J. Math. Anal. Appl. 398 (2013) 221–229

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Compact composition operators on noncompact Lipschitz spaces

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ARTICLE INFO

Article history:
Received 8 August 2012
Available online 4 September 2012
Submitted by K. Jarosz

Keywords:
Composition operator
Compact operator
Lipschitz function
Spectrum of an operator

ABSTRACT

We characterize compact composition operators between different spaces of scalar-valued Lipschitz functions defined on metric spaces, not necessarily compact, and determine their spectra.

1. Introduction and statement of the results

Kamowitz and Scheinberg [1] studied compact composition operators on $\mathrm{Lip}(X, d)$, the Banach space of all bounded Lipschitz functions f from a metric space (X, d) into the field of real or complex numbers $\mathbb{K}$, endowed with the norm

$$\|f\|_d = \max\{\|f\|_\infty, L_d(f)\},$$

where $\|f\|_\infty$ is the supremum norm of f and

$$L_d(f) = \sup\left\{\frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X,\ x \neq y\right\}.$$

They proved that if (X, d) is a compact metric space and $\phi: X \to X$ is a Lipschitz mapping, then the composition operator $C_\phi: \mathrm{Lip}(X, d) \to \mathrm{Lip}(X, d)$, defined by $C_\phi(f) = f \circ \phi$, is compact if and only if ϕ is supercontractive. This means that for each $\varepsilon > 0$, there exists $\delta > 0$ such that $d(\phi(x), \phi(y))/d(x, y) < \varepsilon$ whenever $0 < d(x, y) < \delta$. Really, they stated this result for the space of complex-valued Lipschitz functions f on X with the norm $\|f\|_\infty + L_d(f)$; but the same proof works also in the real-valued case and with the norm $\|f\|_d$.

Our first aim in this paper is to give a more complete characterization of compact composition operators on $\mathrm{Lip}(X, d)$ without assuming compactness on X. Namely we prove the following result.

Theorem 1.1. *Let (X, d) be a metric space and let $\phi: X \to X$ be a Lipschitz mapping. Then the operator $C_\phi: \mathrm{Lip}(X, d) \to \mathrm{Lip}(X, d)$ is compact if and only if ϕ is supercontractive and $\phi(X)$ is totally bounded in X.*

The purpose of removing the compactness hypothesis on the metric space (X, d) is not new when one studies a linear preserver problem in the context of Lipschitz spaces. We can cite in this direction the papers by Araujo and Dubarbie [2]

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2012.08.044

and Weaver [3] in their studies of the representation of surjective linear isometries of $\mathrm{Lip}(X, d)$ by means of weighted composition operators, and those of Araujo and Dubarbie [4] and Leung [5] on biseparating maps of Lipschitz spaces.

Our approach lies in tackling the problem for pointed Lipschitz spaces Lip_0 that generalize the Lipschitz spaces Lip.

Let (X, d) be a pointed metric space, that is, a metric space with a base point $e \in X$. Then $\mathrm{Lip}_0(X, d)$ is the Banach space of all Lipschitz functions $f: X \to \mathbb{K}$ such that $f(e) = 0$ with the norm $L_d(f)$. If X is bounded, $\mathrm{diam}(X)$ denotes the diameter of X.

Our main result is the following characterization of compact composition operators on Lip_0 spaces.

Theorem 1.2. *Let (X, d) be a bounded pointed metric space and let $\phi: X \to X$ be a base point-preserving Lipschitz mapping. Then the operator $C_\phi: \mathrm{Lip}_0(X, d) \to \mathrm{Lip}_0(X, d)$ is compact if and only if ϕ is supercontractive and $\phi(X)$ is totally bounded in X.*

Moreover, we obtain the analogous result for compact composition operators on little Lipschitz spaces $\mathrm{lip}(X, d)$ that satisfy a kind of uniform point separation property.

Given a metric space (X, d), $\mathrm{lip}(X, d)$ is the closed subspace of $\mathrm{Lip}(X, d)$ formed by all those functions f in $\mathrm{Lip}(X, d)$ which are supercontractive, that is,

$$\lim_{t\to 0} \sup_{0<d(x,y)<t} \frac{|f(x) - f(y)|}{d(x, y)} = 0.$$

We refer the reader to the book [6] for a complete study on all these spaces of Lipschitz functions. Following Weaver [6, Definition 3.2.1], we introduce the following property.

Definition 1.1. Let (X, d) be a metric space, not assumed to be compact. It is said that a linear subspace $\mathcal{M}$ of $\mathrm{Lip}(X, d)$ separates points uniformly on bounded subsets of X if for each bounded set $K \subset X$, there exists a constant $a \geq 1$ (which may depend on K) such that for every $x, y \in K$, some $f \in \mathcal{M}$ satisfies $\|f\|_d \leq a$ and $|f(x) - f(y)| = d(x, y)$.

Note that $\mathrm{Lip}(X, d)$ separates points uniformly on bounded subsets $K \subset X$ with $a = \max\{1, \mathrm{diam}(K)\}$ by taking, for each $x \in K$, the function $f(z) = \min\{\mathrm{diam}(K), d(x, z)\}$ defined on X. On the other hand, if X is a connected and complete Riemannian manifold, then $\mathrm{lip}(X, d)$ does not separate points because it contains only constant functions [6, Example 3.1.5]. However, $\mathrm{lip}(X, d)$ satisfies the aforementioned uniform separation property when X is uniformly discrete (that is, $\inf\{d(x, y): x \neq y\} > 0$) since $\mathrm{lip}(X, d) = \mathrm{Lip}(X, d)$, or when X is a totally disconnected compact metric space [6, Example 3.1.6].

Moreover, replacing the metric d by a metric d^α where $0 < \alpha < 1$, the spaces $\mathrm{lip}(X, d^\alpha)$ have the uniform separation property on bounded subsets $K \subset X$ with $a = 2^{(1-\alpha)} \max\{1, \mathrm{diam}(K)\}$. It suffices to notice that $\mathrm{Lip}(X, d) \subset \mathrm{lip}(X, d^\alpha)$ and $\|f\|_{d^\alpha} \leq 2^{(1-\alpha)} \|f\|_d$ for all $f \in \mathrm{Lip}(X, d)$.

Theorem 1.3. *Let (X, d) be a metric space and $\phi: X \to X$ a bounded Lipschitz mapping. Assume that $\mathrm{lip}(X, d)$ separates points uniformly on bounded subsets of X. Then the operator $C_\phi: \mathrm{lip}(X, d) \to \mathrm{lip}(X, d)$ is compact if and only if ϕ is supercontractive and $\phi(X)$ is totally bounded in X.*

We must point out that Theorem 1.3 extends also the result obtained by Kamowitz and Scheinberg in [1] for $\mathrm{lip}(X, d^\alpha)$ with X compact and $0 < \alpha < 1$. Furthermore, Theorems 1.1–1.3 hold for composition operators C_ϕ from $\mathrm{Lip}(X, d_X)$ ($\mathrm{Lip}_0(X, d_X)$, $\mathrm{lip}(X, d_X)$) into $\mathrm{Lip}(Y, d_Y)$ (respectively, $\mathrm{Lip}_0(Y, d_Y)$, $\mathrm{lip}(Y, d_Y)$).

