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Determining elements in C^*-algebras through spectral properties

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ABSTRACT

Let A be a unital C^*-algebra and A'' its second dual. By $\sigma(a)$ and $r(a)$ we denote the spectrum and the spectral radius of $a \in A$, respectively. The following two statements hold for arbitrary $a, b \in A$: (1) $\sigma(ac) \subseteq \sigma(bc) \cup \{0\}$ for every $c \in A$ if and only if there exists a central projection $z \in A''$ such that $a = zb$, (2) $r(ac) \le r(bc)$ for every $c \in A$ if and only if there exists a central element z in A'' such that $a = zb$ and $\|z\| \le 1$.

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1. Introduction

The goal of this note is to generalize and complete the main results from the recent paper [4], to which we refer for motivation and applications concerning the conditions that we are going to study. Let us just mention here that applications are connected with the well-known problem, initiated by Kaplansky in [6], of characterizing multiplicative maps through their spectral properties. We also refer to the paper [3], which also continues the line of investigation started in [4], but in a different direction as we do here.

While the general setting of [4] is Banach algebras, the two main results concern (unital) C^*-algebras. The first one says that the elements a, b from a C^*-algebra A must be equal if $\sigma(ac) = \sigma(bc)$ holds for every $c \in A$, and the second one says that if A is a prime C^*-algebra, then $r(ac) \le r(bc)$ holds for every $c \in A$ if and only if there exists $\lambda \in \mathbb{C}$ such that $a = \lambda b$ and $|\lambda| \le 1$; here, $\sigma(\cdot)$ and $r(\cdot)$ stand for the spectrum and the spectral radius, respectively. We will generalize the first result by treating the inclusion instead of the equality of the spectra (Theorem 2.3), and extend the second result to general C^*-algebras (Theorem 3.6). These higher levels of generality make the problems technically quite more involved. Therefore we have to add new methods to those already used in [4].

We introduce some notation. We write A'' for the second dual of a C^*-algebra A. The spectrum of $a \in A$ will be usually denoted by $\sigma(a)$, but sometimes, when it will be appropriate to emphasize the algebra with respect to which we are considering the spectrum, by $\sigma_A(a)$. The center of A will be denoted by $\mathcal{Z}(A)$.

2. Spectrum

First we state a lemma which is evident from the proofs of Claims 1 and 2 of [4, Theorem 2.6].

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Lemma 2.1. *Let A be a unital C^*-algebra and let $a, b \in A$ be such that $\sigma(ac) \subseteq \sigma(bc) \cup \{0\}$ for every $c \in A$. Suppose that $b^* = b$. Then $a^* = a$ and $ab = ba$.*

We continue by treating the commutative case.

Lemma 2.2. *Let K be a compact Hausdorff space and let $f, g \in C(K)$ be such that $\sigma(fh) \subseteq \sigma(gh) \cup \{0\}$ for every $h \in C(K)$. Then $\operatorname{supp}(f) \subseteq \operatorname{supp}(g)$ and $f = g$ on $\operatorname{supp}(f)$.*

Proof. Let $t \in K \setminus \operatorname{supp}(g)$. Then there exists $h \in C(K)$ with $h(t) = 1$ and $h(\operatorname{supp}(g)) = \{0\}$. Consequently, $gh = 0$ and

$$f(t) = f(t)h(t) \in \sigma(fh) \subseteq \sigma(gh) \cup \{0\} = \{0\}.$$

We thus get $K \setminus \operatorname{supp}(g) \subseteq \{t \in K : f(t) = 0\}$, which obviously implies that $\operatorname{supp}(f) \subseteq \operatorname{supp}(g)$.

