



# Weak solutions to a nonlinear variational sine–Gordon equation<sup>☆</sup>

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## ABSTRACT

This paper considers a nonlinear variational sine–Gordon equation  $u_{tt} - c(u)[c(u)u_x]_x + \frac{\lambda^2}{2} \sin(2u) = 0$  which describes the motion of long waves on a neutral dipole chain in the continuum limit and a few other physical phenomena. We establish the global existence of dissipative weak solutions to its Cauchy problem for initial data in the space  $H^1 \times L^2$  by using the Young measure theory in the setting of  $L^p$  spaces.

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## 1. Introduction

In this paper, we study the existence and regularity properties of weak solutions to the Cauchy problem for the nonlinear variational sine–Gordon equation

$$\begin{cases} u_{tt} - c(u)[c(u)u_x]_x + \frac{1}{2}\lambda^2 \sin(2u) = 0, \\ u|_{t=0} = u_0(x), \quad u_t|_{t=0} = u_1(x), \end{cases} \quad (1.1)$$

where  $\lambda$  is a constant, the wave speed  $c$  is a given smooth, bounded and positive function of  $u$  alone,  $u_0(x) \in H^1(\mathbb{R})$  and  $u_1(x) \in L^2(\mathbb{R})$ . When  $c(\cdot)$  is constant, Eq. (1.1) is reduced to the nonlinear sine–Gordon equation and then some exact solutions are presented among others in [1,7].

Eq. (1.1) arises in a number of various physical contexts, for example, it can be derived from the non-local sine–Gordon equation; see [19] for details. Most importantly, it can model, to the first-order, the motion of long waves on a neutral dipole chain in the continuum limit. In [31], Zorski and Infeld described long waves on a neutral dipole chain by a variational principle of the form

$$\frac{\delta}{\delta \mathbf{n}} \int \left( \mathbf{n}_t \cdot \mathbf{n}_t - (\beta - \alpha)(\mathbf{e} \cdot \mathbf{n}_x)^2 - \alpha \mathbf{n}_x^2 + \lambda^2(\mathbf{e} \cdot \mathbf{n})^2 \right) dx dt = 0, \quad \mathbf{n} \cdot \mathbf{n} = 1, \quad (1.2)$$

where  $\alpha, \beta$  and  $\lambda$  are constant scalar parameters,  $\mathbf{e}$  is a constant unit vector. Consider the special case that the director field  $\mathbf{n}$  takes of the form

$$\mathbf{n} = \mathbf{n}(x, t) = \cos u(x, t)\mathbf{e}_x + \sin u(x, t)\mathbf{e}_y, \quad (1.3)$$

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where  $u(x, t) \in \mathbb{R}^1$  is the dependent variable,  $\mathbf{e}_x$  and  $\mathbf{e}_y$  are, respectively, unit vectors in the  $x$  and  $y$  directions. Geometrically, this special case means that the dipoles are lined up parallel to the  $z$  axis, and  $u(x, t)$  measures the angle of the director field to the  $x$ -direction at each column. For planar deformations (1.3) with  $\mathbf{e} = \mathbf{e}_x$ , the variational principle (1.2) reduces to

$$\frac{\delta}{\delta u} \int [u_t^2 - c^2(u)u_x^2 + \lambda^2 \cos^2 u] dx dt = 0,$$

with the wave speed  $c$  given by  $c^2(u) = \alpha \cos^2 u + \beta \sin^2 u$ . The Euler–Lagrange equation corresponding to the above simplified variational principle is the nonlinear variational sine–Gordon equation (1.1).

We notice that Eq. (1.1) is a “sine–Gordon” version of nonlinear variational wave equation

$$u_{tt} - c(u)[c(u)u_x]_x = 0, \quad (1.4)$$

which is derived from the theory of nematic liquid crystals by Saxton [16] and Hunter and Saxton [12]. In [6], Glassey, Hunter and Zheng have shown that, even for smooth initial data, cusp-type singularities can form in finite time for Eq. (1.4). It is therefore necessary to study the global existence of weak solutions. There are at least two natural distinct classes of admissible weak solutions, which are called dissipative and conservative solutions. The existence of a dissipative weak solution to the Cauchy problem of (1.4), as well as for related asymptotic models, has been extensively studied by Zhang and Zheng [21–28] and Hunter and Zheng [13]. The relevant existence results of conservative weak solutions for (1.4) and its generalized equations are presented in [2,3,8,10,11,29,30].

In [17], Song proved that smooth solutions of (1.1) may breakdown in finite time. Recently, we [9] have established an energy-conservative weak solution to its Cauchy problem for initial data of finite energy. In the present paper, we study the existence of dissipative type weak solutions for (1.1).

The purpose of this paper is to establish the global existence of dissipative weak solutions for (1.1) under the condition that the wave speed  $c(u)$  satisfies  $c'(\cdot) \geq 0$  and  $c'(u_0(\cdot)) > 0$ . The main difficulty is, as the variational wave equation (1.4), that the potential oscillations get amplified unboundedly by quadratic growth terms of the equation and the possible concentrations in the approximate solutions. We shall use the Young measure method and the relevant techniques used in Zhang and Zheng [27,28] to handle this difficulty. Comparing with Eq. (1.4), the  $\sin(2u)$  term in Eq. (1.1) may lead to the problem more complicated, for instance, we should establish estimates to control the effect caused by this term. However, we will see that the  $\sin(2u)$  term does not affect the existence and regularity of solutions of Eq. (1.1).

We here introduce some notations.  $\text{Lip}$  stands for Lipschitz,  $\mathbb{R}^+ = (0, \infty)$ ,

$$\begin{aligned} R &:= \partial_t u + c(u)\partial_x u, & R_0(x) &= R(0, x), \\ S &:= \partial_t u - c(u)\partial_x u, & S_0(x) &= S(0, x), \end{aligned} \quad (1.5)$$

and  $\tilde{c}(\cdot) := \frac{1}{4} \ln c(\cdot)$ . With the above notations, Eq. (1.1) can be rewritten as

$$\begin{cases} \partial_t R - c(u)\partial_x R = \tilde{c}'(u)(R^2 - S^2) - \frac{1}{2}\lambda^2 \sin(2u), \\ \partial_t S + c(u)\partial_x S = \tilde{c}'(u)(S^2 - R^2) - \frac{1}{2}\lambda^2 \sin(2u), \\ \partial_x u = \frac{R - S}{2c(u)}, \\ R|_{t=0} = R_0(x), \quad S|_{t=0} = S_0(x), \quad u|_{t=0} = u_0(x). \end{cases} \quad (1.6)$$

Before we state our main result, let us first recall the definition of weak solutions introduced by Zhang and Zheng [27,28].