Our second aim is to study the spectrum of compact composition operators on Lipschitz spaces $\mathrm{Lip}(X, d)$ and $\mathrm{lip}(X, d)$. In [1, Theorem 2], Kamowitz and Scheinberg claim that if (X, d) is a compact metric space and $C_\phi: \mathrm{Lip}(X, d) \to \mathrm{Lip}(X, d)$ is a nonzero compact composition operator, then the spectrum $\sigma(C_\phi)$ of C_ϕ has only two points, 0 and 1. However, when X is not connected, this need not be true as the following modification of an example due to Kamowitz and Scheinberg [1] shows.

Example 1.1. Take the sets $Z = [-1, -1/2] \cup [1/2, 1]$ and $Y = [-1/2, -1/4] \cup [1/4, 1/2]$ endowed, respectively, with the metrics

$$d_Z(x, y) = |x - y|, \quad \forall x, y \in Z; \qquad d_Y(x, y) = \sqrt{|x - y|}, \quad \forall x, y \in Y.$$

Let $X = Y \cup Z$ and let $d: X \times X \to \mathbb{R}$ be the distance on X given by

$$d(x, y) = \begin{cases} d_Z(x, y) & \text{if } x, y \in Z; \\ d_Y(x, y) & \text{if } x, y \in Y; \\ d_Z(x, -1/2) + d_Y(-1/2, y) & \text{if } x \in [-1, -1/2],\ y \in Y; \\ d_Z(y, -1/2) + d_Y(-1/2, x) & \text{if } y \in [-1, -1/2],\ x \in Y; \\ d_Z(x, 1/2) + d_Y(1/2, y) & \text{if } x \in [1/2, 1],\ y \in Y; \\ d_Z(y, 1/2) + d_Y(1/2, x) & \text{if } y \in [1/2, 1],\ x \in Y. \end{cases}$$

Notice that X is compact since the topology generated by d is the usual topology of X. Consider now the map $\phi: X \to X$ defined by

$$\phi(x) = \begin{cases} -2x & \text{if } x \in Y, \\ 1 & \text{if } x \in [-1, -1/2], \\ -1 & \text{if } x \in [1/2, 1]. \end{cases}$$

It is not hard to check that ϕ is Lipschitz and supercontractive, and thus $C_\phi: \mathrm{Lip}(X, d) \to \mathrm{Lip}(X, d)$ is compact by Theorem 1.1. However, -1 is in the spectrum of C_ϕ since the function $f: X \to \mathbb{R}$ given by

$$f(x) = \begin{cases} -1 & \text{if } x \in [-1, -1/4], \\ 1 & \text{if } x \in [1/4, 1], \end{cases}$$

belongs to $\mathrm{Lip}(X, d)$ and $f \circ \phi = -f$.

Let us recall that if X is a set, $n \in \mathbb{N}$ and $\phi: X \to X$, then a point $x_0 \in X$ is called a fixed point of ϕ of order n if $\phi^n(x_0) = x_0$ and $\phi^k(x_0) \neq x_0$ for all $k = 1, \ldots, n-1$. Using these points, we correct the flaw in [1, Theorem 2] as follows.

Theorem 1.4. *Let (X, d) be a metric space, $\phi: X \to X$ a Lipschitz mapping, $\widetilde{\phi}$ its extension to the completion $\widetilde{X}$ of X and A the set of natural numbers n such that $\widetilde{\phi}$ has a fixed point of order n.*

(i) *If $C_\phi: \mathrm{Lip}(X, d) \to \mathrm{Lip}(X, d)$ is a compact operator, then A is finite and*

$$\sigma(C_\phi) \setminus \{0\} = \bigcup_{n \in A} \{\lambda \in \mathbb{K}: \lambda^n = 1\}.$$

(ii) *Assume further that ϕ is bounded and $\mathrm{lip}(X, d)$ separates points uniformly on bounded subsets of X. If $C_\phi: \mathrm{lip}(X, d) \to \mathrm{lip}(X, d)$ is a compact operator, then A is finite and*

$$\sigma(C_\phi) \setminus \{0\} = \bigcup_{n \in A} \{\lambda \in \mathbb{K}: \lambda^n = 1\}.$$

In both cases, if X is in addition infinite and connected, then $\sigma(C_\phi) = \{0, 1\}$.

The r-connectedness of a metric space provides a very useful tool for analysing the linear isometries of spaces $\mathrm{Lip}(X, d)$ and $\mathrm{lip}(X, d)$ (see [6]). We use this type of connectedness in the proof of Theorem 1.4. The argument given in the proof of Theorem 2 in [1] fails because, unlike the r-connected components, the number of connected components of a compact metric space need not be finite.

2. Proofs

Before going into the proof of Theorem 1.2, we deduce Theorem 1.1 from Theorem 1.2. To do this, we recall that every Lip space is isometrically isomorphic to a certain Lip_0 space.

Lemma 2.1. *(See Proposition 1.7.1 and Theorem 1.7.2 in [6]). Let (X, d) be a metric space, $e \notin X$ and $X_0 = X \cup \{e\}$.*

(i) *The mapping $d_0: X_0 \times X_0 \to \mathbb{R}$ given by*

$$d_0(x, y) = \min\{d(x, y), 2\}, \qquad d_0(x, e) = d_0(e, y) = 1 \qquad (x, y \in X), \qquad d_0(e, e) = 0,$$

is a distance on X_0.

(ii) *The mapping $\Phi: \mathrm{Lip}(X, d) \to \mathrm{Lip}_0(X_0, d_0)$ defined by*

$$\Phi(f)(x) = f(x) \quad (x \in X), \qquad \Phi(f)(e) = 0,$$

is an isometric isomorphism.

We also will need the next easy result.

Lemma 2.2. *Let (X, d) be a metric space and let $e \notin X$. Let (X_0, d_0) be the metric space given above. For each mapping $\phi: X \to X$, define $\phi_0: X_0 \to X_0$ by*

$$\phi_0(x) = \phi(x) \quad (x \in X), \qquad \phi_0(e) = e.$$

Then the following hold.

(i) *If ϕ is Lipschitz, then ϕ_0 is Lipschitz.*
(ii) *ϕ is supercontractive if and only if ϕ_0 is supercontractive.*
(iii) *$\phi(X)$ is totally bounded in X if and only if $\phi_0(X_0)$ is totally bounded in X_0.*

Let us recall that a linear operator between Banach spaces $T: E \to F$ is compact if T takes bounded sets in E into relatively compact sets in F. If E, F and G are Banach spaces and $S: E \to F$ and $T: F \to G$ are bounded linear operators, then TS is compact if either S or T is compact.

