + \sqrt{a+b(\alpha-1)})g(X,u)\ {}^H u. \end{cases} \tag{3.7}$$

for all $X,\ Y \in \Im^1_0(M)$ and we call it $J_{a,b}$. Also note that $J_a({}^Hu) = -\frac{1}{\sqrt{a+b(\alpha-1)}}{}^V u$ and $J_a({}^Vu) = -\sqrt{a+b(\alpha-1)}\ {}^H u$. Hence, we have the result as follows.

**Theorem 3.** *Let $(M, g)$ be a Riemannian manifold and $TM$ be its tangent bundle equipped with the Riemannian metric $g_{a,b}$ defined by (3.6) and the paracomplex structure $J_{a,b}$ defined by (3.7). The triple $(TM, J_{a,b}, g_{a,b})$ is an almost paracomplex Norden manifold.*

Now we give the next proposition.

**Proposition 2** ([11]). *Let $(M, g)$ be a Riemannian manifold and let $TM$ be its tangent bundle equipped with the Riemannian metric $g_{a,b}$. Then the corresponding Levi–Civita connection $\nabla^{a,b}$ satisfies the following relations*

$$\begin{cases} \nabla^{a,b}_{{}^H X}{}^H Y = {}^H(\nabla_X Y) - \frac{1}{2}{}^V(R(X,Y)u), \\ \nabla^{a,b}_{{}^H X}{}^V Y = {}^V(\nabla_X Y) + \frac{a}{2}{}^H(R(u,Y)X), \\ \nabla^{a,b}_{{}^V X}{}^H Y = \frac{a}{2}{}^H(R(u,X)Y), \\ \nabla^{a,b}_{{}^V X}{}^V Y = L\left\{g(X,u)^V Y + g(Y,u)^V X\right\} + Rg(X,Y)^V u + Ng(X,u)g(Y,u)^V u, \end{cases} \tag{3.8}$$

*where* $L = \frac{a'(r^2)}{2a(r^2)}$, $R = \frac{2b(r^2) - a'(r^2)}{2(a(r^2) + (\alpha-1)b(r^2))}$ *and* $N = \frac{a(r^2)b'(r^2) - 2a'(r^2)b(r^2)}{2a(r^2)(a(r^2) + (\alpha-1)b(r^2))}$.

Having determined Levi-Civita connection $\nabla^{a,b}$, we can compute the covariant derivative of $J_{a,b}$. By direct computation, we obtain the following relations

$$\begin{cases} \text{(i) } (\nabla^{a,b}_{{}^H X} J_{a,b})^H Y = \frac{\sqrt{a}}{2}{}^H(R(X,Y)u + R(u,Y)X), \\ \text{(ii) } (\nabla^{a,b}_{{}^H X} J_{a,b})^V Y = -\frac{\sqrt{a}}{2}{}^V(R(X,Y)u + R(u,Y)X) - \left(\frac{\sqrt{a} + \sqrt{a + b(\alpha-1)}}{2(\alpha-1)}\right) g(Y,u)^V(R(u,X)u) \\ \quad + \frac{\sqrt{a}(\sqrt{a} + \sqrt{a + b(\alpha-1)})}{2(\alpha-1)\sqrt{a + b(\alpha-1)}} g(R(u,Y)X,u)^V u, \\ \text{(iii) } (\nabla^{a,b}_{{}^V X} J_{a,b})^H Y = \frac{-a'}{2a\sqrt{a}} g(X,u)^V Y + \frac{a'}{2a\sqrt{a}} g(Y,u)^V X \\ \quad + \frac{-(\alpha-1)a' - 2(-a + \sqrt{a^2 + ab(\alpha-1)})}{2\sqrt{a}(\alpha-1)(a + b(\alpha-1))} g(X,Y)^V u - \frac{\sqrt{a}}{2}{}^V(R(u,X)Y) \\ \quad + \frac{\left[ab' - 2a'b + \left(2a' - \frac{4a}{\alpha-1}\right)(a + b(\alpha-1))\right]\sqrt{a + b(\alpha-1)} + \left[a' + b' + 3b + \frac{2a}{\alpha-1}\right] 2a\sqrt{a}}{2a\sqrt{a}(a + b(\alpha-1))\sqrt{a + b(\alpha-1)}} \\ \quad \times g(X,u)g(Y,u)^V u + \frac{a}{2(\alpha-1)}\left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{a + b(\alpha-1)}}\right) g(R(u,X)Y,u)^V u, \\ \text{(iv) } (\nabla^{a,b}_{{}^V X} J_{a,b})^V Y = \frac{a'}{2\sqrt{a}} g(X,u)^H Y - \frac{a'}{2\sqrt{a}} g(Y,u)^H X \\ \quad + \frac{a\sqrt{a}}{2}{}^H(R(u,X)Y) - \frac{a(\sqrt{a} + \sqrt{a + b(\alpha-1)})}{2(\alpha-1)} g(Y,u)\,{}^H(R(u,X)u) \\ \quad + \frac{-a'(\alpha-1) - 2(-a + \sqrt{a^2 + ab(\alpha-1)})}{2(\alpha-1)\sqrt{a + b(\alpha-1)}} g(X,Y)^H u \\ \quad + \frac{(4a^2 - 4a\sqrt{a^2 + ab(\alpha-1)}) - (4a'\sqrt{a + b(\alpha-1)} + 4\sqrt{a}a' + 3\sqrt{a}b'(\alpha-1) - 2\sqrt{a}b)\sqrt{a}(\alpha-1)}{2a(\alpha-1)^2\sqrt{a + b(\alpha-1)}} \\ \quad \times g(X,u)g(Y,u)^H u. \end{cases} \tag{3.9}$$

If $a$ is a positive constant and $b$ vanishes, from (3.9), we write $\nabla^{a,b} J_{a,b} = 0$ if and only if the Riemannian manifold $(M, g)$ is flat. Otherwise, $\nabla^{a,b} J_{a,b} \neq 0$, i.e. $(TM, J_{a,b}, g_{a,b})$ is never a paraholomorphic Norden manifold (or a para-Kähler-Norden manifold). Thus, we have the next theorem.

**Theorem 4.** *Let $(M, g)$ be a Riemannian manifold and TM be its tangent bundle equipped with the Riemannian metric $g_{a,b}$ defined by (3.6) and the paracomplex structure $J_{a,b}$ defined by (3.7). The triple $(TM, J_{a,b}, g_{a,b})$ is a paraholomorphic Norden manifold if and only if $a = C$(positive const.), $b = 0$ and the Riemannian manifold $(M, g)$ is flat.*

**Remark 1.** Let $(M, g)$ be a Riemannian manifold and $TM$ its tangent bundle. If $a = 1$ and $b = 0$ in (3.6) and (3.7), the Riemannian metric $g_{a,b}$ is the Sasaki metric ${}^S g$ and the paracomplex structure $J_{a,b}$ is the paracomplex structure $J_S$ being compatible with the Sasaki metric. In [9], it is proved that the triple $(TM, J_S, {}^S g)$ is a paraholomorphic Norden manifold if and only if $(M, g)$ is flat.