We now claim that

$$\left(f(t)\overline{g(t)}\right)^2 = f(t)\overline{g(t)}|g(t)|^2 \tag{2.1}$$

for each $t \in K$. Let $h = g^* + ig^*|g|^2$. Then

$$\sigma\left(fg^* + ifg^*|g|^2\right) = \sigma(fh) \subseteq \sigma(gh) \cup \{0\} = \sigma\left(|g|^2 + i|g|^4\right) \cup \{0\}.$$

Hence, for each $t \in K$, either $f(t)\overline{g(t)} + if(t)\overline{g(t)}\,|g(t)|^2 = 0$, in which case $f(t)\overline{g(t)} = 0$, or there exists $s_t \in K$ such that $f(t)\overline{g(t)} + if(t)\overline{g(t)}\,|g(t)|^2 = |g(s_t)|^2 + i\,|g(s_t)|^4$. Using $\sigma(fg^*) \subseteq \sigma(|g|^2) \cup \{0\} \subseteq \mathbb{R}$ and $\sigma(fg^*|g|^2) \subseteq \sigma(|g|^4) \cup \{0\} \subseteq \mathbb{R}$ we see that in the latter case we have $f(t)\overline{g(t)} = |g(s_t)|^2$ and $f(t)\overline{g(t)}\,|g(t)|^2 = |g(s_t)|^4$. This proves (2.1).

On account of (2.1), we have $f(t) = g(t)$ for each $t \in K$ with $f(t), g(t) \neq 0$. We claim that $f = g$ on $\operatorname{supp}(f)$. Of course, it suffices to prove that $f = g$ on the set $U = \{t \in K : f(t) \neq 0\}$. Pick $t \in U$. Since $\operatorname{supp}(f) \subseteq \operatorname{supp}(g)$, it follows that $t \in \operatorname{supp}(g)$ and so there exists a net (t_λ) in K with $g(t_\lambda) \neq 0$ for every λ and $\lim t_\lambda = t$. Since U is a neighborhood of t we can certainly assume that $t_\lambda \in U$ for each λ. Therefore $f(t_\lambda) = g(t_\lambda)$ for each λ, which gives $f(t) = \lim f(t_\lambda) = \lim g(t_\lambda) = g(t)$. □

Theorem 2.3. *Let A be a unital C^*-algebra and let $a, b \in A$. Then the following properties are equivalent.*

(1) $\sigma(ac) \subseteq \sigma(bc) \cup \{0\}$ *for every* $c \in A$.
(2) *There exists a central projection $z \in A''$ such that $a = zb$.*

Proof. We begin by assuming that (1) holds. By applying the hypothesis with c replaced by b^*c we arrive at

$$\sigma(ab^*c) \subseteq \sigma(bb^*c) \cup \{0\} \quad (c \in A).$$

Lemma 2.1 then shows that $ab^* = ba^*$ commutes with bb^*. We now apply the hypothesis with c replaced by a^*c to get

$$\sigma(aa^*c) \subseteq \sigma(ba^*c) \cup \{0\} \quad (c \in A).$$

Since we already know that ba^* is self-adjoint, Lemma 2.1 now shows that ba^* commutes with aa^*. Moreover, the preceding inclusions yield

$$\sigma(aa^*c) \subseteq \sigma(bb^*c) \cup \{0\} \quad (c \in A)$$

and Lemma 2.1 then shows that aa^* commutes with bb^*. Consequently, the C^*-subalgebra B of A generated by $\mathbf{1}$, aa^*, $ab^* = ba^*$, and bb^* is commutative. Further, we have

$$\sigma_B(aa^*c) \subseteq \sigma_B(ab^*c) \cup \{0\} \subseteq \sigma_B(bb^*c) \cup \{0\} \quad (c \in B).$$

On account of Lemma 2.2, we have

$$\operatorname{supp}(aa^*) \subseteq \operatorname{supp}(ab^*) \subseteq \operatorname{supp}(bb^*),$$