Proof of Theorem 1.1. We have a metric space (X, d) and a Lipschitz mapping $\phi: X \to X$. Let us choose a point $e \notin X$ and define X_0, d_0, Φ and ϕ_0 as in Lemmas 2.1 and 2.2. Clearly, (X_0, d_0) is a bounded pointed metric space and $\phi_0: X_0 \to X_0$ a base point-preserving Lipschitz mapping. Hence we can apply Theorem 1.2 to the operator $C_{\phi_0}: \mathrm{Lip}_0(X_0, d_0) \to \mathrm{Lip}_0(X_0, d_0)$.

On the one hand, an easy verification yields $\Phi C_\phi = C_{\phi_0}\Phi$ and then it is deduced readily that $C_\phi: \mathrm{Lip}(X, d) \to \mathrm{Lip}(X, d)$ is compact if and only if $C_{\phi_0}: \mathrm{Lip}_0(X_0, d_0) \to \mathrm{Lip}_0(X_0, d_0)$ is compact. On the other hand, $C_{\phi_0}: \mathrm{Lip}_0(X_0, d_0) \to \mathrm{Lip}_0(X_0, d_0)$ is compact if and only if ϕ_0 is supercontractive and $\phi(X_0)$ is totally bounded in X_0 by Theorem 1.2. This is equivalent to that ϕ is supercontractive and $\phi(X)$ is totally bounded in X by Lemma 2.2, and the proof is complete. □

Remark 2.1. Observe that if $\phi: X \to X$ is a Lipschitz mapping, then $\phi_0: X_0 \to X_0$ is the unique base point-preserving Lipschitz mapping making the equality $\Phi C_\phi = C_{\phi_0}\Phi$ holds.

In the proof of Theorem 1.2, we use the following characterization of the compactness of the composition operators on $\mathrm{Lip}_0(X, d)$.

Proposition 2.3. *Let (X, d) be a pointed metric space and let $\phi: X \to X$ be a base point-preserving Lipschitz mapping. Then the operator $C_\phi: \mathrm{Lip}_0(X, d) \to \mathrm{Lip}_0(X, d)$ is compact if and only if for each bounded sequence $\{f_n\}$ in $\mathrm{Lip}_0(X, d)$ which converges to zero uniformly on totally bounded subsets of X, there exists a subsequence $\{f_{n_k}\}$ such that $L_d(f_{n_k} \circ \phi) \to 0$ as $k \to \infty$.*

For the proof we need two lemmas.

Lemma 2.4. *Let (X, d) be a pointed metric space and let $\{f_n\}$ be a sequence in $\mathrm{Lip}_0(X, d)$ that converges to a function f in $\mathrm{Lip}_0(X, d)$. Then $\{f_n\}$ converges to f pointwise on X.*

Proof. Let $x \in X$. If $x = e$, we have $f_n(x) = 0$ for all n and so $\lim_{n\to\infty} f_n(x) = 0 = f(x)$. Assume now $x \neq e$ and let $\varepsilon > 0$ be given. Then there exists a $m \in \mathbb{N}$ such that $L_d(f_n - f) < \varepsilon/d(x, e)$ whenever $n \geq m$. It follows that $|f_n(x) - f(x)| \leq L_d(f_n - f)d(x, e) < \varepsilon$ if $n \geq m$, and thus $\{f_n(x)\}$ converges to $f(x)$. □

Lemma 2.5. *Let (X, d) be a pointed metric space. Then every bounded sequence $\{f_n\}$ in $\mathrm{Lip}_0(X, d)$ has a subsequence that converges pointwise on X to a function $f \in \mathrm{Lip}_0(X, d)$. Moreover, this convergence is uniform on each totally bounded subset of X.*

Proof. According to [6, Theorem 2.2.2], $\mathrm{Lip}_0(X, d)$ is isometrically isomorphic to a dual space. Then, by the Banach–Alaoglu Theorem, there exist a function $f \in \mathrm{Lip}_0(X, d)$ and a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}\}$ converges to f in the weak* topology on $\mathrm{Lip}_0(X, d)$. Since the weak* topology agrees with the topology of pointwise convergence on bounded subsets of $\mathrm{Lip}_0(X, d)$ [6, Theorem 2.2.2], it follows that $\{f_{n_k}\}$ converges pointwise on X to f. Finally, let K be any totally bounded subset of X and let $\varepsilon > 0$ be given. Let $M = \sup\{L_d(f_{n_k}): k \in \mathbb{N}\} \cup \{L_d(f)\}$. Since K is totally bounded, there exists a finite set $\{x_1, \dots, x_m\} \subset K$ such that $K \subset \bigcup_{i=1}^m B(x_i, \varepsilon/3M)$. Now choose k_0 so large that $|f_{n_k}(x_i) - f(x_i)| < \varepsilon/3$ whenever $k \geq k_0$ and $1 \leq i \leq m$. Now, given $x \in K$, choose i such that $x \in B(x_i, \varepsilon/3M)$, and for all $k \geq k_0$ we have

$$\begin{aligned}|f_{n_k}(x) - f(x)| &\leq |f_{n_k}(x) - f_{n_k}(x_i)| + |f_{n_k}(x_i) - f(x_i)| + |f(x_i) - f(x)| \\ &< Md(x, x_i) + \frac{\varepsilon}{3} + Md(x, x_i) < \varepsilon.\end{aligned}$$

Thus $|f_{n_k}(x) - f(x)| < \varepsilon$ for all $x \in K$ whenever $k \geq k_0$. This proves that $\{f_{n_k}\}$ converges to f uniformly on K. □

Let us recall that a linear operator between Banach spaces $T: E \to F$ is compact if and only if every bounded sequence $\{x_n\}$ in E has a subsequence $\{x_{n_k}\}$ such that the sequence $\{T(x_{n_k})\}$ converges in F.

Proof of Proposition 2.3. Let us assume that $C_\phi: \mathrm{Lip}_0(X, d) \to \mathrm{Lip}_0(X, d)$ is compact and let $\{f_n\}$ be a bounded sequence in $\mathrm{Lip}_0(X, d)$ that converges uniformly to 0 on totally bounded subsets of X. Since C_ϕ is compact, there exist a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and a function $f \in \mathrm{Lip}_0(X, d)$ such that $L_d(f_{n_k} \circ \phi - f) \to 0$ as $k \to \infty$. By Lemma 2.4, for each $x \in X$ the sequence $\{f_{n_k}(\phi(x))\}$ converges to $f(x)$ as $k \to \infty$, but, on the other hand, this sequence converges to 0 as $k \to \infty$. Hence $f(x) = 0$ for every $x \in X$ and so $f = 0$ as desired.

