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Orthogonality with respect to a Jacobi differential operator and applications

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ABSTRACT

Let μ be a finite positive Borel measure on $[-1, 1]$, m a fixed natural number and $\mathcal{L}^{(\alpha,\beta)}[f] = (1-x^2)f'' + (\beta - \alpha - (\alpha+\beta+2)x)f'$, with $\alpha, \beta > -1$. We study algebraic and analytic properties of the sequence of monic polynomials $(Q_n)_{n>m}$ that satisfy the orthogonality relations

$$\int \mathcal{L}^{(\alpha,\beta)}[Q_n](x)x^k d\mu(x) = 0 \quad \text{for all } 0 \le k \le n-1.$$

A fluid dynamics model for source points location of a flow of an incompressible fluid with preassigned stagnation points is also considered.

1. Introduction

Let $\mathbb{P}$ be the space of all polynomials. We say that a sequence $(p_n)_{n\in\mathbb{Z}_+} \subset \mathbb{P}$, is a polynomial system if for each n the nth polynomial p_n is of degree n.

Let μ be a finite positive Borel measure on $[-1, 1]$ and $(L_n)_{n\in\mathbb{Z}_+}$ the corresponding system of monic orthogonal polynomials; i.e. $L_n(z) = x^n + \cdots$ and

$$\langle L_n, L_k\rangle_\mu = \int L_n(x)L_k(x)d\mu(x) \begin{cases} \neq 0 & \text{if } n = k, \\ = 0 & \text{if } n \neq k. \end{cases} \tag{1.1}$$

Denote by $\mathcal{L}^{(\alpha,\beta)}$ the Jacobi differential operator on the space $\mathbb{P}$, with $\alpha, \beta > -1$, where

$$\mathcal{L}^{(\alpha,\beta)}[f] = (1-x^2)f'' + (\beta - \alpha - (\alpha+\beta+2)x)f', \quad f \in \mathbb{P}, \tag{1.2}$$

or equivalently (cf. [9, (4.2.2)])

$$\mathcal{L}^{(\alpha,\beta)}[f] = \frac{\left((1-x)^{\alpha+1}(1+x)^{\beta+1}f'\right)'}{(1-x)^{\alpha}(1+x)^{\beta}}, \quad f \in \mathbb{P}. \tag{1.3}$$

From (1.2) it follows that f and $\mathcal{L}^{(\alpha,\beta)}[f]$ are polynomials of the same degree. It is straightforward that integrating (1.3) with respect to the (α, β)-Jacobi measure $d\mu_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta\, dx$ on $[-1, 1]$, we obtain

$$\int \mathcal{L}^{(\alpha,\beta)}[f](x)\, d\mu_{\alpha,\beta}(x) = 0, \quad f \in \mathbb{P}. \tag{1.4}$$

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We say that Q_n is the nth monic orthogonal polynomial with respect to the pair $(\mathcal{L}^{(\alpha,\beta)}, \mu)$ if $\deg[Q_n] \leq n$ and

$$\int \mathcal{L}^{(\alpha,\beta)}[Q_n](x)x^k d\mu(x) = 0 \quad \text{for all } 0 \leq k \leq n-1. \tag{1.5}$$

This nonstandard orthogonality with respect to a differential operator was introduced in [1], where the authors prove conditions of existence and uniqueness. The starting points of this work are [6,2], where the orthogonality with respect to the differential operator $\mathcal{L}_\zeta[f] = f + (z-\zeta)f'$, $\zeta \in \mathbb{C}$, was analyzed.

From (1.1), we have that a monic polynomial Q_n of degree n is orthogonal with respect to $(\mathcal{L}^{(\alpha,\beta)}, \mu)$ if and only if it is a polynomial solution of the differential equation

$$\mathcal{L}^{(\alpha,\beta)}[Q_n] = \lambda_n L_n, \quad \text{where } \lambda_n = \lambda_n^{(\alpha,\beta)} = -n(1+n+\alpha+\beta). \tag{1.6}$$

As will be shown, it is not always possible to guarantee the existence of a system of polynomials $(Q_n)_{n\in\mathbb{Z}_+}$ orthogonal with respect to the pair $(\mathcal{L}^{(\alpha,\beta)}, \mu)$. Let $m \in \mathbb{N}$ be fixed, a fundamental role in this paper is played by the class $\mathcal{P}_m(\alpha,\beta)$ defined as the finite positive Borel measures μ supported on $[-1,1]$ for which there exist a non negative polynomial ρ of degree m, such that $d\mu(x) = \rho^{-1}(x)d\mu_{\alpha,\beta}(x)$.

This manuscript deals with some algebraic and analytic aspects of the sequence of orthogonal polynomials $(Q_n)_{n>m}$ orthogonal with respect to the pair $(\mathcal{L}^{(\alpha,\beta)}, \mu)$. We provide asymptotic results for the sequence of the orthogonal polynomials with respect to $(\mathcal{L}^{(\alpha,\beta)}, \mu)$ and study the set of accumulation points of their zeros as well. In particular, we prove

Theorem 1.1. *Let $\mu \in \mathcal{P}_m(\alpha,\beta)$, where $m \in \mathbb{N}$ and $\alpha, \beta > -1$. If $(\zeta_n)_{n>m}$ is a sequence of complex numbers with limit $\zeta \in \mathbb{C}\setminus[-1,1]$ and $(Q_n)_{n>m}$ the sequence of monic orthogonal polynomials with respect to the pair $(\mathcal{L}^{(\alpha,\beta)}, \mu)$ such that $Q_n(\zeta_n) = 0$, then the accumulation points of zeros of $(Q_n)_{n>m}$ are located on the set $E = \mathcal{E}(\zeta)\bigcup[-1,1]$, where $\mathcal{E}(\zeta)$ is the ellipse*

$$\mathcal{E}(\zeta) := \left\{z \in \mathbb{C} : z = \cosh(\eta_\zeta + i\theta), 0 \leq \theta < 2\pi\right\}, \tag{1.7}$$

and $\eta_\zeta := \ln|\varphi(\zeta)| = \ln|\zeta + \sqrt{\zeta^2-1}|$. If $\delta(\zeta) = \inf_{-1\leq x\leq 1}|\zeta - x| > 2$ then $E = \mathcal{E}(\zeta)$.

Let $\varphi(z) = z + \sqrt{z^2-1}$ be the function which maps the complement of $[-1,1]$ onto the exterior of the unit circle, where we take the branch of $\sqrt{z^2-1}$ for which $|\varphi(z)| > 1$ whenever $z \in \mathbb{C}\setminus[-1,1]$. If $\mu \in \mathcal{P}_m(\alpha,\beta)$, let $\nu_1, \nu_2, \ldots, \nu_m \in \mathbb{C}$ be the m zeros of the polynomial $\rho(z) = r\prod_{i=1}^m (z-\nu_i)$, for which $d\mu(x) = \rho^{-1}(x)d\mu_{\alpha,\beta}(x)$.