$aa^* = ab^*$ on $\operatorname{supp}(aa^*)$, and $ab^* = bb^*$ on $\operatorname{supp}(ab^*)$. Let e be the projection in A'' corresponding to the characteristic function of the set $\operatorname{supp}(aa^*)$. It is immediate to check that $(a - eb)(a - eb)^* = 0$, which implies

$$a = eb. \tag{2.2}$$

Let s be a self-adjoint element in A. Then $\sigma(asb^*c) \subseteq \sigma(bsb^*c) \cup \{0\}$ for each $c \in A$. Since bsb^* is self-adjoint, Lemma 2.1 shows that $asb^* = bsa^*$. This clearly implies that $acb^* = bca^*$ for each $c \in A$ and therefore that $axb^* = bxa^*$ for each $x \in A''$. On account of (2.2), we have

$$ebxb^* = bxb^*e \quad (x \in A'')$$

and this clearly gives

$$(1-e)bA''b^*e = \{0\}.$$

Therefore, there exists a central projection $z \in A''$ such that $zb^*e = b^*e$ and $z(1-e)b = 0$ (see for example [2, Proposition III.1.1.7]). The first identity now yields $zeb = eb = a$, while the second one gives $zeb = zb$. Consequently, $a = zb$, as required.

Finally, assume that (2) holds. Let $c \in A$. Since $\sigma_{A''}(z) \subseteq \{0, 1\}$, it follows that

$$\sigma(ac) = \sigma_{A''}(ac) \subseteq \sigma_{A''}(z)\sigma_{A''}(bc) \subseteq \sigma_{A''}(bc) \cup \{0\} = \sigma(bc) \cup \{0\}. \quad \square$$

3. Spectral radius

Let a be an element in a von Neumann algebra $\mathcal{M}$. The smallest projection p in $\mathcal{M}$ such that $pa = a$ ($ap = a$) is the *left support* (resp. *right support*) of a. If a is self-adjoint, then both supports coincide and this common projection, called the *support* of a, is denoted by $s(a)$. We refer the reader to [7, Section 1.10] for the basic properties of the support.

We continue with a series of technical lemmas.

Lemma 3.1. *Let $\mathcal{M}$ be a von Neumann algebra and let $b, w \in \mathcal{M}$. Suppose that $b^* = b$ and that $wbub^2ub = bubwbub$ for every self-adjoint element $u \in \mathcal{M}$. Then $ws(b)xs(b) = s(b)xws(b)$ for each $x \in \mathcal{M}$.*

Proof. Replacing u by $u + v$ with both u and v self-adjoint elements in $\mathcal{M}$ it follows that

$$wbub^2vb + wbvb^2ub = bubwbvb + bvbwbub.$$

Since every element in $\mathcal{M}$ is a linear combination of two self-adjoint elements, it follows that

$$wbxb^2yb + wbyb^2xb = bxbwbyb + bybwbxb \quad (x, y \in \mathcal{M}). \tag{3.1}$$

On account of [7, Proposition 1.10.4], $s(b)$ belongs to the von Neumann subalgebra of $\mathcal{M}$ generated by b. Consequently, there is a net $(P_i)_{i\in I}$ of polynomials with $P_i(0) = 0$ ($i \in I$) such that $s(b)$ is the limit with respect to the weak* topology on $\mathcal{M}$ of the net $(P_i(b))$. For each $i \in I$ we write $P_i(\lambda) = \lambda Q_i(\lambda)$ for some polynomial Q_i. Replacing x by $Q_i(b)x$ in (3.1) it follows that

$$wP_i(b)xb^2yb + wbybP_i(b)xb = P_i(b)xbwbyb + bybwP_i(b)xb \quad (x, y \in \mathcal{M}). \tag{3.2}$$

Taking the limit with respect to the weak*-topology on $\mathcal{M}$ in (3.2) and taking into account the separate weak*-continuity of the product we arrive at

$$ws(b)xb^2yb + wbybxb = s(b)xbwbyb + bybws(b)xb \quad (x, y \in \mathcal{M}).$$

The same reasoning starting with x replaced by $xQ_i(b)$ gives

$$ws(b)xbyb + wbybxs(b) = s(b)xs(b)wbyb + bybws(b)xs(b) \quad (x, y \in \mathcal{M}).$$

We now apply this argument once again, with respect to y instead of x, to obtain

$$ws(b)xs(b)yb + ws(b)ybxs(b) = s(b)xs(b)ws(b)yb + s(b)ybws(b)xs(b)$$

and then

$$ws(b)xs(b)ys(b) + ws(b)ys(b)xs(b) = s(b)xs(b)ws(b)ys(b) + s(b)ys(b)ws(b)xs(b)$$

for all $x, y \in \mathcal{M}$.