Conversely, let us assume that every bounded sequence $\{f_n\}$ in $\mathrm{Lip}_0(X, d)$ which converges to zero uniformly on totally bounded subsets of X has a subsequence $\{f_{n_k}\}$ satisfying $L_d(f_{n_k} \circ \phi) \to 0$ as $k \to \infty$. To prove that $C_\phi: \mathrm{Lip}_0(X, d) \to \mathrm{Lip}_0(X, d)$ is compact, take a bounded sequence $\{f_n\}$ in $\mathrm{Lip}_0(X, d)$. By Lemma 2.5, we have a subsequence $\{f_{n_k}\}$ and a function $f \in \mathrm{Lip}_0(X, d)$ such that $\{f_{n_k}\}$ converges uniformly to f on totally bounded subsets of X. By our assumption, $\{f_{n_k} - f\}$ has a subsequence, denoted next also by $\{f_{n_k} - f\}$, satisfying $L_d(f_{n_k} \circ \phi - f \circ \phi) \to 0$ as $k \to \infty$. This proves that $C_\phi: \mathrm{Lip}_0(X, d) \to \mathrm{Lip}_0(X, d)$ is compact. □

Let (X, d) be a pointed metric space. For each $x \in X$, let $\delta_x \colon \mathrm{Lip}_0(X, d) \to \mathbb{K}$ be the evaluation functional defined by $\delta_x(f) = f(x)$ for all $f \in \mathrm{Lip}_0(X, d)$. It is known that $\delta_x \in \mathrm{Lip}_0(X, d)^*$ and $\|\delta_x\| = d(x, e)$ (see [6, p. 27]). Moreover, it is easy to show that $\left\|\delta_x - \delta_y\right\| = d(x, y)$ for all $x, y \in X$.

Now, we are in a position to prove our main theorem.

Proof of Theorem 1.2 (*Necessity*)**.** Let us assume first that $\phi(X)$ is not totally bounded in X. Then for some $\varepsilon > 0$, there is a sequence $\{x_n\}$ in X such that $d(\phi(x_n), \phi(x_m)) \geq \varepsilon$ for all $n, m \in \mathbb{N}$ such that $n \neq m$. It follows that $\left\|\delta_{\phi(x_n)} - \delta_{\phi(x_m)}\right\| \geq \varepsilon$ whenever $n \neq m$, and this says us that the set

$$\left\{\delta_{\phi(x)} \colon x \in X\right\} = \left\{C_\phi^*(\delta_x) \colon x \in X\right\}$$

is not relatively compact in $\mathrm{Lip}_0(X, d)^*$. Since $\{\delta_x \colon x \in X\} \subset \mathrm{Lip}_0(X, d)^*$ and $\|\delta_x\| \leq \mathrm{diam}(X)$ for all $x \in X$, we infer that $C_\phi^* \colon \mathrm{Lip}_0(X, d)^* \to \mathrm{Lip}_0(X, d)^*$ is not compact and, by Schauder's Theorem, the same is true for the operator $C_\phi \colon \mathrm{Lip}_0(X, d) \to \mathrm{Lip}_0(X, d)$ as required.

Suppose now that ϕ is not supercontractive. Then there exist $\varepsilon > 0$ and, for each $n \in \mathbb{N}$, $x_n, y_n \in X$ such that

$$0 < d(x_n, y_n) < \frac{1}{n}, \qquad \frac{d(\phi(x_n), \phi(y_n))}{d(x_n, y_n)} \geq \varepsilon.$$

For each $n \in \mathbb{N}$, let $f_n \colon X \to \mathbb{R}$ be defined by

$$f_n(x) = \frac{\exp(-nd(e, \phi(y_n))) - \exp(-nd(x, \phi(y_n)))}{n}.$$

It is clear that $f_n(e) = 0$ and, for every $x, y \in X$ with $x \neq y$,

$$\frac{|f_n(x) - f_n(y)|}{d(x, y)} = \frac{|\exp(-nd(y, \phi(y_n))) - \exp(-nd(x, \phi(y_n)))|}{nd(x, y)}.$$

If $d(x, \phi(y_n)) \neq d(y, \phi(y_n))$, the Mean Value Theorem applied to the function $g_n(t) = \exp(-nt)$ on the interval with extremes $d(y, \phi(y_n))$ and $d(x, \phi(y_n))$ guarantees the existence of a real number $\theta_n > 0$ between $d(y, \phi(y_n))$ and $d(x, \phi(y_n))$ satisfying

$$\frac{|\exp(-nd(y, \phi(y_n))) - \exp(-nd(x, \phi(y_n)))|}{nd(x, y)} = \frac{\exp(-n\theta_n)\, |d(y, \phi(y_n)) - d(x, \phi(y_n))|}{d(x, y)}.$$

Since $|d(y, \phi(y_n)) - d(x, \phi(y_n))| \leq d(x, y)$ and $\exp(-n\theta_n) < 1$, it follows that

$$\frac{|f_n(x) - f_n(y)|}{d(x, y)} < 1.$$

This inequality holds clearly if $d(x, \phi(y_n)) = d(y, \phi(y_n))$. Therefore f_n belongs to $\mathrm{Lip}_0(X, d)$ and $L_d(f_n) \leq 1$. Hence $\{f_n\}$ is a bounded sequence in $\mathrm{Lip}_0(X, d)$ and, in addition, $\{f_n\}$ converges to 0 uniformly on X since $\|f_n\|_\infty \leq 2/n$ for all $n \in \mathbb{N}$.

However, it is not possible to find a subsequence $\left\{f_{n_k}\right\}$ for which $L_d(f_{n_k} \circ \phi) \to 0$ as $k \to \infty$ because

$$\begin{aligned} L_d(f_n \circ \phi) &\geq \frac{|f_n(\phi(x_n)) - f_n(\phi(y_n))|}{d(x_n, y_n)} \\ &= \frac{|1 - \exp(-nd(\phi(x_n), \phi(y_n)))|}{nd(x_n, y_n)} \\ &= \exp(-n\rho_n)\frac{d(\phi(x_n), \phi(y_n))}{d(x_n, y_n)} \\ &\geq \exp(-L_d(\phi))\varepsilon \end{aligned}$$

for all $n \in \mathbb{N}$, where we now have applied the Mean Value Theorem to $g_n(t) = \exp(-nt)$ on the interval $[0, d(\phi(x_n), \phi(y_n))]$ and the estimate $\exp(-n\rho_n) \geq \exp(-L_d(\phi))$ which follows because g is decreasing and

$$0 < \rho_n < d(\phi(x_n), \phi(y_n)) \leq L_d(\phi)d(x_n, y_n) < \frac{1}{n}L_d(\phi).$$

Hence $C_\phi \colon \mathrm{Lip}_0(X, d) \to \mathrm{Lip}_0(X, d)$ is not compact by Proposition 2.3, and this completes the proof of the necessary condition of the theorem. Some steps in the preceding argument have been motivated by ideas from the proof of [1, Theorem 1].