If $a = C$(positive const.) and $b = 0$ in (3.6), the Riemannian metric $g_{a,b}$ is a Sasaki type metric. By virtue of Theorem 3.6 in [9] and Theorems 2 and 4 in the present paper, we have the following result.

**Corollary 1.** *There is no Cheeger Gromoll type structure on TM such that TM is a paraholomorphic Norden manifold (or a para-Kähler-Norden manifold).*

## 4. Killing vector fields with respect to the Riemannian metric $g_{a,b}$

With a torsion-free affine connection $\nabla$ given on $M$, we can introduce on each induced coordinate neighborhood $\pi^{-1}(U)$ of $TM$ a frame field which is very useful in our computation. In each local chart $U \subset M$, we put $X_{(j)} = \frac{\partial}{\partial x^j}$, $j = 1, \ldots, n$. Then from (2.1) and (2.2), we see that these vector fields have, respectively, local expressions

$$^{H}X_{(j)} = \delta_j^h \partial_h + (-\Gamma_{sj}^h x^s)\partial_{\overline{h}}$$
$$^{V}X_{(j)} = \delta_j^h \partial_{\overline{h}}$$

with respect to the natural frame $\left\{\partial_h, \partial_{\overline{h}}\right\}$, where $\delta_j^h$-Kronecker delta. These $2n$ vector fields are linear independent and generate, respectively, the horizontal distribution of $\nabla$ and the vertical distribution of $TM$. We have called the set $\left\{^{H}X_{(j)}, {}^{V}X_{(j)}\right\}$ the frame adapted to the affine connection $\nabla$ in $\pi^{-1}(U) \subset TM$. On putting

$$E_j = {}^{H}X_{(j)},$$
$$E_{\overline{j}} = {}^{V}X_{(j)},$$

we write the adapted frame as $\{E_\lambda\} = \left\{E_j, E_{\overline{j}}\right\}$. $\left\{dx^h, \delta y^h\right\}$ is the dual frame of $\left\{E_i, E_{\overline{i}}\right\}$, where $\delta y^h = dy^h + y^b \Gamma_{ba}^h dx^a$. By straightforward calculation, we have the following lemma.

**Lemma 1** ([21,35]). *The Lie brackets of the adapted frame of TM satisfy the following identities*

$$\begin{cases} \left[E_j, E_i\right] = y^b R_{ijb}{}^{a} E_{\overline{c}} \\ \left[E_j, E_{\overline{i}}\right] = \Gamma_{ji}^a E_{\overline{a}} \\ \left[E_{\overline{j}}, E_{\overline{i}}\right] = 0 \end{cases}$$

*where $R_{ijb}{}^{a}$ denote the components of the curvature tensor of $M$.*

Using (2.1)–(2.3), we have

$$^{H}X = \begin{pmatrix} X^j \delta_j^h \\ -X^j \Gamma_{sj}^h y^s \end{pmatrix} = X^j \begin{pmatrix} \delta_j^h \\ -\Gamma_{sj}^h y^s \end{pmatrix} = X^j E_j$$
$$^{V}X = \begin{pmatrix} 0 \\ X^h \end{pmatrix} = \begin{pmatrix} 0 \\ X^j \delta_j^h \end{pmatrix} = X^j \begin{pmatrix} 0 \\ \delta_j^h \end{pmatrix} = X^j E_{\overline{j}},$$

and

$$^{C}X = \begin{pmatrix} X^j \delta_j^h \\ y^s \partial_s X^j \end{pmatrix}$$
$$= X^j \begin{pmatrix} \delta_j^h \\ -\Gamma_{sj}^h y^s \end{pmatrix} + y^m \nabla_m X^j \begin{pmatrix} 0 \\ \delta_j^h \end{pmatrix} = X^j E_j + y^m \nabla_m X^j E_{\overline{j}}$$

with respect to the adapted frame $\{E_\lambda\}$.

We shall need a new vector field on $TM$. For any vector field $Y \in \Im_0^1(M)$ with the components $(Y^h)$, let $Y_A$ be a vector field on $TM$ defined by

$$Y_A = \{-a(r^2) y^r \nabla^i Y_r\} E_i + \left\{ \frac{(\alpha - 1)(a'(r^2) - b(r^2)) + 2a(r^2)}{2a(r^2)} Y^i - \frac{a'(r^2)}{a(r^2)} g_{ks} Y^k y^s y^i \right\} E_{\overline{i}},$$

with respect to the adapted frame $\{E_\lambda\}$, where $\alpha = 1 + r^2$ and $r^2 = g_{ij} y^i y^j$. Clearly the lift $Y_A$ is a smooth vector field on $TM$.

Let $L_{\widetilde{X}}$ be the Lie derivation with respect to the vector field $\widetilde{X}$, then we have the following lemma.

**Lemma 2** (See [36]). *The Lie derivations of the adapted frame and its dual basis with respect to $\widetilde{X} = v^h E_h + v^{\overline{h}} E_{\overline{h}}$ are given as follows*

$$(1)\ L_{\widetilde{X}} E_h = -(E_h v^a) E_a + \left\{ y^b v^c R_{hcb}{}^{a} - v^{\overline{b}} \Gamma_{b\,h}^{a} - E_h(v^{\overline{a}}) \right\} E_{\overline{a}}$$
$$(2)\ L_{\widetilde{X}} E_{\overline{h}} = -(E_{\overline{h}} v^a) E_a + \left\{ v^b \Gamma_{b\,h}^{a} - E_{\overline{h}} v^{\overline{a}} \right\} E_{\overline{a}}$$
$$(3)\ L_{\widetilde{X}} dx^h = (E_a v^h) dx^a + (E_{\overline{a}} v^h) \delta y^a$$
$$(4)\ L_{\widetilde{X}} \delta y^h = \left\{ y^c v^b R_{bac}{}^{h} + v^{\overline{b}} \Gamma_{b\,a}^{h} + E_a v^{\overline{h}} \right\} dx^a - \left\{ v^b \Gamma_{b\,a}^{h} - E_{\overline{a}} v^{\overline{h}} \right\} \delta y^a.$$

If $g = g_{ij}dx^i dx^j$ is the expression of the Riemannian metric $g$, the metric $g_{a,b}$ is expressed in the adapted local frame by

$$g_{a,b} = g_{ij}dx^i dx^j + h_{ij}\delta y^i \delta y^j,$$

where $h_{ij}$ is the function on $\pi^{-1}(U)$ defined by $h_{ij} = a(r^2)g_{ij} + b(r^2)g_{is}g_{tj}y^s y^t$.