**Definition 1.1.** A function  $u(t, x) ((t, x) \in \mathbb{R}^+ \times \mathbb{R})$  is called an admissible weak solution to (1.1) if the following hold.

(i)  $u(t, x) \in L^\infty(\mathbb{R}^+, H^1(\mathbb{R})) \cap \text{Lip}(\mathbb{R}^+, L^2(\mathbb{R}))$ , and

$$\int_{\mathbb{R}} \left( |\partial_t u|^2 + |c(u)\partial_x u|^2 + \lambda^2 \sin^2 u \right) dx \leq \int_{\mathbb{R}} \left( |u_1|^2 + |c(u_0)\partial_x u_0| + \lambda^2 \sin^2 u_0 \right) dx. \quad (1.7)$$

(ii) For all test functions  $\phi(t, x) \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$ ,

$$\iint_{\mathbb{R}^+ \times \mathbb{R}} \left\{ \partial_t \phi \partial_t u - \partial_x \phi c^2(u) \partial_x u - \phi \left[ c(u)c'(u)(\partial_x u)^2 + \frac{1}{2}\lambda^2 \sin(2u) \right] \right\} dx dt = 0. \quad (1.8)$$

(iii) (The entropy condition). For any  $(t_0, x_0)$  with  $t_0 > 0$ , there always exists a positive constant  $M(t_0, x_0)$  such that

$$R(t, x) \geq -M(t_0, x_0), \quad S(t, x) \geq -M(t_0, x_0) \quad (1.9)$$

hold in a neighborhood  $\mathcal{N}(t_0, x_0)$  of  $(t_0, x_0)$ .

(iv)  $u(t, x) \rightarrow u_0(x)$  in  $L^2(\mathbb{R})$  and  $\partial_t u(t, x) \rightarrow u_1(x)$  in the distributional sense as  $t \rightarrow 0^+$ .

Throughout this paper, we always assume that the wave speed  $c$  satisfies

$$0 < \dot{c} \leq c(\cdot) \leq \ddot{c}, \quad \text{and} \quad |c^{(l)}(\cdot)| \leq M_l, \quad l \geq 1 \quad (1.10)$$

for some positive constants  $\dot{c} < \ddot{c}$  and  $M_l$ . Our main result can be stated as follows.

**Theorem 1.1.** Suppose that  $c$  satisfies  $c'(\cdot) \geq 0$  and  $c'(u_0(\cdot)) > 0$ . Let  $u_0 \in H^1(\mathbb{R})$  and  $u_1 \in L^2(\mathbb{R})$ . Then (1.1) has a global admissible weak solution  $u$  in the sense of Definition 1.1. Moreover, one has

$$\iint_{\Omega} |\partial_x u|^{2+\theta} dx dt \leq C_{\Omega, \theta}, \quad \text{for any } \theta \in (0, 1), \quad (1.11)$$

where  $\Omega$  is a small neighborhood of any point  $(t, x) \in \overline{\mathbb{R}^+} \times \mathbb{R}$  at which  $c'(u(t, x)) \neq 0$ , and  $C_{\Omega, \theta}$  is a positive constant depending only on  $\Omega$ ,  $\theta$ ,  $\|u_0\|_{H^1}$  and  $\|u_1\|_{L^2}$ .

We point out that, by using the inequalities  $R(t, x) \leq M(t_0, x_0)$  and  $S(t, x) \leq M(t_0, x_0)$  to replace the entropy condition (1.9) in Definition 1.1, similar results also hold if we assume  $c'(\cdot) \leq 0$  and  $c'(u_0(\cdot)) < 0$ .

**Remark 1.2.** Comparing with the results of (1.4) in Zhang and Zheng [28], we observe that the  $\sin(2u)$  term in (1.1) does not affect the existence and regularity properties of solutions.

## 2. Approximate solutions and uniform estimates

Similar to the previous works by Zhang and Zheng [25,27,28], we define for  $\varepsilon > 0$

$$Q_\varepsilon(\xi) := \begin{cases} \frac{1}{2}\xi^2, & \xi < \frac{1}{\varepsilon}, \\ \frac{1}{\varepsilon}(\xi - \frac{1}{2\varepsilon}), & \xi \geq \frac{1}{\varepsilon}. \end{cases} \quad (2.1)$$

Now we define the approximate solution sequence by the following equations

$$\begin{cases} \partial_t R^\varepsilon - c(u^\varepsilon) \partial_x R^\varepsilon = \tilde{c}'(u^\varepsilon)(2Q_\varepsilon(R^\varepsilon) - S^{\varepsilon 2}) - \frac{1}{2}\lambda^2 \sin(2u^\varepsilon), \\ \partial_t S^\varepsilon + c(u^\varepsilon) \partial_x S^\varepsilon = \tilde{c}'(u^\varepsilon)(2Q_\varepsilon(S^\varepsilon) - R^{\varepsilon 2}) - \frac{1}{2}\lambda^2 \sin(2u^\varepsilon), \\ \partial_x u^\varepsilon = \frac{R^\varepsilon - S^\varepsilon}{2c(u^\varepsilon)}, \\ \lim_{x \rightarrow -\infty} u^\varepsilon(t, x) = 0, \\ R^\varepsilon|_{t=0} = R_0(x), \quad S^\varepsilon|_{t=0} = S_0(x), \quad u^\varepsilon|_{t=0} = u_0(x). \end{cases} \quad (2.2)$$

In what follows, we sometimes omit the superscript  $\varepsilon$  in the approximate solution sequence  $\{(R^\varepsilon, S^\varepsilon, u^\varepsilon)\}_{\varepsilon>0}$  for writing convenience.

**Lemma 2.1** (Solution of (2.2) With Smooth Data). For any given  $(R_0, S_0) \in C_c^\infty(\mathbb{R})$ , the problem (2.2) has a global smooth solution

$$(R, S)(t, x) \in L^\infty(\mathbb{R}^+, W^{1,\infty}(\mathbb{R})), \quad u(t, x) \in L^\infty(\mathbb{R}^+, W^{2,\infty}(\mathbb{R})),$$

which satisfies the energy inequalities

$$\int (R^2 + S^2 + 2\lambda^2 \sin^2 u)(t, x) dx \leq \int (R_0^2 + S_0^2 + 2\lambda^2 \sin^2 u_0)(x) dx \quad (2.3)$$

and

$$\int_0^\infty \int_{\mathbb{R}} c'(u^\varepsilon) G_\varepsilon dx dt \leq \int (R_0^2 + S_0^2 + 2\lambda^2 \sin^2 u_0)(x) dx, \quad (2.4)$$

where

$$G_\varepsilon := R[R^2 - 2Q_\varepsilon(R)] + S[S^2 - 2Q_\varepsilon(S)] \geq 0.$$

Moreover, if the plus and minus characteristics  $\Phi_t^\pm(x)$  are introduced as

$$\frac{d}{dt} \Phi_t^\pm = \pm c(u(t, \Phi_t^\pm)), \quad \Phi_t^\pm|_{t=0} = x, \quad (2.5)$$

then one has the energy inequality in a characteristic cone

$$\int_a^d (R^2 + \lambda^2 \sin^2 u)(t_a^+(y), y) dy + \int_d^b (S^2 + \lambda^2 \sin^2 u)(t_b^-(y), y) dy \leq \frac{1}{2} \int_a^b (R_0^2 + S_0^2 + 2\lambda^2 \sin^2 u_0)(x) dx, \quad (2.6)$$

where  $a < b$ ,  $d$  is where the two characteristics  $\Phi_t^+(a)$  and  $\Phi_t^-(b)$  meet at some positive time,  $t = t_a^+(y)$  and  $t = t_b^-(y)$  are, respectively, the inverse functions of  $y = \Phi_t^+(a)$  and  $y = \Phi_t^-(b)$ .