Taking $y = s(b)$ we get the identity

$$2ws(b)xs(b) = s(b)xs(b)ws(b) + s(b)ws(b)xs(b) \quad (x \in \mathcal{M}).$$

Taking $x = s(b)$ in the preceding identity we arrive at $ws(b) = s(b)ws(b)$. We now use this property in the previous identity to get $2ws(b)xs(b) = s(b)xws(b) + ws(b)xs(b)$ and therefore $ws(b)xs(b) = s(b)xws(b)$ for each $x \in \mathcal{M}$, as required. $\square$

Lemma 3.2. *Let $\mathcal{M}$ be a von Neumann algebra, let w be a normal element in $\mathcal{M}$, and let p be a projection in $\mathcal{M}$. Suppose that $wpxp = pxpw$ for each $x \in \mathcal{M}$. Then there exists $z \in \mathcal{Z}(\mathcal{M})$ such that $zp = wp$ and $\|z\| \leq \|w\|$.*

Proof. Let $\mathcal{E}$ be the spectral measure on $\sigma(w)$ such that

$$w = \int_{\sigma(w)} \lambda \, d\mathcal{E}(\lambda).$$

Since $wpxp = pxpw$ for all $x \in \mathcal{M}$, it follows that

$$\mathcal{E}(\Delta)pxp = pxp\mathcal{E}(\Delta) \quad (x \in \mathcal{M}) \tag{3.3}$$

and, in particular, we have $\mathcal{E}(\Delta)p = p\mathcal{E}(\Delta)$ for each Borel subset Δ of $\sigma(w)$.

Every projection q in $\mathcal{M}$ has a central carrier, the smallest projection $\theta(q)$ in $\mathcal{Z}(\mathcal{M})$ majorizing q. We refer the reader to [2, Section III.1.1] and [7, Section 1.10] for the basic facts about the central carrier. For every Borel subset Δ of $\sigma(w)$ we define

$$\mathcal{F}(\Delta)=\theta\big(\mathcal{E}(\Delta)p\big).$$

Let Δ be a Borel subset Δ of $\sigma(w)$. We claim that $\mathcal{F}(\Delta)\in\theta(p)\mathcal{Z}(\mathcal{M})$. It is clear that $\mathcal{E}(\Delta)p\le\theta(\mathcal{E}(\Delta))\theta(p)\in\mathcal{Z}(\mathcal{M})$ and so $\theta\big(\mathcal{E}(\Delta)p\big)\le\theta(\mathcal{E}(\Delta))\theta(p)$. According to [2, Proposition II.3.3.1], we have

$$\begin{aligned}\mathcal{F}(\Delta)&=\theta\big(\mathcal{E}(\Delta)p\big)=\theta\big(\mathcal{E}(\Delta)p\big)\Big(\theta(\mathcal{E}(\Delta))\theta(p)\Big)\\&=\Big(\theta\big(\mathcal{E}(\Delta)p\big)\theta(\mathcal{E}(\Delta))\Big)\theta(p)\in\mathcal{Z}(\mathcal{M})\theta(p).\end{aligned}$$

Our next objective is to show that if Δ_1 and Δ_2 are disjoint Borel subsets of $\sigma(w)$, then

$$\mathcal{F}(\Delta_1)\mathcal{F}(\Delta_2)=0. \tag{3.4}$$

On account of (3.3), we have $\mathcal{E}(\Delta_1)px\mathcal{E}(\Delta_2)p=0$ and [7, Proposition 1.10.7] then gives (3.4).

