Flat functions in Carleman ultraholomorphic classes via proximate orders

Javier Sanz

Departamento de Álgebra, Análisis Matemático, Geometría y Topología, Instituto de Investigación en Matemáticas de la Universidad de Valladolid, IMUVA, Facultad de Ciencias, Universidad de Valladolid, 47011 Valladolid, Spain

A R T I C L E   I N F O

A B S T R A C T

Whenever the defining sequence of a Carleman ultraholomorphic class (in the sense of H. Komatsu) is strongly regular and associated with a proximate order, flat functions are constructed in the class on sectors of optimal opening. As consequences, we obtain analogues of both Borel–Ritt–Gevrey theorem and Watson's lemma in this general situation.

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1. Introduction

The Carleman ultraholomorphic classes $\mathcal{A}_{\mathbb{M}}(S)$ in a sector S of the Riemann surface of the logarithm consist of those holomorphic functions f in S whose derivatives of order $n \geqslant 0$ are uniformly bounded there by, essentially, the values $n!M_n$, where $\mathbb{M} = (M_n)_{n\in\mathbb{N}_0}$ is a sequence of positive real numbers. In case bounds are not uniform on S but are valid and depend on every proper subsector of S to which the function is restricted, we obtain the class $\tilde{\mathcal{A}}_{\mathbb{M}}(S)$ of functions with a (non-uniform) $\mathbb{M}$-asymptotic expansion at 0 in S, given by a formal power series $\hat{f} = \sum_{n\geqslant 0} a_n z^n/n!$ whose coefficients are again suitably bounded in terms of $\mathbb{M}$ (we write $f \sim_{\mathbb{M}} \hat{f}$ and $(a_n)_{n\in\mathbb{N}_0} \in \Lambda_{\mathbb{M}}$). The map sending f to $(a_n)_{n\in\mathbb{N}_0}$ is the asymptotic Borel map $\tilde{\mathcal{B}}$, and f is said to be flat if $\tilde{\mathcal{B}}(f)$ is the null sequence. See Section 2.2 for the precise definitions of all these classes and concepts.

In order to obtain good properties for these classes, the sequence $\mathbb{M}$ is usually subject to some standard conditions; in particular, we will only consider strongly regular sequences as defined by V. Thilliez [27], see Section 2.3. The best known example is that of Gevrey classes, appearing when the sequence is chosen to be $\mathbb{M}_\alpha = (n!^\alpha)_{n\in\mathbb{N}_0}$, $\alpha > 0$, and for which we use the notations $\mathcal{A}_\alpha(S)$, $\tilde{\mathcal{A}}_\alpha(S)$, Λ_α, $f \sim_\alpha \hat{f}$ and so on, for

E-mail address: jsanzg@am.uva.es.

simplicity. Let us denote by S_γ the sector bisected by the direction $d = 0$ and with opening $\pi\gamma$. It is well known that $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_\alpha(S_\gamma) \to \Lambda_\alpha$ is surjective if, and only if, $\gamma \leqslant \alpha$ (Borel–Ritt–Gevrey theorem, see [22,23,20], [1, Thm. 2.2.1]). It is natural to call this an extension result, and to think then about the possibility of obtaining linear and continuous right inverses for $\tilde{\mathcal{B}}$ in suitably topologized classes. On the other hand, $\tilde{\mathcal{B}}$ is injective (i.e., $\tilde{\mathcal{A}}_\alpha(S_\gamma)$ does not contain nontrivial flat functions, and then the class $\tilde{\mathcal{A}}_\alpha(S_\gamma)$ is said to be quasianalytic) if, and only if, $\gamma > \alpha$ (Watson's lemma, see for example [2, Prop. 11]). Our main aim in this paper is to provide generalizations of this kind of results in the framework of Carleman ultraholomorphic classes associated with strongly regular sequences inducing a proximate order. Let us start with an overview of the existing literature in this respect.

In 1995 V. Thilliez [26] obtained right inverses in the Gevrey case when $\gamma < \alpha$ by applying techniques from the ultradifferentiable setting (i.e. regarding extension results for classes of smooth functions on open subsets of $\mathbb{R}^n$, determined by imposing a suitable growth of the derivatives), and the same was done by the author in [24] by adapting the truncated Laplace transform procedure already used by J.P. Ramis in Borel–Ritt–Gevrey theorem [22] (this second solution was well-suited for the extension of this result to the several variable case). Regarding general classes, J. Schmets and M. Valdivia [25] extended some results of H.-J. Petzsche [21] for ultradifferentiable classes, and applied them in order to provide the first powerful results in the present framework. Subsequently, V. Thilliez [27] improved the results in [25] in several respects (see Section 3.1 in his paper for the details) by relying on a double application of suitable Whitney's extension results for Whitney ultradifferentiable jets on compact sets with Lipschitz boundary appearing in [21,3,5]. In particular, he introduced a growth index $\gamma(\mathbb{M}) \in (0, \infty)$ for every strongly regular sequence $\mathbb{M}$ (which for $\mathbb{M}_\alpha$ equals α), and proved the following facts: if $\gamma < \gamma(\mathbb{M})$, then $\mathcal{A}_\mathbb{M}(S_\gamma)$ is not quasianalytic, and there exist right inverses for $\tilde{\mathcal{B}}$, which are obtained due to the explicit construction of nontrivial flat functions in the class $\mathcal{A}_\mathbb{M}(S_\gamma)$. Indeed, these flat functions allowed A. Lastra, S. Malek and the author [15] to define suitable kernels and moment sequences by means of which to obtain again right inverses by the classical truncated Laplace transform technique. Because of the integral form of the solution, this procedure admits an easy generalization to the several variable case, and does not rest on any result from the ultradifferentiable setting.

However, the preceding results for general classes are not fully satisfactory. Firstly, the equivalences stated in Borel–Ritt–Gevrey theorem and Watson's lemma for the Gevrey case are now only one-side implications. Secondly, and strongly related to the previous remark, the need to restrict the opening of the sector S_γ to $\gamma < \gamma(\mathbb{M})$ in order to obtain flat functions in $\mathcal{A}_\mathbb{M}(S_\gamma)$ does not allow one to treat the apparently limit situation in which $\gamma = \gamma(\mathbb{M})$. Note that, in the Gevrey case, the function $e^{-z^{-1/\alpha}}$ is flat in the class $\tilde{\mathcal{A}}_\alpha(S_\alpha)$, and of course in every $\mathcal{A}_\alpha(S_\gamma)$ for $\gamma < \alpha$. So, our main objective will be to obtain flat functions in sectors of optimal opening.

In this sense, we first introduce for every strongly regular sequence $\mathbb{M}$ a new constant $\omega(\mathbb{M})$, measuring the rate of growth of the sequence $\mathbb{M}$, in terms of which quasianalyticity in the classes $\mathcal{A}_\mathbb{M}(S_\gamma)$ may be properly characterized due to a classical result of B.I. Korenbljum ([11]; see Theorem 3.2). This constant is easily computed in concrete situations (see (10)), and indeed it is the inverse of the order of growth of the classical function $M(t)$ associated with $\mathbb{M}$, namely $M(t) = \sup_{n \in \mathbb{N}_0} \log(t^n/M_n)$, $t > 0$ (see (11)). Regarding the construction of flat functions, V. Thilliez [28] had characterized flatness in $\tilde{\mathcal{A}}_\mathbb{M}(S_\gamma)$ in terms of the existence of non-uniform estimates governed by the function $e^{-M(1/|z|)}$, much in the same way as the function $e^{-z^{-1/\alpha}}$ expresses flatness in the Gevrey case. So, it became clear to us the need to construct functions in sectors whose growth is accurately specified by the function $M(t)$. The classical theory of growth for holomorphic functions defined in sectorial regions, based on the notion of (constant) exponential order, showed itself not profound enough to deal with the general case. Luckily, the theory of proximate orders, allowing to change the constant order $\rho > 0$ into a function $\rho(r)$ more closely specifying the desired rate of growth, is available since the 1920s, and some of its quite recent developments, mainly due to L.S. Maergoiz [18], have been the key for our success. The problem of characterizing those sequences $\mathbb{M}$ associated with a proximate order has

been solved (Proposition 4.9), and it turns out that all the interesting examples of strongly regular sequences appearing in the literature belong to this class. Whenever this is the case, the results of L.S. Maergoiz allow us to obtain the desired flat functions in $\tilde{\mathcal{A}}_{\mathbb{M}}(S_{\omega(\mathbb{M})})$ (see Theorem 4.7) and, immediately, we may generalize Watson's lemma, see Corollary 4.12. Subsequently, in Section 5 suitable kernels and moment sequences are introduced, by means of which we may prove that $\tilde{\mathcal{B}}$ is surjective in $\tilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma)$ if, and only if, $\gamma \leqslant \omega(\mathbb{M})$, so generalizing Borel–Ritt–Gevrey theorem (see Theorem 6.1).

It should be mentioned that for the standard strongly regular sequences appearing in the literature, the value of the constants $\gamma(\mathbb{M})$ and $\omega(\mathbb{M})$ agree. However, we have only been able to prove that $\gamma(\mathbb{M}) \leqslant \omega(\mathbb{M})$ in general. In case $\gamma(\mathbb{M}) < \omega(\mathbb{M})$ can actually occur for some sequences, our results would definitely improve those of V. Thilliez by enlarging the sectors for which non-quasianalyticity holds or right inverses exist. In any case, the equivalences stated in Theorem 6.1 and Corollary 4.12 are new.