**Proof.** The local existence of smooth solutions to system (2.2) with smooth initial data can be obtained by a standard procedure. Now let  $T^*$  be the life span of a smooth solution to (2.2). We can prove by the characteristic smooth method and Gronwall's inequality that for any  $t < T^*$ ,

$$\begin{aligned} \left( \|R(t, \cdot)\|_{L^\infty} + \|S(t, \cdot)\|_{L^\infty} \right) &< +\infty \text{ implies} \\ \left( \|R(t, \cdot)\|_{W^{2,\infty}} + \|S(t, \cdot)\|_{W^{2,\infty}} \right) &< +\infty. \end{aligned} \quad (2.7)$$

Therefore, in order to establish the global existence, one needs only to show that  $\|R(t, \cdot)\|_{L^\infty} + \|S(t, \cdot)\|_{L^\infty} < +\infty$  for any  $t < +\infty$ .

We first establish the energy inequalities for any  $0 \leq t < T^*$ . Multiplying the first equation of (2.2) by  $R$  and the second by  $S$  and adding them, we obtain with some rearrangement

$$\partial_t (R^2 + S^2 + 2\lambda^2 \sin^2 u) - \partial_x (c(u)(R^2 - S^2)) = -2\tilde{c}'(u)\{R[R^2 - 2Q_\varepsilon(R)] + S[S^2 - 2Q_\varepsilon(S)]\} \leq 0. \quad (2.8)$$

Integrating (2.8) over  $\mathbb{R}$  with respect to  $x$  gets (2.3) and (2.4). we derive (2.5) by integrating (2.8) over the characteristic cone  $\Delta := \{(t, x) | \Phi_t^+(a) \leq x \leq \Phi_t^-(b), 0 \leq t < T^*\}$ .

Now we establish  $L^\infty$  bounds for  $R$  and  $S$ . From the first equation of (2.2), we find

$$\frac{d}{dt} R(t, \Phi_t^-(x)) \geq -\tilde{c}'(u)S^2 - \frac{1}{2}\lambda^2 \sin(2u), \quad (2.9)$$

and

$$\frac{d}{dt} R(t, \Phi_t^-(x)) \leq 2\tilde{c}'(u)Q_\varepsilon(R) - \frac{1}{2}\lambda^2 \sin(2u), \quad (2.10)$$

which, by using (2.5), lead to

$$|R(t, \Phi_t^-(x))| \leq \|R_0\|_{L^\infty} + \frac{1}{2}\lambda^2 t + C \int (R_0^2 + S_0^2 + 2\lambda^2 \sin^2 u_0)(x) dx \quad (2.11)$$

for  $0 \leq t < T^*$ , where constant  $C$  depends on  $\dot{c}$ ,  $\ddot{c}$ ,  $M_1$  and  $\varepsilon$ . Similar inequality also holds for  $S$ . Thus the proof of Lemma 2.1 is completed.  $\square$

Based on Lemma 2.1, we can prove exactly as that in the proof of Lemma 3 of Zhang and Zheng [24] that there exists a subsequence of the approximate solutions  $\{u^\varepsilon\}$ , which we still denote by  $\{u^\varepsilon\}$ , such that

$$u^\varepsilon(t, x) \text{ converges uniformly to a continuous function } u(t, x) \quad (2.12)$$

on any compact subset of  $(0, \infty) \times \mathbb{R}$ .

We next use the continuity of  $u$  to establish the  $L^{2+\theta}$  estimate for  $\partial_x u$  and the entropy condition for  $\{R^\varepsilon, S^\varepsilon\}$ .

**Lemma 2.2** (Local  $L^{2+\theta}$  Estimate). Let  $(R_0, S_0) \in L^2$ . Then the solutions of (2.2) satisfy

$$\iint_{\Omega} (R - S)^2 (R^\theta + S^\theta) dx dt \leq C_{\Omega, \theta} \quad (2.13)$$

for any  $\theta \in (0, 1)$ , where  $\Omega$  is a small neighborhood of any point  $(t, x)$  at which  $c'(u(t, x)) \neq 0$ ,  $C_{\Omega, \theta}$  depends only on  $\Omega$ ,  $\theta$  and the  $L^2$  norms of  $R_0, S_0, u_0$ .

**Proof.** Multiplying the first equation of (2.2) by  $R^\theta$ , where  $\theta = d_2/d_1 \in (0, 1)$  in which  $d_1$  is an odd positive integer and  $d_2$  an even positive integer, and rearranging the resulting equation yields,

$$\begin{aligned} \frac{1-\theta}{1+\theta} \tilde{c}'(u)(R-S)R^{1+\theta} + \tilde{c}'(u)(R^\theta S^2 - SR^{1+\theta}) \\ = \tilde{c}'(u)[2Q_\varepsilon(R) - R^2]R^\theta - \frac{1}{2}\lambda^2 \sin(2u)R^\theta - \frac{1}{1+\theta} \{\partial_t R^{1+\theta} - \partial_x (c(u)R^{1+\theta})\}. \end{aligned} \quad (2.14)$$

Similar arguments for  $S$  gives

$$\begin{aligned} & \frac{1-\theta}{1+\theta} \tilde{c}'(u)(S-R)S^{1+\theta} + \tilde{c}'(u)(S^\theta R^2 - RS^{1+\theta}) \\ &= \tilde{c}'(u)[2Q_\varepsilon(S) - S^2]S^\theta - \frac{1}{2}\lambda^2 \sin(2u)S^\theta - \frac{1}{1+\theta} \{\partial_t S^{1+\theta} + \partial_x(c(u)S^{1+\theta})\}. \end{aligned} \quad (2.15)$$