Sufficiency. Let us suppose that ϕ is supercontractive and $\phi(X)$ is totally bounded in X. Let $\{f_n\}$ be a bounded sequence in $\mathrm{Lip}_0(X, d)$ that converges uniformly to 0 on totally bounded subsets of X. Let $b > 0$ be such that $L_d(f_n) < b$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ be given. We can take a $\delta > 0$ such that if $x, y \in X$ and $0 < d(x, y) < \delta$, then

$$\frac{d(\phi(x), \phi(y))}{d(x, y)} < \frac{\varepsilon}{2b}.$$

Let $x, y \in X$. If $0 < d(x, y) < \delta$, we have

$$\frac{|f_n(\phi(x)) - f_n(\phi(y))|}{d(x,y)} = \frac{|f_n(\phi(x)) - f_n(\phi(y))|}{d(\phi(x),\phi(y))}\frac{d(\phi(x),\phi(y))}{d(x,y)} < \frac{\varepsilon}{2}.$$

If $d(x, y) \geq \delta$, we get

$$\frac{|f_n(\phi(x)) - f_n(\phi(y))|}{d(x,y)} \leq \frac{2\,\|f_n \circ \phi\|_\infty}{\delta}.$$

Since $\{f_n\} \to 0$ uniformly on $\phi(X)$, there is a $m \in \mathbb{N}$ such that for $n \geq m$, we have $\|f_n \circ \phi\|_\infty < \varepsilon\delta/4$. It follows that

$$\frac{|f_n(\phi(x)) - f_n(\phi(y))|}{d(x,y)} < \frac{\varepsilon}{2}$$

whenever $n \geq m$ and $d(x, y) \geq \delta$. So we have proved that $L_d(f_n \circ \phi) < \varepsilon$ for all $n \geq m$. We conclude that $C_\phi\colon \mathrm{Lip}_0(X, d) \to \mathrm{Lip}_0(X, d)$ is compact by Proposition 2.3 and this proves the theorem. $\square$

We now prove Theorem 1.3.

Proof of Theorem 1.3. Assume that ϕ is supercontractive and $\phi(X)$ is totally bounded in X. Let $\{f_n\}$ be a bounded sequence in $\mathrm{lip}(X, d)$. Then $\{f_n\}$ is bounded in $\mathrm{Lip}(X, d)$ and since C_ϕ is a compact operator from $\mathrm{Lip}(X, d)$ into itself by Theorem 1.1, there exist a subsequence $\{f_{n_k}\}$ and a function $f \in \mathrm{Lip}(X, d)$ such that $\left\|f_{n_k} \circ \phi - f\right\|_d \to 0$ as $k \to \infty$. Since $f_{n_k} \circ \phi \in \mathrm{lip}(X, d)$ for all $k \in \mathbb{N}$ and $\mathrm{lip}(X, d)$ is closed in $\mathrm{Lip}(X, d)$, then $f \in \mathrm{lip}(X, d)$. This shows that $C_\phi\colon \mathrm{lip}(X, d) \to \mathrm{lip}(X, d)$ is compact.

In order to prove the converse, note first that as $\mathrm{lip}(X, d)$ separates points uniformly on bounded subsets of X and $\phi(X) \subset X$ is bounded, there is a constant $a \geq 1$ (depending perhaps on $\phi(X)$) such for any pair $x, y \in X$, some function $f_{xy} \in \mathrm{lip}(X, d)$ satisfies $\left\|f_{xy}\right\|_d \leq a$ and $\left|f_{xy}(\phi(x)) - f_{xy}(\phi(y))\right| = d(\phi(x), \phi(y))$.

Suppose that $\phi(X)$ is not totally bounded in X. Then for some $\varepsilon > 0$, there exists a sequence $\{x_n\}$ in X so that $d(\phi(x_n), \phi(x_m)) \geq \varepsilon$ whenever $n \neq m$. We can take a function $f_{nm} \in \mathrm{lip}(X, d)$ satisfying the conditions $\|f_{nm}\|_d \leq a$ and $|f_{nm}(\phi(x_n)) - f_{nm}(\phi(x_m))| = d(\phi(x_n), \phi(x_m))$. It follows that

$$\begin{aligned}
\varepsilon &\leq d(\phi(x_n), \phi(x_m)) \\
&= |f_{nm}(\phi(x_n)) - f_{nm}(\phi(x_m))| \\
&\leq \left\|\delta_{\phi(x_n)} - \delta_{\phi(x_m)}\right\| \|f_{nm}\|_d \\
&\leq a\left\|\delta_{\phi(x_n)} - \delta_{\phi(x_m)}\right\|
\end{aligned}$$

whenever $n \neq m$, and this says us that the set

$$\left\{\delta_{\phi(x)}\colon x \in X\right\} = \left\{C_\phi^*(\delta_x)\colon x \in X\right\}$$

is not relatively compact in $\mathrm{lip}(X, d)^*$. It can be seen easily that $\{\delta_x\colon x \in X\} \subset \mathrm{lip}(X, d)^*$ and $\|\delta_x\| = 1$ for all $x \in X$. Then we deduce that $C_\phi^*\colon \mathrm{lip}(X, d)^* \to \mathrm{lip}(X, d)^*$ is not compact. Hence $C_\phi\colon \mathrm{lip}(X, d) \to \mathrm{lip}(X, d)$ is not compact by Schauder's Theorem, as we desired.

Suppose now that $C_\phi\colon \mathrm{lip}(X, d) \to \mathrm{lip}(X, d)$ is compact. Then the set $C_\phi(B) = \{f \circ \phi\colon f \in B\}$ is relatively compact in $\mathrm{lip}(X, d)$, where B denotes the unit ball of $\mathrm{lip}(X, d)$. Let $\varepsilon > 0$ be given. For each $f \in B$, let

$$B\left(f \circ \phi, \frac{\varepsilon}{4a}\right) = \left\{g \in \mathrm{lip}(X, d)\colon \|g - f \circ \phi\|_d < \frac{\varepsilon}{4a}\right\}.$$