We shall first state the following lemma, which is needed later on.

**Lemma 3.** *The Lie derivatives $L_{\widetilde{X}}g_{a,b}$ with respect to the vector field $\widetilde{X} = v^h E_h + v^{\bar{h}} E_{\bar{h}}$ are given as follows*

$$L_{\widetilde{X}}g_{a,b} = (v^a \partial_a g_{ij} + g_{aj}E_i v^a + g_{ia}E_j v^a)dx^i dx^j + \left\{2h_{aj}(y^c v^b R^{\ a}_{bic} + v^{\bar{b}}\Gamma^a_{b\,i} + E_i v^{\bar{a}}) + 2g_{ai}E_{\bar{j}}v^a\right\} dx^i \delta y^j$$
$$+ \left\{v^a E_a h_{ij} + v^{\bar{a}}E_{\bar{a}}h_{ij} - 2h_{ai}(v^b \Gamma^a_{bj} - E_{\bar{j}}v^{\bar{a}})\right\}\delta y^i \delta y^j.$$

**Proof.** Proof of Lemma 3 is similar to the proof of Proposition 2.3 of Yamauchi [35]. □

The general forms of Killing vector fields on $(TM, g_{a,b})$ are given by

**Theorem 5.** *Let $(TM, g_{a,b})$ be the tangent bundle with the Riemannian metric $g_{a,b}$ of a Riemannian manifold $(M, g)$. Let*

(i) *$X$ be a Killing vector field on $(M, g)$;*

(ii) *$P$ be a $(1, 1)$ tensor field on $M$ which satisfies the following*

  (ii-1) *parallel with respect to $g$, i.e. $\nabla_k P^i_j = 0$, and*

  (ii-2) *skew-symmetric with respect to $g$; $P^a_i g_{aj} + P^a_j g_{ia} = 0$,*

(iii) *$Y$ be a vector field on $M$ which satisfies the following*

  (iii-1) $\nabla_i\nabla_j Y^k + \nabla_j\nabla_i Y^k = 0$, *and*

  (iii-2) $a(r^2)(R_{jrb}^{\ \ \ l}\,\nabla^b Y_s + R_{jsb}^{\ \ \ l}\,\nabla^b Y_r) = -\frac{1}{3}(\nabla^l Y^b)(\frac{4a'(r^2)-2b(r^2)}{a(r^2)}g_{rs}g_{bj} - \frac{2a'(r^2)-b(r^2)}{a(r^2)}(g_{bs}g_{rj} + g_{br}g_{sj}))$.

*Then the vector field on TM defined by*

$$(\sharp) \quad \widetilde{X} = {}^C X + \gamma P + Y_A$$

*is a Killing vector field on $(TM, g_{a,b})$.*

*Conversely, every Killing vector field on $(TM, g_{a,b})$ is of the form $(\sharp)$.*

We shall employ the natural method proposed by Tanno in [37] and prove the above result. Let $TM$ be the tangent bundle over $M$ with the Riemannian metric $g_{a,b}$, and let $\widetilde{X}$ be a Killing vector field on $(TM, g_{a,b})$ such that $L_{\widetilde{X}}g_{a,b} = 0$. By means of Lemma 3, we obtain the following three relations

$$v^a \partial_a g_{ij} + g_{aj}E_i v^a + g_{ia}E_j v^a = 0 \tag{4.1}$$

$$2h_{aj}(y^c v^b R^{\ a}_{bic} + v^{\bar{b}}\Gamma^a_{b\,i} + E_i v^{\bar{a}}) + 2g_{ai}E_{\bar{j}}v^a = 0 \tag{4.2}$$

$$v^a E_a h_{ij} + v^{\bar{a}}E_{\bar{a}}h_{ij} - 2h_{ai}(v^b \Gamma^a_{bj} - E_{\bar{j}}v^{\bar{a}}) = 0. \tag{4.3}$$

First all, we shall study the particular cases ${}^C X$, $\gamma A$ and $Y_A$. Using (4.1)–(4.3) and the local expressions of ${}^C X$, $\gamma P$, $Y_A$ with respect to the adapted frame, one can easily prove, by direct computation, the following lemmas.

**Lemma 4.** *In order that a complete lift ${}^C X$ to TM of a vector field $X$ on $M$ be a Killing vector field of $(TM, g_{a,b})$, it is necessary and sufficient that $X$ itself is a Killing vector field of $(M, g)$.*

**Lemma 5.** *Let $P$ be a $(1, 1)$-tensor field on $(M, g)$ satisfying the conditions* (ii − 1) *and* (ii − 2) *in Theorem 5. Then $\gamma P$ is a Killing vector field on $(TM, g_{a,b})$.*

**Lemma 6.** *Let $Y$ be a vector field on $(M, g)$ satisfying the conditions* (iii − 1) *and* (iii − 2) *in Theorem 5. Then $Y_A$ is a Killing vector field on $(TM, g_{a,b})$.*

**Proof.** Since sufficiency is shown by Lemmas 4–6, we now show necessity. We consider the 0-section $(y^i = 0)$ in the coordinate neighborhood $\pi^{-1}(U)$ in $TM$ and its neighborhood $W$. For a vector field $\widetilde{X} = v^i E_i + v^{\bar{i}} E_{\bar{i}}$ on $TM$, and $(x, y) = (x^i, y^i)$ in $W$, we can write, by Taylor's theorem,

$$v^i(x, y) = v^i(x, 0) + (\partial_{\bar{r}} v^i)(x, 0)y^r + \frac{1}{2}(\partial_{\bar{r}}\partial_{\bar{s}} v^i)(x, 0)y^r y^s + \cdots + [*]^i_\lambda, \tag{4.4}$$

$$v^{\bar{i}}(x, y) = v^{\bar{i}}(x, 0) + (\partial_{\bar{r}} v^{\bar{i}})(x, 0)y^r + \frac{1}{2}(\partial_{\bar{r}}\partial_{\bar{s}} v^{\bar{i}})(x, 0)y^r y^s + \cdots + [*]^{\bar{i}}_\lambda, \tag{4.5}$$

where $[*]^I_\lambda$ $(I = 1, 2, \ldots, 2n)$ is of the form:

$$[*]^I_\lambda = \frac{1}{\lambda!}(\partial^\lambda v^I/\partial y^{i_1}\partial y^{i_2}\ldots\partial y^{i_\lambda})(x^a, \theta(x, y)y^b)y^{i_1}y^{i_2}\ldots y^{i_\lambda}; \quad 1 \le i_1, \ldots, i_\lambda \le n.$$

The following lemma is valid.