For all $z \in \mathbb{C}\setminus[-1,1]$ we define the function $\Phi(\rho, z)$ and a constant ϕ_m as

$$\Phi(\rho,z) = \prod_{k=1}^{m}\frac{z-\nu_i}{\varphi(z)-\varphi(\nu_i)}, \qquad \phi_m = 2^m \exp\left(\frac{1}{2\pi}\int\frac{\log(\rho(t))}{\sqrt{1-t^2}}dt\right).$$

The function $\phi_m\,\Phi(\rho,z)$ is a particular case of the well known *Szegő's function* (cf. [5, Section 6.1]).

For the sequence of monic orthogonal polynomials with respect to the pair $(\mathcal{L}^{(\alpha,\beta)}, \mu)$ the following asymptotic behavior holds

Theorem 1.2. *Let $(\zeta_n)_{n>m}$ be a sequence of complex numbers with limit $\zeta \in \mathbb{C}\setminus[-1,1]$, $m \in \mathbb{N}$, $\mu \in \mathcal{P}_m(\alpha,\beta)$, where $\alpha, \beta > -1$ and $(Q_n)_{n>m}$ the sequence of monic orthogonal polynomials with respect to the pair $(\mathcal{L}^{(\alpha,\beta)}, \mu)$ such that $Q_n(\zeta_n) = 0$, then:*

1. *Uniformly on compact subsets of $\Omega = \{z \in \mathbb{C} : |\varphi(z)| > |\varphi(\zeta)|\}$, (i.e. the exterior of the ellipse $\mathcal{E}(\zeta)$)*

$$\frac{Q_n(z)}{P_n^{(\alpha,\beta)}(z)} \underset{n\to\infty}{\rightrightarrows} \phi_m^2\,\Phi(\rho,z). \tag{1.8}$$

2. *Uniformly on compact subsets of $\Omega = \{z \in \mathbb{C} : |\varphi(z)| < |\varphi(\zeta)|\}\setminus[-1,1]$*

$$\frac{Q_n(z)}{P_n^{(\alpha,\beta)}(\zeta_n)} \underset{n\to\infty}{\rightrightarrows} -\phi_m^2\,\Phi(\rho,\zeta). \tag{1.9}$$

If $\delta(\zeta) > 2$ then (1.9) holds for $\Omega = \{z \in \mathbb{C} : |\varphi(z)| < |\varphi(\zeta)|\}$ (i.e. the interior of the ellipse $\mathcal{E}(\zeta)$).

The paper is organized as follows. In Section 2 we study the existence of a system of polynomials $(Q_n)_{n>m}$, orthogonal with respect to the pair $(\mathcal{L}^{(\alpha,\beta)}, \mu)$. Section 3 is devoted to the study of recurrence relations and location of zeros of the polynomials $(Q_n')_{n>m}$. In Sections 4 and 5 we study the asymptotic behavior of the polynomials $(Q_n)_{n>m}$, $(\widehat{Q}_n)_{n>m}$ and its zeros respectively. In the last section, we introduce a fluid dynamics model for the interpretation of the critical points of Q_n.

2. Existence and uniqueness

It is well known that the differential operator $\mathcal{L}^{(\alpha,\beta)}$ has a system of monic eigenpolynomials $(P_n^{(\alpha,\beta)})_{n\in\mathbb{Z}_+}$ and a sequence of constant $(\lambda_n)_{n\in\mathbb{Z}_+}$ (eigenvalues), such that

$$\mathcal{L}^{(\alpha,\beta)}[P_n^{(\alpha,\beta)}] = \lambda_n P_n^{(\alpha,\beta)}, \tag{2.1}$$

where the eigenpolynomial $P_n^{(\alpha,\beta)}$ is the nth monic orthogonal polynomial with respect to the (α,β)-measure of Jacobi $d\mu_{\alpha,\beta}(x) = (1-x)^{\alpha}\,(1+x)^{\beta}\,dx$ on $[-1,1]$, with $\alpha,\beta > -1$, i.e.

$$\left\langle P_n^{(\alpha,\beta)}, x^k\right\rangle_{\alpha,\beta} = \int P_n^{(\alpha,\beta)}(x)\,x^k d\mu_{\alpha,\beta}(x) = 0, \quad \text{for all } 0\le k\le n-1. \tag{2.2}$$

Furthermore, from [9, (4.21.6) and (4.3.3)]

$$\begin{aligned}\tau_n &= \left\langle P_n^{(\alpha,\beta)}, P_n^{(\alpha,\beta)}\right\rangle_{\alpha,\beta}\\ &= \frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)\,\Gamma(n+1)\,\Gamma(n+\alpha+\beta+1)}{2^{-(2n+\alpha+\beta+1)}\,\Gamma(2n+\alpha+\beta+2)\,\Gamma(2n+\alpha+\beta+1)}.\end{aligned} \tag{2.3}$$

Theorem 2.1. *Let n be a fixed natural number and μ a finite positive Borel measure on $[-1,1]$. Then, the differential equation (1.6) has a monic polynomial solution Q_n of degree n, which is unique up to an additive constant, if and only if*

$$\int L_n(x)d\mu_{\alpha,\beta}(x) = 0, \tag{2.4}$$

where L_n is as (1.1).

Proof. Let L_n be the nth monic orthogonal polynomial for μ and suppose that there exists a polynomial Q_n of degree n, such that $\mathcal{L}^{(\alpha,\beta)}[Q_n] = -n(1+n+\alpha+\beta)\,L_n$. Then (2.4) is straightforward from (1.4).

Conversely, suppose that L_n, the nth monic orthogonal polynomial with respect to μ satisfies (2.4). Let Q_n be the polynomial of degree n defined by

$$Q_n(z) = P_n^{(\alpha,\beta)}(z) + \sum_{k=0}^{n-1} a_{n,k} P_k^{(\alpha,\beta)}(z), \tag{2.5}$$

where $a_{n,0}$ is an arbitrary constant and

$$a_{n,k} = \frac{\lambda_n}{\lambda_k}\,b_{n,k} = \frac{\lambda_n}{\lambda_k\,\tau_k}\left\langle L_n, P_k^{(\alpha,\beta)}\right\rangle_{\alpha,\beta}, \quad k=1,\ldots,n-1.$$

From the linearity of $\mathcal{L}^{(\alpha,\beta)}[\cdot]$ and (2.1) we get that $\mathcal{L}^{(\alpha,\beta)}[Q_n] = -n(1+n+\alpha+\beta)\,L_n$. □

We are interested in discussing systems of polynomials such that for all n sufficiently large they are solutions of (1.6). In this sense, the next corollary is fundamental.