Finally, in Theorem 6.2 we gather the information concerning the existence of right inverses for $\tilde{\mathcal{B}}$ in $\mathcal{A}_{\mathbb{M}}(S_\gamma)$: they exist whenever $\gamma < \omega(\mathbb{M})$, and their existence, under some specific condition (satisfied, for instance, in the Gevrey case), implies that $\gamma < \omega(\mathbb{M})$.

2. Preliminaries

2.1. Notation

We set $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. $\mathcal{R}$ stands for the Riemann surface of the logarithm, and $\mathbb{C}[\![z]\!]$ is the space of formal power series in z with complex coefficients.

For $\gamma > 0$, we consider unbounded sectors

$$S_\gamma := \left\{ z \in \mathcal{R} \colon \; \big|\arg(z)\big| < \frac{\gamma\pi}{2} \right\}$$

or, in general, bounded or unbounded sectors

$$S(d, \alpha, r) := \left\{ z \in \mathcal{R} \colon \; \big|\arg(z) - d\big| < \frac{\alpha\pi}{2}, \; |z| < r \right\}, \qquad S(d, \alpha) := \left\{ z \in \mathcal{R} \colon \; \big|\arg(z) - d\big| < \frac{\alpha\pi}{2} \right\}$$

with bisecting direction $d \in \mathbb{R}$, opening $\alpha\pi$ and (in the first case) radius $r \in (0, \infty)$.

A sectorial region $G(d, \alpha)$ with bisecting direction $d \in \mathbb{R}$ and opening $\alpha\pi$ will be a domain in $\mathcal{R}$ such that $G(d, \alpha) \subset S(d, \alpha)$, and for every $\beta \in (0, \alpha)$ there exists $\rho = \rho(\beta) > 0$ with $S(d, \beta, \rho) \subset G(d, \alpha)$. In particular, sectors are sectorial regions.

A sector T is a bounded proper subsector of a sectorial region G (denoted by $T \ll G$) whenever the radius of T is finite and $\overline{T} \setminus \{0\} \subset G$. Given two unbounded sectors T and S, we say T is an unbounded proper subsector of S, and we write $T \prec S$, if $\overline{T} \setminus \{0\} \subset S$.

$\mathcal{H}(U)$ denotes the space of holomorphic functions in an open set $U \subset \mathcal{R}$.

$D(z_0, r)$ stands for the disk centered at z_0 with radius $r > 0$.

2.2. Asymptotic expansions and ultraholomorphic classes

Given a sequence of positive real numbers $\mathbb{M} = (M_n)_{n \in \mathbb{N}_0}$, a constant $A > 0$ and a sector S, we define

$$\mathcal{A}_{\mathbb{M}, A}(S) = \left\{ f \in \mathcal{H}(S) \colon \; \|f\|_{\mathbb{M}, A} := \sup_{z \in S, \, n \in \mathbb{N}_0} \frac{|f^{(n)}(z)|}{A^n n! M_n} < \infty \right\}.$$

$(\mathcal{A}_{\mathbb{M}, A}(S), \| \; \|_{\mathbb{M}, A})$ is a Banach space, and $\mathcal{A}_{\mathbb{M}}(S) := \bigcup_{A > 0} \mathcal{A}_{\mathbb{M}, A}(S)$ is called a *Carleman ultraholomorphic class* in the sector S.

One may accordingly define classes of sequences

$$\Lambda_{\mathbb{M},A} = \left\{ \boldsymbol{\mu} = (\mu_n)_{n \in \mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0} \colon |\boldsymbol{\mu}|_{\mathbb{M},A} := \sup_{n \in \mathbb{N}_0} \frac{|\mu_n|}{A^n n! M_n} < \infty \right\}.$$

$(\Lambda_{\mathbb{M},A}, |\ |_{\mathbb{M},A})$ is again a Banach space, and we put $\Lambda_{\mathbb{M}} := \bigcup_{A>0} \Lambda_{\mathbb{M},A}$.

Since the derivatives of $f \in \mathcal{A}_{\mathbb{M},A}(S)$ are Lipschitzian, for every $n \in \mathbb{N}_0$ one may define

$$f^{(n)}(0) := \lim_{z \in S, z \to 0} f^{(n)}(z) \in \mathbb{C},$$

and it is clear that the sequence

$$\tilde{\mathcal{B}}(f) := \left(f^{(n)}(0) \right)_{n \in \mathbb{N}_0} \in \Lambda_{\mathbb{M},A}, \quad f \in \mathcal{A}_{\mathbb{M},A}(S).$$

The map $\tilde{\mathcal{B}} : \mathcal{A}_{\mathbb{M}}(S) \longrightarrow \Lambda_{\mathbb{M}}$ so defined is the *asymptotic Borel map*.

Next, we will recall the relationship between these classes and the concept of asymptotic expansion.

Definition 2.1. We say a holomorphic function f in a sectorial region G admits the formal power series $\hat{f} = \sum_{p=0}^{\infty} a_p z^p \in \mathbb{C}[[z]]$ as its $\mathbb{M}$-*asymptotic expansion* in G (when the variable tends to 0) if for every $T \ll G$ there exist $C_T, A_T > 0$ such that for every $n \in \mathbb{N}$, one has

$$\left| f(z) - \sum_{p=0}^{n-1} a_p z^p \right| \leqslant C_T A_T^n M_n |z|^n, \quad z \in T.$$

We will write $f \sim_{\mathbb{M}} \sum_{p=0}^{\infty} a_p z^p$ in G. $\tilde{\mathcal{A}}_{\mathbb{M}}(G)$ stands for the space of functions admitting $\mathbb{M}$-asymptotic expansion in G.

Definition 2.2. Given a sector S, we say $f \in \mathcal{H}(S)$ admits $\hat{f}$ as its *uniform $\mathbb{M}$-asymptotic expansion in S of type* $A > 0$ if there exists $C > 0$ such that for every $n \in \mathbb{N}$, one has

$$\left| f(z) - \sum_{p=0}^{n-1} a_p z^p \right| \leqslant C A^n M_n |z|^n, \quad z \in S. \tag{1}$$

As a consequence of Taylor's formula and Cauchy's integral formula for the derivatives, we have the following result (see [2,6]).

Proposition 2.3. *Let S be a sector and G a sectorial region.*

(i) *If $f \in \mathcal{A}_{\mathbb{M},A}(S)$, then f admits $\hat{f} = \sum_{p \in \mathbb{N}_0} \frac{1}{p!} f^{(p)}(0) z^p$ as its uniform $\mathbb{M}$-asymptotic expansion in S of type A.*

(ii) *$f \in \tilde{\mathcal{A}}_{\mathbb{M}}(G)$ if, and only if, for every $T \ll G$ there exists $A_T > 0$ such that $f|_T \in \mathcal{A}_{\mathbb{M},A_T}(T)$. Hence, the map $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}}(G) \longrightarrow \Lambda_{\mathbb{M}}$ is also well defined.*

Definition 2.4. A function f in any of the previous classes is said to be *flat* if $\tilde{\mathcal{B}}(f)$ is the null sequence or, in other words, $f \sim_{\mathbb{M}} \hat{0}$, where $\hat{0}$ denotes the null power series.

Remark 2.5. As a consequence of Cauchy's integral formula for the derivatives, given a sector S one can prove that whenever $T \ll S$, there exists a constant $c = c(T,S) > 0$ such that the restriction to T, f_T, of functions f defined on S and admitting uniform $\mathbb{M}$-asymptotic expansion in S of type $A > 0$, belongs to $\mathcal{A}_{\mathbb{M},cA}(T)$, and moreover, if one has (1) then $\|f_T\|_{\mathbb{M},cA} \leqslant C$.

2.3. Strongly regular sequences and associated functions

Most of the information in this subsection is taken from the works of A.A. Goldberg and I.V. Ostrovskii [8], H. Komatsu [10] and V. Thilliez [27], which we refer to for further details and proofs. In what follows, $\mathbb{M} = (M_p)_{p \in \mathbb{N}_0}$ will always stand for a sequence of positive real numbers, and we will always assume that $M_0 = 1$.

Definition 2.6. We say $\mathbb{M}$ is *strongly regular* if the following hold:

(α_0) $\mathbb{M}$ is *logarithmically convex*: $M_p^2 \leqslant M_{p-1}M_{p+1}$ for every $p \in \mathbb{N}$.

(μ) $\mathbb{M}$ is of *moderate growth*: there exists $A > 0$ such that

$$M_{p+\ell} \leqslant A^{p+\ell} M_p M_\ell, \quad p, \ell \in \mathbb{N}_0.$$

(γ_1) $\mathbb{M}$ satisfies the *strong non-quasianalyticity condition*: there exists $B > 0$ such that

$$\sum_{\ell \geqslant p} \frac{M_\ell}{(\ell+1)M_{\ell+1}} \leqslant B \frac{M_p}{M_{p+1}}, \quad p \in \mathbb{N}_0.$$

Remark 2.7. In the literature a different set of conditions appears frequently when dealing with ultraholomorphic or ultradifferentiable classes of functions. Let us clarify the relationship between these two approaches: If $\mathbb{M}$ is strongly regular, then $\mathbb{M}' = (n! M_n)_{n \in \mathbb{N}_0}$ verifies the standard conditions (M.1), (M.2) and (M.3) of H. Komatsu (see [10,19]). On the other hand, if a sequence of positive real numbers $\mathbb{M}' = (M_n')_{n \in \mathbb{N}_0}$, with $M_0' = 1$, verifies (M.2) and (M.3) of H. Komatsu, and moreover $\mathbb{M} := (M_n'/n!)_{n \in \mathbb{N}_0}$ is logarithmically convex, then $\mathbb{M}$ is strongly regular.