Adding (2.14) and (2.15), we obtain

$$\begin{aligned} & \frac{1-\theta}{1+\theta} \tilde{c}'(u)(R-S)(R^{1+\theta} - S^{1+\theta}) + \tilde{c}'(u)R^\theta S^\theta (R-S)(R^{1-\theta} - S^{1-\theta}) \\ &= \tilde{c}'(u) \left\{ [2Q_\varepsilon(R) - R^2]R^\theta + [2Q_\varepsilon(S) - S^2]S^\theta \right\} \\ & \quad - \frac{1}{2}\lambda^2 \sin(2u)(R^\theta + S^\theta) - \frac{1}{1+\theta} \left\{ \partial_t(R^{1+\theta} + S^{1+\theta}) + \partial_x(c(u)(S^{1+\theta} - R^{1+\theta})) \right\}. \end{aligned} \quad (2.16)$$

For any point  $(t_0, x_0)$  satisfying  $c'(u(t_0, x_0)) \neq 0$ , there exists a small neighborhood  $\Omega_0$  of  $(t_0, x_0)$  such that  $c'(u^\varepsilon(t, x)) \geq c'(u(t_0, x_0))/2$  in  $\Omega$  for all sufficiently small  $\varepsilon$  by the continuity of function  $u$ . We integrate (2.16) over  $\Omega \subset \subset \Omega_0$  to complete the proof. Here the  $\sin(2u)(R^\theta + S^\theta)$  term in (2.16) can be controlled by applying Cauchy's inequality and the energy inequality (2.3).  $\square$

We now establish the entropy condition for the approximate solutions  $\{R^\varepsilon, S^\varepsilon\}$ , which is needed to prove their precompactness.

**Lemma 2.3.** *Let  $(R_0, S_0) \in L^2(\mathbb{R})$ . For any point  $(t_0, x_0)$  with  $t_0 > 0$  and  $c'(u(t_0, x_0)) \neq 0$ , there exists a neighborhood  $\mathcal{N}(t_0, x_0)$  of  $(t_0, x_0)$  such that for all  $(t, x) \in \mathcal{N}(t_0, x_0)$*

$$R^\varepsilon(t, x) \geq -M(t_0, x_0), \quad S^\varepsilon(t, x) \geq -M(t_0, x_0) \quad (2.17)$$

for some positive constant  $M(t_0, x_0)$  independent of  $\varepsilon$ .

**Proof.** By the continuity of  $u$ , there exists a small enough number  $\eta$  such that for all sufficiently small  $\varepsilon$ ,

$$c'(u(t, x)) \geq \frac{c'(u(t_0, x_0))}{2} > 0, \quad c'(u^\varepsilon(t, x)) \geq \frac{c'(u(t_0, x_0))}{4} > 0 \quad (2.18)$$

in the ball  $B_\eta(t_0, x_0) := \{(t, x) \mid |t - t_0| + |x - x_0| \leq \eta^2\}$ , where  $(t_0, x_0)$  is a point satisfying the assumptions of the lemma.

We notice that the plus and minus characteristics  $\Phi_t^{\varepsilon, \pm}$  defined by

$$\begin{cases} \frac{d}{dt} \Phi_t^{\varepsilon, \pm}(y) = \pm c(u^\varepsilon(t, \Phi_t^{\varepsilon, \pm}(y))), \\ \Phi_t^{\varepsilon, \pm}(y)|_{t=0} = y, \end{cases}$$

can pass through any point in the upper plane. Then from any point  $(t, x) \in B_{\eta/2}(t_0, x_0)$ , we draw the minus characteristic  $x = \Phi_t^{\varepsilon, -}(y)$  for some fixed  $y$ , which intersects the boundary of  $B_\eta(t_0, x_0)$  at two points  $(t_1, \Phi_{t_1}^{\varepsilon, -}(y))$ ,  $(t_2, \Phi_{t_2}^{\varepsilon, -}(y))$ , and the boundary of  $B_{\eta/2}(t_0, x_0)$  at two points  $(t_3, \Phi_{t_3}^{\varepsilon, -}(y))$ ,  $(t_4, \Phi_{t_4}^{\varepsilon, -}(y))$  satisfying  $t_1 < t_3 \leq t_4 < t_2$ . It is easy to see by (1.10) and (2.18) that  $t_3 - t_1 \geq \delta\eta$  for some small positive number  $\delta$ . Moreover, it follows from (2.6) that

$$\int_0^\infty (\tilde{c}'(u)S^{\varepsilon 2})(s, \Phi_s^{\varepsilon, -}(y)) \, ds \leq C_1, \quad (2.19)$$

where  $C_1$  is a constant depending only on the  $L^2$  norms of  $R_0, S_0$  and  $u_0$ . Let  $C_2 = 4\sqrt{(C_1 + \lambda^2\eta)/\delta\eta c'(u(t_0, x_0))}$ . If  $R^\varepsilon(t_1, \Phi_{t_1}^{\varepsilon, -}(y)) \geq -C_2$ , then integrating the first equation of (2.2) along the minus characteristic from  $(t_1, \Phi_{t_1}^{\varepsilon, -}(y))$  to  $(t, \Phi_t^{\varepsilon, -}(y))$  yields

$$\begin{aligned} R^\varepsilon(t, \Phi_t^{\varepsilon, -}(y)) &= R^\varepsilon(t_1, \Phi_{t_1}^{\varepsilon, -}(y)) + \int_{t_1}^t \left\{ \tilde{c}'(u^\varepsilon)[2Q_\varepsilon(R^\varepsilon) - S^{\varepsilon 2}] - \frac{1}{2}\lambda^2 \sin(2u) \right\}(s, \Phi_s^{\varepsilon, -}(y)) \, ds \\ &\geq -C_2 - C_1 - \frac{1}{2}\lambda^2(t - t_1). \end{aligned} \quad (2.20)$$

Otherwise, we may without loss of generality assume that there exists a number  $\tau \in [t_1, t_2]$  such that  $R^\varepsilon(t, \Phi_t^{\varepsilon, -}(y)) \leq -C_2$  for  $t \in [t_1, \tau]$  and  $R^\varepsilon(\tau, \Phi_\tau^{\varepsilon, -}(y)) = -C_2$ . Obviously, one has  $R^\varepsilon(t, \Phi_t^{\varepsilon, -}(y)) \geq -C_2 - C_1 - \lambda^2\eta$  for  $t \in [\tau, t_2]$  by repeating the above process. Now we consider the interval  $[t_1, \tau]$ . Dividing the first equation of (2.2) by  $R^{\varepsilon 2}$  and integrating the