Corollary 2.2. *Let μ be a finite positive Borel measure on $[-1,1]$ such that $d\mu(x) = r(x)d\mu_{\alpha,\beta}(x)$, where $r\in L^2(\mu_{\alpha,\beta})$. Then, m is the smallest natural number such that for each $n>m$ there exists a monic polynomial Q_n of degree n, unique up to an additive constant and orthogonal with respect to $(\mathcal{L}^{(\alpha,\beta)},\mu)$, if and only if r^{-1} is a polynomial of degree m.*

Proof. Suppose that m is the smallest natural number such that for each $n>m$ there exists a monic polynomial Q_n of degree n, unique up to an additive constant and orthogonal with respect to $(\mathcal{L}^{(\alpha,\beta)},\mu)$. According to Theorem 2.1

$$\int \frac{1}{r(x)}L_n(x)d\mu(x) = \int L_n(x)d\mu_{\alpha,\beta}(x)\begin{cases} = 0 & \text{if } n>m\\ \neq 0 & \text{if } n=m.\end{cases}$$

But this is equivalent to saying that $\frac{1}{r(x)} = \sum_{k=0}^{m} c_k L_k(x)$ with $c_m \neq 0$. The converse is straightforward. □

From the previous corollary, if $\mu\in\mathcal{P}_m(\alpha,\beta)$ then the differential equation (1.6) has an unique monic polynomial solution Q_n of degree n for all $n>m$, except for an additive constant.

Let $(\zeta_n)_{n>m}$ be a sequence of complex numbers, where $m\in\mathbb{N}$ is fixed, and assume that $\mu\in\mathcal{P}_m(\alpha,\beta)$. We complement the definition of the sequence $(Q_n)_{n>m}$ in (1.5) by considering that henceforth Q_n for each $n>m$ is the polynomial solution of the initial value problem

$$\begin{cases}\mathcal{L}^{(\alpha,\beta)}[y] = \lambda_n\,L_n, & n>m,\\ y(\zeta_n) = 0.\end{cases} \tag{2.6}$$

We say that $(Q_n)_{n>m}$, is the sequence of monic orthogonal polynomials with respect to the pair $(\mathcal{L}^{(\alpha,\beta)}, \mu)$ such that $Q_n(\zeta_n) = 0$.

If $\widehat{Q}_n$ is the monic polynomial of degree n defined by the formula

$$\widehat{Q}_n(z) = \lambda_n \sum_{k=0}^{m} \frac{b_{n,n-k}}{\lambda_{n-k}} P_{n-k}^{(\alpha,\beta)}(z), \quad b_{n,n-k} = \frac{1}{\tau_{n-k}} \left\langle L_n, P_{n-k}^{(\alpha,\beta)} \right\rangle_{\alpha,\beta}, \tag{2.7}$$

then, the initial value problem (2.6) has the unique polynomial solution

$$y(z) = Q_n(z) = \widehat{Q}_n(z) - \widehat{Q}_n(\zeta_n). \tag{2.8}$$

3. The polynomial Q_n'

Let $m \in \mathbb{N}$, $(\zeta_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers, and $\mu \in \mathcal{P}_m(\alpha, \beta)$ be fixed, then for all $n > m$ the polynomials Q_n (solution of (2.6)) are uniquely determined by (2.7)–(2.8). Without loss of generality, we will complete the sequence of polynomials Q_n for all $n \in \mathbb{N}$ as follows

$$Q_n(z) = \left(P_n^{(\alpha,\beta)}(z) - P_n^{(\alpha,\beta)}(\zeta_n)\right) + \lambda_n \sum_{k=1}^{\min(m,n)} \frac{b_{n,n-k}}{\lambda_{n-k}} \left(P_{n-k}^{(\alpha,\beta)}(z) - P_{n-k}^{(\alpha,\beta)}(\zeta_n)\right), \quad n \geq 1 \tag{3.1}$$

$$Q_0(z) = 1.$$

For convenience, only in the previous formula we consider $\lambda_0 = 1$. Note that $(Q_n)_{n\in\mathbb{N}}$ defined by (3.1) is a system of polynomials, such that $Q_n(\zeta_n) = 0$ for all $n \geq 1$. Let us remark that if $n \leq m$, in general, $\mathcal{L}^{(\alpha,\beta)}[Q_n] \neq \lambda_n L_n$.

Additionally, as the degree of a polynomial is invariant under the operator $\mathcal{L}^{(\alpha,\beta)}[\cdot]$ and the polynomial Q_n, for all $n \leq m$, is of degree n (see (3.1)),

$$(1, \mathcal{L}^{(\alpha,\beta)}[Q_1], \ldots, \mathcal{L}^{(\alpha,\beta)}[Q_m], L_{m+1}, \ldots, L_n, \ldots) \tag{3.2}$$

is a polynomial system.

In the following result, we show that for $n > (2m+1)$ the derivatives of the system of polynomials Q_n satisfy a recurrence relation with a fixed finite number of terms.

Theorem 3.1. *Let $m \in \mathbb{N}$, $\mu \in \mathcal{P}_m(\alpha, \beta)$, where $\alpha, \beta > -1$. Then if R is any primitive of ρ, for each $n > (2m+1)$ the sequence of polynomials Q_n' satisfies the relation*

$$R(z)Q_n'(z) = \sum_{k=-m-1}^{m+1} \theta_{R,n,n-k}\, Q_{n-k}'(z), \tag{3.3}$$

where the initial values $Q_{m+1}', \ldots, Q_{2m+2}'$ are given by the derivatives of (2.7) and

$$\theta_{R,n,n-k} = \frac{1}{\lambda_{n-k}} (\lambda_n\, e_{R,n,n-k} + d_{n,n-k}),$$

$$e_{R,n,n-k} = \frac{1}{l_{n-k}} \int R(x)\, L_{n-k}(x)\, L_n(x)\, d\mu(x), \tag{3.4}$$

$$l_i = \int L_i^2(x) d\mu(x), \tag{3.5}$$

$$d_{n,n-k} = \frac{1}{l_{n-k}} \sum_{j=j_1(k)}^{j_2(k)} \tau_{n-j}\, \widetilde{c}_{n,n-j}\, b_{n-k,n-j}, \quad j_1(k) = \max\{-1, k\} \text{ and } j_2(k) = \min\{m+1, m+k\},$$

$$\widetilde{c}_{n,n-k} = \lambda_n \sum_{j=j_3(k)}^{j_4(k)} \frac{b_{n,n-j}\, c_{n-j,j-k}}{\lambda_{n-j}}, \quad j_3(k) = \max\{0, k-1\} \text{ and } j_4(k) = \min\{m, k+1\},$$

$$c_{n,1} = -n,$$

$$c_{n,0} = \frac{2n\,(\alpha-\beta)\,(n+\alpha+\beta+1)}{(2n+\alpha+\beta)\,(2n+\alpha+\beta+2)},$$

$$c_{n,-1} = 4n\,\frac{(n+\alpha)\,(n+\beta)\,(n+\alpha+\beta)\,(n+\alpha+\beta+1)}{(2n+\alpha+\beta)^2\,((2n+\alpha+\beta)^2-1)}.$$

Before starting the proof of Theorem 3.1, we will state and prove some lemmas.