Example 2.8.

(i) The best known example of strongly regular sequence is $\mathbb{M}_\alpha = (n!^\alpha)_{n \in \mathbb{N}_0}$, called the *Gevrey sequence of order* $\alpha > 0$.

(ii) The sequences $\mathbb{M}_{\alpha,\beta} = (n!^\alpha \prod_{m=0}^{n} \log^\beta(e+m))_{n \in \mathbb{N}_0}$, where $\alpha > 0$ and $\beta \in \mathbb{R}$, are strongly regular.

(iii) For $q > 1$, $\mathbb{M} = (q^{n^2})_{n \in \mathbb{N}_0}$ is logarithmically convex and strongly non-quasianalytic, but not of moderate growth.

For a sequence $\mathbb{M} = (M_p)_{p \in \mathbb{N}_0}$ verifying properties (α_0) and (γ_1) one has that the associated *sequence of quotients*, $\boldsymbol{m} = (m_p := M_{p+1}/M_p)_{p \in \mathbb{N}_0}$, is an increasing sequence to infinity, so that the map $h_{\mathbb{M}} : [0, \infty) \to \mathbb{R}$, defined by

$$h_{\mathbb{M}}(t) := \inf_{p \in \mathbb{N}_0} M_p t^p, \quad t > 0; \qquad h_{\mathbb{M}}(0) = 0,$$

turns out to be a non-decreasing continuous map in $[0, \infty)$ onto $[0, 1]$. In fact

$$h_{\mathbb{M}}(t) = \begin{cases} t^p M_p & \text{if } t \in [\frac{1}{m_p}, \frac{1}{m_{p-1}}), \ p = 1, 2, \dots, \\ 1 & \text{if } t \geqslant 1/m_0. \end{cases}$$

Definition 2.9. (See [21,5].) Two sequences $\mathbb{M} = (M_p)_{p \in \mathbb{N}_0}$ and $\mathbb{M}' = (M_p')_{p \in \mathbb{N}_0}$ of positive real numbers are said to be *equivalent* if there exist positive constants L, H such that

$$L^p M_p \leqslant M_p' \leqslant H^p M_p, \quad p \in \mathbb{N}_0.$$

In this case, it is straightforward to check that

$$h_{\mathbb{M}}(Lt) \leqslant h_{\mathbb{M}'}(t) \leqslant h_{\mathbb{M}}(Ht), \quad t \geqslant 0. \tag{2}$$

One may also associate to a strongly regular sequence $\mathbb{M}$ the function

$$M(t) := \sup_{p \in \mathbb{N}_0} \log\left(\frac{t^p}{M_p}\right) = -\log\left(h_{\mathbb{M}}(1/t)\right), \quad t > 0; \qquad M(0) = 0, \tag{3}$$

which is a non-decreasing continuous map in $[0, \infty)$ with $\lim_{t \to \infty} M(t) = \infty$. Indeed,

$$M(t) = \begin{cases} p \log t - \log(M_p) & \text{if } t \in [m_{p-1}, m_p), \ p = 1, 2, \ldots, \\ 0 & \text{if } t \in [0, m_0), \end{cases}$$

and one can easily check that M is convex in $\log t$, i.e., the map $t \mapsto M(e^t)$ is convex in $\mathbb{R}$.

Some additional properties of strongly regular sequences needed in the present work are the following ones.

Lemma 2.10. *(See [27].) Let $\mathbb{M} = (M_p)_{p \in \mathbb{N}_0}$ be a strongly regular sequence and $A > 0$ the constant appearing in the property (μ) in Definition 2.6. Then,*

$$m_p \leqslant A^2 M_p^{1/p} \leqslant A^2 m_p \quad \text{for every } p \in \mathbb{N}_0. \tag{4}$$

Let s be a real number with $s \geqslant 1$. There exists $\rho(s) \geqslant 1$ (only depending on s and $\mathbb{M}$) such that

$$h_{\mathbb{M}}(t) \leqslant \left(h_{\mathbb{M}}\left(\rho(s)t\right)\right)^s \quad \text{for } t \geqslant 0. \tag{5}$$

Remark 2.11.

(i) The condition of moderate growth (μ) plays a fundamental role in the proof of (5), which will in turn be crucial in many of our arguments.

(ii) From property (4) we deduce that $\mathbb{M}$ and $(m_p^p)_{p \in \mathbb{N}_0}$ are equivalent.

(iii) For every $p \in \mathbb{N}_0$, the continuity of M at m_p amounts to the trivial equality $m_p^p/M_p = m_p^{p+1}/M_{p+1}$.

(iv) Moreover, since the sequence $\boldsymbol{m} = (m_p)_{p \in \mathbb{N}_0}$ (respectively, the function $M(t)$) increases to infinity as p (resp. t) tends to infinity, the sequence $(M(m_p))_{p \in \mathbb{N}_0} = (\log(m_p^p/M_p))_{p \in \mathbb{N}_0}$, and consequently also $(m_p^p/M_p)_{p \in \mathbb{N}_0}$, increase to infinity, starting at the value 0 and 1, respectively. Note also that the p-th and $(p+1)$-th terms of any of these two sequences are equal if, and only if, $m_p = m_{p+1}$.

We now recall the following definitions and facts, mainly taken from the book of A.A. Goldberg and I.V. Ostrovskii [8].

Definition 2.12. (See [8, p. 43].) Let $\alpha(r)$ be a nonnegative and nondecreasing function in (c, ∞) for some $c \geqslant 0$ (we write $\alpha \in \Lambda$). The *order* of α is defined as

$$\rho = \rho[\alpha] := \limsup_{r \to \infty} \frac{\log^+ \alpha(r)}{\log r} \in [0, \infty],$$

where $\log^+ = \max(\log, 0)$. $\alpha(r)$ is said to have finite order if $\rho < \infty$.

We are firstly interested in determining the order of the function $M(r) \in \Lambda$ defined in (3) and associated with a strongly regular sequence $\mathbb{M}$. To this end, we need to recall now the definition of exponent of convergence of a sequence and how it may be computed.

Proposition 2.13. *(See [9, p. 65].) Let $(c_n)_{n \in \mathbb{N}_0}$ be a nondecreasing sequence of positive real numbers tending to infinity. The exponent of convergence of $(c_n)_n$ is defined as*

$$\lambda_{(c_n)} := \inf \left\{ \mu > 0 \colon \sum_{n=0}^{\infty} \frac{1}{c_n^\mu} \ converges \right\}$$

(if the previous set is empty, we put $\lambda_{(c_n)} = \infty$). Then, one has

$$\lambda_{(c_n)} = \limsup_{n \to \infty} \frac{\log(n)}{\log(c_n)}. \tag{6}$$

We will also need the following fact, which can be found in [19]: if we consider the *counting function* for the sequence of quotients $\boldsymbol{m}$, $\nu : (0, \infty) \to \mathbb{N}_0$ given by

$$\nu(r) := \#\{j \colon m_j \leqslant r\}, \tag{7}$$

then one has that

$$M(t) = \int_0^t \frac{\nu(r)}{r} \, dr, \quad t > 0. \tag{8}$$

We may now state our first result.

Theorem 2.14. *Let $\mathbb{M}$ be strongly regular, $\boldsymbol{m}$ the sequence of its quotients and $M(r)$ its associated function. Then, the order of $M(r)$ is given by*

$$\rho[M] = \lim_{r \to \infty} \frac{\log M(r)}{\log r} = \limsup_{n \to \infty} \frac{\log(n)}{\log(m_n)}. \tag{9}$$

Remark 2.15. The function d, defined for $r > \max\{1, m_0\}$ by $d(r) = \log(M(r))/\log r$, will play an important role in what follows. It is clearly continuous and piecewise continuously differentiable in its domain (meaning that it is differentiable except at a sequence of points, tending to infinity, at any of which it is continuous and has distinct finite lateral derivatives).

Proof of Theorem 2.14. The first equality for $\rho[M]$ is due to the fact that the function $d(r)$ is eventually strictly increasing, as we now show. It is enough to prove that $d'(r) > 0$ for $r \in (m_{p-1}, m_p)$ and p large enough. This is best seen by considering the auxiliary function

$$D(t) := d(e^t) = \frac{\log(pt - \log(M_p))}{t}, \quad t \in \big(\log(m_{p-1}), \log(m_p)\big), \ p \in \mathbb{N},$$

and then proving that $D'(t) > 0$ for $t \in (\log(m_{p-1}), \log(m_p))$ and large enough. We have

$$D'(t) = \frac{1}{t^2} \left(1 + \frac{\log(M_p)}{pt - \log(M_p)} - \log\big(pt - \log(M_p)\big) \right), \quad t \in \big(\log(m_{p-1}), \log(m_p)\big).$$

When t runs over $(\log(m_{p-1}), \log(m_p))$, the value $pt - \log(M_p)$ runs over

$$\big(\log\big(m_{p-1}^{p-1}/M_{p-1}\big), \log\big(m_p^p/M_p\big)\big),$$

which, as long as $m_{p-1} < m_p$, is a nonempty interval contained in $(0, \infty)$ (see Remark 2.11(iv)).