**Lemma 7.** *In the above situation, the following*

$$X = (X^i(x)) = (v^i(x, 0)),$$
$$Y = (Y^i(x)) = (v^{\bar{i}}(x, 0)),$$
$$K = (K^i_r(x)) = ((\partial_{\bar{r}} v^i)(x, 0)),$$
$$E = (E^i_{rs}(x)) = ((\partial_{\bar{r}}\partial_{\bar{s}} v^i)(x, 0)),$$
$$P = (P^i_r(x)) = ((\partial_{\bar{r}} v^{\bar{i}})(x, 0) - (\partial_r v^i)(x, 0))$$

*are tensor fields on M* [37].

For a Killing vector field $\widetilde{X} = v^iE_i + v^{\bar{i}}E_{\bar{i}}$ on $TM$, with the notations of Lemma 7, we can write:

$$v^i(x, y) = X^i + K^i_r y^r + \frac{1}{2}E^i_{rs}y^r y^s + \cdots + [*]^i_\lambda, \tag{4.6}$$

$$v^{\bar{i}}(x, y) = Y^i + \tilde{P}^i_r y^r + \frac{1}{2}Q^i_{rs}y^r y^s + \cdots + [*]^{\bar{i}}_\lambda, \tag{4.7}$$

where $\tilde{P}^i_r$ and $Q^i_{rs}$ are given by $\tilde{P}^i_r = (\partial_{\bar{r}} v^{\bar{i}})(x, 0)$ and $Q^i_{rs} = (\partial_{\bar{r}}\partial_{\bar{s}} v^{\bar{i}})(x, 0)$. Then we have

$$(\partial_{\bar{j}} v^i)(x, y) = K^i_j + E^i_{rj}y^r + \cdots + < * >^i_{\lambda-1}, \tag{4.8}$$

$$(\partial_{\bar{j}} v^{\bar{i}})(x, y) = \tilde{P}^i_j + Q^i_{rj}y^r + \cdots + \langle * \rangle^{\bar{i}}_{\lambda-1}, \ldots \text{etc.} \tag{4.9}$$

When we apply Taylor's theorem to the left hand sides of (4.1)–(4.3) to some order $\lambda$, the results are the same as one obtains by substituting (4.6)–(4.9), etc. into (4.1)–(4.3) up to order $\lambda - 1$. Furthermore the vanishing of the right hand sides of the Eqs. (4.1)–(4.3) implies the vanishing of each coefficient (up to order $\lambda - 1$).

Substituting (4.6) into (4.1) and taking the part which does not contain $y^r$, we have

$$X^a\partial_a g_{ij} + (\partial_i X^a)g_{aj} + (\partial_j X^a)g_{ia} = 0. \tag{4.10}$$

Hence, the vector field $X$ with the components $(X^i)$ is a Killing vector on $(M, g)$. Since, by Lemma 4, ${}^CX = X^aE_a + (y^m\nabla_m X^a)E_{\bar{a}}$ is a Killing vector on $(TM, g_{a,b})$, $\widetilde{X} - {}^CX$ is also a Killing vector. Therefore, in the following, denoting $\widetilde{X} - {}^CX$ by the same letter $\widetilde{X}$, one may assume that $X^i = 0$ in (4.6). Then $(\tilde{P}^r_i) = (P^r_i)$ is a tensor field on $M$ by Lemma 7.

Putting (4.6) and (4.7) into (4.2) (from now on, we omit this statement) and taking the part which does not contain $y^r$, we get:

$$g_{ai}K^a_j = -a(r^2)g_{aj}\nabla_i Y^a = -a(r^2)\nabla_i Y_j \tag{4.11}$$

which gives

$$K^i_j = -a(r^2)\nabla^i Y_j. \tag{4.12}$$

Taking the coefficient of $y^r$ in (4.1), we get

$$K^a_r\partial_a g_{ij} + g_{aj}(\partial_i K^a_r) - g_{aj}\Gamma^m_{ri}K^a_m + g_{ia}(\partial_j K^a_r) - g_{ia}\Gamma^m_{rj}K^a_m = 0.$$

Using the equality $\partial_a g_{ij} = \Gamma^m_{aj}g_{im} + \Gamma^m_{ia}g_{mj}$ and (4.11), we see that the last equation can be simplified to

$$\nabla_i\nabla_j Y_r + \nabla_j\nabla_i Y_r = 0. \tag{4.13}$$

Taking the part which does not contain $y^r$ in (4.3), we have

$$P^a_i g_{aj} + P^a_j g_{ia} = 0. \tag{4.14}$$

Taking the coefficient of $y^r$ in (4.2), we get

$$a(r^2)g_{aj}(\Gamma^a_{bi}P^b_r + \partial_i P^a_r - \Gamma^b_{ri}P^a_b) + g_{ai}E^a_{jr} = 0,$$
$$g_{ai}E^a_{jr} + a(r^2)g_{aj}\nabla_i P^a_r = 0.$$

By (4.14), the last equation is simplified to

$$g_{ai}E^a_{jr} + a(r^2)\nabla_i P_{rj} = 0,$$

where $P_{rj} = P^a_r g_{aj}$.

Since $E^a_{jr}$ is symmetric in $j$ and $r$, and $(\nabla_i P^a_r)g_{aj} = \nabla_i P_{rj}$ is skew-symmetric in $j$ and $r$ by (4.14), we have

$$E^a_{jr} = 0 \quad \text{and} \quad \nabla_i P^a_r = 0. \tag{4.15}$$

Taking the coefficient of $y^r$ in (4.3), we get

$$Q^a_{rj}g_{ia} + Q^a_{ri}g_{aj} + Y^a\left\{\frac{2a'(r^2)}{a(r^2)}g_{ar}g_{ij} + \frac{b(r^2)}{a(r^2)}(g_{ia}g_{rj} + g_{ir}g_{aj})\right\} = 0. \tag{4.16}$$

We put $Q^i_{rs} = \frac{a'(r^2)-b(r^2)}{a(r^2)}Y^i g_{rs} - \frac{a'(r^2)}{a(r^2)}(Y^k g_{kr}\delta^i_s + Y^k g_{ks}\delta^i_r) + T^i_{rs}$. By a simple calculation, using (4.16), we can verify that $g_{ia}T^a_{rj} + g_{aj}T^a_{ir} = 0$. If we put $T_{irj} = g_{aj}T^a_{ir}$, then $T_{irj}$ is symmetric in $i$ and $r$, and skew-symmetric $i$ and $j$. Hence $T_{irj} = 0$. That is

$$Q^i_{rs} = \frac{a'(r^2) - b(r^2)}{a(r^2)}Y^i g_{rs} - \frac{a'(r^2)}{a(r^2)}(Y^k g_{kr}\delta^i_s + Y^k g_{ks}\delta^i_r). \tag{4.17}$$

