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Decay of the non-isentropic Navier–Stokes–Poisson equations

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ABSTRACT

We establish the time decay rates of the solution to the Cauchy problem for the non-isentropic compressible Navier–Stokes–Poisson system via a refined pure energy method. In particular, the optimal decay rates of the higher-order spatial derivatives of the solution are obtained. As a corollary, we also obtain the usual $L^p - L^2 (1 < p \leq 2)$ type of the optimal decay rates. The $\dot{H}^{-s} (0 \leq s < 3/2)$ negative Sobolev norms are shown to be preserved along time evolution and enhance the decay rates. We use a family of scaled energy estimates with minimum derivative counts and interpolations among them without linear decay analysis.

1. Introduction

In this paper, we consider the non-isentropic compressible Navier–Stokes–Poisson (NSP) system for $(x, t) \in \mathbb{R}^3 \times \mathbb{R}^+$

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0 \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p = -\rho \nabla \Phi + \nabla \cdot \mathbb{T} \\ \partial_t (\rho \mathcal{E}) + \nabla \cdot (\rho u \mathcal{E} + up) = -\rho u \cdot \nabla \Phi + \nabla \cdot (u \mathbb{T}) + k \Delta \theta \\ -\lambda^2 \Delta \Phi = \rho - \bar{\rho}, \qquad \lim\limits_{|x| \to \infty} \Phi(x, t) = 0. \end{cases} \tag{1.1}$$

which governs the charge transport in semiconductor devices [1], where $\rho, u = (u^1, u^2, u^3), \theta, p = p(\rho, \theta)$, and Φ represent the density, the velocity, the absolute temperature, the pressure and the electrostatic potential respectively. The total energy $\mathcal{E} = \frac{1}{2}|u|^2 + e$ with $e = C_v \theta$, the stress tensor $\mathbb{T} = \mu(\nabla u + (\nabla u)^T) + \nu(\nabla \cdot u) I$ with I the identity matrix, and the constants $C_v > 0, \kappa > 0$ are the heat capacity at constant volume and the coefficient of heat conductivity respectively. $\lambda > 0$ is the scaled Debye length and $\bar{\rho} > 0$ is the background doping profile [1].

We first review some previous works on the global existence of the solutions to the non-isentropic NSP system. The local and global existence of the multi-dimension re-normalized solution was obtained in [2,3], and the global existence and uniqueness of the strong solution in hybrid Besov space was shown in [4]. Later, Tan–Wu [5] extended these results to the non-isentropic case in hybrid Besov space. The decay rate of solutions to the NS system has been investigated extensively since the works [6,7]. When the initial perturbation $\rho_0 - 1, u_0 \in L^p \cap H^N$ with $p \in [1, 2]$ (Indeed, in those references p is near 1 and $N \geq 3$ is a large enough integer for the nonlinear system.), the L^2 optimal decay rate of the solution to the NS system is

$$\|(\rho - 1, u)(t)\|_{L^2} \leq C(1 + t)^{-\frac{3}{2}\left(\frac{1}{p} - \frac{1}{2}\right)}. \tag{1.2}$$

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But, Guo and Wang get a different result in [8] when the initial perturbation $\rho_0 - 1, u_0 \in H^N \cap H^{-s}$, which is $\|(\rho-1,u)(t)\|_{L^2} \leq C(1+t)^{-\frac{s}{2}}$ $(0 < s < 3/2)$.

Recently, the decay rate of solutions to the isentropic NSP system was investigated in [9–12] and the references therein. It is observed that the electric field has significant effects on the large time behavior of the solution. When the initial perturbation $\rho_0 - 1, u_0 \in L^p \cap H^N$ with $p \in [1,2]$, then the L^2 optimal decay rate of the solution to the NSP system is

$$\|(\rho-1)(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)} \quad \text{and} \quad \|u(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)+\frac{1}{2}}. \tag{1.3}$$

This implies that the presence of the electric field slows down the decay rate of the velocity of the NSP system with the factor 1/2 compared to the NS system. For the isentropic NSP system, Wang [11] gave a different comprehension of the effect of the electric field on the time decay rates of the solutions to the compressible NSP system and showed that both the dispersion effect of the electric field and the viscous dissipation contribute to enhance the decay rate of the density by introducing the novel negative Sobolev space H^{-s}, with the delay rate $\|u, \theta-1, \nabla\Phi(t)\| \leq C(1+t)^{-\frac{s}{2}}$ and $\|\rho-1\| \leq C(1+t)^{-\frac{s+1}{2}}$.

Notation. When $\ell < 0$ or ℓ is not a positive integer, ∇^ℓ stands for Λ^ℓ defined by (1.4). We use $\dot{H}^s(\mathbb{R}^3)$, $s \in \mathbb{R}$ to denote the homogeneous Sobolev spaces on $\mathbb{R}^3$ with norm $\|\cdot\|_{\dot{H}^s}$ defined by (1.5), and we use $H^s(\mathbb{R}^3)$ to denote the usual Sobolev spaces with norm $\|\cdot\|_{H^s}$ and $L^p(\mathbb{R}^3)$, $1 \leq p \leq \infty$ to denote the usual L^p spaces with norm $\|\cdot\|_{L^p}$. Specially, $\|g\|$ stands for $\|g\|_{L^2}$. $\|f,g\|^2$ stands for $\|f\|^2 + \|g\|^2$. We also use C_0 for a positive constant depending additionally on the initial data. We define the operator Λ^s, $s \in \mathbb{R}$ by

$$\Lambda^s f(x) = \int_{\mathbb{R}^3} |\xi|^s \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi, \tag{1.4}$$

where $\hat{f}$ is the Fourier transform of f. We define the homogeneous Sobolev space $\dot{H}^s$ of all f for which $\|f\|_{\dot{H}^s}$ is finite, where

$$\|f\|_{\dot{H}^s} := \left\|\Lambda^s f\right\|_{L^2} = \left\| |\xi|^s \hat{f} \right\|_{L^2}. \tag{1.5}$$

Our main results are stated in the following theorem.

Theorem 1.1. *Assume that* $\rho_0 - 1, u_0, \nabla\Phi_0 \in H^N$, $N \geq 3$ *and*

$$\int_{\mathbb{R}^3} (\rho_0 - 1)\, dx = 0 \quad (\textit{neutrality}). \tag{1.6}$$

Then there exists a constant δ_0 *such that if*

$$\|\rho_0 - 1\|_{H^3} + \|u_0\|_{H^3} + \|\nabla\Phi_0\|_{H^3} + \|q_0 - 1\|_{H^3} \leq \delta_0, \tag{1.7}$$

then the problem (2.1) *admits a unique global solution* $(\rho, u, \nabla\Phi, q)$ *satisfying that for all* $t \geq 0$,

$$\begin{aligned} &\|(\rho-1)(t)\|_{H^N}^2 + \|u(t)\|_{H^N}^2 + \|\nabla\Phi(t)\|_{H^N}^2 + \|(q-1)(t)\|_{H^N}^2 \\ &\quad + \int_0^t \|(\rho-1)(\tau)\|_{H^N}^2 + \|\nabla u(\tau)\|_{H^N}^2 + \|\nabla q(\tau)\|_{H^N}^2 + \|\nabla\nabla\Phi(\tau)\|_{H^N}^2\, d\tau \\ &\leq C\left(\|\rho_0 - 1\|_{H^N}^2 + \|u_0\|_{H^N}^2 + \|\nabla\Phi_0\|_{H^N}^2 + \|q_0 - 1\|_{H^N}^2\right). \end{aligned} \tag{1.8}$$

