



# Asymptotic formulas associated with psi function with applications



Long Lin

School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City 454003, Henan Province, People's Republic of China

## ARTICLE INFO

### Article history:

Received 31 July 2012

Available online 28 March 2013

Submitted by Michael J. Schlosser

### Keywords:

Psi function

Euler–Mascheroni constant

Bernoulli numbers and polynomials

Asymptotic formula

## ABSTRACT

We prove several asymptotic formulas associated with the psi function, and then apply them to derive the asymptotic formulas for the Euler–Mascheroni constant. Also, we give another proof of an open problem of Chen and Mortici concerning the Euler–Mascheroni constant first proved by S. Yang.

© 2013 Elsevier Inc. All rights reserved.

## 1. Introduction

The classical Euler's gamma function may be defined for  $x > 0$  by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

Some inequalities and asymptotic formulas for the gamma function can be found (see, for example, [3,12,11,13,10]). The logarithmic derivative of  $\Gamma(x)$ , denoted by  $\psi(x) = \Gamma'(x)/\Gamma(x)$ , is known as the psi (or digamma) function. The psi function has the following asymptotic expansion (see [9, p. 32]):

$$\psi(x+t) \sim \ln x - \sum_{n=1}^{\infty} \frac{(-1)^n B_n(t)}{nx^n} \quad \text{as } x \rightarrow \infty, \quad (1.1)$$

where  $B_n(t)$  stands for the Bernoulli polynomials defined by the following generating function:

$$\frac{xe^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!}. \quad (1.2)$$

Note that the Bernoulli numbers  $B_n$  ( $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ ,  $\mathbb{N} := \{1, 2, 3, \dots\}$ ) are defined by  $B_n := B_n(0)$  in (1.2). Setting  $t = 1$  in (1.1) and noting that

$$B_n(0) = (-1)^n B_n(1) = B_n \quad \text{for } n \in \mathbb{N}_0$$

E-mail address: [linlong454000@163.com](mailto:linlong454000@163.com).

(see [1, p. 805]), we obtain from (1.1) that

$$\begin{aligned}\psi(x+1) &\sim \ln x - \sum_{n=1}^{\infty} \frac{B_n}{nx^n} \\ &= \ln x + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \frac{1}{240x^8} - \frac{1}{132x^{10}} + \cdots \quad \text{as } x \rightarrow \infty.\end{aligned}\quad (1.3)$$

By using  $e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$ , we deduce from (1.3) that

$$\exp(\psi(x+1)) \sim x + \frac{1}{2} + \frac{1}{24x} - \frac{1}{48x^2} + \frac{23}{5760x^3} + \frac{17}{3840x^4} - \frac{10099}{2903040x^5} - \frac{2501}{1161216x^6} + \cdots \quad \text{as } x \rightarrow \infty. \quad (1.4)$$

The main object of this paper is to give an explicit formula for determining the coefficients in the asymptotic expansion (1.4) (see Section 2), and then apply it to give another proof of an open problem of Chen and Mortici [4] concerning the Euler–Mascheroni constant first proved by S. Yang [15] (see Section 3).

## 2. Asymptotic expansions associated with psi function

**Theorem 2.1.** *The function  $\exp(\psi(x+1))$  has the following asymptotic expansion:*

$$\exp(\psi(x+1)) \sim x \left( 1 + \sum_{j=1}^{\infty} \frac{p_j}{x^j} \right) \quad \text{as } x \rightarrow \infty, \quad (2.1)$$

with the coefficients  $p_j$  (for  $j \in \mathbb{N}$ ) given by

$$p_j = \sum_{k_1+2k_2+\cdots+jk_j=j} \frac{(-1)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!} \left(\frac{B_1}{1}\right)^{k_1} \left(\frac{B_2}{2}\right)^{k_2} \cdots \left(\frac{B_j}{j}\right)^{k_j}, \quad (2.2)$$

where  $B_j$  are the Bernoulli numbers, summed over all nonnegative integers  $k_j$  satisfying the equation

$$k_1 + 2k_2 + \cdots + jk_j = j.$$

**Proof.** To determine  $p_j$  (for  $j \in \mathbb{N}$ ) in (2.1), we first express (2.1) as follows:

$$\psi(x+1) - \ln x = \ln \left( 1 + \sum_{j=1}^m \frac{p_j}{x^j} \right) + O(x^{-m-1}) \quad \text{as } x \rightarrow \infty. \quad (2.3)$$

By using the fundamental theorem of algebra, we see that there exist unique complex numbers  $\lambda_1, \dots, \lambda_m$  such that

$$1 + \frac{p_1}{x} + \cdots + \frac{p_m}{x^m} = \left( 1 + \frac{\lambda_1}{x} \right) \cdots \left( 1 + \frac{\lambda_m}{x} \right). \quad (2.4)$$