Since $C_\phi(B)$ is relatively compact, there exist $f_1, \ldots, f_n \in B$ such that $C_\phi(B) \subset \bigcup_{k=1}^n B(f_k \circ \phi, \varepsilon/4a)$. For every $f \in B$, we have $L_d(f \circ \phi - f_k \circ \phi) < \varepsilon/4a$ for some $k \in \{1, \ldots, n\}$ and, consequently,

$$\begin{aligned}
\frac{|f(\phi(x)) - f(\phi(y))|}{d(x,y)} &\leq \frac{|(f - f_k)(\phi(x)) - (f - f_k)(\phi(y))|}{d(x,y)} + \frac{|f_k(\phi(x)) - f_k(\phi(y))|}{d(x,y)} \\
&\leq L_d((f - f_k) \circ \phi) + \frac{|f_k(\phi(x)) - f_k(\phi(y))|}{d(x,y)} \\
&< \frac{\varepsilon}{4a} + \frac{|f_k(\phi(x)) - f_k(\phi(y))|}{d(x,y)}
\end{aligned}$$

for any $x, y \in X$ with $x \neq y$. For every $k = 1, \ldots, n$, there is a $\delta_k > 0$ such that

$$\frac{|f_k(\phi(x)) - f_k(\phi(y))|}{d(x,y)} < \frac{\varepsilon}{4a}$$

whenever $0 < d(x, y) < \delta_k$. Taking $\delta = \min\{\delta_1, \ldots, \delta_n\}$, we have

$$\frac{|f(\phi(x)) - f(\phi(y))|}{d(x,y)} < \frac{\varepsilon}{2a}$$

if $0 < d(x, y) < \delta$. Since $\left\|\delta_{\phi(x)} - \delta_{\phi(y)}\right\| := \sup\{|f(\phi(x)) - f(\phi(y))| : f \in B\}$, we conclude that

$$\frac{\left\|\delta_{\phi(x)} - \delta_{\phi(y)}\right\|}{d(x, y)} < \frac{\varepsilon}{a}$$

whenever $0 < d(x, y) < \delta$.

If $x, y \in X$ and $0 < d(x, y) < \delta$, we may take a function $f_{xy} \in \mathrm{lip}(X, d)$ satisfying $\left\|f_{xy}\right\|_d \leq a$ and $\left|f_{xy}(\phi(x)) - f_{xy}(\phi(y))\right| = d(\phi(x), \phi(y))$. Then we have

$$\frac{d(\phi(x), \phi(y))}{d(x, y)} = \frac{\left|f_{xy}(\phi(x)) - f_{xy}(\phi(y))\right|}{d(x, y)} \leq \frac{a\left\|\delta_{\phi(x)} - \delta_{\phi(y)}\right\|}{d(x, y)} < \varepsilon,$$

and hence ϕ is supercontractive. This completes the proof of the theorem. □

It remains to prove Theorem 1.4. First, we recall that if X is a metric space, $\widetilde{X}$ is its completion and Y is a complete metric space, then every Lipschitz mapping ϕ from X to Y has a Lipschitz extension $\widetilde{\phi}$ from $\widetilde{X}$ to Y, and the Lipschitz constant does not change [6, Proposition 1.7.1]. For the case that $\phi: X \to X$ is also supercontractive, it is straightforward to obtain the following fact.

Lemma 2.6. *Let X be a metric space, $\widetilde{X}$ its completion and $\phi: X \to X$ a supercontractive Lipschitz mapping. Then the extension of ϕ to $\widetilde{X}$, $\widetilde{\phi}: \widetilde{X} \to \widetilde{X}$, is supercontractive.*

We now give a sufficient condition for a scalar to be in the spectrum of a (not necessarily compact) operator C_ϕ.

Proposition 2.7. *Let (X, d) be a metric space, $\phi: X \to X$ a Lipschitz mapping and C_ϕ the composition operator induced by ϕ on* $\mathrm{Lip}(X, d)$ *(respectively, on* $\mathrm{lip}(X, d)$*, where* $\mathrm{lip}(X, d)$ *separates points uniformly on bounded subsets of X). If $\widetilde{\phi}$ has a fixed point of order $n \in \mathbb{N}$ and $\lambda \in \mathbb{K}$ satisfies $\lambda^n = 1$, then $\lambda \in \sigma(C_\phi)$.*

Proof. Let $x_0 \in \widetilde{X}$ be a fixed point of $\widetilde{\phi}$ of order $n \in \mathbb{N}$ and let $\lambda \in \mathbb{K}$ be such that $\lambda^n = 1$. If $n = 1$, then $\lambda = 1 \in \sigma(C_\phi)$. Assume now $n \geq 2$. Since $\mathrm{Lip}(X, d)$ and $\mathrm{lip}(X, d)$ separate points uniformly on bounded subsets of X, we can take the constant $a > 1$ involved in Definition 1.1. Denote $B = \left\{\widetilde{\phi}(x_0), \ldots, \widetilde{\phi}^{n-1}(x_0)\right\}$ and pick $\varepsilon = (1/5a)d(x_0, B)$. Consider $z, y_1, \ldots, y_{n-1} \in X$ such that $d(z, x_0) < \varepsilon$ and $d(y_k, \widetilde{\phi}^k(x_0)) < \varepsilon/(n-1)$ for $k = 1, \ldots, n-1$. Notice that $d(z, \{y_1, \ldots, y_{n-1}\}) \geq d(x_0, B)/2$. For each $k \in \{1, \ldots, n-1\}$ there exists a function $f_k \in \mathrm{Lip}(X, d)$ (respectively, $f_k \in \mathrm{lip}(X, d)$) with $L_d(f_k) \leq a$ for which $|f_k(z) - f_k(y_k)| = d(z, y_k)$. Take $g \in \mathrm{Lip}(X, d)$ (respectively, $g \in \mathrm{lip}(X, d)$) defined by

$$g(x) = \min\left\{\frac{|f_k(x) - f_k(y_k)|}{d(z, y_k)} : k = 1, \ldots, n-1\right\} \quad (x \in X).$$

Clearly, $g(z) = 1$, $g(y_1) = \cdots = g(y_{n-1}) = 0$ and $L_d(g) \leq 2a/d(x_0, B)$. To obtain a contradiction, suppose $\lambda f - f \circ \phi = g$ for some $f \in \mathrm{Lip}(X, d)$ (respectively, $f \in \mathrm{lip}(X, d)$). It is easy to prove by induction that

$$\lambda^n f - f \circ \phi^n = \lambda^{n-1} g + \sum_{k=1}^{n-1} \lambda^{n-1-k} g \circ \phi^k,$$

and therefore

$$\lambda^n \widetilde{f}(x_0) - \widetilde{f}\left(\widetilde{\phi}^n(x_0)\right) = \lambda^{n-1}\widetilde{g}(x_0) + \sum_{k=1}^{n-1} \lambda^{n-1-k}\widetilde{g}\left(\widetilde{\phi}^k(x_0)\right).$$

As the first member of this equality vanishes, it follows that

$$-\lambda^{n-1}\widetilde{g}(x_0) = \sum_{k=1}^{n-1} \lambda^{n-1-k}\widetilde{g}\left(\widetilde{\phi}^k(x_0)\right).$$

Taking modulus, we have

$$|\widetilde{g}(x_0)| \leq \sum_{k=1}^{n-1} \left|\widetilde{g}\left(\widetilde{\phi}^k(x_0)\right) - \widetilde{g}(y_k)\right| \leq \sum_{k=1}^{n-1} L_d(g) d\left(\widetilde{\phi}^k(x_0), y_k\right) < L_d(g)\varepsilon.$$

On the other hand, from the inequality $|\widetilde{g}(x_0) - \widetilde{g}(z)| \leq L_d(g) d(x_0, z)$, we infer that $1 - L_d(g)\varepsilon \leq \widetilde{g}(x_0)$. Hence $1 \leq 2L_d(g)\varepsilon$. Then $1 \leq 4a\varepsilon/d(x_0, B) = 4/5$, which is impossible. This proves that $\lambda I - C_\phi$ is not surjective and thus $\lambda \in \sigma(C_\phi)$. □

Second, we recall the notion of r-connectedness (see [6]). Given $r > 0$, it is said that a metric space (X, d) is r-connected if there are not two nonempty disjoint sets $A, B \subset X$ such that $X = A \cup B$ and $d(A, B) \geq r$. We gather some properties about r-connectedness.