If further, $\rho_0 - 1, u_0, \nabla\Phi_0, q_0 - 1 \in \dot{H}^{-s}$ *for some* $s \in [0, 3/2)$, *then for all* $t \geq 0$,

$$\|(\rho-1)(t)\|_{\dot{H}^{-s}}^2 + \|u(t)\|_{\dot{H}^{-s}}^2 + \|(q-1)(t)\|_{\dot{H}^{-s}}^2 + \|\nabla\Phi(t)\|_{\dot{H}^{-s}}^2 \leq C_0, \tag{1.9}$$

and for $\ell = 0, 1, \ldots, N-1$, *the following decay results hold:*

$$\left\|\nabla^\ell(\rho-1)(t)\right\|_{H^{N-\ell}} + \left\|\nabla^\ell u(t)\right\|_{H^{N-\ell}} + \left\|\nabla^\ell(q-1)(t)\right\|_{H^{N-\ell}} + \left\|\nabla^\ell\nabla\Phi(t)\right\|_{H^{N-\ell}} \leq C_0(1+t)^{-\frac{\ell+s}{2}} \tag{1.10}$$

and

$$\left\|\nabla^\ell(\rho-1)(t)\right\|_{L^2} \leq C_0(1+t)^{-\frac{\ell+s+1}{2}} \quad \textit{for } \ell = 0, 1, \ldots, N-2. \tag{1.11}$$

Note that the Hardy–Littlewood–Sobolev theorem (cf. Lemma 3.1) implies that for $p \in (1,2]$, $L^p \subset \dot{H}^{-s}$ with $s = 3(\frac{1}{p} - \frac{1}{2}) \in [0, 3/2)$. Then by Theorem 1.1, we have the following corollary of the usual L^p–L^2 type of the optimal decay results:

Corollary 1.2. *Under the assumptions of Theorem 1.1 except that we replace the $\dot{H}^{-s}$ assumption by that $\rho_0 - 1, q_0 - 1, u_0, \nabla\Phi_0 \in L^p$ for some $p \in (1, 2]$, then the following decay results hold for $\ell = 0, 1, \dots, N-1$:*

$$\left\|\nabla^\ell(\rho-1)(t)\right\|_{H^{N-\ell}} + \left\|\nabla^\ell u(t)\right\|_{H^{N-\ell}} + \left\|\nabla^\ell\nabla\Phi(t)\right\|_{H^{N+1-\ell}} + \left\|\nabla^\ell(q-1)(t)\right\|_{H^{N-\ell}} \le C_0(1+t)^{-\sigma_{p,\ell}} \tag{1.12}$$

and

$$\left\|\nabla^\ell(\rho-1)(t)\right\|_{L^2} \le C_0(1+t)^{-\left(\sigma_{p,\ell}+\frac{1}{2}\right)} \quad \text{for } \ell = 0, 1, \dots, N-2. \tag{1.13}$$

Here the number $\sigma_{p,\ell}$ is defined by

$$\sigma_{p,\ell} := \frac{3}{2}\left(\frac{1}{p} - \frac{1}{2}\right) + \frac{\ell}{2}. \tag{1.14}$$

The present paper is structured as follows. Section 2 is devoted to establish the Energy estimates of (2.1). Section 3 is devoted to establish negative Sobolev estimates. In Section 4, we prove Theorem 1.1 and Corollary 1.2.

2. Energy estimates

Denoting $n = \rho - 1, u = u, q = \theta - 1, f(n) = \frac{n}{n+1}, g(n,q) = \frac{p_n(n+1,q+1)}{n+1} - 1, h(n,q) = \frac{p_q(n+1,q+1)}{n+1} - 1, B(n,q) = \frac{p(n+1,q+1)}{n+1} - 1$. Without loss of generality, we assume $P_p(1,1) = P_\theta(1,1) = C_v = k = \lambda = 1$. We can write the Eq. (1.1) as:

$$\begin{cases} \partial_t n + \operatorname{div} u = g_1, \\ \partial_t u - \mu\Delta u - (\mu+\lambda)\nabla \operatorname{div} u + \nabla n + \nabla q - \nabla\Phi = g_2, \\ \partial_t q - \nabla q + \nabla u = g_3, \\ \Delta\Phi = n, \qquad \lim\limits_{|x|\to+\infty} \Phi(x,t) = 0, \end{cases} \tag{2.1}$$

where

$$\begin{cases} g_1 = -\operatorname{div}(nu), \\ g_2 = -u\nabla u - f(n)(\mu\Delta u + (\mu+\lambda)\nabla \operatorname{div} u) - g(n,q)\nabla n - h(n,q)\nabla q, \\ g_3 = -u\nabla q + f(n)\Delta q - B(n,q)\nabla u + \dfrac{1}{n+1}[2\mu D(u) : D(u) + \nu(\nabla u)^2]. \end{cases} \tag{2.2}$$

In this section, we will derive the a priori energy estimates for the equivalent system (2.1). Hence we assume a priori that for sufficiently small $\delta > 0$,

$$\sqrt{\varepsilon_0(t)} = \|\varrho(t)\|_{H^3} + \|u(t)\|_{H^3} + \|q(t)\|_{H^3} + \|\nabla\Phi(t)\|_{H^3} \le \delta. \tag{2.3}$$

First of all, by (2.3) and Sobolev's inequality, we obtain

$$1/2 \le n+1 \le 2. \tag{2.4}$$

Hence, we immediately have

$$|f(n)|,\ |g(n,q)|,\ |h(n,q)|,\ |g(n,q)| \le C(|n| + |q|). \tag{2.5}$$

We will extensively use the Sobolev interpolation of the Gagliardo–Nirenberg inequality.

Lemma 2.1. *Let $0 \le m, \alpha \le \ell$, then we have*

$$\|\nabla^\alpha f\|_{L^p} \le C\left\|\nabla^m f\right\|_{L^q}^{1-\theta}\left\|\nabla^\ell f\right\|_{L^r}^{\theta} \tag{2.6}$$

where α satisfies

$$\frac{\alpha}{3} - \frac{1}{p} = \left(\frac{m}{3} - \frac{1}{q}\right)(1-\theta) + \left(\frac{\ell}{3} - \frac{1}{r}\right)\theta. \tag{2.7}$$

Proof. This is a special case of [13, p. 125, Theorem]. □

Lemma 2.2. *Assume that $\|n, q\|_{H^2} \le 1$. Let $g(n,q)$ be a smooth function of n, q with bounded derivatives, then for any integer $m \ge 1$ we have*

$$\left\|\nabla^m(g(n,q))\right\|_{L^\infty} \le C\left(\left\|\nabla^m n\right\|_{L^2}^{1/4}\left\|\nabla^{m+2} n\right\|_{L^2}^{3/4} + \left\|\nabla^m q\right\|_{L^2}^{1/4}\left\|\nabla^{m+2} q\right\|_{L^2}^{3/4}\right). \tag{2.8}$$