Let (Δ_n) be a sequence of pairwise disjoint Borel subsets of $\sigma(w)$. Then

$$\mathcal{F}\left(\bigcup_{n=1}^{\infty}\Delta_n\right)=\theta\left(\mathcal{E}\left(\bigcup_{n=1}^{\infty}\Delta_n\right)p\right)=\theta\left(\sum_{n=1}^{\infty}\mathcal{E}(\Delta_n)p\right).$$

Since $\big(\mathcal{E}(\Delta_n)p\big)$ is a sequence of pairwise orthogonal projections and, according to (3.4), the sequence $\big(\theta(\mathcal{E}(\Delta_n)p)\big)$ also consists of pairwise orthogonal projections, [5, Propositions 2.5.8 and 5.5.3] show that

$$\theta\left(\sum_{n=1}^{\infty}\mathcal{E}(\Delta_n)p\right)=\sum_{n=1}^{\infty}\theta\big(\mathcal{E}(\Delta_n)p\big).$$

We thus get

$$\mathcal{F}\left(\bigcup_{n=1}^{\infty}\Delta_n\right)=\sum_{n=1}^{\infty}\mathcal{F}(\Delta_n).$$

Consequently, $\mathcal{F}$ is a spectral measure on $\sigma(w)$ with range in the von Neumann algebra $\theta(p)\mathcal{Z}(\mathcal{M})$.

We now define

$$z=\int_{\sigma(w)}\lambda\,d\mathcal{F}(\lambda).$$

Then $z\in\mathcal{Z}(\mathcal{M})$ and it is immediate to check that $\|z\|\le\|w\|$. Our final goal is to show that $zp=wp$. To this end it suffices to show that $\mathcal{F}(\Delta)p=\mathcal{E}(\Delta)p$ for each Borel subset Δ of $\sigma(w)$. Let Δ be a Borel subset of $\sigma(w)$. On the one hand, we have $\mathcal{E}(\Delta)p\le\mathcal{F}(\Delta)$ and hence

$$\mathcal{F}(\Delta)\big(\mathcal{E}(\Delta)p\big)=\mathcal{E}(\Delta)p. \tag{3.5}$$

On the other hand, on account of (3.4) and (3.5) (with Δ replaced by $\sigma(w)\setminus\Delta$), we have

$$\mathcal{F}(\Delta)\big(\mathcal{E}(\sigma(w)\setminus\Delta)p\big)=\mathcal{F}(\Delta)\mathcal{F}(\sigma(w)\setminus\Delta)\big(\mathcal{E}(\sigma(w)\setminus\Delta)p\big)=0. \tag{3.6}$$

From (3.5) and (3.6) we deduce that

$$\mathcal{F}(\Delta)p=\mathcal{F}(\Delta)\big(\mathcal{E}(\Delta)p+\mathcal{E}(\sigma(w)\setminus\Delta)p\big)=\mathcal{E}(\Delta)p,$$

as required. □

Lemma 3.3. *Let A be a unital C^*-algebra and let $a,b\in A$ be such that $r(ac)\le r(bc)$ for every $c\in A$. Suppose that $b^*=b$. Then a is normal and $ac=ca$ for every $c\in A$ such that $bc=cb$.*

Proof. Let $B=\{u\in A: bu=ub\}$. Then B is a C^*-algebra containing b. Define $\varphi: B\times B\to A$ by $\varphi(u,v)=uav$ for all $u,v\in B$. Suppose that $u,v\in B$ are such that $uv=0$. Then $ubv=buv=0$ and [4, Lemma 3.2] then yields $uav=0$. On account of [1, Theorem 2.11 and Example 1.3.2], we have $\varphi(uv,w)=\varphi(u,vw)$ for all $u,v,w\in B$. By taking $u=w=\mathbf{1}$ we get $va=av$ for each $v\in B$, as claimed.