resulting over  $[t_1, t]$  gives

$$\begin{aligned} \frac{1}{R^\varepsilon(t, \Phi_t^{\varepsilon,-}(y))} &= \frac{1}{R^\varepsilon(t_1, \Phi_{t_1}^{\varepsilon,-}(y))} - \int_{t_1}^t \left\{ \tilde{c}'(u^\varepsilon) \left( \frac{S^\varepsilon}{R^\varepsilon} \right)^2 + \frac{1}{2} \lambda^2 \frac{\sin(2u)}{R^{\varepsilon 2}} - \tilde{c}'(u^\varepsilon) \right\} (s, \Phi_s^{\varepsilon,-}(y)) \, ds \\ &\leq -\frac{1}{4} c'(u(t_0, x_0))(t - t_1) + \frac{C_1}{C_2^2} + \frac{1}{2} \lambda^2 \frac{t - t_1}{C_2^2} \\ &\leq -\frac{\delta\eta}{8} c'(u(t_0, x_0)), \end{aligned}$$

if  $t - t_1 \geq \delta\eta$ . Thus we have

$$R^\varepsilon(t, \Phi_t^{\varepsilon,-}(y)) \geq -\frac{8}{\delta\eta c'(u(t_0, x_0))} \quad \text{for } t \in [t_1 + \delta\eta, \tau]. \quad (2.21)$$

Denote  $M(t_0, x_0) := \max(C_1 + C_2 + \lambda^2\eta, 8/\delta\eta c'(u(t_0, x)))$ , then we get by summing up (2.20) and (2.21)

$$R^\varepsilon(t, \Phi_t^{\varepsilon,-}(y)) \geq -M(t_0, x_0) \quad \text{for } t \in [t_1 + \delta\eta, t_2].$$

The above inequality thus holds for  $(t, x) \in B_{\eta/2}(t_0, x_0)$  by the fact that  $t_3 \geq t_1 + \delta\eta$ . The above argument is also valid for  $S^\varepsilon$ . Hence we can take  $\mathcal{N}(t_0, x_0) = B_{\eta/2}(t_0, x_0)$  to finish the proof.  $\square$

The above lemma and the assumptions  $c'(\cdot) \geq 0$ ,  $c'(u_0(\cdot)) > 0$  directly lead to the following lemma.

**Lemma 2.4.** Assume that  $c'(\cdot) \geq 0$ ,  $c'(u_0(\cdot)) > 0$ . Let  $(R_0, S_0) \in L^2(\mathbb{R})$  and let  $t_0$  be any sufficiently small positive number. Then for any point  $(\bar{t}, \bar{x})$  with  $\bar{t} > t_0$ , there exists a neighborhood  $\mathcal{N}(\bar{t}, \bar{x})$  such that for all  $(t, x) \in \mathcal{N}(\bar{t}, \bar{x})$

$$R^\varepsilon(t, x) \geq -\bar{M}, \quad S^\varepsilon(t, x) \geq -\bar{M} \quad (2.22)$$

for some positive constant  $\bar{M}$  depending only on  $t_0, c'(u_0)$  and the  $L^2$  norm of  $(R_0, S_0, u_0)$ .

### 3. Precompactness

In this section, we establish the precompactness of the approximate solutions by mollifying the initial data and then passing the limit  $\varepsilon \rightarrow 0$  in system (2.2).

Let  $(R_0, S_0) \in L^2(\mathbb{R})$ . Let  $j_\varepsilon(x)$  be the standard Friedrichs' mollifier. We define  $R_0^\varepsilon = (R_0 \chi_\varepsilon) * j_\varepsilon$  and  $S_0^\varepsilon = (S_0 \chi_\varepsilon) * j_\varepsilon$ , where  $\chi_\varepsilon(x) = \chi(\frac{x}{\varepsilon})$  with  $\chi(x) \in C_c^\infty(\mathbb{R})$  and  $\chi(x) = 1$  around  $x = 0$ . Then Lemma 2.1 states that the problem (2.2) has a global smooth solution  $(R^\varepsilon, S^\varepsilon, u^\varepsilon)$  with the initial data  $(R_0^\varepsilon, S_0^\varepsilon)$ . Moreover, we have by (2.3)

$$\int_{\mathbb{R}} (R^{\varepsilon 2} + S^{\varepsilon 2} + 2\lambda^2 \sin^2 u^\varepsilon)(t, x) \, dx \leq \int_{\mathbb{R}} (R_0^2 + S_0^2 + 2\lambda^2 \sin^2 u_0)(x) \, dx. \quad (3.1)$$

We shall also use energy estimate (2.6) and (2.13) in this new setting.

To prove the strong compactness of the approximate solution sequence, we need to investigate the structure of the Young measure associated with the sequence  $\{R^\varepsilon, S^\varepsilon\}$ . For convenience of the reader, we state the existence of the Young measure as follows.

**Lemma 3.1** (Time-Distinguished Young Measure [20,18,5,14]). There exists a subsequence of the solution sequence  $\{R^\varepsilon(t, x), S^\varepsilon(t, x)\}$ , still denoted by the same notation for convenience, and three families of Young measures  $\nu_{tx}^1(\xi)$  on  $\mathbb{R}$ ,  $\nu_{tx}^2(\eta)$  on  $\mathbb{R}$  and  $\mu_{tx}(\xi, \eta)$  on  $\mathbb{R}^2$ , such that for all continuous functions  $f(\lambda) \in C_c^\infty(\mathbb{R})$ ,  $\psi(x) \in C_c^\infty(\mathbb{R})$ ,  $g(\xi, \eta) \in C^\infty(\mathbb{R}^2)$  with  $g(\xi, \eta) = o(|\xi| + |\eta|)^p$  as  $|\xi| + |\eta| \rightarrow \infty$  for some  $p < 2$ , and  $\varphi(t, x) \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$ , there hold

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} f(R^\varepsilon(t, x)) \psi(x) \, dx &= \iint_{\mathbb{R} \times \mathbb{R}} f(\xi) \psi(x) \, d\nu_{tx}^1(\xi) \, dx, \\ \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} f(S^\varepsilon(t, x)) \psi(x) \, dx &= \iint_{\mathbb{R} \times \mathbb{R}} f(\eta) \psi(x) \, d\nu_{tx}^2(\eta) \, dx, \end{aligned} \quad (3.2)$$

uniformly in every compact subset of  $[0, \infty)$ , and

$$\lim_{\varepsilon \rightarrow 0} \iint_{\mathbb{R}^+ \times \mathbb{R}} g(R^\varepsilon(t, x), S^\varepsilon(t, x)) \varphi(t, x) \, dx \, dt = \iint_{\mathbb{R}^+ \times \mathbb{R}} \iint_{\mathbb{R} \times \mathbb{R}} g(\xi, \eta) \varphi(t, x) \, d\mu_{tx}(\xi, \eta) \, dx \, dt. \quad (3.3)$$

Moreover,

$$\begin{aligned} t \in [0, \infty) &\mapsto \iint_{\mathbb{R} \times \mathbb{R}} f(\xi) \psi(x) \, dv_{tx}^1(\xi) dx \text{ is continuous,} \\ t \in [0, \infty) &\mapsto \iint_{\mathbb{R} \times \mathbb{R}} f(\eta) \psi(x) \, dv_{tx}^2(\eta) dx \text{ is continuous,} \end{aligned} \quad (3.4)$$

and

$$\mu_{tx}(\xi, \eta) = v_{tx}^1(\xi) \otimes v_{tx}^2(\eta). \quad (3.5)$$

Denote

$$\overline{g(R, S)} := \int_{\mathbb{R}} g(\xi, \eta) \, d\mu_{tx}(\xi, \eta).$$

Thus,  $(\bar{R}, \bar{S})$  represents the weak-star limit of  $\{R^\varepsilon, S^\varepsilon\}$  in  $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}))$  or the weak limit in  $L^2((0, T) \times \mathbb{R})$  for all  $T < \infty$ .