Lemma 3.2. *Let $m \in \mathbb{N}$ and $\mu \in \mathcal{P}_m(\alpha,\beta)$, where $\alpha,\beta > -1$. Then for $n > m$*

$$L_n(z) = \sum_{k=0}^{m} b_{n,n-k}\, P_{n-k}^{(\alpha,\beta)}(z), \tag{3.6}$$

$$\rho(z)P_n^{(\alpha,\beta)}(z) = \tau_n \sum_{k=0}^{m} \frac{b_{n+k,n}}{l_{n+k}}\, L_{n+k}(z), \tag{3.7}$$

where $b_{i,j} = \frac{1}{\tau_j}\left\langle L_i, P_j^{(\alpha,\beta)}\right\rangle_{\alpha,\beta}$, τ_j *as in* (2.3) *and* l_i *as in* (3.5).

Proof. As $\mu \in \mathcal{P}_m(\alpha,\beta)$ then $\langle L_n, x^k\rangle_{\alpha,\beta} = 0$ for all $n > m$ and $k < n-m$. Hence, $b_{n,k} = 0$ for $k = 0, 1, \ldots, n-m-1$ and (3.6) is established.

From the orthogonality relations (1.1) and (2.2), if $i < j$ or $i > j+m$, we have

$$\int L_i(x)P_j^{(\alpha,\beta)}(x)d\mu_{\alpha,\beta}(x) = \int L_i(x)P_j^{(\alpha,\beta)}(x)\,\rho(x)\,d\mu(x) = 0. \tag{3.8}$$

The relation (3.7) is straightforward from the Fourier expansion of $\rho\, P_n^{(\alpha,\beta)}$ in terms of the (L_k), $k = 0, 1, \ldots, n+m$ and (3.8). $\square$

From the lemma above, the polynomials $\widehat{Q}_n$ defined in (2.7) and its derivatives can be written as a linear combination of the polynomials $(L_n)_{n\in\mathbb{Z}_+}$ as we show in the next lemma.

Lemma 3.3. *Under the conditions of Lemma 3.2, for $n > m$ the polynomials $\widehat{Q}_n$ satisfy the following relations:*

$$\int \widehat{Q}_n(x)\, x^k\, d\mu_{\alpha,\beta}(x) = 0, \quad k = 0, 1, \ldots, n-m-1, \tag{3.9}$$

$$(1-z^2)\rho(z)Q_n'(z) = \sum_{k=-m-1}^{m+1} d_{n-k,k}\, L_{n-k}(z). \tag{3.10}$$

Proof. From (2.7), (3.9) and the relation

$$(1-z^2)\,Q_n'(z) = (1-z^2)\,\widehat{Q}_n'(z) = \lambda_n \sum_{k=0}^{m} \frac{b_{n,n-k}}{\lambda_{n-k}}\,(1-z^2)\left(P_{n-k}^{(\alpha,\beta)}(z)\right)'. \tag{3.11}$$

Using the structure relation fulfilled by Jacobi polynomials (see [9, (4.5.5)–(4.5.6)]), we have

$$(1-z^2)\left(P_{n-k}^{(\alpha,\beta)}(z)\right)' = c_{n-k,1}\, P_{n-k+1}^{(\alpha,\beta)}(z) + c_{n-k,0}\, P_{n-k}^{(\alpha,\beta)}(z) + c_{n-k,-1}\, P_{n-k-1}^{(\alpha,\beta)}(z).$$

Substituting this formula into (3.11), we obtain

$$(1-z^2)Q_n'(z) = \sum_{k=-1}^{m+1} \widetilde{c}_{n,n-k}\, P_{n-k}^{(\alpha,\beta)}(z),$$

and from (3.7), (3.10) immediately follows. $\square$

Proof of Theorem 3.1. As the sequence $(Q_n)_{n\in\mathbb{Z}_+}$ is a system of polynomials, then the sequence of its derivatives $(Q_n')_{n\in\mathbb{N}}$ is also system of polynomials. Hence, the polynomial $R\,Q_n'$ can be expanded as linear combination of the polynomials $(\widehat{Q}_n')_{n\in\mathbb{N}}$, i.e. there exist $(n+m)$ constants $\theta_{R,n,1}, \ldots, \theta_{R,n,n+m}$ such that

$$R(z)Q_n'(z) = \sum_{k=-m}^{n-1} \theta_{R,n,n-k}\, \widehat{Q}_{n-k}'(z). \tag{3.12}$$

Let $\widetilde{\mathcal{L}}^{(\alpha,\beta)}$ be the linear differential operator on the space of all polynomials $\mathbb{P}$ given by $\widetilde{\mathcal{L}}^{(\alpha,\beta)}[f'] = \mathcal{L}^{(\alpha,\beta)}[f]$ for all $f \in \mathbb{P}$, i.e.

$$\widetilde{\mathcal{L}}^{(\alpha,\beta)}[f] = (1-x^2)f' + (\beta-\alpha-(\alpha+\beta+2)x)f.$$

Since $(L_n)_{n\in\mathbb{Z}_+}$ is a system of polynomials and $\widetilde{\mathcal{L}}^{(\alpha,\beta)}[\cdot]$ is a linear operator, the polynomial $\widetilde{\mathcal{L}}^{(\alpha,\beta)}[RQ_n']$ can be written as a linear combination of the system of polynomials (3.2) as follows

$$\widetilde{\mathcal{L}}^{(\alpha,\beta)}\left[RQ_n'\right](z) = \sum_{k=-m-1}^{n-m-1} \theta_{R,n,n-k}\,\lambda_{n-k}\, L_{n-k}(z) + \sum_{k=n-m}^{n-1} \theta_{R,n,n-k}\, \mathcal{L}^{(\alpha,\beta)}\left[Q_{n-k}(z)\right]. \tag{3.13}$$