Proof. Notice that for $m \geq 1$,

$$\nabla^m(g(n,q)) = \text{a sum of products } g^{\gamma_1,\dots,\gamma_n;\gamma_1',\dots,\gamma_n'}(n,q)\nabla^{\gamma_1}n\cdots\nabla^{\gamma_n}n\nabla^{\gamma_1'}q\cdots\nabla^{\gamma_n'}q, \tag{2.9}$$

where the functions $g^{\gamma_1,\dots,\gamma_n}(n)$ are some derivatives of $g(n,q)$ and $1 \leq \gamma_i \leq m,\ i = 1,\dots,n$ with $\gamma_1+\cdots+\gamma_n+\gamma_1'+\cdots+\gamma_n' = m$. We then use the Sobolev interpolation of Lemma 2.1 to bound

$$\begin{aligned}
\left\|\nabla^m(g(n,q))\right\|_{L^\infty} &\leq C\|\nabla^{\gamma_1}n\|_{L^\infty}\cdots\|\nabla^{\gamma_n}n\|_{L^\infty}\left\|\nabla^{\gamma_1'}q\right\|_{L^\infty}\cdots\left\|\nabla^{\gamma_n'}q\right\|_{L^\infty}\\
&\leq C\left(\|\nabla^{\gamma_1}n\|_{L^2}\cdots\|\nabla^{\gamma_n}n\|_{L^2}\right)^{1/4}\left(\left\|\nabla^2\nabla^{\gamma_1}n\right\|_{L^2}\cdots\left\|\nabla^2\nabla^{\gamma_n}n\right\|_{L^2}\right)^{3/4}\\
&\quad\times\left(\left\|\nabla^{\gamma_1'}q\right\|_{L^2}\cdots\left\|\nabla^{\gamma_n'}q\right\|_{L^2}\right)^{1/4}\left(\left\|\nabla^2\nabla^{\gamma_1'}q\right\|_{L^2}\cdots\left\|\nabla^2\nabla^{\gamma_n'}q\right\|_{L^2}\right)^{3/4}\\
&\leq C\left(\|n\|_{L^2}^{1-\gamma_1/m}\left\|\nabla^m n\right\|_{L^2}^{\gamma_1/m}\cdots\|n\|_{L^2}^{1-\gamma_n/m}\left\|\nabla^m n\right\|_{L^2}^{\gamma_n/m}\right)^{1/4}\\
&\quad\times\left(\left\|\nabla^2 n\right\|_{L^2}^{1-\gamma_1/m}\left\|\nabla^{m+2}n\right\|_{L^2}^{\gamma_1/m}\cdots\left\|\nabla^2 n\right\|_{L^2}^{1-\gamma_n/m}\left\|\nabla^{m+2}n\right\|_{L^2}^{\gamma_n/m}\right)^{3/4}\\
&\quad\times\left(\|q\|_{L^2}^{1-\gamma_1'/m}\left\|\nabla^m q\right\|_{L^2}^{\gamma_1'/m}\cdots\|q\|_{L^2}^{1-\gamma_n'/m}\left\|\nabla^m q\right\|_{L^2}^{\gamma_n'/m}\right)^{1/4}\\
&\quad\times\left(\left\|\nabla^2 q\right\|_{L^2}^{1-\gamma_1'/m}\left\|\nabla^{m+2}q\right\|_{L^2}^{\gamma_1'/m}\cdots\left\|\nabla^2 q\right\|_{L^2}^{1-\gamma_n'/m}\left\|\nabla^{m+2}q\right\|_{L^2}^{\gamma_n'/m}\right)^{3/4}\\
&\leq C\|n,q\|_{H^2}^{n-1}\left(\left\|\nabla^m n\right\|_{L^2}^{1/4}\left\|\nabla^{m+2}n\right\|_{L^2}^{3/4}+\left\|\nabla^m q\right\|_{L^2}^{1/4}\left\|\nabla^{m+2}q\right\|_{L^2}^{3/4}\right).
\end{aligned}\tag{2.10}$$

Hence, we conclude our lemma since $\|n,q\|_{H^2} \leq 1$. □

We first derive the following energy estimates which contains the dissipation estimate for u.

Lemma 2.3. *If $\sqrt{\varepsilon_0(t)} \leq \delta$, then for $k = 0, 1, 2, \dots, N$, we have*

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla^k n|^2+|\nabla^k u|^2+|\nabla^k q|^2+|\nabla^k\nabla\Phi|^2\,dx + C\left(\left\|\nabla^{k+1}u\right\|_{L^2}^2+\left\|\nabla^{k+1}q\right\|_{L^2}^2\right)\\
&\quad\leq C\sqrt{\varepsilon_0}\left(\left\|\nabla^k n\right\|_{L^2}^2+\left\|\nabla^k q\right\|_{L^2}^2+\left\|\nabla^{k+1}u\right\|_{L^2}^2+\left\|\nabla^{k+1}q\right\|_{L^2}^2+\left\|\nabla^{k+1}\nabla\Phi\right\|_{L^2}^2\right).
\end{aligned}\tag{2.11}$$

Proof. Applying ∇^k to (2.1) $_1$, (2.1) $_2$, (2.1) $_3$, multiplying $(\nabla^k n, \nabla^k u, \nabla^k q)$ and integrating by part, we get:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int(|\nabla^k n|^2+|\nabla^k u|^2+|\nabla^k q|^2+|\nabla^k\nabla\Phi|^2)\,dx+\int_{\mathbb{R}^3}u|\nabla^{k+1}u|^2+(u+\lambda)|\nabla^k\operatorname{div}u|^2+|\nabla\nabla^k q|^2\,dx\\
&\quad-\int_{\mathbb{R}^3}\nabla\nabla^k\Phi\cdot\nabla^k u\,dx\\
&\quad=\int\nabla^k g_1\nabla^k n dx+\int\nabla^k g_2\nabla^k u dx+\int\nabla^k g_3\nabla^k q dx\\
&\quad:= w_1+w_2+w_3.
\end{aligned}\tag{2.12}$$

$$\begin{aligned}
w_2 &= \int\nabla^k\{-(u\nabla)u-f(n)(\mu\Delta u+(\mu+\lambda)\nabla\operatorname{div}u)-g(n,q)\nabla n-h(n,q)\nabla q\}\nabla^k u dx\\
&= w_{21}+w_{22}+w_{23}+w_{24},
\end{aligned}\tag{2.13}$$

$$\begin{aligned}
w_3 &= \int\nabla^k\left(-u\cdot\nabla q+f(n)\Delta q-B(n,q)\nabla u+\frac{1}{n+1}[2uD(u):D(u)+v(\nabla u^2)]\right)\nabla^k q dx\\
&= w_{31}+w_{32}+w_{33}+w_{34}.
\end{aligned}\tag{2.14}$$

We will estimate w_i $(i = 1, 2, 3)$. We only estimate w_1 for example, the rest are similar. Besides, in the course of estimating w_{23} and w_{24}, we use Lemma 2.2 in addition. When $l \geq \frac{[k]}{2}+1$,

$$\begin{aligned}
w_1 &:= -\int_{\mathbb{R}^3}\nabla^k(\operatorname{div}(nu))\nabla^k n\,dx\\
&= -\frac{1}{2}\int\nabla u|\nabla^k n|^2 dx-\sum_{l=1}^k c_l\int\nabla^l u\nabla^{k-1}\nabla n\nabla^k n dx-\sum_{j=1}^k\int\nabla^j(\nabla u)\nabla^{k-j}n\nabla^k n dx
\end{aligned}\tag{2.15}$$