Finally, we consider the coefficient of $y^r y^s$ in (4.2), we get by virtue of (4.17)

$$\begin{aligned} &K^b_r R_{bisj} + K^b_s R_{rjbi} + (\nabla_i Y^a)\left(\frac{a'(r^2) - b(r^2)}{a(r^2)}g_{rs}g_{aj} + \frac{2b(r^2) - a'(r^2)}{a(r^2)}g_{ar}g_{sj} - \frac{a'(r^2)}{a(r^2)}g_{as}g_{rj}\right) \\ &\quad + 2g_{ai}(\partial_{\bar{s}}\partial_{\bar{r}}\partial_{\bar{j}}v^a)(x, 0) = 0. \end{aligned} \tag{4.18}$$

Taking the skew-symmetric part $s$ and $j$ of (4.18), we get

$$2K^b_r R_{bisj} + K^b_s R_{rjbi} - K^b_j R_{rsbi} = -(\nabla_i Y^a)\left(\frac{2a'(r^2) - b(r^2)}{a(r^2)}g_{rs}g_{aj} - \frac{2a'(r^2) - b(r^2)}{a(r^2)}g_{as}g_{rj}\right). \tag{4.19}$$

Taking the symmetric part $s$ and $r$ of (4.19), we get

$$K^b_s R_{rjbi} + K^b_r R_{bisj} = -\frac{1}{3}(\nabla_i Y^a)\left(\frac{4a'(r^2) - 2b(r^2)}{a(r^2)}g_{rs}g_{aj} - \frac{2a'(r^2) - b(r^2)}{a(r^2)}(g_{as}g_{rj} + g_{ar}g_{sj})\right). \tag{4.20}$$

Now, by (4.14) and (4.15), we see that $\gamma P$ is a Killing vector field on $(TM, g_{a,b})$ by Lemma 5. By (4.12), (4.13), (4.20) and Lemma 6, $Y_A$ is a Killing vector field on $(TM, g_{a,b})$.

Summing up we find that $\widetilde{X} \in \Im^1_0(TM)$ is a Killing vector field with respect to the Riemannian metric $g_{a,b}$ of Cheeger Gromoll type iff

$$\begin{aligned} \tilde{X} &= \{X^i - a(r^2)y^r\nabla^i Y_r\}E_i + \left\{y^s(\nabla_s X^i + P^i_s) + \frac{(\alpha-1)(a'(r^2) - b(r^2)) + 2a(r^2)}{2a(r^2)}Y^i - \frac{a'(r^2)}{a(r^2)}g_{ks}Y^k y^s y^i\right\}E_{\bar{i}} \\ &= {}^C X + \gamma P + Y_A \end{aligned}$$

for each local coordinate systems $(x^i)$, $i = 1, \ldots, n$ on $M$. This proves the assertion and the conditions (i), (ii-1), (ii-2), (iii-1), (iii-2) are direct consequences of (4.10), (4.12)–(4.15) and (4.20). □

Let $\widetilde{X}$ be a vector field on $TM$ with components $(v^h, v^{\bar{h}})$ with respect to the adapted frame $\{E_h, E_{\bar{h}}\}$. Then $\widetilde{X}$ is a fibre-preserving vector field on $TM$ if and only if $v^h$ depend only on the variables $(x^h)$. In the case, the vector field $\widetilde{X}$ in Theorem 5 reduces $\widetilde{Z} = {}^C X + \gamma P + {}^{V'}Y$, where ${}^{V'}Y = \{\frac{(\alpha-1)(a'(r^2)-b(r^2))+2a(r^2)}{2a(r^2)}Y^i - \frac{a'(r^2)}{a(r^2)}g_{ks}Y^k y^s y^i\}E_{\bar{i}}$. Note that ${}^{V'}Y$ is a vertical vector field on $TM$. In fact, $f \in \Im^0_0(M)$; ${}^{V'}Y\ ({}^V f) = 0$. Also, note that ${}^C X$ and $\gamma P$ are fibre-preserving vector fields on $TM$, respectively. By virtue of Theorem 5 and its proof, we have the following result.

**Theorem 6.** *Let $(TM, g_{a,b})$ be the tangent bundle with the Riemannian metric $g_{a,b}$ of a Riemannian manifold $(M, g)$. Let*

(i) *$X$ be a Killing vector field on $(M, g)$;*
(ii) *$P$ be a $(1, 1)$ tensor field on $M$ which satisfies the following*
  (ii-1) *parallel with respect to $g$, i.e. $\nabla_k P^i_j = 0$, and*
  (ii-2) *skew-symmetric with respect to $g$; $P^a_i g_{aj} + P^a_j g_{ia} = 0$,*
(iii) *$Y$ be a vector field on $M$ which is parallel with respect to $g$.*

*Then the fibre-preserving vector field on TM defined by*

$$(*)\ \widetilde{Z} = {}^{C}X + \gamma P + {}^{V'}Y$$

*is a fibre-preserving Killing vector field on* $(TM, g_{a,b})$.

*Conversely, every fibre-preserving Killing vector field on* $(TM, g_{a,b})$ *is of the form* $(*)$.

In Theorem 5, if we consider the manifold $(M, g)$ is compact and if necessary, orientable, then $Y$ satisfying (iii-1 and 2) is parallel [8,37]. Hence we have the following theorem.

**Theorem 7.** *In Theorem 5, if* $(M, g)$ *is compact, then* $(\sharp)$ *is* $\widetilde{X} = {}^{C}X + \gamma P + {}^{V'}Y$.

**Theorem 8.** *In Theorem 5, if* $(M, g)$ *is an Einstein with non-zero scalar curvature with the condition* $2a(r^2)\frac{S}{n} + (n-1)\frac{4a'(r^2)-2b(r^2)}{3a(r^2)} \neq 0$, *then Y satisfying* (iii − 1 *and* 2) *vanishes, in which case* $(\sharp)$ *is* $\widetilde{X} = {}^{C}X + \gamma P$.