With the above preparation, we now show the precompactness of  $\{R^\varepsilon, S^\varepsilon\}$ .

**Lemma 3.2** (Precompactness of  $\{R^\varepsilon, S^\varepsilon\}$ ). Assume that  $c'(\cdot) \geq 0$  and  $c'(u_0(\cdot)) > 0$ . Let  $(R_0, S_0) \in L^2(\mathbb{R})$ . Then  $v_{tx}^1(\xi) = \delta_{\bar{R}(t, x)}(\xi)$  and  $v_{tx}^2(\eta) = \delta_{\bar{S}(t, x)}(\eta)$ .

**Proof.** The idea, as in Zhang and Zheng [24,25,27,28], is to derive the evolution equations (inequalities) for the quantities  $(\bar{R}^2 - \bar{R}^2)$  and  $(\bar{S}^2 - \bar{S}^2)$ , so that they are zero for all positive time if all these quantities vanish at time zero which are true in our case. The derivation process is valid only in the weak sense.

We first derive the equation for  $\bar{R}^2$ . By direct calculation, the equation for  $R^\varepsilon$  in (2.2) can be rewritten as

$$\partial_t R^\varepsilon - \partial_x(c(u^\varepsilon)R^\varepsilon) = -\tilde{c}'(u^\varepsilon)(R^\varepsilon - S^\varepsilon)^2 - \tilde{c}'(u^\varepsilon)[R^{\varepsilon 2} - 2Q_\varepsilon(R^\varepsilon)] - \frac{1}{2}\lambda^2 \sin(2u^\varepsilon), \quad (3.6)$$

which, by taking  $\varepsilon \rightarrow 0$  and noticing the facts

$$\tilde{c}'(u^\varepsilon)(R^\varepsilon - S^\varepsilon)^2 \rightharpoonup \tilde{c}'(u)(\bar{R}^2 - 2\bar{R} \cdot \bar{S} + \bar{S}^2), \quad (3.7)$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\mathbb{R}} \tilde{c}'(u^\varepsilon)[R^{\varepsilon 2} - 2Q_\varepsilon(R^\varepsilon)] \, dx dt = 0, \quad (3.8)$$

yields

$$\partial_t \bar{R} - \partial_x(c(u)\bar{R}) = -\tilde{c}'(u)(\bar{R}^2 - 2\bar{R} \cdot \bar{S} + \bar{S}^2) - \frac{1}{2}\lambda^2 \sin(2u). \quad (3.9)$$

Here (3.7) can be proved as in [27,28] by Lemma 3.1 and (2.13), while (3.8) holds by (2.4). Convolution (3.9) with the standard Friedrichs' mollifier  $j_\varepsilon(x)$ , one has

$$\partial_t \bar{R}^\varepsilon - \partial_x(c(u)\bar{R}^\varepsilon) = -\left\{ \tilde{c}'(u)(\bar{R} - \bar{S})^2 + \frac{1}{2}\lambda^2 \sin(2u) \right\} * j_\varepsilon + \gamma_\varepsilon, \quad (3.10)$$

where

$$\bar{R}^\varepsilon = \int_{\mathbb{R}} \bar{R}(t, y) \cdot j_\varepsilon(x - y) \, dy, \quad \gamma_\varepsilon = j_\varepsilon * \partial_x(c(u)\bar{R}) - \partial_x(c(u)\bar{R}^\varepsilon).$$

The DiPerna–Lions folklore lemma (see Lemma 2.1 in DiPerna and Lions [4] or Lemma 2.3 in Lions [15]) and the Lebesgue dominated convergence theorem in the time direction lead to  $\gamma_\varepsilon \rightarrow 0$  in  $L_{loc}^1(\mathbb{R}^+ \times \mathbb{R})$ .

Define for  $k > 0$

$$T_k(\xi) := \begin{cases} -k, & \xi \leq -k, \\ \xi, & |\xi| \leq k, \\ k, & \xi \geq k, \end{cases} \quad S_k(\xi) := \begin{cases} -k(\xi + \frac{1}{2}k), & \xi \leq -k, \\ \frac{1}{2}\xi^2, & |\xi| \leq k, \\ k(\xi - \frac{1}{2}k), & \xi \geq k. \end{cases}$$

Then multiplying (3.10) by  $T_k(\bar{R}^\varepsilon)$  and taking  $\varepsilon \rightarrow 0$  in the resulting equation, we get after some simplifications

$$\begin{aligned} \partial_t S_k(\bar{R}) - \partial_x(c(u)S_k(\bar{R})) &= \tilde{c}'(u) \left\{ -2\bar{R}S_k(\bar{R}) + T_k(\bar{R})\bar{R}^2 + 2\bar{S}S_k(\bar{R}) - T_k(\bar{R})\bar{S}^2 - T_k(\bar{R})(\bar{R}^2 - \bar{R}^2) \right\} \\ &\quad - \frac{1}{2}\lambda^2 \sin(2u)T_k(\bar{R}). \end{aligned} \quad (3.11)$$

On the other hand, we multiply (3.6) by  $T_k(R^\varepsilon)$  to obtain

$$\begin{aligned} \partial_t S_k(R^\varepsilon) - \partial_x(c(u^\varepsilon)S_k(R^\varepsilon)) \\ = -2\tilde{c}'(u^\varepsilon)(R^\varepsilon - S^\varepsilon)S_k(R^\varepsilon) + \tilde{c}'(u^\varepsilon)T_k(R^\varepsilon)[2Q_\varepsilon(R^\varepsilon) - S^{\varepsilon 2}] - \frac{1}{2}\lambda^2 \sin(2u^\varepsilon)T_k(R^\varepsilon). \end{aligned} \quad (3.12)$$