Observe that $M_p > 1$ for p large enough, what we assume from now on. Since for any $A > 0$ the function $y \in (0, \infty) \mapsto 1 + A/y - \log(y)$ is strictly decreasing, we will conclude that $D'(t) > 0$ for $t \in (\log(m_{p-1}), \log(m_p))$ if we have that

$$1 + \frac{\log(M_p)}{\log(m_p^p/M_p)} - \log\big(\log\big(m_p^p/M_p\big)\big) > 0.$$

But

$$1 + \frac{\log(M_p)}{\log(m_p^p/M_p)} - \log\big(\log\big(m_p^p/M_p\big)\big) = \frac{\log\big(\frac{m_p^p}{\log(m_p^p/M_p)}\big)}{\log(m_p^p/M_p)},$$

whose denominator is positive; finally, note that

$$m_p^p > \log\big(m_p^p\big) > \log\big(m_p^p\big) - \log(M_p) = \log\big(m_p^p/M_p\big) > 0,$$

so that the numerator is also positive and we are done.

For the second expression of $\rho[M]$, we take into account the link given in (8) between $M(r)$ and the counting function $\nu(r)$ for $\boldsymbol{m}$ (as defined in (7)), which also belongs to Λ. We may apply Theorem 2.1.1 in [8] to deduce that the order of $M(r)$ equals that of $\nu(r)$. Now, from Theorem 2.1.8 in [8] we know that the order of $\nu(r)$ is in turn the exponent of convergence of $\boldsymbol{m}$, given by the formula in (6). $\square$

Remark 2.16.

(i) Let $\mathbb{M}_\alpha$ be the Gevrey sequence of order $\alpha > 0$, and $M_\alpha(r)$ its associated function. By means of (9), it is obvious that $\rho[M_\alpha] = 1/\alpha$. The same is true for any sequence of the form $(a^n n!^\alpha)_{n \in \mathbb{N}_0}$, with $a, \alpha > 0$.

(ii) Let $\mathbb{M} = (M_n)_{n \in \mathbb{N}_0}$ and $\mathbb{M}^* = (M_n^*)_{n \in \mathbb{N}_0}$ be strongly regular sequences such that $M_n \leqslant M_n^*$ for every $n \in \mathbb{N}_0$. By the very definition of the respective associated functions $M(r)$ and $M^*(r)$, one has that $M(r) \geqslant M^*(r)$ for every $r \geqslant 0$, and consequently $\rho[M] \geqslant \rho[M^*]$.

(iii) By Lemma 1.3.2 in Thilliez [27], for every strongly regular sequence $\mathbb{M}$ there exist positive constants a_1, a_2, γ, δ, with $\gamma < \delta$, such that

$$a_1^n n!^\gamma \leqslant M_n \leqslant a_2^n n!^\delta, \quad n \in \mathbb{N}_0.$$

From the two previous remarks we deduce that $1/\delta \leqslant \rho[M] \leqslant 1/\gamma$, and, in particular, $\rho[M] \in (0, \infty)$.

We next recall the notion of growth index defined and studied by V. Thilliez [27, Sect. 1.3].

Definition 2.17. Let $\mathbb{M} = (M_p)_{p \in \mathbb{N}_0}$ be a strongly regular sequence and $\gamma > 0$. We say $\mathbb{M}$ satisfies property (P_γ) if there exist a sequence of real numbers $m' = (m'_p)_{p \in \mathbb{N}_0}$ and a constant $a \geqslant 1$ such that: (i) $a^{-1} m_p \leqslant m'_p \leqslant a m_p$, $p \in \mathbb{N}$, and (ii) $((p+1)^{-\gamma} m'_p)_{p \in \mathbb{N}_0}$ is increasing.

The *growth index* of $\mathbb{M}$ is

$$\gamma(\mathbb{M}) := \sup\big\{\gamma \in \mathbb{R}: (P_\gamma) \text{ is fulfilled}\big\} \in (0, \infty).$$

Example 2.18.

(i) For the Gevrey sequence of order $\alpha > 0$, one has $\gamma(\mathbb{M}_\alpha) = \alpha$.

(ii) For the sequences $\mathbb{M}_{\alpha, \beta}$ in Example 2.8(ii) one can check that $\gamma(\mathbb{M}_{\alpha, \beta}) = \alpha$.

3. Results on quasianalyticity in ultraholomorphic classes

We are interested in characterizing those classes in which the asymptotic Borel map is injective. First, quasianalytic Carleman classes are defined.

Definition 3.1. Let S be a sector and $\mathbb{M} = (M_p)_{p \in \mathbb{N}_0}$ be a sequence of positive numbers. We say that $\mathcal{A}_{\mathbb{M}}(S)$ is *quasianalytic* if it does not contain nontrivial flat functions.

Characterizations of quasianalyticity for general sequences $\mathbb{M}$ in one and several variables are available in [13], generalizing the work of B.I. Korenbljum [11]. In this paper, we restrict our attention to the one-variable case, and focus on strongly regular sequences, although in many of the results in this section weaker assumptions on $\mathbb{M}$ suffice. As shown in the next result, quasianalyticity is governed by the opening of the sector.

Theorem 3.2. *(See [11].) Let $\mathbb{M}$ be strongly regular and $\gamma > 0$. The following statements are equivalent:*

(i) *The class $\mathcal{A}_{\mathbb{M}}(S_\gamma)$ is quasianalytic.*
(ii) $\sum_{n=0}^{\infty} \left(\frac{M_n}{(n+1)M_{n+1}} \right)^{1/(\gamma+1)} = \infty.$

Accordingly, we introduce a new quantity regarding quasianalyticity.

Definition 3.3. For a strongly regular sequence $\mathbb{M}$, we put

$$Q_{\mathbb{M}} = \big\{ \gamma > 0 \colon \mathcal{A}_{\mathbb{M}}(S_\gamma) \text{ is quasianalytic} \big\}.$$

The *order of quasianalyticity* of $\mathbb{M}$ is defined as $\omega(\mathbb{M}) := \inf Q_{\mathbb{M}}$.

We can obtain its value due to the following result.

Theorem 3.4. *For a strongly regular sequence $\mathbb{M}$ with associated function $M(r)$, we have*

$$\omega(\mathbb{M}) = \liminf_{n \to \infty} \frac{\log(m_n)}{\log(n)} = \frac{1}{\lambda_{(m_n)}}, \tag{10}$$

and consequently,

$$\omega(\mathbb{M}) = \frac{1}{\rho[M]} \in (0, \infty). \tag{11}$$

Proof. Since $\mathbb{M}$ is strongly regular, $(n!M_n)_{n \in \mathbb{N}_0}$ is logarithmically convex. So, the sequence of its quotients, $\left(\frac{(n+1)M_{n+1}}{M_n} \right)_{n \in \mathbb{N}_0}$, is nondecreasing and, moreover, tends to infinity because of property (γ_1) in Definition 2.6. In view of (6), the exponent of convergence of the sequence $((n+1)M_{n+1}/M_n)_{n \in \mathbb{N}_0} = ((n+1)m_n)_{n \in \mathbb{N}_0}$ and that of the sequence $(m_n)_{n \in \mathbb{N}_0}$ are related as follows:

$$\lambda_{((n+1)m_n)} = \limsup_{n \to \infty} \frac{\log(n)}{\log((n+1)m_n)} = \frac{1}{1 + \liminf_{n \to \infty} \frac{\log(m_n)}{\log(n)}} = \frac{1}{1 + 1/\lambda_{(m_n)}}.$$

On the other hand, from Theorem 3.2 and the definition of $\omega(\mathbb{M})$ it is clear that

$$\frac{1}{\omega(\mathbb{M}) + 1} = \lambda_{((n+1)m_n)},$$

hence

$$\omega(\mathbb{M}) = \liminf_{n\to\infty} \frac{\log(m_n)}{\log(n)} = \frac{1}{\lambda_{(m_n)}}.$$

Comparing this to (9), and by Remark 2.16(iii), we conclude. □

Remark 3.5. Observe that $\pi\omega(\mathbb{M})$ is the optimal opening for quasianalyticity, in the sense that the class $\mathcal{A}_{\mathbb{M}}(S)$ is (respectively, is not) quasianalytic whenever the opening of S exceeds (respectively is less than) this quantity. When the opening of the sector equals $\pi\omega(\mathbb{M})$ both cases are possible, as shown in the forthcoming Example 3.10.

Remark 3.6.

(i) Consider a pair of equivalent sequences $\mathbb{M}$ and $\mathbb{M}'$. Given a sector S, the spaces $\mathcal{A}_{\mathbb{M}}(S)$ and $\mathcal{A}_{\mathbb{M}'}(S)$ coincide. For a sectorial region G, also $\tilde{\mathcal{A}}_{\mathbb{M}}(G)$ and $\tilde{\mathcal{A}}_{\mathbb{M}'}(G)$ agree. So, it is clear that $\omega(\mathbb{M}) = \omega(\mathbb{M}')$, and from (11), also $\rho[M] = \rho[M']$ for the associated functions (this last fact can also be deduced from (2)).

(ii) If the strongly regular sequences $\mathbb{M}$ and $\mathbb{M}^*$ are such that $M_n \leqslant M_n^*$ for every n, then $\mathcal{A}_{\mathbb{M}}(S) \subset \mathcal{A}_{\mathbb{M}^*}(S)$ for any sector S, and so $Q_{\mathbb{M}^*} \subset Q_{\mathbb{M}}$ and $\omega(\mathbb{M}) \leqslant \omega(\mathbb{M}^*)$. Note that this fact is not at all clear from the formula (10).