Since $b\in B$ it follows that $ab=ba$ and therefore $ba^*=a^*b$. This shows that $a^*\in B$ and therefore $aa^*=a^*a$. □

Lemma 3.4. *Let K be a compact Hausdorff space and let $f,g\in C(K)$ be such that $r(fh)\le r(gh)$ for each $h\in C(K)$. Then $|f|\le|g|$.*

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Proof. On the contrary, suppose that $|f(t_0)| > |g(t_0)|$ for some $t_0 \in K$. Then $U = \{t \in K : |g(t)| < |f(t_0)|\}$ is an open neighborhood of t_0. We take a continuous function $h: K \to [0, 1]$ with $\mathrm{supp}(h) \subseteq U$ and $h(t_0) = 1$. Then $|(gh)(t)| < |f(t_0)| = |(fh)(t_0)|$, which shows that $r(gh) < r(fh)$, a contradiction. □

Lemma 3.5. *Let A be a unital C^*-algebra and let $a, b \in A$ such that $r(ac) \le r(bc)$ for each $c \in A$. Suppose that $b^* = b$. Then there exists $z \in \mathcal{Z}(A'')$ such that $a = zb$ and $\|z\| \le 1$.*

Proof. Let B be the C^*-subalgebra of A generated by $\mathbf{1}$, a, and b. By Lemma 3.3, the algebra B is commutative so that it can be identified with $C(K)$ for some compact Hausdorff space K. From Lemma 3.4 it follows that $|a(t)| \le |b(t)|$ for each $t \in K$. We now define $w \in A''$ by $w(t) = a(t)/b(t)$ whenever $t \in K$ is such that $b(t) \ne 0$ and $w(t) = 0$ elsewhere. Then $a = wb$ and $\|w\| \le 1$. Our purpose is to show that w can be replaced by an appropriate element in $\mathcal{Z}(A'')$.

Pick a self-adjoint element $u \in A$. Replacing c by ubc in $r(wbc) \le r(bc)$ we get $r(wbubc) \le r(bubc)$ for each $c \in A$. Since bub is self-adjoint, Lemma 3.3 shows that $(wbub)(bub) = (bub)(wbub)$. Lemma 3.1 now yields $ws(b)xs(b) = s(b)xws(b)$ for every $x \in A''$. We now observe that $ws(b) = s(b)w$ and therefore Lemma 3.2 gives $z \in \mathcal{Z}(A'')$ such that $zs(b) = ws(b)$ and $\|z\| \le \|w\| \le 1$. Finally, we observe that

$$zb = z(s(b)b) = (zs(b))b = (ws(b))b = w(s(b)b) = wb = a. \quad \square$$

Theorem 3.6. *Let A be a unital C^*-algebra and let $a, b \in A$. Then the following properties are equivalent.*

(1) *$r(ac) \le r(bc)$ for every $c \in A$.*
(2) *There exists $z \in \mathcal{Z}(A'')$ such that $a = zb$ and $\|z\| \le 1$.*

Proof. Assume that (1) holds. Then

$$r(a^*c) \le r(b^*c) \quad (c \in A). \tag{3.7}$$

Indeed,

$$\begin{aligned} r(a^*c) &= r(ca^*) = r\big((ac^*)^*\big) = r(ac^*) \\ &\le r(bc^*) = r\big((cb^*)^*\big) = r(cb^*) = r(b^*c). \end{aligned}$$

Taking b^*c for c in (1) we get

$$r(ab^*c) \le r(bb^*c) \quad (c \in A).$$

Since bb^* is self-adjoint, Lemma 3.5 yields $z \in \mathcal{Z}(A'')$ such that $ab^* = zbb^*$ and $\|z\| \le 1$. Our goal is to show that $a = zb$.

By (3.7)

$$r(aa^*c) \le r(ba^*c) = r(a^*cb) \le r(b^*cb) = r(bb^*c)$$

and Lemma 3.5 now gives $w \in \mathcal{Z}(A'')$ such that $aa^* = wbb^*$ and $\|w\| \le 1$.