$$
\begin{aligned}
|w_{11}| &= \left|\sum_{\ell=1}^{k} c_\ell \int \nabla^\ell u \nabla^{k-1}\nabla n \nabla^k n dx\right| \\
&\le c_1 \int \nabla u \nabla^{k-1}\nabla n \nabla^k n dx + \sum_{\ell=2}^{k} c_\ell \left|\int \nabla^\ell u \nabla^{k-1}\nabla n \nabla^k n dx\right| \\
&\le c_1\sqrt{\varepsilon_0}\left\|\nabla^k n\right\|^2 + \sum_{\ell=2}^{k} c_\ell| \left\|\nabla^\ell u\right\|_{L^6}\left\|\nabla^{k-\ell+1} n\right\|_{L^3}\left\|\nabla^k n\right\| \\
&\le c_1\sqrt{\varepsilon_0}\left\|\nabla^k n\right\|^2 + \sum_{\ell=2}^{k} c_\ell| \left\|\nabla^\alpha u\right\|^{1-\frac{\ell}{k+1}}\left\|\nabla^{k+1} u\right\|^{\frac{\ell}{k+1}}\left\|\nabla^\beta n\right\|^{\frac{\ell}{k+1}}\left\|\nabla^k n\right\|^{1-\frac{\ell}{k+1}}\left\|\nabla^k n\right\|
\end{aligned}
\tag{2.16}
$$

where

$$
\begin{cases}
\dfrac{1}{3}-\dfrac{1}{6}=\left(\dfrac{\alpha}{3}-\dfrac{1}{2}\right)\times\left(1-\dfrac{\ell}{k+1}\right)+\left(\dfrac{k+1}{3}-\dfrac{1}{2}\right)\times\dfrac{\ell}{k+1}, \\
\dfrac{k-\ell+1}{3}-\dfrac{1}{3}=\left(\dfrac{\beta}{3}-\dfrac{1}{2}\right)\times\dfrac{\ell}{k+1}+\left(\dfrac{k}{3}-\dfrac{1}{2}\right)\times\left(1-\dfrac{\ell}{k+1}\right).
\end{cases}
\tag{2.17}
$$

By computing it directly, we get:

$$
\alpha=\frac{k+1}{k-\ell+1}\le 1,\qquad \beta=\frac{3k+3}{2\ell}-1\le 3.
$$

So,

$$
|w_{11}|\le C\sqrt{\varepsilon_0}\left(\left\|\nabla^k n\right\|^2+\left\|\nabla^{k+1}u\right\|^2\right). \tag{2.18}
$$

In the same way, when $l\le\frac{[k]}{2}+1$, we have also have

$$
|w_{11}|\le C\sqrt{\varepsilon_0}\left(\left\|\nabla^k n\right\|^2+\left\|\nabla^{k+1}u\right\|^2\right). \tag{2.19}
$$

From (2.18) and (2.19), we get

$$
|w_{11}|\le C\sqrt{\varepsilon_0}\left(\left\|\nabla^k n\right\|^2+\left\|\nabla^{k+1}u\right\|^2\right). \tag{2.20}
$$

Similarly, we have

$$
|w_{12}|\le C\sqrt{\varepsilon_0}\left(\left\|\nabla^k n\right\|^2+\left\|\nabla^{k+1}u\right\|^2\right), \tag{2.21}
$$

$$
|w_{21}|\le C\sqrt{\varepsilon_0}\left\|\nabla^{k+1}u\right\|^2, \tag{2.22}
$$

$$
|w_{22}|\le C\sqrt{\varepsilon_0}\left(\left\|\nabla^k n\right\|^2+\left\|\nabla^{k+1}u\right\|^2\right), \tag{2.23}
$$

$$
|w_{23}|\le C\sqrt{\varepsilon_0}(\|\nabla^k n\|^3+\|\nabla^k q\|^2+\|\nabla^{k+1}u\|^2), \tag{2.24}
$$

$$
|w_{24}|\le C\sqrt{\varepsilon_0}(\|\nabla^k n\|^3+\|\nabla^k q\|^2+\|\nabla^{k+1}u\|^2), \tag{2.25}
$$

$$
|w_{31}|\le C\sqrt{\varepsilon_0}(\|\nabla^k q\|^2+\|\nabla^k u\|^2+\|\nabla^{k+1}q\|^2), \tag{2.26}
$$

$$
|w_{32}|\le C\sqrt{\varepsilon_0}(\|\nabla^k q\|^2+\|\nabla^k n\|^2+\|\nabla^{k+1}q\|^2), \tag{2.27}
$$

$$
|w_{33}|\le C\sqrt{\varepsilon_0}(\|\nabla^k q\|^2+\|\nabla^k n\|^2+\|\nabla^{k+1}q\|^2), \tag{2.28}
$$

$$
|w_{34}|\le C\sqrt{\varepsilon_0}\left(\|\nabla^k n\|^2+\|\nabla^k u\|^2+\|\nabla^{k+1}q\|^2\right). \tag{2.29}
$$

So, from (2.20), (2.21), we have

$$
|w_1|\le C\sqrt{\varepsilon_0}\left(\left\|\nabla^k n\right\|^2+\left\|\nabla^{k+1}u\right\|^2\right). \tag{2.30}
$$

So, from (2.22)–(2.25), we have

$$
|w_2|\le C\sqrt{\varepsilon_0}\left(\|\nabla^k n\|^2+\|\nabla^k q\|^2+\|\nabla^{k+1}u\|^2\right). \tag{2.31}
$$

So, from (2.26)–(2.29), we have

$$|w_3| \le C\sqrt{\varepsilon_0}\left(\|\nabla^k n\|^2 + \|\nabla^k q\|^2 + \|\nabla^{k+1} q\|^2 + \|\nabla^{k+1} u\|^2\right). \tag{2.32}$$

Finally, from (2.30)–(2.32), we have completed the proof of Lemma 2.3. □

The following lemma provides the dissipation estimate for n and $\nabla\Phi$.

Lemma 2.4. *If $\sqrt{\varepsilon_0(t)} \le \delta$, then for $k = 0, 1, \ldots, N-1$, we have*

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3} \nabla^k u \cdot \nabla\nabla^k n\, dx + C\left(\left\|\nabla^k n\right\|_{L^2}^2 + \left\|\nabla^{k+1} n\right\|_{L^2}^2 + \left\|\nabla^{k+1}\nabla\Phi\right\|_{L^2}^2 + \left\|\nabla^{k+2}\nabla\Phi\right\|_{L^2}^2\right)\\ &\quad\le C\left(\left\|\nabla^{k+1} u\right\|_{L^2}^2 + \left\|\nabla^{k+2} u\right\|_{L^2}^2 + \sqrt{\varepsilon_0}\left\|\nabla^{k+1} q\right\|^2\right).\end{aligned} \tag{2.33}$$

Proof. Applying ∇^k to (2.1) $_2$ and then taking the L^2 inner product with $\nabla\nabla^k n$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^3} |\nabla\nabla^k n|^2\, dx - \int_{\mathbb{R}^3} \nabla^k\nabla\Phi \cdot \nabla\nabla^k n\, dx\\ &\quad\le -\int_{\mathbb{R}^3} \nabla^k\partial_t u \cdot \nabla\nabla^k n\, dx + C\left\|\nabla^{k+2} u\right\|_{L^2}\left\|\nabla^{k+1} n\right\|_{L^2}\\ &\qquad + \left\|\nabla^k\left(u\cdot\nabla u + h(n)(\mu\Delta u + (\mu+\lambda)\nabla\operatorname{div} u) + g(n,q)\nabla n + h(n,q)\nabla q\right)\right\|_{L^2}\left\|\nabla^{k+1} n\right\|_{L^2}.\end{aligned} \tag{2.34}$$

$$\begin{aligned}-\int_{\mathbb{R}^3} \nabla^k u_t \cdot \nabla\nabla^k n\, dx &= -\frac{d}{dt}\int_{\mathbb{R}^3} \nabla^k u \cdot \nabla\nabla^k n\, dx - \int_{\mathbb{R}^3} \nabla^k \operatorname{div} u \cdot \nabla^k n_t\, dx\\ &= -\frac{d}{dt}\int_{\mathbb{R}^3} \nabla^k u \cdot \nabla\nabla^k n\, dx + \left\|\nabla^k \operatorname{div} u\right\|_{L^2}^2 + \int_{\mathbb{R}^3} \nabla^k \operatorname{div} u \cdot \nabla^k \operatorname{div}(nu)\, dx.\end{aligned} \tag{2.35}$$