By Lemma 3.1 and (2.13), one also has

$$\begin{aligned} \tilde{c}'(u^\varepsilon)(R^\varepsilon - S^\varepsilon)S_k(R^\varepsilon) &\rightharpoonup \tilde{c}'(u)(\bar{R} - \bar{S})\overline{S_k(\bar{R})}, \\ \tilde{c}'(u^\varepsilon)T_k(R^\varepsilon)[2Q_\varepsilon(R^\varepsilon) - S^{\varepsilon 2}] &\rightharpoonup \tilde{c}'(u)\overline{T_k(\bar{R})(\bar{R}^2 - \bar{S}^2)}. \end{aligned}$$

Then taking  $\varepsilon \rightarrow 0$  in (3.12) and using the above facts, we get

$$\begin{aligned} \partial_t \overline{S_k(\bar{R})} - \partial_x(c(u)\overline{S_k(\bar{R})}) &= \tilde{c}'(u) \left\{ -2\bar{R}\overline{S_k(\bar{R})} + \overline{T_k(\bar{R})\bar{R}^2} + 2\bar{S} \cdot \overline{S_k(\bar{R})} - \overline{T_k(\bar{R}) \cdot \bar{S}^2} \right\} \\ &\quad - \frac{1}{2}\lambda^2 \sin(2u)\overline{T_k(\bar{R})}. \end{aligned} \quad (3.13)$$

We subtract (3.11) from (3.13) to obtain after some rearrangement

$$\begin{aligned} \partial_t \left( \overline{S_k(\bar{R})} - S_k(\bar{R}) \right) - \partial_x \left( c(u)(\overline{S_k(\bar{R})} - S_k(\bar{R})) \right) &= \tilde{c}'(u) \left\{ 2(\bar{S} + T_k(\bar{R}))(\overline{S_k(\bar{R})} - S_k(\bar{R})) + (T_k(\bar{R}) - \overline{T_k(\bar{R})})\bar{S}^2 \right\} \\ &\quad - \frac{1}{2}\lambda^2 \sin(2u)(\overline{T_k(\bar{R})} - T_k(\bar{R})) + G_k, \end{aligned} \quad (3.14)$$

where

$$G_k := \tilde{c}'(u) \left\{ (T_k(\bar{R}) - k)\overline{(R - k)^2 1_{R \geq k}} + (T_k(\bar{R}) + k)\overline{(R + k)^2 1_{R \leq -k}} - (T_k(\bar{R}) - \overline{T_k(\bar{R})})k^2 \right\}.$$

We claim that, for  $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}$  with  $c'(u(t_0, x_0)) > 0$ , there exists a neighborhood  $\mathcal{N}(t_0, x_0)$  of  $(t_0, x)$  and some positive constant  $M(t_0, x_0)$  such that

$$G_k \leq 0 \quad \text{for } (t, x) \in \mathcal{N}(t_0, x_0), \quad (3.15)$$

when  $k \geq M(t_0, x_0)$ . The proof is based on Lemma 2.3 and the detailed structure of terms in  $G_k$ . We here omit its presentation for brevity. (See Zhang and Zheng [28] if needed.)

By letting  $f_k(t, x) := \overline{S_k(\bar{R})} - S_k(\bar{R})$ , we rewrite (3.14) as

$$\begin{aligned} \partial_t f_k - \partial_x(c(u)f_k) &= \tilde{c}'(u) \left\{ 2(\bar{S} + T_k(\bar{R}))f_k + (T_k(\bar{R}) - \overline{T_k(\bar{R})})\bar{S}^2 \right\} \\ &\quad - \frac{1}{2}\lambda^2 \sin(2u)(\overline{T_k(\bar{R})} - T_k(\bar{R})) + G_k. \end{aligned}$$

Notice that  $f_k \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}))$  for any fixed  $k$ . Thus we have by the Diperna–Lions folklore lemma and the Lebesgue dominated convergence theorem in the time direction

$$\begin{aligned} \partial_t f_k^\varepsilon - \partial_x(c(u)f_k^\varepsilon) &= \tilde{c}'(u) \left\{ 2(\bar{S} + T_k(\bar{R}))f_k^\varepsilon + (T_k(\bar{R}) - \overline{T_k(\bar{R})})\bar{S}^2 \right\} \\ &\quad - \frac{1}{2}\lambda^2 \sin(2u)(\overline{T_k(\bar{R})} - T_k(\bar{R})) + G_k * j_\varepsilon + \gamma_\varepsilon, \end{aligned} \quad (3.16)$$



where  $f_k^\varepsilon(t, x) := \int_{\mathbb{R}} f_k(t, y) j_\varepsilon(x - y) dy$  and  $\gamma_\varepsilon \rightarrow 0$  in  $L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})$ . Multiplying (3.16) by  $\frac{1}{2}(f_k^\varepsilon + \eta)^{-1/2}$  for any  $\eta > 0$  gets

$$\begin{aligned} \partial_t(f_k^\varepsilon + \eta)^{1/2} - \partial_x(c(u)(f_k^\varepsilon + \eta)^{1/2}) &= \left\{ \tilde{c}'(u)(\bar{R} + T_k(\bar{R}))f_k^\varepsilon + \frac{1}{2}G_k * j_\varepsilon \right\} (f_k^\varepsilon + \eta)^{-1/2} \\ &\quad - 2\tilde{c}'(u)(\bar{R} - \bar{S})(f_k^\varepsilon + \eta)^{1/2} + \frac{1}{2}\tilde{c}'(u)\bar{S}^2(T_k(\bar{R}) - \overline{T_k(\bar{R})})(f_k^\varepsilon + \eta)^{-1/2} \\ &\quad - \frac{1}{4}\lambda^2 \sin(2u)(\overline{T_k(\bar{R})} - T_k(\bar{R}))(f_k^\varepsilon + \eta)^{-1/2} + \frac{1}{2}(f_k^\varepsilon + \eta)^{-1/2}\gamma_\varepsilon, \end{aligned} \quad (3.17)$$

which, by taking  $\varepsilon \rightarrow 0$ , gives

$$\begin{aligned} \partial_t(f_k + \eta)^{1/2} - \partial_x(c(u)(f_k + \eta)^{1/2}) &= \left\{ \tilde{c}'(u)(\bar{R} + T_k(\bar{R}))f_k + \frac{1}{2}G_k \right\} (f_k + \eta)^{-1/2} \\ &\quad - 2\tilde{c}'(u)(\bar{R} - \bar{S})(f_k + \eta)^{1/2} + \frac{1}{2}\tilde{c}'(u)\bar{S}^2(T_k(\bar{R}) - \overline{T_k(\bar{R})})(f_k + \eta)^{-1/2} \\ &\quad - \frac{1}{4}\lambda^2 \sin(2u)(\overline{T_k(\bar{R})} - T_k(\bar{R}))(f_k + \eta)^{-1/2}. \end{aligned} \quad (3.18)$$