By using the following series expansion:

$$\ln \left( 1 + \frac{z}{x} \right) = \sum_{j=1}^q \frac{(-1)^{j-1} z^j}{j x^j} + O(x^{-q-1}) \quad \text{for } |z| < |x| \text{ and } x \rightarrow \infty,$$

we obtain

$$\ln \left( 1 + \frac{p_1}{x} + \cdots + \frac{p_m}{x^m} \right) = \sum_{j=1}^m \frac{(-1)^{j-1} \sigma_j}{j x^j} + O(x^{-m-1}) \quad \text{as } x \rightarrow \infty, \quad (2.5)$$

where

$$\sigma_j = \lambda_1^j + \cdots + \lambda_m^j \quad \text{for } j = 1, \dots, m.$$

We then find from (1.3) and (2.5) that

$$\sigma_j = (-1)^j B_j \quad \text{for } j = 1, \dots, m, \quad (2.6)$$

that is,

$$\begin{cases} \lambda_1 + \cdots + \lambda_m = -B_1, \\ \lambda_1^2 + \cdots + \lambda_m^2 = B_2, \\ \cdots \\ \lambda_1^m + \cdots + \lambda_m^m = (-1)^m B_m. \end{cases} \quad (2.7)$$

Let

$$Q_m(x) = x^m + d_1 x^{m-1} + \cdots + d_{m-1} x + d_m$$

be a polynomial with zeros  $\lambda_1, \dots, \lambda_m$  satisfying the system of Eqs. (2.7). So we have

$$Q_m(x) = (x - \lambda_1) \cdots (x - \lambda_m). \quad (2.8)$$

The Newton formulas (see, for example, [7] and references therein) give the connection between the coefficients  $d_j$  and the power sums  $\sigma_j$ :

$$\sigma_j + \sigma_{j-1} d_1 + \sigma_{j-2} d_2 + \cdots + \sigma_1 d_{j-1} + j d_j = 0 \quad \text{for } j = 1, \dots, m. \quad (2.9)$$

It is known (see [7]) that  $d_j$  can be expressed in terms of  $\sigma_j$  as

$$d_j = \sum_{k_1+2k_2+\cdots+jk_j=j} \frac{(-1)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!} \left(\frac{\sigma_1}{1}\right)^{k_1} \left(\frac{\sigma_2}{2}\right)^{k_2} \cdots \left(\frac{\sigma_j}{j}\right)^{k_j}.$$

From (2.8) we obtain

$$\frac{(-1)^m}{x^m} Q_m(-x) = \left(1 + \frac{\lambda_1}{x}\right) \cdots \left(1 + \frac{\lambda_m}{x}\right).$$

We thus have

$$1 + \frac{(-1)d_1}{x} + \frac{(-1)^2 d_2}{x^2} + \cdots + \frac{(-1)^m d_m}{x^m} = \left(1 + \frac{\lambda_1}{x}\right) \cdots \left(1 + \frac{\lambda_m}{x}\right). \quad (2.10)$$

We see from (2.4) and (2.10) that the coefficients  $p_j$  are given by

$$p_j = (-1)^j d_j = (-1)^j \sum_{k_1+2k_2+\cdots+jk_j=j} \frac{(-1)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!} \left(\frac{\sigma_1}{1}\right)^{k_1} \left(\frac{\sigma_2}{2}\right)^{k_2} \cdots \left(\frac{\sigma_j}{j}\right)^{k_j}, \quad (2.11)$$

where  $\sigma_j$  are given by (2.6). Finally, substituting the expression (2.6) into (2.11) yields (2.2). The proof is complete.  $\square$

**Remark 2.1.** Following the same method used in the proof of Theorem 2.1, we can prove the following results.

(i) The function  $\ln x - \psi(x)$  has the following asymptotic formula:

$$\ln x - \psi(x) \sim \ln \left(1 + \sum_{j=1}^{\infty} \frac{a_j}{x^j}\right) \quad \text{as } x \rightarrow \infty, \quad (2.12)$$

with the coefficients  $a_j$  (for  $j \in \mathbb{N}$ ) given by

$$a_j = (-1)^j \sum_{k_1+2k_2+\cdots+jk_j=j} \frac{(-1)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!} \left(\frac{S_1}{1}\right)^{k_1} \left(\frac{S_2}{2}\right)^{k_2} \cdots \left(\frac{S_j}{j}\right)^{k_j}, \quad (2.13)$$

where

$$S_1 = \frac{1}{2} \quad \text{and} \quad S_j = (-1)^{j-1} B_j \quad \text{for } j = 2, 3, \dots$$