By Hölder's inequality, we have

$$\int_{\mathbb{R}^3} \nabla^k \operatorname{div} u \cdot \nabla^k \operatorname{div}(nu)\, dx \lesssim \sum_{0\le\ell\le k+1}\left\|\nabla^\ell n\nabla^{k+1-\ell} u\right\|_{L^2}\left\|\nabla^{k+1} u\right\|_{L^2}. \tag{2.36}$$

So, we only need to estimate $\left\|\nabla^\ell n\nabla^{k+1-\ell} u\right\|_{L^2}$. By the method in Part two, we have

$$\left\|\nabla^\ell n\nabla^{k+1-\ell} u\right\|_{L^2} \le C\varepsilon_0\left(\left\|\nabla^k n\right\|_{L^2} + \left\|\nabla^{k+1} u\right\|_{L^2}\right). \tag{2.37}$$

Thus, in view of (2.35), we obtain

$$-\int_{\mathbb{R}^3} \nabla^k u_t \cdot \nabla\nabla^k n\, dx \le -\frac{d}{dt}\int_{\mathbb{R}^3} \nabla^k u \cdot \nabla\nabla^k n\, dx + C\left\|\nabla^{k+1} u\right\|_{L^2}^2 + C\sqrt{\varepsilon_0^3}\left\|\nabla^{k+1} n\right\|_{L^2}^2. \tag{2.38}$$

Next, note that it has been already proved along the proof of Lemma 2.3 that

$$\begin{aligned}&\left\|\nabla^k\left(u\cdot\nabla u + h(n)(\mu\Delta u + (\mu+\lambda)\nabla\operatorname{div} u) + f(n)\nabla n\right)\right\|_{L^2}\\ &\quad\le C\sqrt{\varepsilon_0}\left(\left\|\nabla^{k+1} n\right\|_{L^2} + \left\|\nabla^{k+1} q\right\|_{L^2} + \left\|\nabla^{k+2} u\right\|_{L^2}\right).\end{aligned} \tag{2.39}$$

We now use the integration by parts and the Poisson equation (2.1) $_3$ to have

$$-\int_{\mathbb{R}^3} \nabla^k\nabla\Phi \cdot \nabla\nabla^k n\, dx = \int_{\mathbb{R}^3} \nabla^k\Delta\Phi\nabla^k n\, dx = \int_{\mathbb{R}^3} |\nabla^k n|^2\, dx. \tag{2.40}$$

On the other hand, it follows from the Poisson equation that

$$\left\|\nabla^{k+1}\nabla\Phi\right\|_{L^2}^2 = \left\|\nabla^k\Delta\Phi\right\|_{L^2}^2 = \left\|\nabla^k n\right\|_{L^2}^2 \quad \text{and} \quad \left\|\nabla^{k+2}\nabla\Phi\right\|_{L^2}^2 = \left\|\nabla^{k+1} n\right\|_{L^2}^2. \tag{2.41}$$

Consequently, by (2.38)–(2.41), together with Cauchy's inequality, since $\sqrt{\varepsilon_0} \le \delta$ is small, we then deduce (2.33) from (2.34). □

3. Negative Sobolev estimates

In this section, we will derive the evolution of the negative Sobolev norms of the solution to Lemma 2.3.

If $s \in (0,3)$, $\Lambda^{-s}f$ defined by (1.4) is the Riesz potential. The Hardy–Littlewood–Sobolev theorem implies the following L^p type inequality for the Riesz potential:

Lemma 3.1. *Let $0<s<3,\ 1<p<q<\infty,\ 1/q+s/3=1/p$, then*

$$\left\|\Lambda^{-s}f\right\|_{L^q} \le C\,\|f\|_{L^p}. \tag{3.1}$$

Proof. See [14, p. 119, Theorem 1]. □

In order to estimate the nonlinear terms, we need to restrict ourselves to that $s \in (0,3/2)$. We will establish the following lemma.

Lemma 3.2. *If $\sqrt{\varepsilon_0} \le \delta$, then for $s \in (0,1/2]$, we have*

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} |\Lambda^{-s}n|^2 + |\Lambda^{-s}u|^2 + |\Lambda^{-s}\nabla\Phi|^2 + |\Lambda^{-s}q|^2\,dx + C(\left\|\nabla\Lambda^{-s}u\right\|_{L^2}^2 + \left\|\nabla\Lambda^{-s}q\right\|_{L^2}^2)\\
&\quad\le C\left(\|n,q\|_{H^2}^2 + \|\nabla u\|_{H^1}^2\right)\left(\left\|\Lambda^{-s}n\right\|_{L^2} + \left\|\Lambda^{-s}u\right\|_{L^2} + \left\|\Lambda^{-s}\nabla\Phi\right\|_{L^2} + |\Lambda^{-s}q|\right),
\end{aligned} \tag{3.2}$$

and for $s \in (1/2,3/2)$, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} |\Lambda^{-s}n|^2 + |\Lambda^{-s}u|^2 + |\Lambda^{-s}\nabla\Phi|^2 + |\Lambda^{-s}q|^2\,dx + C(\left\|\nabla\Lambda^{-s}u\right\|_{L^2}^2 + \left\|\nabla\Lambda^{-s}q\right\|_{L^2}^2 + \left\|\nabla\Lambda^{-s}q\right\|_{L^2}^2)\\
&\quad\le C\,\|(n,u,q,\nabla u)\|_{L^2}^{s-1/2}\left(\|n,q\|_{H^2} + \|\nabla u\|_{H^1}\right)^{5/2-s}\left(\left\|\Lambda^{-s}n,\Lambda^{-s}u,\Lambda^{-s}q,\Lambda^{-s}\nabla\Phi\right\|_{L^2}\right).
\end{aligned} \tag{3.3}$$

Proof. Applying Λ^{-s} to (2.1) $_1$, (2.1) $_2$, (2.1) $_3$ and multiplying the resulting identities by $\Lambda^{-s}n$, $\Lambda^{-s}u$, $\Lambda^{-s}q$ respectively, summing them up and then integrating over $\mathbb{R}^3$ by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |\Lambda^{-s}n|^2 + |\Lambda^{-s}u|^2 + |\Lambda^{-s}q|^2\,dx + \int_{\mathbb{R}^3}\mu|\nabla\Lambda^{-s}u|^2 + (\mu+\lambda)|\operatorname{div}\Lambda^{-s}u|^2\,dx\\
&\quad\times\int_{\mathbb{R}^3}|\Lambda^{-s}q|^2 - \int_{\mathbb{R}^3}\Lambda^{-s}\nabla\Phi\cdot\Lambda^{-s}u\,dx = \int_{\mathbb{R}^3}\Lambda^{-s}g_1\Lambda^{-s}n + \Lambda^{-s}g_2\Lambda^{-s}u + \Lambda^{-s}g_3\Lambda^{-s}q\,dx\\
&\quad:= T_1+T_2+T_3,
\end{aligned} \tag{3.4}$$

$$T_1 = \int_{\mathbb{R}^3}\Lambda^{-s}\left(-n\operatorname{div}u - u\cdot\nabla n\right)\Lambda^{-s}n\,dx = T_{11}+T_{12}. \tag{3.5}$$