Take a self-adjoint element $u \in A$. Replacing c by ubc in (3.7) we get $r(a^*ubc) \le r(b^*ubc)$ for each $c \in A$. Since b^*ub is self-adjoint, Lemma 3.3 shows that $(a^*ub)(b^*ub) = (b^*ub)(a^*ub)$. Linearizing this identity we get $a^*ubb^*vb + a^*vbb^*ub = b^*uba^*vb + b^*vba^*ub$ for all self-adjoint elements $u, v \in A$. This obviously implies that $a^*cbb^*db + a^*dbb^*cb = b^*cba^*db + b^*dba^*cb$ for all $c, d \in A$, which gives

$$a^*xbb^*yb + a^*ybb^*xb = b^*xba^*yb + b^*yba^*xb \quad (x, y \in A'').$$

Taking into account that $ab^* = zbb^*$ we arrive at

$$a^*xbb^*yb + a^*ybb^*xb = b^*xz^*bb^*yb + b^*yz^*bb^*xb \quad (x, y \in A'')$$

and therefore

$$(a - zb)^*xbb^*yb + (a - zb)^*ybb^*xb = 0 \quad (x, y \in A'').$$

In particular, we have

$$(a - zb)^*xbb^*xb = 0 \quad (x \in A'').$$

The last two identities yield

$$\begin{aligned} \big((a - zb)^*xbb^*\big)y\big((a - zb)^*xbb^*\big) &= (a - zb)^*xbb^*\big(y(a - zb)^*x\big)bb^* \\ &= -(a - zb)^*\big(y(a - zb)^*x\big)bb^*xbb^* \\ &= -(a - zb)^*y\big((a - zb)^*xbb^*xb\big)b^* \\ &= 0 \end{aligned}$$

for all $x, y \in A''$. By taking $y = \big((a - zb)^* xbb^*\big)^*$ with $x \in A''$ we arrive at

$$\big((a - zb)^* xbb^*\big)\big((a - zb)^* xbb^*\big)^*\big((a - zb)^* xbb^*\big) = 0$$

and multiplying by $\big((a - zb)^* xbb^*\big)^*$ on the right we obtain

$$\Big(\big((a - zb)^* xbb^*\big)\big((a - zb)^* xbb^*\big)^*\Big)^2 = 0.$$

This implies that $(a - zb)^* xbb^* = 0$ for each $x \in A''$. Equivalently,

$$(bb^*)x(a - zb) = 0 \quad (x \in A'').$$

Suppose that $a \neq zb$. Then there exists an irreducible representation π of A'' on a Hilbert space with $\pi(a - zb) \neq 0$. Since

$$\pi(bb^*)\pi(A'')\pi(a - zb) = 0$$

and $\pi(A'')$ is prime, it follows that $\pi(bb^*) = 0$ and hence that $\pi(b) = 0$. Since $aa^* = wbb^*$, it follows that $\pi(aa^*) = 0$ and hence that $\pi(a) = 0$. This shows that $\pi(a - zb) = 0$, a contradiction.

Conversely, assume that (2) holds. Since z is central, it follows that

$$r(ac) = r(zbc) \leq r(z)r(bc) \leq r(bc). \quad \square$$

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References

[1] J. Alaminos, M. Brešar, J. Extremera, A.R. Villena, Maps preserving zero products, Studia Math. 193 (2009) 131–159.
[2] B. Blackadar, Operator algebras. Theory of C^*-algebras and von Neumann algebras, in: Operator Algebras and Non-commutative Geometry, III, in: Encyclopaedia of Mathematical Sciences, vol. 122, Springer-Verlag, Berlin, 2006, p. xx+517.
[3] M. Brešar, B. Magajna, Š. Špenko, Identifying derivations through the spectra of their values, Integral Equations Operator Theory 73 (2012) 395–411.
[4] M. Brešar, Š. Špenko, Determining elements in Banach algebras through spectral properties, J. Math. Anal. Appl. 393 (1) (2012) 144–150.
[5] R.V. Kadison, J.R. Ringrose, Fundamentals of the Theory of Operator Algebras. Vol. I. Elementary Theory, in: Pure and Applied Mathematics, vol. 100, Academic Press, Inc., New York, 1983, p. xv+398.
[6] I. Kaplansky, Algebraic and Analytic Aspects of Operator Algebras, in: Regional Conference Series in Mathematics, vol. 1, Amer. Math. Soc., 1970.
[7] S. Sakai, C^*-algebras and W^*-algebras, in: Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 60, Springer-Verlag, New York, Heidelberg, 1971, p. xii+253.