Taking $\widetilde{\mathcal{L}}^{(\alpha,\beta)}[\cdot]$ on the left hand side of the equality (3.12), we get

$$\begin{aligned}\widetilde{\mathcal{L}}^{(\alpha,\beta)}[RQ_n'](z) &= R(z)\widetilde{\mathcal{L}}^{(\alpha,\beta)}[Q_n'(z)] + (1-z^2)\rho(z)Q_n'(z) \\ &= \lambda_n R(z)\, L_n(z) + (1-z^2)\rho(z)Q_n'(z).\end{aligned} \tag{3.14}$$

From (1.1)

$$R(z)\, L_n(z) = \sum_{k=-m-1}^{m+1} e_{R,n,n-k}\, L_{n-k}(z). \tag{3.15}$$

Substituting (3.10) and (3.15) in (3.14), we have

$$\widetilde{\mathcal{L}}^{(\alpha,\beta)}[RQ_n'](z) = \sum_{k=-m-1}^{m+1} \left(\lambda_n\, e_{R,n,n-k} + d_{n,n-k}\right)\, L_{n-k}(z). \tag{3.16}$$

As $n \geq 2(m+1)$, we can assume that (3.16) is the expansion of the polynomial $\widetilde{\mathcal{L}}^{(\alpha,\beta)}[R\,Q_n']$ in terms of the polynomials L_n. Now, identifying coefficients between (3.13) and (3.16) we have that $\theta_{R,n,n-k} = 0$ for all $k = 1, \ldots, n-m-2$, and we have the formulas (3.3)–(3.4). □

4. Asymptotic behavior of the polynomials $\widehat{Q}_n$ and their zeros

In this section we study the asymptotic behavior of the polynomials $\widehat{Q}_n$ and their zeros. Let us denote by $\|\cdot\|_{\alpha,\beta}$ and $\|\cdot\|_{\mu}$ the standard norms on the spaces $L^2(\mu_{\alpha,\beta})$ and $L^2(\mu)$ respectively. The following result is essential in the proof of the theorems in this and the next section.

Theorem 4.1. *Let $m \in \mathbb{N}$ and $\mu \in \mathcal{P}_m(\alpha,\beta)$, where $\alpha, \beta > -1$. Then*

$$\frac{\widehat{Q}_n(z)}{P_n^{(\alpha,\beta)}(z)} \underset{n\to\infty}{\rightrightarrows} \phi_m^2\, \Phi(\rho, z), \tag{4.1}$$

uniformly on closed subsets of $\overline{\mathbb{C}} \setminus [-1,1]$ where $\overline{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}$.

First, we state a preliminary lemma which follows from Theorems 26 and 29 of [5, Section 6.1].

Lemma 4.2. *Let $\mu \in \mathcal{P}_m(\alpha,\beta)$, where $\alpha, \beta > -1$ and $m \in \mathbb{N}$. If $(L_n)_{n\in\mathbb{Z}_+}$ is the sequence of monic orthogonal polynomials with respect to μ*

$$\frac{L_n(z)}{P_n^{(\alpha,\beta)}(z)} \underset{n\to\infty}{\rightrightarrows} \phi_m^2\, \Phi(\rho, z),$$

uniformly on closed subsets of $\overline{\mathbb{C}} \setminus [-1,1]$.

Proof of Theorem 4.1. From (2.7) and (3.6)

$$\frac{\widehat{Q}_n(z) - L_n(z)}{P_n^{(\alpha,\beta)}(z)} = \sum_{k=1}^{m} \left(\frac{\lambda_n}{\lambda_{n-k}} - 1\right) b_{n,n-k} \frac{P_{n-k}^{(\alpha,\beta)}(z)}{P_n^{(\alpha,\beta)}(z)}. \tag{4.2}$$

As $\lambda_n = -n(1+n+\alpha+\beta)$, for each k fixed, $k = 1, 2, \ldots, m$,

$$\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n-k}} = 1. \tag{4.3}$$

Let K be a closed subset of $\overline{\mathbb{C}} \setminus [-1,1]$. From the interlacing property of the zeros of consecutive Jacobi polynomials on $[-1,1]$, it easily follows that there exists a constant M_* such that for all $z \in K$

$$\left|\frac{P_{n-k}^{(\alpha,\beta)}(z)}{P_n^{(\alpha,\beta)}(z)}\right| < M_k \leq M_*, \quad k = 1, \ldots, m, \tag{4.4}$$

where $M_k = \sup_{\substack{z\in K\\ x\in[-1,1]}} |z-x|^{-k}$ and $M_* = \max\{M_1, M_m\}$.

From (2.3), it is not hard to see that there exist two monic polynomials of degree $4(m-k)$ in the variable n, $q^{(\alpha,\beta)}_{1,4(m-k)}(n)$ and $q^{(\alpha,\beta)}_{2,4(m-k)}(n)$, such that

$$\left\|P^{(\alpha,\beta)}_{n-k}\right\|^2_{\alpha,\beta} = 4^{k-m}\frac{q^{(\alpha,\beta)}_{1,4(m-k)}(n)}{q^{(\alpha,\beta)}_{2,4(m-k)}(n)}\left\|P^{(\alpha,\beta)}_{n-m}\right\|^2_{\alpha,\beta}, \quad k=1,2,\ldots,m.$$

Therefore, from the Cauchy–Bunyakovsky–Schwarz inequality and the extremal property of the monic orthogonal polynomials, for n sufficiently large, we get

$$\begin{aligned} |b_{n,n-k}| &\le \frac{\|L_n\|_{\alpha,\beta}}{\sqrt{\tau_{n-k}}} \le \sqrt{\frac{c_1}{\tau_{n-k}}}\,\|L_n\|_\mu \le \sqrt{\frac{c_1}{\tau_{n-k}}}\,\frac{\left\|\rho P^{(\alpha,\beta)}_{n-m}\right\|_\mu}{|r|} \\ &\le \frac{c_1\left\|P^{(\alpha,\beta)}_{n-m}\right\|_{\alpha,\beta}}{|r|\sqrt{\tau_{n-k}}} = \frac{c_1 2^{m-k}}{|r|}\sqrt{\frac{q^{(\alpha,\beta)}_{2,4(m-k)}(n)}{q^{(\alpha,\beta)}_{1,4(m-k)}(n)}} \le \frac{c_1\,2^{m+1}}{|r|} \end{aligned} \tag{4.5}$$

where $1\le k\le m$, $c_1=\sup_{x\in[-1,1]}\rho(x)$ and r is the leading coefficient of ρ. Hence by (4.2), (4.3), (4.4), and (4.5)

$$\left|\frac{\widehat{Q}_n(z)-L_n(z)}{P^{(\alpha,\beta)}_n(z)}\right| \underset{n\to\infty}{\rightrightarrows} 0, \quad \text{uniformly on closed subsets of } \overline{\mathbb{C}}\setminus[-1,1],$$

and from Lemma 4.2 the asymptotic formula (4.1) is established. □

Corollary 4.3. *Let $m\in\mathbb{N}$ and $\mu\in\mathcal{P}_m(\alpha,\beta)$, where $\alpha,\beta>-1$. Then*