Regarding the relationship between $\omega(\mathbb{M})$ and $\gamma(\mathbb{M})$, we have the following result.

Proposition 3.7. *For every strongly regular sequence $\mathbb{M}$ one has $\omega(\mathbb{M}) \geqslant \gamma(\mathbb{M})$.*

Proof. Suppose $\mathbb{M}$ verifies (P_γ) (see Definition 2.17) for some $\gamma > 0$. As indicated in [27, p. 173], this easily implies the existence of a constant $a > 0$ such that $a^n n!^\gamma \leqslant M_n$ for every $n \in \mathbb{N}_0$. Hence, by (ii) in the previous remark we have $\gamma = \omega((a^n n!^\gamma)_{n\in\mathbb{N}_0}) \leqslant \omega(\mathbb{M})$, and the definition of $\gamma(\mathbb{M})$ is enough to conclude. □

According to the very definition of $\omega(\mathbb{M})$, the previous result is indeed equivalent to the following one, proved by V. Thilliez [27] and, subsequently, by A. Lastra and the author [13]. However, the present argument seems to be simpler than the ones involved in the previous proofs of this theorem.

Theorem 3.8. *(See [27,13].) Let $0 < \gamma < \gamma(\mathbb{M})$. Then, the class $\mathcal{A}_{\mathbb{M}}(S_\gamma)$ is not quasianalytic.*

It is an open problem to decide whether $\omega(\mathbb{M}) = \gamma(\mathbb{M})$ in general. At the moment, we have not been able to find an example showing that $\omega(\mathbb{M}) > \gamma(\mathbb{M})$ may occur. However, from the fact that $\omega(\mathbb{M}) = 1/\lambda_{(m_n)}$ we may deduce an easy characterization for the equality of both constants.

Corollary 3.9. *Let $\mathbb{M}$ be strongly regular. The following statements are equivalent:*

(i) $\omega(\mathbb{M}) = \gamma(\mathbb{M})$.

(ii) *For every $\gamma > \gamma(\mathbb{M})$ we have that*

$$\sum_{n=0}^{\infty} \left(\frac{1}{m_n}\right)^{1/\gamma} = \infty.$$

Example 3.10. Consider the sequences $\mathbb{M}_{\alpha,\beta}$, $\alpha > 0$, $\beta \in \mathbb{R}$, introduced in Examples 2.8(ii) and 2.18(ii). Applying Theorem 3.2, it is easy to check that

$$Q_{\mathbb{M}_{\alpha,\beta}} = \begin{cases} [\alpha, \infty) & \text{if } \alpha \geqslant \beta - 1, \\ (\alpha, \infty) & \text{if } \alpha < \beta - 1, \end{cases}$$

so that $\omega(\mathbb{M}_{\alpha,\beta}) = \alpha = \gamma(\mathbb{M}_{\alpha,\beta})$.

4. Flat functions via proximate orders

In this section we show how one can construct flat functions in the classes $\tilde{\mathcal{A}}_{\mathbb{M}}(S_{\omega(\mathbb{M})})$ for a strongly regular sequence $\mathbb{M}$ by relying on the notion of analytic proximate orders, appearing in the theory of growth of entire functions and developed, among others, by E. Lindelöf, G. Valiron, B.Ja. Levin, A.A. Goldberg, I.V. Ostrosvkii and L.S. Maergoiz (see the references [30,17,8,18]).

Definition 4.1. (See [30].) We say a real function $\rho(r)$, defined on (c, ∞) for some $c \geqslant 0$, is a *proximate order* if the following hold:

  (i) $\rho(r)$ is continuous and piecewise continuously differentiable in (c, ∞),
 (ii) $\rho(r) \geqslant 0$ for every $r > c$,
(iii) $\lim_{r \to \infty} \rho(r) = \rho < \infty$,
 (iv) $\lim_{r \to \infty} r\rho'(r) \log(r) = 0$.

Definition 4.2. Two proximate orders $\rho_1(r)$ and $\rho_2(r)$ are said to be *equivalent* if

$$\lim_{r \to \infty} \big(\rho_1(r) - \rho_2(r)\big) \log(r) = 0.$$

Remark 4.3. If $\rho_1(r)$ and $\rho_2(r)$ are equivalent and $\lim_{r \to \infty} \rho_1(r) = \rho$, then $\lim_{r \to \infty} \rho_2(r) = \rho$ and $\lim_{r \to \infty} r^{\rho_1(r)}/r^{\rho_2(r)} = 1$.

From the work of L.S. Maergoiz we have the following result.

Theorem 4.4. *(See [18, Thm. 2.4].) Let $\rho(r)$ be a proximate order with $\rho(r) \to \rho > 0$ as $r \to \infty$. For every $\gamma > 0$ there exists an analytic function $V(z)$ in S_γ such that:*

 (i) *For every $z \in S_\gamma$,*

$$\lim_{r \to \infty} \frac{V(zr)}{V(r)} = z^\rho,$$

uniformly in the compact sets of S_γ.
 (ii) *$\overline{V(z)} = V(\bar{z})$ for every $z \in S_\gamma$ (where, for $z = (|z|, \arg(z))$, we put $\bar{z} = (|z|, -\arg(z))$).*
(iii) *$V(r)$ is positive in $(0, \infty)$, monotone increasing and $\lim_{r \to 0} V(r) = 0$.*
 (iv) *The function $t \in \mathbb{R} \to V(e^t)$ is strictly convex (i.e. V is strictly convex relative to $\log(r)$).*
  (v) *The function $\log(V(r))$ is strictly concave in $(0, \infty)$.*
 (vi) *The function $\rho_0(r) := \log(V(r))/\log(r)$, $r > 0$, is a proximate order equivalent to $\rho(r)$.*

We denote by $\mathfrak{B}(\gamma, \rho(r))$ the class of such functions V. They share a property that will be crucial in the construction of flat functions.

Proposition 4.5. *(See [18, Property 2.9].)* *Let $\rho > 0$, $\rho(r)$ be a proximate order with $\rho(r) \to \rho$, $\gamma \geqslant 2/\rho$ and $V \in \mathfrak{B}(\gamma, \rho(r))$. Then, for every $\alpha \in (0, 1/\rho)$ there exist constants $b > 0$ and $R_0 > 0$ such that*

$$\Re\big(V(z)\big) \geqslant bV\big(|z|\big), \quad z \in S_\alpha, \ |z| \geqslant R_0,$$

where $\Re$ stands for the real part.

We will also make use of the following result of V. Thilliez.

Theorem 4.6. *(See [28, Proposition 4].)* *Let $\mathbb{M}$ be a strongly regular sequence and S a sector. For $f \in \mathcal{H}(S)$, the following are equivalent*:

(i) $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(S)$ *and* $f \sim_{\mathbb{M}} \hat{0}$.
(ii) *For every bounded proper subsector T of S there exist $c_1, c_2 > 0$ with*

$$\big|f(z)\big| \leqslant c_1 h_{\mathbb{M}}\big(c_2|z|\big) = c_1 e^{-M(1/(c_2|z|))}, \quad z \in T.$$

In the next result we obtain the desired flat function in case $\omega(\mathbb{M}) < 2$ and $d(r)$, defined in Remark 2.15, is a proximate order. Subsequently, we will indicate how to deal with the case $\omega(\mathbb{M}) \geqslant 2$. Finally, we will determine conditions on $\mathbb{M}$ amounting to $d(r)$ being a proximate order, or at least guaranteeing that $d(r)$ is a proximate order.

Theorem 4.7. *Suppose $\mathbb{M}$ is a strongly regular sequence with $\omega(\mathbb{M}) < 2$ and such that $d(r)$ is a proximate order. Then, for every $V \in \mathfrak{B}(2\omega(\mathbb{M}), d(r))$ the function G defined in $S_{\omega(\mathbb{M})}$ by*

$$G(z) = \exp\big(-V(1/z)\big)$$

belongs to $\tilde{\mathcal{A}}_{\mathbb{M}}(S_{\omega(\mathbb{M})})$ and it is a (nontrivial) flat function.

Proof. It is enough to reason with sectors $S(0, \omega, r_0) \ll S_{\omega(\mathbb{M})}$, where $0 < \omega < \omega(\mathbb{M})$ and $r_0 > 0$. If $z \in S(0, \omega, r_0)$, we have $1/z \in S_\omega$. By our assumptions, $d(r)$ is a proximate order, and by (9) and (11), we have that

$$\lim_{r \to \infty} d(r) = \rho[M] = \frac{1}{\omega(\mathbb{M})}.$$

We are in a position to apply Proposition 4.5 with $\rho = 1/\omega(\mathbb{M})$, $\rho(r) = d(r)$, $\gamma = 2\omega(\mathbb{M})$ and $\alpha = \omega$, and deduce the existence of constants $R_0 > 0$ and $b > 0$ such that $\Re(V(\zeta)) \geqslant bV(|\zeta|)$ whenever $\zeta \in S_\omega$ with $|\zeta| \geqslant R_0$. Then, for $z \in S(0, \omega, 1/R_0)$ we obtain

$$\big|G(z)\big| = e^{-\Re(V(1/z))} \leqslant e^{-bV(1/|z|)},$$

and for a suitable $C > 0$ we will have $|G(z)| \leqslant Ce^{-bV(1/|z|)}$ for $z \in S(0, \omega, r_0)$. Now observe that, by the definition of $\mathfrak{B}(2\omega(\mathbb{M}), d(r))$, we know that the function $\log(V(r))/\log(r)$ is a proximate order equivalent to $d(r) = \log(M(r))/\log(r)$, so that, as a consequence of Remark 4.3, we have that there exists $c > 0$ such that for $r > 1/r_0$ one has $V(r) > cM(r)$, and

$$\big|G(z)\big| \leqslant Ce^{-bcM(1/|z|)} = C\big(h_{\mathbb{M}}\big(|z|\big)\big)^{bc} \leqslant Ch_{\mathbb{M}}\big(D|z|\big), \quad z \in S(0, \omega, r_0),$$

where $D > 0$ is a positive constant, suitably chosen according to whether $bc > 1$ or not (see property (5)). It suffices to take into account Theorem 4.6 in order to conclude. $\quad\square$