Lemma 2.8. *Let (X, d) be a metric space and $r > 0$.*

(i) *If C_1 and C_2 are r-connected subsets of X such that $C_1 \cap C_2 \neq \emptyset$, then $C_1 \cup C_2$ is r-connected.*
(ii) *The relation $\sim$, defined on X by setting $x \sim y$ if and only if there is a finite sequence of points $x_1, \ldots, x_{k+1} \in X$ such that $x_1 = x$, $x_{k+1} = y$ and $d(x_j, x_{j+1}) < r$ for all $j \in \{1, \ldots, k\}$, is an equivalence relation. The equivalence classes of this relation are called the r-connected components of X.*
(iii) *The r-connected components of X are maximal r-connected subsets in X.*
(iv) *The r-connected components of X are open and closed subsets of X.*

According to the terminology of Weaver [6, p. 80], it is said that a mapping $\phi: X \to X$ is (δ, ε)-flat if $0 < d(x, y) < \delta$ implies $|f(x) - f(y)| < \varepsilon\, d(x, y)$. Thus, if ϕ is supercontractive, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that ϕ is (δ, ε)-flat.

The following lemma will allow us to extend the results from the r-connected case to the general case.

Lemma 2.9. *Let (X, d) be a compact metric space, $r > 0$, C a r-connected component of X and $\phi: X \to X$ a $(r, 1/2)$-flat mapping. Then there are $k, n \in \mathbb{N}$ and a r-connected component C_0 of X such that both $\phi^k(C)$ and $\phi^n(C_0)$ are contained in C_0.*

Proof. Let $x \in C$ and let $m \in \mathbb{N}$ be the number of r-connected components of X. As the set

$$\{\phi(x), \phi^2(x), \ldots, \phi^{m+1}(x)\}$$

is contained in the union of the r-connected components of X, then there exist a r-connected component C_0 and $j, k \in \{1, \ldots, m+1\}$ with $k < j$ such that $\phi^k(x), \phi^j(x) \in C_0$. Thus $\phi^{k+(j-k)}(x) \in \phi^{j-k}(C_0) \cap C_0$ and $\phi^k(x) \in \phi^k(C) \cap C_0$. Taking into account that ϕ is $(r, 1/2)$-flat, it follows easily that $\phi^{j-k}(C_0)$ and $\phi^k(C)$ are r-connected subsets. Since $\phi^{j-k}(C_0)$ and C_0 are r-connected, and $\phi^{j-k}(C_0) \cap C_0 \neq \emptyset$, it has that $\phi^{j-k}(C_0) \cup C_0$ is r-connected. Then $\phi^{j-k}(C_0) \subset \phi^{j-k}(C_0) \cup C_0 = C_0$ because C_0 is a maximal r-connected subset in X. Analogously, $\phi^k(C) \subset C_0$. □

We now prove an auxiliary lemma. Its second part is interesting in its own right because it provides a fixed point theorem for a $(r, 1/2)$-flat mapping on a r-connected compact metric space.

Lemma 2.10. *Let $r > 0$, (X, d) a r-connected compact metric space and $\phi: X \to X$ a $(r, 1/2)$-flat mapping. Then the following assertions hold.*

(i) *The sequence $\{\mathrm{diam}(\phi^n(X))\}$ converges to 0.*
(ii) *ϕ has a unique fixed point $x_0 \in X$ and $\bigcap_{n=1}^{\infty} \phi^n(X) = \{x_0\}$.*
(iii) *If ϕ is supercontractive, $\lambda \in \mathbb{K} \setminus \{0, 1\}$ and $f \in \mathrm{Lip}(X, d)$ satisfies $f \circ \phi = \lambda f$, then $f = 0$.*

Proof. (i) For each $x \in X$, we denote $B(x, r) = \{y \in X: d(x, y) < r\}$. Since X is compact, there exists a finite subset $F \subset X$ such that $X = \bigcup_{x \in F} B(x, r)$. Let m be the number of elements of F and $n \in \mathbb{N}$.

If $y_0, z_0 \in F$, it is easy to check by induction on m that there exist $k \in \{1, \ldots, m\}$ and $x_1, \ldots, x_{k+1} \in F$ such that $x_1 = y_0$, $x_{k+1} = z_0$ and $d(B(x_j, r), B(x_{j+1}, r)) < r$ for all $j \in \{1, \ldots, k\}$. Then, given $j \in \{1, \ldots, k\}$, there are $x \in B(x_j, r)$ and $w \in B(x_{j+1}, r)$ such that $d(x, w) < r$. So, applying the condition of ϕ, we have

$$d(\phi^n(x_j), \phi^n(x_{j+1})) \leq d(\phi^n(x_j), \phi^n(x)) + d(\phi^n(x), \phi^n(w)) + d(\phi^n(w), \phi^n(x_{j+1})) < \frac{3r}{2^n}.$$

Therefore

$$d(\phi^n(y_0), \phi^n(z_0)) \leq \sum_{j=1}^{k} d(\phi^n(x_j), \phi^n(x_{j+1})) < \sum_{j=1}^{k} \frac{3r}{2^n} \leq \frac{3mr}{2^n}.$$

Now, let $y, z \in X$. Then $y \in B(y_0, r)$ and $z \in B(z_0, r)$ for some $y_0, z_0 \in F$. Taking into account what has been stated above, it follows that

$$d(\phi^n(y), \phi^n(z)) \leq d(\phi^n(y), \phi^n(y_0)) + d(\phi^n(y_0), \phi^n(z_0)) + d(\phi^n(z_0), \phi^n(z)) < \frac{(3m+2)r}{2^n}.$$

Since y and z are arbitrary, we have $\mathrm{diam}(\phi^n(X)) \leq (3m+2)r/2^n$. Hence the sequence $\{\mathrm{diam}(\phi^n(X))\}$ converges to 0.