1. $$\lim_{n\to\infty}\left|\widehat{Q}_n(z)\right|^{\frac{1}{n}} = \frac{|z+\sqrt{z^2-1}|}{2}, \tag{4.6}$$

   uniformly on compact subsets of $\overline{\mathbb{C}}\setminus[-1,1]$.
2. *The set of accumulation points of the zeros of the sequence of polynomials $(\widehat{Q}_n)_{n>m}$ is $[-1,1]$, i.e.*

   $$\bigcap_{n\ge m}\overline{\bigcup_{k=n}^{\infty}\left\{z:\widehat{Q}_k(z)=0\right\}} = [-1,1].$$

   For each n at least $(n-m)$ zeros of $\widehat{Q}_n$ are contained on $[-1,1]$.

Proof. The first part of the theorem is an immediate consequence of (4.1) and [9, (8.21.9) and (4.21.6)].

To prove the second part, from (3.9) it easily follows that $\widehat{Q}_n$ has at least $n-m$ zeros contained in $[-1,1]$ (cf. [9, Section 3.3]).

The function in the right-hand side of (4.1) in Theorem 4.1 is holomorphic and does not have zeros in $\overline{\mathbb{C}}\setminus[-1,1]$. Let K be a closed subset of $\overline{\mathbb{C}}\setminus[-1,1]$, from the Rouché's theorem we have that for n large the polynomial $\widehat{Q}_n$ does not have zeros on K, i.e. the zeros of the sequence of polynomials $(\widehat{Q}_n)_{n>m}$ cannot accumulate outside $[-1,1]$.

On the other hand, (4.6) implies the weak star asymptotic of the zero counting measures of the polynomials $(\widehat{Q}_n)_{n>m}$ (cf. [8, Chapter 2]). That is, if we associate to each $\widehat{Q}_n$ the measure $\mu_n=\frac{1}{n}\sum_{Q_n(\omega)=0}\delta_\omega$, where δ_ω is the Dirac measure with mass 1 at the point ω, then $d\mu_n(x)\xrightarrow{*}\frac{1}{\pi}\frac{dx}{\sqrt{1-x^2}}$ (the equilibrium distribution on $[-1,1]$) in the weak-* topology and this implies that the zeros of $(\widehat{Q}_n)_{n>m}$ must be dense in $[-1,1]$. □

5. Asymptotic behavior of the polynomials Q_n and their zeros

Some basic properties of the zeros of Q_n follow from (1.6). For example, the multiplicity of the zeros of Q_n is at most 3, a zero of multiplicity 3 is also a zero of L_n and a zero of multiplicity 2 is a critical point of $\widehat{Q}_n$.

From the second part of Corollary 4.3, we get that Q_n' has at least $(n-m-1)$ zeros of odd multiplicity on the open interval $]-1,1[$. For $m=1$ we have that

Theorem 5.1. *Under the same hypothesis of Theorem 3.1, if $m=1$ the critical points of Q_n interlace the zeros of L_n.*

Proof. If $m=1$ then from (3.9) the polynomial $\widehat{Q}_n$ has at least $(n-1)$ real zeros of odd order on $]-1,1[$. But, $\widehat{Q}_n$ is a polynomial with real coefficients and degree n, consequently the zeros of $\widehat{Q}_n$ are real and simples. As $Q_n'=\widehat{Q}_n'$, from Rolle's theorem all the critical points of Q_n are real, simple and $(n-2)$ of them are contained on $]-1,1[$.

Denote $P(x) = (1-x)^{\alpha+1}(1+x)^{\beta+1}Q_n'(x)$. As α and β are real numbers in general P is not a polynomial. Notice that P is a real-valued differentiable function on $[-1,1]$. Without loss of generality, suppose that there exists $x \in]1,\infty[$ such that $P(x) = 0$, as $P(1) = 0$ from Rolle's theorem there exists $x' \in]1, x[$ such that $P'(x') = 0$. But, from (1.3) and (1.6) $P'(x) = \lambda_n(1-x)^{\alpha}(1+x)^{\beta}L_n(x)$ and all the critical points of P are contained in $[-1,1]$. Hence all the critical points of Q_n are contained in $]-1,1[$. Again using Rolle's theorem, it is straightforward that the critical points of Q_n interlace the zeros of L_n. □

From Corollary 4.3 we have that the set of accumulation points of Q_n' is $[-1,1]$. For $m = 1$, Theorem 5.1 gives that the critical points of Q_n are in $[-1,1]$, interlace the zeros of L_n, and are simple. Numerical experiments also show this behavior for $m > 1$. We conjecture that this always is the case.

For the proof of Theorems 1.1 and 1.2 we will use the following result.

Lemma 5.2. *Given $z \in \mathbb{C}$, define $\Delta(z) = \sup_{x\in[-1,1]}|z-x|$ and $\delta(z) = \inf_{x\in[-1,1]}|z-x|$. If $\mu \in \mathcal{P}_m(\alpha,\beta)$, $(\zeta_n)_{n>m}$ is a sequence of complex number with limit $\zeta \in \mathbb{C}$ and $(Q_n)_{n>m}$ the sequence of monic orthogonal polynomials with respect to the pair $(\mathcal{L}^{(\alpha,\beta)}, \mu)$ such that $Q_n(\zeta_n) = 0$, then:*

1. *For every $d > 1$ there is a positive number N_d, such that $\{z \in \mathbb{C} : Q_n(z) = 0\} \subset \{z \in \mathbb{C} : |z| \le \Delta(\zeta) + d\}$ whenever $n > N_d$.*
2. *If $\delta(\zeta) > 2$, the zeros of Q_n cannot accumulate on $[-1,1]$ and for n sufficiently large they are simple.*

Proof. We already know that $Q_n(\zeta_n) = 0$ and if $Q_n(z) = 0$ then $\widehat{Q}_n(z) = \widehat{Q}_n(\zeta_n)$. From the Gauss–Lucas theorem (cf. [7, Section 2.1.3]), it is known that the critical points of $\widehat{Q}_n$ lie in the convex hull of its zeros and from 2. of Corollary 4.3 the zeros of the polynomials $(\widehat{Q}_n)_{n>m}$ accumulate on $[-1,1]$. Hence from the *bisector theorem* (see the proof of Theorem 5.1 or [7, Section 5.5.7]) $|z| \le \Delta(\zeta_n) + 1$ and the first part of the theorem is established.