**Proof.** Contracting (iii-2) with respect to $l$ and $j$, we obtain

$$a(r^2)(R_{rb}\nabla^b Y_s + R_{sb}\nabla^b Y_r) = -\frac{1}{3}\left(\frac{4a'(r^2)-2b(r^2)}{a(r^2)}(\nabla^b Y_b)g_{rs} - \frac{2a'(r^2)-b(r^2)}{a(r^2)}(\nabla_r Y_s + \nabla_s Y_r)\right)$$

where $R_{ij}$ denotes the Ricci curvature tensor of $(M, g)$. Since $(M, g)$ satisfies $R_{rb} = \frac{S}{n}g_{rb}$ for non-zero scalar curvature $S$, we have

$$a(r^2)\left(\frac{S}{n}g_{rb}\nabla^b Y_s + \frac{S}{n}g_{sb}\nabla^b Y_r\right) = -\frac{1}{3}\left(\frac{4a'(r^2)-2b(r^2)}{a(r^2)}(\nabla^b Y_b)g_{rs} - \frac{2a'(r^2)-b(r^2)}{a(r^2)}(\nabla_r Y_s + \nabla_s Y_r)\right),$$

from which

$$\left\{a(r^2)\frac{S}{n} - \frac{2a'(r^2)-b(r^2)}{3a(r^2)}\right\}(\nabla_r Y_s + \nabla_s Y_r) = -\frac{4a'(r^2)-2b(r^2)}{3a(r^2)}(\nabla^b Y_b)g_{rs}.$$

Multiplying both sides of the last equation by $g^{rs}$, and summing over $r$ and $s$, we get

$$\left\{2a(r^2)\frac{S}{n} + (n-1)\frac{4a'(r^2)-2b(r^2)}{3a(r^2)}\right\}\nabla^s Y_s = 0.$$

Since $2a(r^2)\frac{S}{n} + (n-1)\frac{4a'(r^2)-2b(r^2)}{3a(r^2)} \neq 0$, we have $\nabla^s Y_s = 0$, which means that $\nabla_r Y^r = 0$.

Now, by virtue of (iii-1) and the Ricci identity

$$\nabla_i\nabla_j Y^r - \nabla_j\nabla_i Y^r = R_{ijs}{}^{r}Y^s,$$

we get

$$\nabla_i\nabla_j Y^r = -\frac{1}{2}R_{jis}{}^{r}Y^s.$$

Contracting with respect to $r$ and $j$, and applying $\nabla_r Y^r = 0$, we have $R_{is}Y^s = 0$, that is $\frac{S}{n}g_{is}Y^s = \frac{S}{n}Y_s = 0$. Hence $Y = 0$, since $S \neq 0$. □

We at last come to the following result.

**Theorem 9.** *Let* $(TM, g_{a,b})$ *be the tangent bundle with the Riemannian metric* $g_{a,b}$ *of Cheeger Gromoll type of a Riemannian manifold* $(M, g)$. *If* $(M, g)$ *is compact and orientable, and if the first and second Betti numbers vanish, then the Lie algebra of Killing vectors on* $(M, g)$ *and the Lie algebra of Killing vectors on* $(TM, g_{a,b})$ *are isomorphic, via the correspondence* $X \to {}^{C}X$.

**Proof.** $M$ being compact, $Y$ satisfying (iii-1 and 2) is parallel. Applying Hodge's theorem [38], by virtue of $b_1(M) = 0$, we have $Y = 0$. Furthermore, if the (1, 1)-tensor field $P$ is parallel then, by $b_2(M) = 0$ and Hodge's theorem, we have $P = 0$. Hence every Killing vector field on $(TM, g_{a,b})$ is of the form ${}^{C}X$ for some Killing vector field $X$ on $(M, g)$. On the other hand, for any vector fields $X$ and $Y$ on $M$, it is known that ${}^{C}[X, Y] = [{}^{C}X, {}^{C}Y]$. This proves the theorem. □

If a Riemannian manifold $M$ is isometrically immersed in the Euclidean $E_{n+r}$, then there exist on $M$, $r$ symmetric tensors $b_{ij}^{(\rho)}$, $\rho = 1, 2, \ldots, r$, such that the curvature tensor on $M$ has the represantation

$$R_{ijkl} = \sum_{\rho=1}^{r}(b_{jk}^{(\rho)}b_{il}^{(\rho)} - b_{jl}^{(\rho)}b_{ik}^{(\rho)}).$$

Now, let us consider that for the Riemannian manifold $M$, the curvature tensor has such a representation in the neighborhood of every point, with tensors $b_{ij}^{(\rho)}$ defined in each neighborhood only. The Riemannian manifold $M$ is called intrinsically semi-convex if all $b_{ij}^{(\rho)}$ are positive semi-definite, and the Riemannian manifold is called intrinsically convex if at least one tensor is positive definite. In [39, p. 172], it is proven that if a compact orientable Riemannian manifold $M$ is intrinsically convex, then the first two Betti numbers are zero, i.e. $b_1(M) = b_2(M) = 0$. Using the result, we obtain, as a corollary to Theorem 9, the following conclusion.

**Corollary 2.** *Let the Riemannian manifold $(M, g)$ be compact, orientable and intrinsically convex, and TM its tangent bundle with the Riemannian metric $g_{a,b}$. Then the Lie algebra of Killing vectors on $(M, g)$ and the Lie algebra of Killing vectors on $(TM, g_{a,b})$ are isomorphic, via the correspondence $X \to {}^{C}X$.*

## 5. Geodesics in the tangent bundle $(TM, g_{a,b})$

An important geometric problem is to find the geodesics on the smooth manifolds with respect to the Riemannian metrics (see [21,40–44]). In [21], Yano and Ishihara proved that the curves on the tangent bundles of Riemannian manifolds are geodesics with respect to certain lifts of the metric from the base manifold, if and only if the curves are obtained as certain types of lifts of the geodesics from the base manifold. In [44], Salimov and his collaborators studied the analogous problem for the tensor bundles.

Let $\widetilde{C} : [0, 1] \to TM$ be a curve on $TM$ and suppose that $\widetilde{C}$ is expressed locally by $x^A = x^A(t)$, i.e., $x^h = x^h(t)$, $x^{\bar{h}} = x^{\bar{h}}(t) = y^h(t)$ with respect to induced coordinates $(x^h, x^{\bar{h}})$ in $\pi^{-1}(U) \subset TM$, $t$ being a parameter. Then the curve $C = \pi \circ \widetilde{C}$ on $M$ is called the projection of the curve $\widetilde{C}$ and denoted by $\pi\widetilde{C}$, which is expressed locally by $x^h = x^h(t)$. Let $X^h(t)$ be a vector field along $C$. Then, on $TM$ we define a curve $\widetilde{C}$ by

$$\begin{cases} x^h = x^h(t) \\ x^{\bar{h}} = X^h(t). \end{cases} \tag{5.1}$$

If the curve (5.1) satisfies at all points the relation

$$\frac{\delta X^h}{dt} = \frac{dX^h}{dt} + \Gamma_{ji}^h \frac{dx^j}{dt} X^i = 0,$$

then the curve $\widetilde{C}$ is said to be a horizontal lift of the curve $C$ and denoted by ${}^{H}C$ [21, p. 172]. If $X^h$ is the tangent vector field $\frac{dx^h}{dt}$ to $C$, then the curve $\widetilde{C}$ defined by (5.1) is called the natural lift of the curve $C$ and denoted by $C^*$.