For $s \in (0,\frac{1}{2})$,

$$\begin{aligned}
T_{11} &= -\int_{\mathbb{R}^3}\Lambda^{-s}(n\operatorname{div}u)\Lambda^{-s}n\,dx \le C\left\|\Lambda^{-s}(n\operatorname{div}u)\right\|_{L^2}\left\|\Lambda^{-s}n\right\|_{L^2}\\
&\le C\,\|n\operatorname{div}u\|_{L^{\frac{1}{1/2+s/3}}}\left\|\Lambda^{-s}n\right\|_{L^2} \le C\,\|n\|_{L^{3/s}}\|\nabla u\|_{L^2}\left\|\Lambda^{-s}n\right\|_{L^2}\\
&\le C\,\|\nabla n\|_{L^2}^{1/2-s}\left\|\nabla^2 n\right\|_{L^2}^{1/2+s}\|\nabla u\|_{L^2}\left\|\Lambda^{-s}n\right\|_{L^2}\\
&\le C\left(\|\nabla n\|_{H^1}^2 + \|\nabla u\|_{L^2}^2\right)\left\|\Lambda^{-s}n\right\|_{L^2}.
\end{aligned} \tag{3.6}$$

Similarly, we can bound the remaining terms by

$$T_{12} = -\int_{\mathbb{R}^3}\Lambda^{-s}(u\cdot\nabla n)\Lambda^{-s}n\,dx \le C\left(\|\nabla u\|_{H^1}^2 + \|\nabla n\|_{L^2}^2\right)\left\|\Lambda^{-s}n\right\|_{L^2}. \tag{3.7}$$

$$T_2 = -\int_{\mathbb{R}^3}\Lambda^{-s}g_2\Lambda^{-s}u\,dx \le C\left(\|\nabla u\|_{H^1}^2 + \|\nabla n,u\|_{L^2}^2\right)\left\|\Lambda^{-s}u\right\|_{L^2}. \tag{3.8}$$

$$T_3 = -\int_{\mathbb{R}^3}\Lambda^{-s}g_3\Lambda^{-s}q\,dx \le C\left(\|\nabla u,q\|_{H^1}^2 + \|\nabla u\|_{L^2}^2\right)\left\|\Lambda^{-s}n\right\|_{L^2} \tag{3.9}$$

when $s \in (1/2, 3/2)$, we obtain

$$
\begin{aligned}
T_{11} &= -\int_{\mathbb{R}^3} \Lambda^{-s}(n \operatorname{div} u)\Lambda^{-s} n\, dx \le C \left\| \Lambda^{-s}(n \operatorname{div} u)\right\|_{L^2} \left\|\Lambda^{-s} n\right\|_{L^2} \\
&\le C \left\| n \operatorname{div} u\right\|_{L^{\frac{1}{1/2+s/3}}} \left\|\Lambda^{-s} n\right\|_{L^2} \le C \|n\|_{L^{3/s}} \|\nabla u\|_{L^2} \left\|\Lambda^{-s} n\right\|_{L^2} \\
&\le C \|n\|_{L^2}^{s-1/2} \|\nabla n\|_{L^2}^{3/2-s} \|\nabla u\|_{L^2} \left\|\Lambda^{-s} n\right\|_{L^2}. \qquad (3.10)
\end{aligned}
$$

Similarly, we can bound the remaining terms by

$$
\begin{aligned}
T_{12} &= \int_{\mathbb{R}^3} \Lambda^{-s}(u\cdot\nabla n)\Lambda^{-s} n\, dx \le C \|u\|_{L^2}^{s-1/2} \|\nabla u\|_{L^2}^{3/2-s} \|\nabla n\|_{L^2} \left\|\Lambda^{-s} n\right\|_{L^2}. \\
T_2 &= \int_{\mathbb{R}^3} \Lambda^{-s} g_2 \Lambda^{-s} u\, dx \le C \|n, q\|_{L^2}^{s-1/2} \|\nabla n, \nabla q\|_{L^2}^{3/2-s} \left\|\nabla^2 q, \nabla^2 n\right\|_{L^2} \left\|\Lambda^{-s} u\right\|_{L^2}. \\
T_2 &= \int_{\mathbb{R}^3} \Lambda^{-s} g_3 \Lambda^{-s} u\, dx \le C \|n, q\|_{L^2}^{s-1/2} \|\nabla n, \nabla q\|_{L^2}^{3/2-s} \left\|\nabla^2 q, \nabla^2 n\right\|_{L^2} \left\|\Lambda^{-s} q\right\|_{L^2}.
\end{aligned}
$$

Finally, we turn to the left hand side of (3.4). For the second term, we have

$$
\int_{\mathbb{R}^3} \mu|\nabla\Lambda^{-s} u|^2 + (\mu+\lambda)|\operatorname{div}\Lambda^{-s} u|^2\, dx - \int_{\mathbb{R}^3} \Lambda^{-s}\nabla\Phi\cdot\Lambda^{-s} u\, dx \ge \sigma_0 \left\|\nabla\Lambda^{-s} u\right\|_{L^2}^2. \qquad (3.11)
$$

While for the Poisson term, by the continuity equation $(2.1)_1$ and the Poisson equation $(2.1)_3$ and the integration by parts, we get

$$
\begin{aligned}
-\int_{\mathbb{R}^3} \Lambda^{-s}\nabla\Phi\cdot\Lambda^{-s} u\, dx &= \int_{\mathbb{R}^3} \Lambda^{-s}\Phi\Lambda^{-s}\operatorname{div} u\, dx \\
&= \int_{\mathbb{R}^3} -\Lambda^{-s}\Phi\Lambda^{-s}\partial_t n - \Lambda^{-s}\Phi\Lambda^{-s}\operatorname{div}(nu)\, dx \\
&= \int_{\mathbb{R}^3} -\Lambda^{-s}\Phi\Lambda^{-s}\partial_t\Delta\Phi + \Lambda^{-s}\nabla\Phi\cdot\Lambda^{-s}(nu)\, dx \\
&= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |\Lambda^{-s}\nabla\Phi|^2\, dx + \int_{\mathbb{R}^3} \Lambda^{-s}\nabla\Phi\cdot\Lambda^{-s}(nu)\, dx. \qquad (3.12)
\end{aligned}
$$

If $s \in (0, 1/2]$, we use Lemmas 3.1 and 2.1 to obtain

$$
\left\|\Lambda^{-s}(nu)\right\|_{L^2} \le C \|n\|_{L^2} \|u\|_{L^{3/s}} \le C \|n\|_{L^2} \|\nabla u\|_{L^2}^{1/2-s} \left\|\nabla^2 u\right\|_{L^2}^{1/2+s}, \qquad (3.13)
$$

and if $s \in (1/2, 3/2)$, we have

$$
\left\|\Lambda^{-s}(nu)\right\|_{L^2} \le C \|n\|_{L^2} \|u\|_{L^{3/s}} \le C \|n\|_{L^2} \|u\|_{L^2}^{s-1/2} \|\nabla u\|_{L^2}^{3/2-s}. \qquad (3.14)
$$

Consequently, in light of (3.6)–(3.14), we deduce (3.3) from (3.4). □

4. Proof of Theorem 1.1

We will employ the following special Sobolev interpolation:

Lemma 4.1. *Let $s \ge 0$ and $\ell \ge 0$, then we have*

$$
\left\|\nabla^\ell f\right\|_{L^2} \le \left\|\nabla^{\ell+1} f\right\|_{L^2}^{1-\theta} \|f\|_{\dot{H}^{-s}}^{\theta}, \quad \text{where } \theta = \frac{1}{\ell+1+s}. \qquad (4.1)
$$