We notice that

$$|\overline{T_k(\bar{R})} - T_k(\bar{R})|(f_k + \eta)^{-1/2} \leq 2, \quad (3.19)$$

which follows from the equality [27,28]

$$\frac{1}{2}(\overline{T_k(\bar{R})} - T_k(\bar{R}))^2 \leq (\bar{S}_k(\bar{R}) - S_k(\bar{R})).$$

Then we get by (3.19)

$$\bar{S}^2|T_k(\bar{R}) - \overline{T_k(\bar{R})}|(f_k + \eta)^{-1/2} \leq 2\bar{S}^2 \quad (3.20)$$

and

$$|\sin(2u)||\overline{T_k(\bar{R})} - T_k(\bar{R})|(f_k + \eta)^{-1/2} \leq 2. \quad (3.21)$$

Moreover, we claim that

$$\lim_{k \rightarrow \infty} \|\overline{T_k(\bar{R})} - T_k(\bar{R})\|_{L^1([0, T] \times \mathbb{R})} = 0, \quad \forall T < \infty. \quad (3.22)$$

In fact, we first find by the Cauchy–Schwarz inequality that

$$|\bar{R} - \overline{T_k(\bar{R})}| = \int [(\xi - k)1_{\xi \geq k} + (\xi + k)1_{\xi \leq -k}] dv_{\text{ix}}^1(\xi) \leq \frac{1}{k} \int \xi^2 dv_{\text{ix}}^1(\xi),$$

which, together with (3.4), leads to

$$\lim_{k \rightarrow \infty} \|\bar{R} - \overline{T_k(\bar{R})}\|_{L^1([0, T] \times \mathbb{R})} = 0, \quad \forall T < \infty. \quad (3.23)$$

Similarly, one has

$$\lim_{k \rightarrow \infty} \|\bar{R} - T_k(\bar{R})\|_{L^1([0, T] \times \mathbb{R})} = 0, \quad \forall T < \infty. \quad (3.24)$$

Combining (3.23) and (3.24) arrives at (3.22). Thus, by the Lebesgue dominated convergence theorem, we find for any  $T > 0$  that

$$\begin{aligned} \lim_{k \rightarrow \infty} \|\bar{S}^2(T_k(\bar{R}) - \overline{T_k(\bar{R})})(f_k + \eta)^{-1/2}\|_{L^1([0, T] \times \mathbb{R})} &= 0, \\ \lim_{k \rightarrow \infty} \|\sin(2u)(\overline{T_k(\bar{R})} - T_k(\bar{R}))(f_k + \eta)^{-1/2}\|_{L^1([0, T] \times \mathbb{R})} &= 0. \end{aligned} \quad (3.25)$$

Using the Lebesgue dominated convergence theorem again, we have by (3.4)

$$\lim_{k \rightarrow \infty} f_k(t, x) = \frac{1}{2}(\bar{R}^2 - \bar{R}^2) := f(t, x). \quad (3.26)$$

Furthermore, (3.18), (3.20) and (3.21) imply that  $\{G_k(f_k + \eta)^{-1/2}\}$  is uniformly bounded in  $H_{\text{loc}}^{-1}((0, T) \times \mathbb{R}) + L^\infty([0, T] \times \mathbb{R})$  with respect to  $k$  and  $\eta$  for all  $T > 0$ . Hence a diagonal process provides

$$G_{k_i}(f_{k_i} + \eta_j)^{-1/2} \rightharpoonup G \quad (3.27)$$

in the sense of distributions for some two sequences  $\{k_i\}$ ,  $\{\eta_j\}$ , with  $k_i, \eta_j \rightarrow \infty$  as  $i, j \rightarrow \infty$ . By summing up (3.18), (3.25)–(3.27), taking  $k = k_i$ ,  $\eta = \eta_j$  in (3.18), and letting  $i \rightarrow \infty$ , then  $j \rightarrow \infty$ , we finally obtain

$$\partial_t \sqrt{f(t, x)} - \partial_x(c(u) \sqrt{f(t, x)}) = 2\tilde{c}'(u) \bar{S} \sqrt{f(t, x)} + \frac{1}{2}G, \quad (3.28)$$

which is identical to the corresponding equation derived by the nonlinear variational wave equation (1.4); see Eq. (3.47) in Zhang and Zheng [28]. Based on their results, we thus have  $g(t, x) = 0$  a.e.  $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$  and therefore  $v_{\text{bx}}^1(\xi) = \delta_{\bar{R}(t, x)}(\xi)$ . The equality  $v_{\text{bx}}^2(\eta) = \delta_{\bar{S}(t, x)}(\eta)$  can be established in a similar way. The proof of this lemma is complete.  $\square$

**Proof of Theorem 1.1.** We first follow the same procedure as the proof of (3.9) to get

$$\partial_t \bar{S} + \partial_x(c(u) \bar{S}) = -\tilde{c}'(u)(\bar{S}^2 - 2\bar{R} \cdot \bar{S} + \bar{R}^2) - \frac{1}{2}\lambda^2 \sin(2u). \quad (3.29)$$

Subtracting the above from (3.9) yields

$$\partial_t(\bar{R} - \bar{S}) - \partial_x(c(u)(\bar{R} + \bar{S})) = 0. \quad (3.30)$$

Moreover, from the third equation of (2.2), we have

$$2c(u)\partial_x u = \bar{R} - \bar{S}. \quad (3.31)$$

Inserting the above into (3.30) leads to

$$\partial_t\{c(u)(2\partial_t u - (\bar{R} + \bar{S}))\} = 0,$$

which implies, by the initial data, that

$$\partial_t u = \frac{\bar{R} + \bar{S}}{2}. \quad (3.32)$$

Combining (3.31) and (3.32), we obtain

$$\bar{R} = \partial_t u + c(u)\partial_x u, \quad \bar{S} = \partial_t u - c(u)\partial_x u. \quad (3.33)$$

Second, we find by (3.9), (3.29) and Lemma 3.2 that system

$$\begin{cases} \partial_t \bar{R} - \partial_x(c(u)\bar{R}) = -\tilde{c}'(u)(\bar{R} - \bar{S})^2 - \frac{1}{2}\lambda^2 \sin(2u), \\ \partial_t \bar{S} + \partial_x(c(u)\bar{S}) = -\tilde{c}'(u)(\bar{S} - \bar{R})^2 - \frac{1}{2}\lambda^2 \sin(2u) \end{cases} \quad (3.34)$$

holds in the sense of distributions.

Finally, we use (3.1) and (3.33) to get (1.7). By summing up the two equations of (3.34) and applying (3.33), there holds (1.8). The entropy condition (1.9) follows from (2.22). By (2.13) and (3.31), there holds (1.11). This completes the proof of Theorem 1.1.  $\square$

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