Remark 4.8. In case $\omega(\mathbb{M}) \geqslant 2$, we may also construct nontrivial flat functions by taking into account the following facts:

(i) Given a strongly regular sequence $\mathbb{M} = (M_n)_{n \in \mathbb{N}_0}$ and a positive real number $s > 0$, the sequence of s-powers $\mathbb{M}^{(s)} := (M_n^s)_{n \in \mathbb{N}_0}$ is strongly regular (see Lemma 1.3.4 in [27]) and one easily checks that, with self-explaining notation, $\boldsymbol{m}^{(s)} = (m_n^s)_{n \in \mathbb{N}_0}$, $M^{(s)}(t) = sM(t^{1/s})$ for every $t \geqslant 0$, $\omega(\mathbb{M}^{(s)}) = s\omega(\mathbb{M})$, $d^{(s)}(r) = d(r^{1/s})/s + \log(s)/\log(r)$ for r large enough, and

$$r(d^{(s)})'(r)\log(r) = \frac{1}{s}r^{1/s}d'(r^{1/s})\log(r^{1/s}) - \frac{\log(s)}{\log(r)}$$

whenever both sides are defined. So, it is clear that $d(r)$ is a proximate order if, and only if, $d^{(s)}(r)$ is.

(ii) If $\mathbb{M}$ is strongly regular, $\omega(\mathbb{M}) \geqslant 2$ and $d(r)$ is a proximate order, choose $s > 0$ such that $s\omega(\mathbb{M}) < 2$. By (i), we may apply Theorem 4.7 to $\mathbb{M}^{(s)}$ and obtain $G_0 \in \tilde{\mathcal{A}}_{\mathbb{M}^{(s)}}(S_{\omega(\mathbb{M}^{(s)})})$ which is flat. Now, the function G, given in $S_{\omega(\mathbb{M})}$ by $G(z) = G_0(z^s)$, is well-defined and it is plain to see that it is a nontrivial flat element in $\tilde{\mathcal{A}}_{\mathbb{M}}(S_{\omega(\mathbb{M})})$.

Our next objective is to characterize those $\mathbb{M}$ such that $d(r)$ is a proximate order. After looking at Remark 2.15 and Theorems 2.14 and 3.4, it is clear that we only need to care about whether $rd'(r)\log(r) \to 0$ as $r \to \infty$. The following result provides us with statements equivalent to this fact.

Proposition 4.9. *Let $\mathbb{M}$ be a strongly regular sequence, and $d(r)$ its associated function. The following are equivalent*:

(i) $d(r)$ *is a proximate order,*
(ii) $\lim_{p\to\infty} m_p d'(m_p^+)\log(m_p) = 0$,
(iii) $\lim_{p\to\infty} \frac{p+1}{M(m_p)} = \frac{1}{\omega(\mathbb{M})} = \rho[M]$.

Proof. For convenience, write $b(r) = rd'(r)\log(r)$ whenever it exists. It is easy to obtain that

$$b(r) = \frac{rM'(r)}{M(r)} - d(r) = \frac{p}{M(r)} - d(r), \quad r \in (m_{p-1}, m_p), \ p \in \mathbb{N}. \tag{12}$$

Since for sufficiently large r, distinct from every m_p, we know that $d'(r) > 0$, the function $b(r)$ is positive. Moreover, as M and d are both increasing and continuous for large r, we see from (12) that $b(r)$ is decreasing in every interval (m_{p-1}, m_p), and it presents at every m_p a jump of positive height equal to

$$\lim_{r\to m_p^+} b(r) - \lim_{r\to m_p^-} b(r) = \frac{1}{M(m_p)}.$$

From this it is clear that (i) holds if, and only if, $\lim_{p\to\infty} b(m_p^+) = 0$, and this is precisely (ii). Now, observe that

$$b(m_p^+) = \frac{p+1}{M(m_p)} - d(m_p),$$

and recall from Theorem 2.14 that $\lim_{p\to\infty} d(m_p) = \rho[M] = 1/\omega(\mathbb{M})$. So, (ii) amounts to (iii) and we are done. $\quad\square$

Next we obtain some easy condition that ensures that $d(r)$ is a proximate order.

Corollary 4.10. *If*

$$\lim_{p \to \infty} p \log\left(\frac{m_{p+1}}{m_p}\right) \quad exists \ (finite \ or \ not), \tag{13}$$

then its value is a fortiori $\omega(\mathbb{M})$, $d(r)$ *is a proximate order and, moreover,*

$$\omega(\mathbb{M}) = \lim_{p \to \infty} \frac{\log(m_p)}{\log(p)} \quad \left(instead \ of \ \liminf_{p \to \infty}, \ see \ (10)\right).$$

Proof. By Stolz's criterion we have that $\lim_{p \to \infty} \frac{\log(m_p)}{\log(p)}$ exists, since

$$\lim_{p \to \infty} \frac{\log(m_{p+1}/m_p)}{\log((p+1)/p)} = \lim_{p \to \infty} p \log\left(\frac{m_{p+1}}{m_p}\right),$$

and the last limit exists. We take into account (10) in order to deduce that all these limits equal $\omega(\mathbb{M})$. But, again by Stolz's criterion,

$$\lim_{p \to \infty} \frac{M(m_p)}{p+1} = \lim_{p \to \infty} \frac{\log(m_p^p/M_p)}{p+1} = \lim_{p \to \infty} p \log\left(\frac{m_p}{m_{p-1}}\right) = \omega(\mathbb{M}),$$

and this equality amounts to (iii) in Proposition 4.9. $\square$

Remark 4.11.

(i) The previous condition (13) holds for every sequence $\mathbb{M}_{\alpha,\beta}$, so that in any of these cases $d(r)$ is a proximate order and it is possible to construct flat functions in the corresponding classes. Indeed, we have not been able yet to provide an example of a strongly regular sequence for which $d(r)$ is not a proximate order, i.e., for which condition (iii) in Proposition 4.9 does not hold.

(ii) In the Gevrey case, $\mathbb{M}_{1/k} = (p!^{1/k})_{\in \mathbb{N}_0}$, let us put $M_{1/k}(r)$, $d_{1/k}(r)$, and so on, to denote the corresponding associated functions. Then, one can check (see, for example, [7]) that for large r we have $c_2 r^k \leqslant M_{1/k}(r) \leqslant c_1 r^k$ for suitable constants $c_1, c_2 > 0$, so that $\log(c_2) \leqslant (d_{1/k}(r) - k)\log(r) \leqslant \log(c_1)$ eventually. This shows one can work with the constant proximate order $\rho(r) \equiv k$, and any $V \in \mathfrak{B}(2/k, \rho(r))$ will provide us (due to Theorem 4.7, and since $V(r)$ will be bounded above and below by r^k times some suitable constants) with a flat function in the class $\tilde{\mathcal{A}}_{1/k}(S_{1/k})$. It is easy to see that $V(z) = z^k$ belongs to $\mathfrak{B}(2/k, \rho(r))$, and we obtain in this way the classical flat function in this situation, namely $G(z) = \exp(-z^{-k})$.

(iii) If $\mathbb{M}$ is such that $d(r)$ is not a proximate order, but there exist a proximate order $\rho(r)$ and constants $A, B > 0$ such that eventually $A \leqslant (d(r) - \rho(r))\log(r) \leqslant B$, then, by reasoning as indicated in (ii), one may also construct flat functions in $\tilde{\mathcal{A}}_{\mathbb{M}}(S_{\omega(\mathbb{M})})$.

We are in a position to characterize quasianalyticity in the classes $\tilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma)$.

Corollary 4.12 *(Watson's Lemma). Suppose $\mathbb{M}$ is strongly regular and such that $d(r)$ is a proximate order, and let $\gamma > 0$ be given. The following statements are equivalent:*

(i) $\tilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma)$ *is quasianalytic, i.e., it does not contain nontrivial flat functions* (*in other words, the Borel map is injective in this class*).

(ii) $\gamma > \omega(\mathbb{M})$.

Proof. By Theorem 4.7 and Remark 4.8(ii), whenever $\gamma \leqslant \omega(\mathbb{M})$ we have nontrivial flat functions in $\tilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma)$.

Conversely, suppose $\gamma > \omega(\mathbb{M})$ and that $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma)$ is a nontrivial flat function. Choose γ' with $\omega(\mathbb{M}) < \gamma' < \gamma$. By Proposition 2.3(ii), the restriction of f to $S_{\gamma'}$ belongs to $\mathcal{A}_{\mathbb{M}}(S_{\gamma'})$ and it is flat, so that $\mathcal{A}_{\mathbb{M}}(S_{\gamma'})$ is not quasianalytic, contrary to the definition of $\omega(\mathbb{M})$. $\square$

Remark 4.13. One may observe the difference with respect to the classes $\mathcal{A}_{\mathbb{M}}(S_\gamma)$, which could be quasianalytic for $\gamma = \omega(\mathbb{M})$ (see Example 3.10).