(ii) Let $x \in X$. By (i), $\{\phi^n(x)\}$ is a Cauchy sequence in X, and hence $\{\phi^n(x)\}$ converges to a point $x_0 \in X$. Let $m_0 \in \mathbb{N}$ be such that $d(\phi^n(x), x_0) < r$ for all $n \geq m_0$. Since ϕ is $(r, 1/2)$-flat, we obtain that $d(\phi(\phi^n(x)), \phi(x_0)) \leq (1/2)\, d(\phi^n(x), x_0)$ for all $n \geq m_0$. It follows that $\{\phi^{n+1}(x)\}$ converges to $\phi(x_0)$ and therefore $\phi(x_0) = x_0$.

Finally, by Cantor's theorem, the set $\bigcap_{n=1}^{\infty} \phi^n(X)$ contains only one element, so $\bigcap_{n=1}^{\infty} \phi^n(X) = \{x_0\}$. Furthermore, this equality implies that x_0 is the unique fixed point of ϕ.

(iii) We follow [1, Theorem 2]. By (ii), we have $\phi(x_0) = x_0$; hence $f(x_0) = \lambda f(x_0)$, and therefore $f(x_0) = 0$. Using that ϕ is supercontractive, we obtain $\delta > 0$ such that

$$d(\phi(x), x_0) < \frac{|\lambda|}{2}\, d(x, x_0), \quad \forall x \in X,\ 0 < d(x, x_0) < \delta.$$

By (i) there is $m \in \mathbb{N}$ such that $\operatorname{diam}(\phi^m(X)) < \delta$. Given $x \in \phi^m(X)$ and $n \in \mathbb{N}$, we have

$$|\lambda|^n |f(x)| = |f(\phi^n(x)) - f(x_0)| \leq L_d(f)\, d(\phi^n(x), x_0) \leq \frac{|\lambda|^n}{2^n} L_d(f)\, d(x, x_0) < \frac{|\lambda|^n}{2^n} L_d(f)\, \delta,$$

which implies $f(x) = 0$. Now, for each $z \in X$, we have $\lambda^m f(z) = f(\phi^m(z)) = 0$, and so $f(z) = 0$. $\square$

We are ready to demonstrate Theorem 1.4.

Proof of Theorem 1.4. We prove (i) and similar arguments apply to obtain (ii). Let C_ϕ be a compact composition operator of $\mathrm{Lip}(X, d)$ induced by a Lipschitz mapping $\phi: X \to X$. Then ϕ is supercontractive and $\phi(X)$ is totally bounded in X by Theorem 1.1.

Let $\lambda \in \sigma(C_\phi) \setminus \{0\}$. Since C_ϕ is compact, λ is an eigenvalue of C_ϕ, that is, there exists $f \in \mathrm{Lip}(X, d)$ with $f \neq 0$ such that $f \circ \phi = \lambda f$.

Denote by Y the closure of $\phi(X)$ in $\widetilde{X}$. Clearly, Y is compact. Notice that $\widetilde{\phi}(\widetilde{X}) \subset Y$ and therefore $\widetilde{f}|_Y$ is nonzero.

Since $\widetilde{\phi}$ is supercontractive by Lemma 2.6, there is $r > 0$ such that $\widetilde{\phi}$ is $(r, 1/2)$-flat. We want to prove that $\lambda^n = 1$ for some $n \in A$. Suppose the contrary and take a r-connected component C of Y. By Lemma 2.9, there exist another r-connected component C_0 of Y and $k, j \in \mathbb{N}$ such that $\widetilde{\phi}^k(C) \subset C_0$ and $\widetilde{\phi}^j(C_0) \subset C_0$.

By Lemma 2.10 (ii), $\widetilde{\phi}^j|_{C_0}$ has a unique fixed point $x_0 \in C_0$. Then there exists $p \in A$ such that p is the order of x_0. Applying Lemma 2.10 (iii) to the maps $\widetilde{\phi}^p|_{C_0}$ and $\widetilde{f}|_{C_0}$, and the number $\lambda^p \neq 1$, we obtain $\widetilde{f}|_{C_0} = 0$. Then, given $x \in C$, we have $\lambda^k \widetilde{f}(x) = \widetilde{f}(\widetilde{\phi}^k(x)) = 0$, and therefore $\widetilde{f}(x) = 0$. In this way we obtain $\widetilde{f}|_Y = 0$, which is a contradiction. Hence $\lambda^n = 1$ for some $n \in A$ and so we get that

$$\sigma(C_\phi) \setminus \{0\} \subset \bigcup_{n \in A} \{\lambda \in \mathbb{K} : \lambda^n = 1\}.$$

The converse inclusion follows from Proposition 2.7.

Now we have to prove that A is finite. Let F be the set of all $x \in \widetilde{X}$ such that x is a fixed point of $\widetilde{\phi}$ of order n for some $n \in \mathbb{N}$. If we prove that F is finite, then so is A. Observe that $F \subset Y$ since $\widetilde{\phi}(\widetilde{X}) \subset Y$. Let $C_1, \ldots, C_m$ be the r-connected components of Y. Then $F = \cup_{k=1}^m (F \cap C_k)$. Given $k \in \{1, \ldots, m\}$, if $y_0, z_0 \in F \cap C_k$, we have $\widetilde{\phi}^n(y_0) = y_0$ and $\widetilde{\phi}^p(z_0) = z_0$ for some $n, p \in A$, and therefore y_0 and z_0 are fixed points of $\widetilde{\phi}^{np}$. By applying Lemma 2.10 to the mapping $\widetilde{\phi}^{np}|_{C_k}: C_k \to C_k$, we deduce that $y_0 = z_0$. Then $F \cap C_k$ has at most one point and hence F has at most m points. This completes the proof of (i).

Suppose now that X is infinite and connected. Then Y is connected too and $m = 1$. By the above, F contains at most a point. By Lemma 2.10 applied to $\widetilde{\phi}|_Y: Y \to Y$, we deduce that $\widetilde{\phi}$ has a fixed point of order 1. Therefore $A = \{1\}$ and $\sigma(C_\phi) \setminus \{0\} = \{1\}$. If $0 \notin \sigma(C_\phi)$, then C_ϕ is invertible in the algebra of bounded and linear operators on $\mathrm{Lip}(X, d)$. Hence $I = C_\phi \circ C_\phi^{-1}$ is compact and this implies that $\mathrm{Lip}(X, d)$ is finite-dimensional, but this is impossible since X is infinite. Hence $0 \in \sigma(C_\phi)$ and we conclude that $\sigma(C_\phi) = \{0, 1\}$. $\square$

Remark 2.2. In the proof of Theorem 1.4 we have shown that the number of fixed points of $\widetilde{\phi}$ of any order is at most the number of r-connected components of $\phi(X)$.

Acknowledgements

This research was partially supported by MICINN under project MTM 2010-17687, and Junta de Andalucía under grant FQM-3737.

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