We write $\nabla^{a,b}_{E_\alpha} E_\beta = {}^{a,b}\Gamma_{\alpha\beta}^{\gamma} E_\gamma$ with respect to the adapted frame $\{E_\lambda\}$ of $TM$, where ${}^{a,b}\Gamma_{\alpha\beta}^{\gamma}$ denote the Christoffel symbols constructed by $g_{a,b}$. The particular values of ${}^{a,b}\Gamma_{\alpha\beta}^{\gamma}$ for different indices, on taking account of (3.8), are found to be

$$\begin{cases} {}^{a,b}\Gamma_{ji}^{h} = \Gamma_{ji}^{h}, \qquad {}^{a,b}\Gamma_{ji}^{\bar{h}} = -\dfrac{1}{2} y^k R_{jik}{}^{h} \\ {}^{a,b}\Gamma_{\bar{j}i}^{h} = \dfrac{a}{2} y^k R_{kij}{}^{h}, \qquad {}^{a,b}\Gamma_{\bar{j}i}^{\bar{h}} = \Gamma_{ji}^{h} \\ {}^{a,b}\Gamma_{j\bar{i}}^{h} = \dfrac{a}{2} R_{kji}{}^{h}, \qquad {}^{a,b}\Gamma_{j\bar{i}}^{\bar{h}} = 0 \\ {}^{a,b}\Gamma_{\bar{j}\bar{i}}^{h} = 0 \\ {}^{a,b}\Gamma_{\bar{j}\bar{i}}^{\bar{h}} = L(y_j\delta_i^h + y_i\delta_j^h) + Rg_{ji}y^h + Ny_jy_iy^h \end{cases} \tag{5.2}$$

with respect to the adapted frame, where $y_j = g_{ji}y^i$ and $L = \frac{a'(r^2)}{2a(r^2)}$, $R = \frac{2b(r^2)-a'(r^2)}{2(a(r^2)+(\alpha-1)b(r^2))}$, $N = \frac{a(r^2)b'(r^2)-2a'(r^2)b(r^2)}{2a(r^2)(a(r^2)+(\alpha-1)b(r^2))}$.

The geodesics of the connection ${}^{a,b}\nabla$ is given by the differential equations

$$\frac{\delta^2 x^A}{dt^2} = \frac{d^2 x^A}{dt^2} + {}^{a,b}\Gamma_{CB}^{A} \frac{dx^C}{dt}\frac{dx^B}{dt} = 0, \tag{5.3}$$

with respect to the induced coordinates $(x^h, x^{\bar{h}})$, where $t$ is the arc length of a curve on $TM$.

We write down the form equivalent to (5.3), namely,

$$\frac{d}{dt}\left(\frac{\theta^\alpha}{dt}\right) + {}^{a,b}\Gamma_{\gamma\beta}^{\alpha} \frac{\theta^\gamma}{dt}\frac{\theta^\beta}{dt} = 0$$

with respect to adapted frame $\{E_\lambda\}$, where

$$\frac{\theta^h}{dt} = \frac{dx^h}{dt},$$

$$\frac{\theta^{\bar{h}}}{dt} = \frac{\delta y^h}{dt} = \frac{dy^h}{dt} + \Gamma_{ji}^h \frac{dx^j}{dt} y^i$$

along a curve $x^A = x^A(t)$ on $TM$. Taking account of (5.2), then we have

$$\begin{cases} \text{(a)}\ \dfrac{\delta^2 x^h}{dt^2} + \dfrac{a}{2} y^k R_{kji}^h \dfrac{\delta y^j}{dt}\dfrac{dx^i}{dt} = 0, \\ \text{(b)}\ \dfrac{\delta^2 y^h}{dt^2} + [L(y_j\delta_i^h + y_i\delta_j^h) + Rg_{ji}y^h + Ny_jy_iy^h]\dfrac{\delta y^j}{dt}\dfrac{\delta y^i}{dt} = 0. \end{cases} \tag{5.4}$$

Thus we have the following result.

**Theorem 10.** *Let $\tilde{C}$ be a curve on TM and locally expressed by $x^h = x^h(t)$, $x^{\bar{h}} = y^h(t)$ with respect to the induced coordinates $(x^h, x^{\bar{h}})$ in $\pi^{-1}(U) \subset TM$. The curve $\tilde{C}$ is a geodesic of $g_{a,b}$, if it satisfies Eqs. (5.4).*

If a curve $\widetilde{C}$ satisfying (5.4) lies on a fibre given by $x^h = \text{const}$, then by virtue of $\frac{dx^h}{dt} = 0$ and $\frac{\delta y^h}{dt} = \frac{dy^h}{dt} + \Gamma_{ij}^h \frac{dx^i}{dt} y^j = \frac{dy^h}{dt}$, Eqs. (5.4) reduce to

$$\frac{d^2y^h}{dt^2} + [L(y_j\delta_i^h + y_i\delta_j^h) + Rg_{ji}y^h + Ny_jy_iy^h]\frac{dy^j}{dt}\frac{dy^i}{dt} = 0. \tag{5.5}$$

Hence we have the result as follows.

**Theorem 11.** *If a geodesic lies on a fibre of TM with metric $g_{a,b}$, the geodesic is expressed by Eq. (5.5).*

Let $C = \pi \circ {}^H C$ be a geodesic of $\nabla$ on $M$. Then $\frac{\delta^2 x^h}{dt^2} = 0$. Using the condition $\frac{\delta^2 x^h}{dt^2} = 0$ and the condition $\frac{\delta y^j}{dt} = \frac{\delta X^h}{dt} = 0$, we have the theorem below.

**Theorem 12.** *The horizontal lift of a geodesic on M is always geodesic on TM with the metric $g_{a,b}$.*

Let now $C = \pi \circ C^*$ be a geodesic of $\nabla$ on $M$, i.e. $\frac{\delta^2 x^h}{dt^2} = \frac{\delta}{dt}\left(\frac{dx^h}{dt}\right) = 0$. On the other hand, from definition of the natural lift of the curve, we obtain

$$\frac{\delta y^h}{dt} = \frac{\delta}{dt}\left(\frac{dx^h}{dt}\right) = 0. \tag{5.6}$$

By virtue of (5.4) and (5.6) we easily see that the natural lift of a curve on $M$ defined $x^h = x^h(t)$ is geodesic on $TM$ with the metric $g_{a,b}$. Thus we have the last theorem.