Here, from (2.12), we obtain the explicit asymptotic formula

$$\ln x - \psi(x) \sim \ln \left(1 + \frac{1}{2x} + \frac{5}{24x^2} + \frac{1}{16x^3} + \frac{47}{5760x^4} - \frac{1}{2304x^5} + \cdots\right)$$

as  $x \rightarrow \infty$ , which can be written as

$$\frac{1}{\exp(\psi(x))} \sim \frac{1}{x} + \frac{1}{2x^2} + \frac{5}{24x^3} + \frac{1}{16x^4} + \frac{47}{5760x^5} - \frac{1}{2304x^6} + \cdots \quad \text{as } x \rightarrow \infty. \quad (2.14)$$

(ii) The function  $\psi\left(x + \frac{1}{2}\right) - \ln x$  has the following asymptotic formula:

$$\psi\left(x + \frac{1}{2}\right) - \ln x \sim \ln\left(1 + \sum_{j=1}^{\infty} \frac{c_j}{x^j}\right) \quad \text{as } x \rightarrow \infty, \quad (2.15)$$

with the coefficients  $c_j$  (for  $j \in \mathbb{N}$ ) given by

$$c_j = \sum_{k_1+2k_2+\dots+jk_j=j} \frac{(-1)^{k_1+k_2+\dots+k_j}}{k_1!k_2!\dots k_j!} \left(\frac{B_1(1/2)}{1}\right)^{k_1} \left(\frac{B_2(1/2)}{2}\right)^{k_2} \dots \left(\frac{B_j(1/2)}{j}\right)^{k_j} \quad (2.16)$$

where

$$B_j(1/2) = -\left(1 - \frac{1}{2^{j-1}}\right) B_j \quad \text{for } j \in \mathbb{N}. \quad (2.17)$$

Here, from (2.15), we obtain the explicit asymptotic formula

$$\psi\left(x + \frac{1}{2}\right) - \ln x \sim \ln\left(1 + \frac{1}{24x^2} - \frac{37}{5760x^4} + \frac{10313}{2903040x^6} - \frac{5509121}{1393459200x^8} + \dots\right)$$

as  $x \rightarrow \infty$ , which can be written as

$$\exp\left(\psi\left(x + \frac{1}{2}\right)\right) \sim x + \frac{1}{24x} - \frac{37}{5760x^3} + \frac{10313}{2903040x^5} - \frac{5509121}{1393459200x^7} + \dots \quad (2.18)$$

as  $x \rightarrow \infty$ .

The formulas (2.14) and (2.18) can be found (see the website mentioned in [8]).

### 3. A solution to the open problem

The Euler–Mascheroni constant  $\gamma = 0.577215664\dots$  is defined as the limit of the sequence

$$D_n = H_n - \ln n, \quad (3.1)$$

where  $H_n = \sum_{k=1}^n \frac{1}{k}$  is the  $n$ th harmonic number.

Young [16] proved that

$$\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2n} \quad (n \in \mathbb{N}). \quad (3.2)$$

The convergence of the sequence  $D_n$  to  $\gamma$  is very slow. By changing the logarithmic term in (3.1), DeTemple [6], Negoi [14] and Chen et al. [5] have presented, respectively, *faster* and *faster* asymptotic formulas as follows:

$$H_n - \ln\left(n + \frac{1}{2}\right) = \gamma + O(n^{-2}) \quad \text{as } n \rightarrow \infty; \quad (3.3)$$

$$H_n - \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right) = \gamma + O(n^{-3}) \quad \text{as } n \rightarrow \infty; \quad (3.4)$$

$$H_n - \ln\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2}\right) = \gamma + O(n^{-4}) \quad \text{as } n \rightarrow \infty. \quad (3.5)$$

Very recently, Chen and Mortici [4] provided a *faster* asymptotic formula than those in (3.3)–(3.5):

$$H_n - \ln\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3}\right) = \gamma + O(n^{-5}) \quad \text{as } n \rightarrow \infty \quad (3.6)$$

and posed the following natural question.

*Open problem.* For a given positive integer  $q$ , find the constants  $d_j$  ( $j = 0, 1, 2, \dots, q$ ) such that

$$H_n - \ln\left(n + \sum_{j=0}^q \frac{d_j}{n^j}\right) \quad (3.7)$$

is the *fastest* sequence which would converge to  $\gamma$ .