To verify the second assertion of the theorem, note that if z is a zero of Q_n, from (2.8) we get

$$\prod_{k=1}^{n}\left|\frac{z-\widehat{x}_{n,k}}{\zeta_n-\widehat{x}_{n,k}}\right| = 1. \tag{5.1}$$

Let $\mathcal{V}_\varepsilon([-1,1]) = \{z \in \mathbb{C} : \delta(z) < \epsilon\}$ be an ε-neighborhood of $[-1,1]$. On the other hand, as $\lim_{n\to\infty}\zeta_n = \zeta$, then for all $\varepsilon > 0$ there is a $N_\varepsilon > 0$ such that $|\delta(\zeta_n) - \delta(\zeta)| < \varepsilon$ whenever $n > N_\varepsilon$.

If $\delta(\zeta) > 2$, let us choose $\varepsilon = \varepsilon_\zeta = \frac{1}{2}(\delta(\zeta) - 2)$ and suppose that there is a $z_0 \in \mathcal{V}_{\varepsilon_\zeta}([-1,1])$ such that $Q_n(z_0) = 0$ for some $n > N_{\varepsilon_\zeta}$. Hence

$$\prod_{k=1}^{n}\left|\frac{z_0-\widehat{x}_{n,k}}{\zeta_n-\widehat{x}_{n,k}}\right| < \left(\frac{2+\varepsilon_\zeta}{\delta(\zeta_n)}\right)^n < 1, \tag{5.2}$$

which is in contradiction with (5.1). Hence $\{z \in \mathbb{C} : Q_n(z) = 0\} \bigcap \mathcal{V}_{\varepsilon_n}([-1,1]) = \varnothing$ for all $n > N_{\varepsilon_\zeta}$, i.e. the zeros of Q_n cannot accumulate on $\mathcal{V}_{\varepsilon_\zeta}([-1,1])$.

From (2.8) it is straightforward that a multiple zero of Q_n is also a critical point of $\widehat{Q}_n$. But, from 2. of Corollary 4.3 and the Gauss–Lucas theorem the critical point of $\widehat{Q}_n$ accumulate on $[-1,1]$. Thus, we have that for n sufficiently large the zeros of Q_n are simple. □

Proof of Theorem 1.1. From (2.8) the zeros of Q_n satisfy the equation

$$\left|\widehat{Q}_n(z)\right|^{\frac{1}{n}} = \left|\widehat{Q}_n(\zeta_n)\right|^{\frac{1}{n}}. \tag{5.3}$$

If $z \in \mathbb{C}\setminus[-1,1]$, by taking limit when $n \to \infty$, from 1. of Lemma 5.2, and using (4.6) on both sides of (5.3), we have that the zeros of the sequence of polynomials $(Q_n)_{n>m}$ cannot accumulate outside the set

$$\left\{z \in \mathbb{C} : |z + \sqrt{z^2-1}| = e^{\eta_\zeta}\right\} \bigcup [-1,1].$$

Hence $z + \sqrt{z^2-1} = e^{\eta_\zeta + i\theta}$ and $z - \sqrt{z^2-1} = e^{-(\eta_\zeta+i\theta)}$ for $0 \le \theta < 2\pi$, where we have that $2z = e^{\eta_\zeta+i\theta} + e^{-(\eta_\zeta+i\theta)}$.

The assertion for $\delta(\zeta) > 2$ is straightforward from 2. of Lemma 5.2. □

Now, we will state the relative asymptotic between the polynomials $(Q_n)_{n>m}$ and the corresponding Jacobi polynomials $P_n^{(\alpha,\beta)}$.

Proof of Theorem 1.2. 1.- Let us prove first that

$$\frac{Q_n(z)}{\widehat{Q}_n(z)} = 1 - \frac{\widehat{Q}_n(\zeta_n)}{\widehat{Q}_n(z)} \underset{n\to\infty}{\rightrightarrows} 1, \tag{5.4}$$

uniformly on compact subsets K of the set $\{z \in \mathbb{C} : |\varphi(z)| > |\varphi(\zeta)|\}$. In order to prove (5.4) it is sufficient to show that

$$\frac{\widehat{Q}_n(\zeta_n)}{\widehat{Q}_n(z)} \underset{n\to\infty}{\rightrightarrows} 0, \quad \text{uniformly on } K. \tag{5.5}$$

From [9, (8.21.9) and (4.21.6)], we have the well known strong or power asymptotic of the monic Jacobi polynomials

$$\frac{2^n P_n^{(\alpha,\beta)}(z)}{\varphi^n(z)} \underset{n\to\infty}{\rightrightarrows} \left(\frac{\varphi(z)-1}{2(z-1)}\right)^{\alpha} \left(\frac{\varphi(z)+1}{2(z+1)}\right)^{\beta} \sqrt{\frac{\varphi'(z)}{2}}, \tag{5.6}$$

uniformly on compact subsets of $\mathbb{C} \setminus [-1, 1]$. Note that

$$\frac{\widehat{Q}_n(\zeta_n)}{\widehat{Q}_n(z)} = \frac{\widehat{Q}_n(\zeta_n)}{P_n^{(\alpha,\beta)}(\zeta_n)} \frac{P_n^{(\alpha,\beta)}(z)}{\widehat{Q}_n(z)} \frac{2^n P_n^{(\alpha,\beta)}(\zeta_n)}{\varphi^n(\zeta_n)} \frac{\varphi^n(z)}{2^n P_n^{(\alpha,\beta)}(z)} \left(\frac{\varphi(\zeta_n)}{\varphi(z)}\right)^n.$$

From (4.1) and (5.6) the first four factors in the right hand side of the previous formula have finite limits; meanwhile, the last factor tends to 0 when $n \to \infty$, and we get (5.5). Finally assertion 1 is straightforward from (4.1).

2.- For assertion 2 of the theorem it is sufficient to prove that

$$\frac{Q_n(z)}{\widehat{Q}_n(\zeta_n)} = \frac{\widehat{Q}_n(z)}{\widehat{Q}_n(\zeta_n)} - 1 \underset{n\to\infty}{\rightrightarrows} -1, \tag{5.7}$$

uniformly on compact subsets K of the set $\{z \in \mathbb{C} : |\varphi(z)| < |\varphi(\zeta)|\} \setminus [-1, 1]$. Note that

$$\frac{\widehat{Q}_n(z)}{\widehat{Q}_n(\zeta_n)} = \frac{\widehat{Q}_n(z)}{P_n^{(\alpha,\beta)}(z)} \frac{P_n^{(\alpha,\beta)}(\zeta_n)}{\widehat{Q}_n(\zeta_n)} \frac{2^n P_n^{(\alpha,\beta)}(z)}{\varphi^n(z)} \frac{\varphi^n(\zeta_n)}{2^n P_n^{(\alpha,\beta)}(\zeta_n)} \left(\frac{\varphi(z)}{\varphi(\zeta_n)}\right)^n.$$

Now, the first part of the assertion 2 is straightforward from (4.1).