Proof. By the Parseval theorem, the definition of (1.5) and Hölder's inequality, we have

$$
\left\|\nabla^\ell f\right\|_{L^2} = \left\||\xi|^\ell \hat{f}\right\|_{L^2} \le \left\||\xi|^{\ell+1}\hat{f}\right\|_{L^2}^{1-\theta} \left\||\xi|^{-s}\hat{f}\right\|_{L^2}^{\theta} = \left\|\nabla^{\ell+1} f\right\|_{L^2}^{1-\theta} \|f\|_{\dot{H}^{-s}}^{\theta}. \quad \square \qquad (4.2)
$$

Summing up the estimates (2.11) of Lemma 2.3 from $k=\ell$ to m, since $\sqrt{\varepsilon_0^3}\le\delta$ is small, we obtain

$$\frac{d}{dt}\sum_{\ell\le k\le m}\left(\left\|\nabla^k n\right\|_{L^2}^2+\left\|\nabla^k u\right\|_{L^2}^2+\left\|\nabla^k\nabla\Phi\right\|_{L^2}^2+\left\|\nabla^k q\right\|_{L^2}^2\right)+C_1\sum_{\ell+1\le k\le m+1}\left\|\nabla^k u\right\|_{L^2}^2+\left\|\nabla^k q\right\|_{L^2}^2$$
$$\le C_2\delta\left(\sum_{\ell\le k\le m}\left(\left\|\nabla^k n\right\|_{L^2}^2+\left\|\nabla^k q\right\|_{L^2}^2\right)+\sum_{\ell+1\le k\le m+1}\left(\left\|\nabla^k q\right\|_{L^2}^2+\left\|\nabla^k u\right\|_{L^2}^2+\left\|\nabla^k\nabla\Phi\right\|_{L^2}^2\right)\right). \tag{4.3}$$

Summing up the estimates (2.33) of Lemma 2.4 from $k=\ell$ to $m-1$, we have

$$\frac{d}{dt}\sum_{\ell\le k\le m-1}\int_{\mathbb{R}^3}\nabla^k u\cdot\nabla\nabla^k n\,dx+C_3\left(\sum_{\ell\le k\le m}\left\|\nabla^k n\right\|_{L^2}^2+\sum_{\ell+1\le k\le m+1}\left\|\nabla^k\nabla\Phi\right\|_{L^2}^2\right)$$
$$\le C_4\sum_{\ell+1\le k\le m+1}\left\|\nabla^k u\right\|_{L^2}^2+\sqrt{\varepsilon_0}\sum_{\ell+1\le k\le m+1}\left\|\nabla^k q\right\|_{L^2}^2. \tag{4.4}$$

Multiplying (4.4) by $2C_2\delta/C_3$, adding the resulting inequality with (4.3), since $\delta>0$ is small, we deduce that there exists a constant $C_5>0$ such that for $0\le\ell\le m-1$,

$$\frac{d}{dt}\left\{\sum_{\ell\le k\le m}\left(\left\|\nabla^k n\right\|_{L^2}^2+\left\|\nabla^k u\right\|_{L^2}^2+\left\|\nabla^k\nabla\Phi\right\|_{L^2}^2\right)+\frac{2C_2\delta}{C_3}\sum_{\ell\le k\le m-1}\int_{\mathbb{R}^3}\nabla^k u\cdot\nabla\nabla^k n\,dx\right\}$$
$$+C_5\left\{\sum_{\ell\le k\le m}\left\|\nabla^k n\right\|_{L^2}^2+\sum_{\ell+1\le k\le m+1}\left\|\nabla^k u\right\|_{L^2}^2+\sum_{\ell+1\le k\le m+1}\left\|\nabla^k\nabla\Phi\right\|_{L^2}^2\right\}\le 0. \tag{4.5}$$

We define $\mathcal{E}_\ell^m(t)$ to be C_5^{-1} times the expression under the time derivative in (4.5). Observe that since δ is small, $\mathcal{E}_\ell^m(t)=\left\|\nabla^\ell n(t),\nabla^\ell u(t),\nabla^\ell q(t),\nabla^\ell\nabla\Phi(t)\right\|_{H^{m-\ell}}^2$. Then we may write (4.5) as that for $0\le\ell\le m-1$,

$$\frac{d}{dt}\mathcal{E}_\ell^m+\left\|\nabla^\ell n\right\|_{H^{m-\ell}}^2+\left\|\nabla^{\ell+1}u\right\|_{H^{m-\ell}}^2+\left\|\nabla^{\ell+1}u\right\|_{H^{m-\ell}}^2+\left\|\nabla^{\ell+1}\nabla\Phi\right\|_{H^{m-\ell}}^2\le 0. \tag{4.6}$$

Now taking $\ell=0$ and $m=3$ in (4.6) and then integrating directly in time, we get

$$\|n(t),u(t),q(t),\nabla\Phi(t)\|_3^2\le C\varepsilon_0(t)\le\varepsilon_0(0)\le C\left\|n_0,q_0,u_0,\nabla\Phi_0\right\|_3^2. \tag{4.7}$$

By a standard continuity argument, this closes the a priori estimates (2.3) if at the initial time we assume that $\|n_0,q_0,u_0,\nabla\Phi_0\|_3^2\le\delta_0$ is sufficiently small. This in turn allows us to take $\ell=0$ and $m=N$ in (4.6), and then integrate it directly in time to obtain (1.8).

Next, we turn to prove (1.9)–(1.11). However, we are not able to prove them for all $s\in[0,3/2)$ at this moment. We shall first prove them for $s\in[0,1/2]$.

Proof. For $s\in[0,1/2]$. Define $\mathcal{E}_{-s}(t):=\left\|\Lambda^{-s}n(t),\Lambda^{-s}u(t),\Lambda^{-s}q(t),\Lambda^{-s}\nabla\Phi(t)\right\|_{L^2}^2$. Then, integrating in time (3.2), by the bound (1.8), we obtain that for $s\in(0,1/2]$,

$$\mathcal{E}_{-s}(t)\le\mathcal{E}_{-s}(0)+C\int_0^t\left(\|n,q\|_{H^2}^2+\|\nabla u\|_{H^1}^2\right)\sqrt{\mathcal{E}_{-s}(\tau)}\,d\tau$$
$$\le C_0\left(1+\sup_{0\le\tau\le t}\sqrt{\mathcal{E}_{-s}(\tau)}\right). \tag{4.8}$$

This implies (1.9) for $s\in[0,1/2]$, that is,

$$\left\|\Lambda^{-s}n(t)\right\|_{L^2}^2+\left\|\Lambda^{-s}q(t)\right\|_{L^2}^2+\left\|\Lambda^{-s}u(t)\right\|_{L^2}^2+\left\|\Lambda^{-s}\nabla\Phi(t)\right\|_{L^2}^2\le C_0\quad\text{for } s\in[0,1/2]. \tag{4.9}$$

If $\ell=1,2$, we may use Lemma 4.1 to have

$$\left\|\nabla^{\ell+1}f\right\|_{L^2}\ge C\left\|\Lambda^{-s}f\right\|_{L^2}^{-\frac{1}{\ell+s}}\left\|\nabla^\ell f\right\|_{L^2}^{1+\frac{1}{\ell+s}}. \tag{4.10}$$