Yang [15] first published the solution of the Open Problem (3.7). We find from (2.1) that for  $q \in \mathbb{N}_0$ ,

$$\psi(n+1) = \ln \left( n + \sum_{j=0}^q \frac{p_{j+1}}{n^j} \right) + O \left( \frac{1}{n^{q+2}} \right) \quad \text{as } n \rightarrow \infty. \quad (3.8)$$

It is well-known (see [1, p. 258]) that

$$\psi(n+1) = -\gamma + H_n. \quad (3.9)$$

From (3.8) and (3.9), we obtain that for  $q \in \mathbb{N}_0$ ,

$$H_n - \ln \left( n + \sum_{j=0}^q \frac{d_j}{n^j} \right) = \gamma + O \left( \frac{1}{n^{q+2}} \right) \quad \text{as } n \rightarrow \infty \quad (3.10)$$

where  $d_j := p_{j+1}$  and  $p_j$  are given by (2.2). We thus provide the solution of the Open Problem (3.7) again.

**Remark 3.1.** From (2.1) and (3.9) we obtain the following asymptotic formula for the Euler–Mascheroni constant:

$$\gamma \sim H_n - \ln \left( n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3} + \frac{17}{3840n^4} - \frac{10099}{2903040n^5} - \frac{2501}{1161216n^6} + \cdots \right) \quad \text{as } n \rightarrow \infty. \quad (3.11)$$

Very recently, Batir and Chen [2] proved the following approximation:

$$\gamma \sim H_n - \ln \left( n + \frac{1}{2} + \frac{1}{24(n + \frac{1}{2})} - \frac{37}{5760(n + \frac{1}{2})^3} + \frac{10313}{2903040(n + \frac{1}{2})^5} \right) \quad \text{as } n \rightarrow \infty. \quad (3.12)$$

Replacing  $x$  by  $n + \frac{1}{2}$  in (2.15) and applying (3.9), we obtain a complete asymptotic expansion:

$$\begin{aligned} \gamma \sim H_n - \ln \left( n + \frac{1}{2} + \frac{1}{24(n + \frac{1}{2})} - \frac{37}{5760(n + \frac{1}{2})^3} + \frac{10313}{2903040(n + \frac{1}{2})^5} \right. \\ \left. - \frac{5509121}{1393459200(n + \frac{1}{2})^7} + \cdots \right) \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (3.13)$$

## References

- [1] M. Abramowitz, I.A. Stegun (Eds.), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, in: Applied Mathematics Series, vol. 55, National Bureau of Standards, Washington, DC, 1972. Ninth Printing.
- [2] N. Batir, C.-P. Chen, Improving some sequences convergent to Euler–Mascheroni constant, *Proyecciones* 31 (2012) 29–38.
- [3] C.-P. Chen, L. Lin, Remarks on asymptotic expansions for the gamma function, *Appl. Math. Lett.* 25 (2012) 2322–2326.
- [4] C.-P. Chen, C. Mortici, New sequence converging towards the Euler–Mascheroni constant, *Comput. Math. Appl.* 64 (2012) 391–398.
- [5] C.-P. Chen, H.M. Srivastava, L. Li, S. Manyama, Inequalities and monotonicity properties for the psi (or digamma) function and estimates for the Euler–Mascheroni constant, *Integral Transforms Spec. Funct.* 22 (2011) 681–693.
- [6] D.W. DeTemple, A quicker convergence to Euler's constant, *Amer. Math. Monthly* 100 (1993) 468–470.
- [7] H.W. Gould, The Girard–Waring power sum formulas for symmetric functions and Fibonacci sequences, *Fibonacci Quart.* 37 (1999) 135–140.
- [8] [http://en.wikipedia.org/wiki/Digamma\\_function](http://en.wikipedia.org/wiki/Digamma_function).
- [9] Y.L. Luke, The Special Functions and their Approximations, Vol. I, Academic Press, New York, 1969.
- [10] C. Mortici, A new method for establishing and proving new bounds for the Wallis ratio, *Math. Inequal. Appl.* 13 (2010) 803–815.
- [11] C. Mortici, New approximation formulas for evaluating the ratio of gamma functions, *Math. Comput. Modelling* 52 (2010) 425–433.
- [12] C. Mortici, New improvements of the Stirling formula, *Appl. Math. Comput.* 217 (2010) 699–704.
- [13] C. Mortici, Improved asymptotic formulas for the gamma function, *Comput. Math. Appl.* 61 (2011) 3364–3369.
- [14] T. Negoi, A faster convergence to the constant of Euler, *Gaz. Math., Ser. A* 15 (1997) 111–113 (in Romanian).
- [15] S. Yang, On an open problem of Chen and Mortici concerning the Euler–Mascheroni constant, *J. Math. Anal. Appl.* 396 (2012) 689–693.
- [16] R.M. Young, Euler's constant, *Math. Gaz.* 75 (1991) 187–190.