**Theorem 13.** *The natural lift $C^*$ of any geodesic on M is a geodesic on TM with the metric $g_{a,b}$.*

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## References

[1] S. Sasaki, On the differential geometry of tangent bundles of Riemannian manifolds, Tohoku Math. J. 10 (1958) 338–358.
[2] O. Kowalski, Curvature of the induced Riemannian metric on the tangent bundle of a Riemannian manifold, J. Reine Angew. Math. 250 (1971) 124–129.
[3] E. Musso, F. Tricerri, Riemannian metrics on tangent bundles, Ann. Mat. Pura Appl. 150 (4) (1988) 1–19.
[4] J. Cheeger, D. Gromoll, On the structure of complete manifolds of nonnegative curvature, Ann. of Math. 96 (1972) 413–443.
[5] M. Sekizawa, Curvatures of tangent bundles with Cheeger–Gromoll metric, Tokyo J. Math. 14 (1991) 407–417.
[6] S. Gudmundsson, E. Kappos, On the geometry of the tangent bundle with the Cheeger–Gromoll metric, Tokyo J. Math. 25 (1) (2002) 75–83.
[7] S. Gudmundsson, E. Kappos, On the geometry of the tangent bundles, Expo. Math. 20 (2002) 1–41.
[8] M.T.K. Abbassi, M. Sarih, Killing vector fields on tangent bundles with Cheeger–Gromoll metric, Tsukuba J. Math. 27 (2003) 295–306.
[9] A.A. Salimov, A. Gezer, M. Iscan, Para-K ähler-Norden structures on the tangent bundles, Ann. Polon. Math. 103 (2012) 247–261.
[10] M. Anastasiei, Locally conformal Kaehler structures on tangent bundle of a sape form, Libertas Math. 19 (1999) 71–76.
[11] M.I. Munteanu, Some aspects on the geometry of the tangent bundles and tangent sphere bundles of a Riemannian manifold, Mediterr. J. Math. 5 (1) (2008) 43–59.
[12] V. Oproiu, Some new geometric structures on the tangent bundle, Publ. Math. Debrecen 55 (1999) 261–281.
[13] V. Oproiu, A locally symmetric Kaehler Einstein structure on the tangent bundle of a space form, Beiträge Algebra Geom. 40 (2) (1999) 363–372.
[14] V. Oproiu, A Kähler Einstein structure on the tangent bundle of a space form, Int. J. Math. Math. Sci. 25 (3) (2001) 183–195.
[15] V. Oproiu, N. Papaghiuc, Some classes of almost anti-Hermitian structures on the tangent bundle, Mediterr. J. Math. 1 (3) (2004) 269–282.
[16] O. Kowalski, M. Sekizawa, Natural transformation of Riemannian metrics on manifolds to metrics on tangent bundles—a classification, Bull. Tokyo Gakugei Univ. 40 (4) (1988) 1–29.
[17] M.T.K. Abbassi, Note on the classification theorems of $g$-natural metrics on the tangent bundle of a Riemannian manifold $(M, g)$, Comment. Math. Univ. Carolin. 45 (4) (2004) 591–596.
[18] M.T.K. Abbassi, M. Sarih, On some hereditary properties of Riemannian $g$-natural metrics on tangent bundles of Riemannian manifolds, Differential Geom. Appl. 22 (1) (2005) 19–47.
[19] M.T.K. Abbassi, M. Sarih, On natural metrics on tangent bundles of Riemannian manifolds, Arch. Math. 41 (2005) 71–92.

[20] I. Kolar, P.W. Michor, J. Slovak, Natural Operations in Differential Geometry, Springer-Verlang, Berlin, 1993.
[21] K. Yano, S. Ishihara, Tangent and Cotangent Bundles, Marcel Dekker, Inc, New York, 1973.
[22] P. Dombrowski, On the geometry of the tangent bundles, J. Reine Angew. Math. 210 (1962) 73–88.
[23] V. Cruceanu, P. Fortuny, P.M. Gadea, A survey on paracomplex geometry, Rocky Mountain J. Math. 26 (1995) 83–115.
[24] V.V. Vishnevskii, A.P. Shirokov, V.V. Shurygin, Spaces over algebras, Kazan Gos. University, Kazan, 1985 (in Russian).
[25] G.I. Kruchkovich, Hypercomplex structure on manifold,I, Tr. Sem. Vect. Tens. Anal, Moscow Univ. 16 (1972) 174–201.
[26] M. Iscan, A.A. Salimov, On Kä hler–Norden manifolds, Proc. Indian Acad. Sci. Math. Sci. 119 (1) (2009) 71–80.
[27] A.A. Salimov, Generalized Yano–Ako operator and the complete lift of tensor fields, Tensor (NS) 55 (2) (1994) 142–146.
[28] A.A. Salimov, M. Iscan, F. Etayo, Paraholomorphic B-manifold and its properties, Topology Appl. 154 (4) (2007) 925–933.
[29] A.A. Salimov, Nonexistence of para-Kä hler–Norden warped metrics, Int. J. Geom. Methods Mod. Phys. 6 (7) (2009) 1097–1102.
[30] A.A. Salimov, A. Gezer, S. Aslanci, On almost complex structures in the cotangent bundle, Turkish J. Math. 35 (2011) 487–492.
[31] V.V. Vishnevskii, Integrable affinor structures and their plural interpretations, J. Math. Sci. 108 (2) (2002) 151–187.
[32] K. Yano, M. Ako, On certain operators associated with tensor field, Kodai Math. Sem. Rep. 20 (1968) 414–436.
[33] S. Tachibana, Analytic tensor and its generalization, Tohoku Math. J. 12 (2) (1960) 208–221.
[34] M.I. Munteanu, Cheeger Gromoll type metrics on the tangent bundle, Sci. Ann. Univ. Agric. Sci. Vet. Med. 49 (2) (2006) 257–268.
[35] K. Yamauchi, On infinitesimal conformal transformations of the tangent bundles over Riemannian manifolds, Ann Rep. Asahikawa. Med. Coll. 15 (1994) 1–10.
[36] I. Hasegawa, K. Yamauchi, Infinitesimal conformal transformations on tangent bundles with the lift metric I+II, Sci. Math. Jpn. 57 (1) (2003) 129–137.
[37] S. Tanno, Killing vectors and geodesic flow vectors on tangent bundles, J. Reine Angew. Math. 282 (1976) 162–171.
[38] T.J. Willmore, An Introduction to Differential Geometry, Oxford Univ. Press, 1959.
[39] K. Yano, S. Bochner, Curvature and Betti Numbers, Princeton University Press, New Jersey, 1953.
[40] R. Caddeo, A. Gray, Curve e Superfici, Ed. CUEC 2000.
[41] Z. Dusek, Structure of geodesics in a 13-dimensional group of Heisenberg type, in: Proc. Colloquium on Differential Geometry, Debrecen, 2001, pp. 95–103.
[42] Z. Dusek, O. Kowalski, Z. Vlasek, Homogeneous geodesics in homogeneous affine manifolds, Results Math. 54 (3–4) (2009) 273–288.
[43] P.T. Nagy, Geodesics on the tangent sphere bundle of a Riemannian manifold, Geom. Dedicata 7 (1978) 233–243.
[44] A.A. Salimov, A. Gezer, K. Akbulut, Geodesics of Sasakian metrics on tensor bundles, Mediterr. J. Math. 6 (2) (2009) 135–147.