By this fact and (4.9), we may find

$$\left\|\nabla^{\ell+1}u\right\|_{L^2}^2+\left\|\nabla^{\ell+1}q\right\|_{L^2}^2+\left\|\nabla^{\ell+1}\nabla\Phi\right\|_{L^2}^2\ge C_0\left(\left\|\nabla^\ell u\right\|_{L^2}^2+\left\|\nabla^{\ell+1}q\right\|_{L^2}^2+\left\|\nabla^\ell\nabla\Phi\right\|_{L^2}^2\right)^{1+\frac{1}{\ell+s}}. \tag{4.11}$$

This together with (1.8) implies in particular that for $\ell = 1, 2$,

$$\begin{aligned}&\left\|\nabla^\ell n\right\|_{H^{N-\ell}}^2 + \left\|\nabla^{\ell+1} u\right\|_{H^{N-\ell-1}}^2 + \left\|\nabla^{\ell+1} q\right\|_{H^{N-\ell-1}}^2 + \left\|\nabla^{\ell+1}\nabla\Phi\right\|_{H^{N-\ell-1}}^2\\ &\quad\geq C_0\left(\left\|\nabla^\ell n\right\|_{H^{N-\ell}}^2 + \left\|\nabla^\ell q\right\|_{H^{N-\ell}}^2 + \left\|\nabla^\ell u\right\|_{H^{N-\ell}}^2 + \left\|\nabla^\ell\nabla\Phi\right\|_{H^{N-\ell}}^2\right)^{1+\frac{1}{\ell+s}}. \end{aligned}\tag{4.12}$$

Thus, we deduce from (4.6) with $m = N$ the following time differential inequality

$$\frac{d}{dt}\mathcal{E}_\ell^N + C_0\left(\mathcal{E}_\ell^N\right)^{1+\frac{1}{\ell+s}} \leq 0 \quad \text{for } \ell = 1, 2. \tag{4.13}$$

Solving this inequality directly gives

$$\mathcal{E}_\ell^N(t) \leq C_0(1+t)^{-(\ell+s)} \quad \text{for } \ell = 1, 2. \tag{4.14}$$

This implies that for $s \in [0, 1/2]$,

$$\left\|\nabla^\ell(n(t), q(t), u(t))\right\|_{H^{N-\ell}}^2 + \left\|\nabla^\ell\nabla\Phi(t)\right\|_{H^{N+1-\ell}}^2 \leq C_0(1+t)^{-(\ell+s)} \quad \text{for } \ell = 1, 2. \tag{4.15}$$

On the other hand, since $n = \operatorname{div}\nabla\Phi$, we have

$$\left\|\nabla^\ell n(t)\right\|_{L^2}^2 \leq \left\|\nabla^{\ell+1}\nabla\Phi(t)\right\|_{L^2}^2 \leq C_0(1+t)^{-(\ell+1+s)} \quad \text{for } \ell = 0, 1. \tag{4.16}$$

Hence, by (4.15), (4.16), (4.9) and the interpolation, we get (1.10)–(1.11) for $s \in [0, 1/2]$. □

Now we can present the

Proof of (1.9)–(1.11) for $s \in (1/2, 3/2)$.

$$\left\|\nabla^\ell n(t), q(t), u(t)\right\|_{H^{N-\ell}}^2 + \left\|\nabla^\ell\nabla\Phi(t)\right\|_{H^{N+1-\ell}}^2 \leq C_0(1+t)^{-(\ell+1/2)} \quad \text{for } \ell = 0, 1, 2 \tag{4.17}$$

and

$$\left\|\nabla^\ell n(t)\right\|_{L^2}^2 \leq C_0(1+t)^{-(\ell+3/2)} \quad \text{for } \ell = 0, 1. \tag{4.18}$$

Hence, by (4.17)–(4.18), we deduce from (3.3) that for $s \in (1/2, 3/2)$,

$$\begin{aligned}\mathcal{E}_{-s}(t) &\leq \mathcal{E}_{-s}(0) + C\int_0^t \|(n, u, q, \nabla u)\|_{L^2}^{s-1/2}\left(\|n, q\|_{H^2} + \|\nabla u\|_{H^1}\right)^{5/2-s}\sqrt{\mathcal{E}_{-s}(\tau)}\,d\tau\\ &\leq C_0 + C_0\int_0^t (1+\tau)^{-(7/4-s/2)}\,d\tau \sup_{0\leq\tau\leq t}\sqrt{\mathcal{E}_{-s}(\tau)}\\ &\leq C_0\left(1 + \sup_{0\leq\tau\leq t}\sqrt{\mathcal{E}_{-s}(\tau)}\right). \end{aligned}\tag{4.19}$$

This implies (1.9) for $s \in (1/2, 3/2)$, that is,

$$\left\|\Lambda^{-s}n(t)\right\|_{L^2}^2 + \left\|\Lambda^{-s}u(t)\right\|_{L^2}^2 + \left\|\Lambda^{-s}q(t)\right\|_{L^2}^2 + \left\|\Lambda^{-s}\nabla\Phi(t)\right\|_{L^2}^2 \leq C_0 \quad \text{for } s \in (1/2, 3/2). \tag{4.20}$$

Now that we have proved (4.20), we may repeat the arguments leading to (1.10)–(1.11) for $s \in [0, 1/2]$ to prove that they hold also for $s \in (1/2, 3/2)$. □

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References

[1] P.A. Markowich, C.A. Ringhofer, C. Schimeiser, Semiconductor, Springer, 1990.
[2] D. Donatelli, Local and global existence for the coupled Navier–Stokes–Poisson problem, Quart. Appl. Math. 61 (2) (2003) 345–361.
[3] Y.-H. Zhang, Z. Tan, On the existence of solutions to the Navier–Stokes–Poisson equations of a two-dimensional compressible flow, Math. Methods Appl. Sci. 30 (2007) 305–329.
[4] C. Hao, H.-L. Li, Global existence for compressible Navier–Stokes–Poisson equations in three and higher dimensions, J. Differential Equations 246 (2009) 4791–4812.

[5] Z. Tan, G.C. Wu, Global existence for the non-isentropic compressible Navier–Stokes–Poisson system in three and higher dimensions, Nonlinear Anal. RWA 13 (2) (2012) 650–664.
[6] A. Matsumura, T. Nishida, The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids, Proc. Japan Acad. Ser. A 55 (1979) 337–342.
[7] A. Matsumura, T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases, J. Math. Kyoto Univ. 20 (1980) 67–104.
[8] Y. Guo, Y.J. Wang, Decay of dissipative equations and negative Sobolev spaces, Comm. Partial Differential Equations (2012) http://dx.doi.org/10.1080/03605302.2012.696296.
[9] K. Deckelnick, L^2-decay for the compressible Navier–Stokes equations in unbounded domains, Comm. Partial Differential Equations 18 (1993) 1445–1476.
[10] H.-L. Li, T. Zhang, Large time behavior of solutions to 3D compressible Navier–Stokes–Poisson system, Science China Math. (2012) 159–177, http://dx.doi.org/10.1007/s,1425-011-4280-z.
[11] Y.J. Wang, Decay of the Navier–Stokes–Poisson equations, J. Differential Equations 253 (2012) 273–297.
[12] G.-J. Zhang, H.-L. Li, C.-J. Zhu, Optimal decay rate of the non-isentropic compressible Navier–Stokes–Poisson system in $\mathbb{R}^3$, J. Differential Equations 250 (2) (2011) 866–891.
[13] L. Nirenberg, On elliptic partial differential equations, Ann. Sc. Norm. Super. Pisa 13 (1959) 115–162.
[14] V.A. Solonnikov, Evolution free boundary problem for equations of motion viscous compressible selfgravitating fluid, Stability Appl. Anal. Contin. Media 3 (1993) 257–275.