5. Kernels and moment sequences associated with $\mathbb{M}$

As a next step in our study, we now devote ourselves to extend to general Carleman classes $\tilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma)$ the well-known result, named Borel–Ritt–Gevrey theorem, stating that the Borel map in Gevrey classes is surjective if, and only if, the sector is narrow enough. The proof will be constructive, and will rest on the use of truncated Laplace-like transforms whose kernels are intimately related to the nontrivial flat functions obtained in Theorem 4.7. To any such kernel we will associate a sequence of moments which, in turn, will be equivalent to the sequence $\mathbb{M}$ we departed from.

Definition 5.1. Let $\mathbb{M} = (M_p)_{p \in \mathbb{N}_0}$ be a strongly regular sequence such that $d(r)$ is a proximate order, and consider the flat function $G \in \tilde{\mathcal{A}}_{\mathbb{M}}(S_{\omega(\mathbb{M})})$ constructed in Theorem 4.7 for a given $V \in \mathfrak{B}(2\omega(\mathbb{M}), d(r))$. We define the *kernel associated with V* as $e_V : S_{\omega(\mathbb{M})} \to \mathbb{C}$ given by

$$e_V(z) := zG(1/z) = ze^{-V(z)}, \quad z \in S_{\omega(\mathbb{M})}.$$

Remark 5.2.

(i) In a previous paper by A. Lastra, S. Malek and the author [15], similar kernels were obtained from flat functions constructed by V. Thilliez in [27]. The main difference with respect to the present one, which will be extremely important in forthcoming applications of these ideas to summability theory of formal power series, is that Thilliez needed to slightly restrict the opening of the optimal sector in order to construct such flat functions, while here we have been able to do it in the whole of $S_{\omega(\mathbb{M})}$.

(ii) The factor z appearing in e_V takes care of the integrability of $z^{-1}e_V(z)$ at the origin (see (i) in the next lemma). Indeed, it could be changed into any power z^α for positive α, where the principal branch of the power is to be considered. Our choice aims at simplicity.

Lemma 5.3. *The function e_V enjoys the following properties*:

(i) $z^{-1}e_V(z)$ *is integrable at the origin, it is to say, for any $t_0 > 0$ and $\tau \in \mathbb{R}$ with $|\tau| < \frac{\pi\omega(\mathbb{M})}{2}$ the integral $\int_0^{t_0} t^{-1}|e_V(te^{i\tau})|\,dt$ is finite.*

(ii) *For every $T \prec S_{\omega(\mathbb{M})}$ there exist $C, K > 0$ such that*

$$\left|e_V(z)\right| \leqslant Ch_{\mathbb{M}}\left(\frac{K}{|z|}\right), \quad z \in T. \tag{14}$$

(iii) *For every $x \in \mathbb{R}$, $x > 0$, the value $e_V(x)$ is positive real.*

Proof. (i) Let $t_0 > 0$ and $\tau \in \mathbb{R}$ with $|\tau| < \frac{\pi\omega(\mathbb{M})}{2}$. Since G is flat, from Theorem 4.6 we obtain $c_1, c_2 > 0$ (depending on τ and t_0) such that

$$\int\limits_0^{t_0} \frac{|e_V(te^{i\tau})|}{t}\, dt \leqslant \int\limits_0^{t_0} c_1 h_{\mathbb{M}}(c_2/t)\, dt.$$

As $h_{\mathbb{M}}$ is continuous and $h_{\mathbb{M}}(s) \equiv 1$ when $s \geqslant \frac{1}{m_1}$, this integral converges.

(ii) As before, given $T \prec S_{\omega(\mathbb{M})}$ and $R > 0$ there exist $c_1, c_2 > 0$ (depending on T and R) such that

$$\left|e_V(z)\right| \leqslant c_1|z|h_{\mathbb{M}}\big(c_2/|z|\big), \quad z \in T,\ |z| \geqslant R.$$

If $|z| \geqslant R$, we may apply (5) for $s = 2$ and the definition of $h_{\mathbb{M}}$ to deduce that

$$\left|e_V(z)\right| \leqslant c_1|z|\left(h_{\mathbb{M}}\left(\frac{\rho(2)c_2}{|z|}\right)\right)^2 \leqslant c_1|z|h_{\mathbb{M}}\left(\frac{\rho(2)c_2}{|z|}\right)M_2\left(\frac{\rho(2)c_2}{|z|}\right)^2 \leqslant \frac{\rho(2)^2 c_1 c_2^2 M_2}{R} h_{\mathbb{M}}\left(\frac{\rho(2)c_2}{|z|}\right).$$

On the other hand, since V is bounded at the origin (because of property (iii) in Theorem 4.4), for $z \in T$ with $|z| < R$ we deduce that $e_V(z) = ze^{-V(z)}$ is bounded, and, in order to conclude, it suffices to observe that $h_{\mathbb{M}}(c_2/|z|)$ is bounded below by some positive constant for $|z| < R$.

(iii) $V(x)$ is real if $x > 0$, so $e_V(x) = xe^{-V(x)} > 0$. $\square$

Remark 5.4. As suggested in Remarks 4.11(ii) and 5.2(ii), in the Gevrey case $\mathbb{M}_{1/k}$, $k > 0$, it is natural and standard to consider the kernel

$$e_k(z) = kz^k \exp\big(-z^k\big), \quad z \in S_{1/k}.$$

Definition 5.5. Let $V \in \mathfrak{B}(2\omega(\mathbb{M}), d(r))$. We define the *moment function associated with V* (or to e_V) as

$$m_V(\lambda) := \int\limits_0^\infty t^{\lambda-1} e_V(t)\, dt = \int\limits_0^\infty t^\lambda G_V(1/t)\, dt = \int\limits_0^\infty t^\lambda e^{-V(t)}\, dt.$$

From Lemma 5.3 we see that m_V, well defined in $\{\mathrm{Re}(\lambda) \geqslant 0\}$, is continuous in its domain, and holomorphic in $\{\mathrm{Re}(\lambda) > 0\}$. Moreover, $m_V(x) > 0$ for every $x \geqslant 0$. So, the following definition makes sense.

Definition 5.6. The sequence of positive real numbers $\mathfrak{m}_V = (m_V(p))_{p \in \mathbb{N}_0}$ is the *sequence of moments associated with V* (or to e_V).

Proposition 5.7. *Let e_V be a kernel associated with the strongly regular sequence $\mathbb{M}$, and $\mathfrak{m}_V = (m_V(p))_{p \in \mathbb{N}_0}$ the sequence of moments associated with V. Then $\mathbb{M}$ and $\mathfrak{m}_V$ are equivalent.*

Proof. It suffices to work with $p \geqslant 1$. From (14) we have $C, K > 0$ such that

$$m_V(p) \leqslant C\int\limits_0^\infty t^{p-1}h_{\mathbb{M}}(K/t)\, dt = C\int\limits_0^{m_p} t^{p-1}h_{\mathbb{M}}(K/t)\, dt + C\int\limits_{m_p}^\infty t^{p-1}h_{\mathbb{M}}(K/t)\, dt.$$

In the first integral of the right-hand side we take into account that $h_{\mathbb{M}}$ is bounded by 1, while in the second one we use the definition of $h_{\mathbb{M}}$ to obtain that $h_{\mathbb{M}}(K/t) \leqslant K^{p+1}M_{p+1}/t^{p+1}$, $t > m_p$. This yields

$$m_V(p) \leqslant C\frac{t^p}{p}\bigg|_0^{m_p} - CK^{p+1}M_{p+1}\frac{1}{t}\bigg|_{m_p}^\infty = \frac{Cm_p^p}{p} + CK^{p+1}\frac{M_{p+1}}{m_p}.$$

We have $M_{p+1} = m_p M_p$, and we may apply (4) to obtain that

$$m_V(p) \leqslant C\big(A^{2p} + K^{p+1}\big) M_p \leqslant 2C \max\{1, K\} \big(\max\{A^2, K\}\big)^p M_p,$$

what concludes the first part of the proof.

On the other hand, L.S. Maergoiz [18, Thm. 3.3] has shown that the function

$$F_V(z) = \sum_{n=0}^{\infty} \frac{z^n}{m_V(n)}, \quad z \in \mathbb{C}, \tag{15}$$

is entire and such that

$$\limsup_{r \to \infty} \frac{\log \max_{|z|=r} |F_V(z)|}{V(r)} \in (0, \infty).$$

From this fact we deduce that there exist constants $C_1, K_1 > 0$ such that for every $z \in \mathbb{C}$ one has

$$\big|F_V(z)\big| \leqslant C_1 \exp\big(K_1 V(|z|)\big).$$

Now, recall that $\log(V(r))/\log(r)$ is a proximate order equivalent to $d(r) = \log(M(r))/\log(r)$. Consequently, by Remark 4.3 there exists $K_2 > 0$ such that $V(r) \leqslant K_2 M(r)$ for large r, and so we have

$$\big|F_V(z)\big| \leqslant \tilde{C} \exp\big(\tilde{K} M(|z|)\big) \tag{16}$$

for every $z \in \mathbb{C}$ and suitably large constants $\tilde{C}, \tilde{K} > 0$. Finally, we take into account the following result by H. Komatsu [10, Prop. 4.5]:

Let $M(r)$ be the function associated with $\mathbb{M}$. Given an entire function $F(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{C}$, the following statements are equivalent:

(i) There exist $C, K > 0$ such that $|F(z)| \leqslant C e^{M(K|z|)}$, $z \in \mathbb{C}$.
(ii) There exist $c, k > 0$ such that for every $n \in \mathbb{N}_0$, $|a_n| \leqslant ck^n/M_n$.