If $\delta(\zeta) > 2$, let $\mathcal{V}_\varepsilon([-1, 1]) = \{z \in \mathbb{C} : \delta(z) < \epsilon\}$ be a ε-neighborhood of $[-1, 1]$, where $\varepsilon = \varepsilon_\zeta = \frac{\delta(\zeta)}{2} - 1$. By the same reasoning that was deduced (5.2) we get

$$\left|\frac{\widehat{Q}_n(z)}{\widehat{Q}_n(\zeta_n)}\right| < \kappa^n, \quad \text{for all } z \in \mathcal{V}_\varepsilon([-1, 1]), \kappa < 1. \tag{5.8}$$

Hence from the first part of the assertion 2 and (5.8) we get the second part of the assertion 2. □

6. Fluid dynamics model of sources and stagnation points

The fluid dynamic interpretation that we will consider in this section was introduced by H. Pijeira et al. in [2]. In that paper the hydrodynamic model was a reinterpretation of the electrostatic model studied by H. Pijeira et al. in [6]. The difference between the fluid dynamic model in [2] and the model introduced in the present paper is the complex potential used.

Let us consider a flow of an incompressible fluid in the complex plane, due to a system of $n-1$ source points ($n > 1$) fixed at w_i, $1 \le i \le n-1$, with unitary rate of fluid emission per unit time (*strength of the source*), and two additional source points at 1 and -1 with strength $a > 0$ and $b > 0$ respectively. Here, a *source* is a point in which the fluid is continuously created and uniformly distributed in all directions with constant strength (*steady source*). Let us call *flow field generated by a Jacobi set of sources* to a flow of a fluid under the above conditions, or simple a *flow field*.

The complex potential of a flow field at any point z (cf. [3, Chapter 10] and [4, Vol. II—Chapter 6]), by the superposition principle of solutions, is given by

$$\begin{aligned}\Upsilon(z) &= \sum_{i=1}^{n-1} \log(z - w_i) + a \log(z-1) + b \log(z+1), \\ &= \log\left((z-1)^a (z+1)^b \prod_{i=1}^{n-1}(z - w_i)\right).\end{aligned} \tag{6.1}$$

From a complex potential Υ, a *complex velocity* $\mathcal{V}$ can be derived by differentiation ($\mathcal{V}(z) = \frac{d\Upsilon}{dz}(z)$). A standard problem associated with the complex velocity is to find the zeros, that correspond to the set of *stagnation points*, i.e. points where the fluid has zero velocity.

We are interested in an inverse problem in the following sense, build a flow field such that the stagnation points are at preassigned points with *nice* properties. As it is well known, the zeros of orthogonal polynomials with respect to a finite positive Borel measure on $[-1, 1]$ have a rich set of *nice* properties [9, Chapter VI], and will be taken as preassigned stagnation points. Here, we consider that $\mu \in \mathcal{P}_1(\alpha, \beta)$. In the next paragraph the statement of the problem will be established.

Problem. Let $\{x_1, x_2, \ldots, x_n\}$ be the set of zeros of the nth orthogonal polynomial (L_n) with respect to $\mu \in \mathcal{P}_1(\alpha, \beta)$ with $1 < n$. Build a flow field (location of the source points $w_1, \ldots, w_{n-1}$) such that the stagnation points are attained at the points x_i, $i = 1, 2, \ldots, n$.

Let Q_n be a monic polynomial of degree n, whose set of critical points is $\{w_1, w_2, \ldots, w_{n-1}\}$, thus

$$Q_n'(z) = n \prod_{i=1}^{n-1}(z - w_i), \qquad \Upsilon(z) = \log\left(\frac{1}{n}(z-1)^a (z+1)^b Q_n'(z)\right),$$

$$\mathcal{V}(z) = \frac{\partial \Upsilon}{\partial z}(z) = \frac{\left((z-1)^a (z+1)^b Q_n'(z)\right)'}{(z-1)^a (z+1)^b Q_n'(z)} = \frac{\mathcal{L}^{(a-1,b-1)}[Q_n](z)}{(z-1)(z+1) Q_n'(z)}.$$

From (6.1) and Theorem 5.1, $\frac{\partial \mathcal{V}}{\partial z}(x_k) = 0$ for each stagnation point x_k (zeros of L_n), $k = 1, 2, \ldots, n$, i.e.

$$\mathcal{L}^{(a-1,b-1)}[Q_n](x_k) = 0, \quad k = 1, 2, \ldots, n. \tag{6.2}$$

From Corollary 2.2, there exists a monic polynomial Q_n of degree n, unique up to an additive constant, satisfying Eq. (6.2), i.e.

$$\mathcal{L}^{(a-1,b-1)}[Q_n](z) = \lambda_n L_n(z), \quad \lambda_n = -n(n + a + b - 1). \tag{6.3}$$

Note that (6.3) is the same as (1.6) with $\alpha = a - 1$ and $\beta = b - 1$. Therefore the $n - 1$ source point of the flow field $\{w_1, \ldots, w_{n-1}\}$ are the critical point of the nth orthogonal polynomial with respect to the differential operator $\mathcal{L}^{(a-1,b-1)}$.

Answer. *A flow fields generated by a Jacobi set of sources with complex potential (6.1) and preassigned stagnation points at the zeros of the nth orthogonal polynomial with respect to the measure $\mu \in \mathcal{P}_1(\alpha, \beta)$ with $n > 1$, has its sources points (with unitary strength) located at the critical points of the nth orthogonal polynomial with respect to $(\mathcal{L}^{(\alpha,\beta)}, \mu)$.*

In Theorem 5.1, we proved that for $m = 1$ all the critical points of Q_n are simple, contained in $[-1, 1]$ and interlace the zeros of L_n. At the beginning of the Section 5, we conjectured that this theorem is true for all $m \in \mathbb{N}$. If this were true, then it is not difficult to see that the above model holds for $m \in \mathbb{N}$.

Note that, if we consider a system of electrostatic charges with potential given by (6.1) instead of a system of source points with the same potential function, then we have an analogous electrostatic interpretation.

As it is known, the zeros of the Jacobi polynomials have an electrostatic interpretation (see [9, Section 6.7]) as the equilibrium points of a certain potential function. For the case of orthogonality with respect to a differential operator the electrostatic interpretation is an inverse problem in the sense that the equilibrium points are known and the question is to build the electrostatic field.

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