It suffices to apply this equivalence to the function F_V, by virtue of (16), and we end the second part of the proof. $\square$

Remark 5.8.

(i) We record for the future that, as a consequence of the first part of the previous proof, given $K > 0$ there exist $C, D > 0$ such that for every $p \in \mathbb{N}$ one has

$$\int_0^{\infty} t^{p-1} h_{\mathbb{M}}(K/t) \, dt \leqslant CD^p M_p. \tag{17}$$

(ii) In the Gevrey case $\mathbb{M}_{1/k}$ and with the kernel e_k introduced in Remark 5.4, we obtain the moment function $m_{1/k}(\lambda) = \Gamma(1 + \lambda/k)$ for $\Re(\lambda) \geqslant 0$, and we immediately check that $\mathbb{M}_{1/k}$ and $\mathfrak{m}_{1/k} = (m_{1/k}(p))_{p \in \mathbb{N}_0}$ are equivalent.

6. A generalization of Borel–Ritt–Gevrey theorem. Right inverses for the asymptotic Borel map

The proof of the next result, a generalization of the classical Borel–Ritt–Gevrey theorem, will only be sketched, since it is similar to the original one in the Gevrey case (see [22,29,4,2]; in the several variables

case, see [24]). Indeed, in a previous work by A. Lastra, S. Malek and the author [15, Thm. 4.1], this same technique was applied by using kernels derived from the flat functions of V. Thilliez [27], what obliged us to work in sectors of non-optimal opening. This drawback is now overcome under the additional assumption that the sequence $\mathbb{M}$ defines a proximate order $d(r)$, which is the case for all the examples we have been able to provide.

Theorem 6.1 *(Generalized Borel–Ritt–Gevrey theorem). Let $\mathbb{M}$ be a strongly regular sequence such that $d(r)$ is a proximate order, and let $\gamma > 0$ be given. The following statements are equivalent:*

(i) $\gamma \leqslant \omega(\mathbb{M})$.

(ii) *For every $\boldsymbol{a} = (a_p)_{p \in \mathbb{N}_0} \in \Lambda_{\mathbb{M}}$ there exists a function $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma)$ such that*

$$f \sim_{\mathbb{M}} \hat{f} = \sum_{p \in \mathbb{N}_0} \frac{a_p}{p!} z^p,$$

i.e., $\tilde{\mathcal{B}}(f) = \boldsymbol{a}$. In other words, the Borel map $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma) \longrightarrow \Lambda_{\mathbb{M}}$ is surjective.

Proof. (i) $\Longrightarrow$ (ii) It is enough to treat the case $\gamma = \omega(\mathbb{M})$. Choose $V \in \mathfrak{B}(2\omega(\mathbb{M}), d(r))$, and consider the associated kernel e_V (see Definition 5.1) and sequence of moments $\mathfrak{m}_V = (m_V(p))_{p \in \mathbb{N}_0}$ (see Definition 5.6). Given $(a_p)_{p \in \mathbb{N}_0} \in \Lambda_{\mathbb{M}}$, there exist $C_1, D_1 > 0$ such that

$$|a_p| \leqslant C_1 D_1^p p! M_p, \quad p \in \mathbb{N}_0,$$

so that, by Proposition 5.7, the series

$$\hat{g} = \sum_{p \in \mathbb{N}_0} \frac{a_p}{p! m_V(p)} z^p \tag{18}$$

converges in a disc $D(0, R)$ for some $R > 0$, to a holomorphic function g. Choose $0 < R_0 < R$, and define

$$f(z) := \int_0^{R_0} e_V\left(\frac{u}{z}\right) g(u) \frac{du}{u}, \quad z \in S_{\omega(\mathbb{M})}, \tag{19}$$

which turns out to be a holomorphic function in $S_{\omega(\mathbb{M})}$. Given $T \prec S_{\omega(\mathbb{M})}$, $N \in \mathbb{N}$ and $z \in T$, by standard arguments we have

$$f(z) - \sum_{p=0}^{N-1} a_p \frac{z^p}{p!} = f(z) - \sum_{p=0}^{N-1} \frac{a_p}{m_V(p)} m_V(p) \frac{z^p}{p!}$$

$$= \int_0^{R_0} e_V\left(\frac{u}{z}\right) \sum_{k=0}^{\infty} \frac{a_k}{m_V(k)} \frac{u^k}{k!} \frac{du}{u} - \sum_{p=0}^{N-1} \frac{a_p}{m_V(p)} \int_0^{\infty} u^{p-1} e_V(u) \, du \frac{z^p}{p!}$$

$$= \int_0^{R_0} e_V\left(\frac{u}{z}\right) \sum_{k=N}^{\infty} \frac{a_k}{m_V(k)} \frac{u^k}{k!} \frac{du}{u} - \int_{R_0}^{\infty} e_V\left(\frac{u}{z}\right) \sum_{p=0}^{N-1} \frac{a_p}{m_V(p)} \frac{u^p}{p!} \frac{du}{u}$$

$$= f_1(z) + f_2(z).$$

By Proposition 5.7 there exist $C_2, D_2 > 0$ such that

$$\frac{|a_k|}{m_V(k) k!} \leqslant \frac{C_1 D_1^k k! M_k}{m_V(k) k!} \leqslant C_2 D_2^k \tag{20}$$

for all $k \in \mathbb{N}_0$, and so, taking $R_0 \leqslant (1-\epsilon)/D_2$ for some $\epsilon > 0$ if necessary, we get

$$\left| f_1(z) \right| \leqslant C_2 \int_0^{R_0} \left| e_V\left(\frac{u}{z}\right) \right| \sum_{k=N}^{\infty} (D_2 u)^k \frac{du}{u} \leqslant \epsilon C_2 D_2^N \int_0^{R_0} \left| e_V\left(\frac{u}{z}\right) \right| u^{N-1} \, du. \tag{21}$$

On the other hand, we have $u^p \leqslant R_0^p u^N / R_0^N$ for $u \geqslant R_0$ and $0 \leqslant p \leqslant N-1$. So, according to (20), we may write

$$\sum_{p=0}^{N-1} \frac{|a_p| u^p}{m_V(p) p!} \leqslant \sum_{p=0}^{N-1} \frac{C_1 D_1^p p! M_p u^p}{m_V(p) p!} \leqslant \sum_{p=0}^{N-1} C_1 D_1^p C_2 D_2^p u^p \leqslant \frac{u^N}{R_0^N} \sum_{p=0}^{N-1} C_1 D_1^p C_2 D_2^p R_0^p \leqslant C_3 D_3^N u^N$$

for some positive constants C_3, D_3, and deduce that

$$\left| f_2(z) \right| \leqslant C_3 D_3^N \int_{R_0}^{\infty} \left| e_V\left(\frac{u}{z}\right) \right| u^{N-1} \, du. \tag{22}$$

In view of (21) and (22), we are done if we prove that

$$\int_0^{\infty} \left| e_V\left(\frac{u}{z}\right) \right| u^{N-1} \, du \leqslant C_4 D_4^N m_V(N) |z|^N$$

for every $z \in T$ and for suitable $C_4, D_4 > 0$. But this is a straightforward consequence of Lemma 5.3(ii) and the estimates in (17).

(ii) $\Longrightarrow$ (i) We will not provide all the details, but the argument could be completed easily with some of the results in the preprint [16]. Anyway, the idea is similar to the one in the Gevrey case, see [2, p. 99]. For $\gamma > \omega(\mathbb{M})$, consider a path $\delta_{\omega(\mathbb{M})}$ in S_γ like the ones used in the classical Borel transform, consisting of a segment from the origin to a point z_0 with $\arg(z_0) = \omega(\mathbb{M})(\pi + \varepsilon)/2$ (for some $\varepsilon \in (0, \pi)$), then the circular arc $|z| = |z_0|$ from z_0 to the point z_1 on the ray $\arg(z) = -\omega(\mathbb{M})(\pi + \varepsilon)/2$, and finally the segment from z_1 to the origin. Choose any lacunary series $\hat{g} = \sum_{p=0}^{\infty} b_p z^p / p!$ convergent in the unit disc to a function g that has no analytic continuation beyond that disc (for example, $\hat{g} = \sum_{p=0}^{\infty} z^{2^p}$). Then, the equivalence of $\mathbb{M}$ and $\mathfrak{m}_V$ implies that $\boldsymbol{a} = (m_V(p) b_p)_{p \in \mathbb{N}_0}$ belongs to $\Lambda_{\mathbb{M}}$. If there would exist a function $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma)$ such that $f \sim_{\mathbb{M}} \hat{f} := \sum_{p \in \mathbb{N}_0} m_V(p) b_p z^p / p!$, one may check that the function

$$G(u) := \frac{-1}{2\pi i} \int_{\delta_{\omega(\mathbb{M})}} F_V(u/z) f(z) \frac{dz}{z}, \quad u \in S_\varepsilon,$$

where F_V is the function introduced in (15), is an analytic continuation of g into the unbounded sector S_ε. Since this is not possible, we deduce $\tilde{\mathcal{B}}$ is not surjective in this case. $\square$

Finally, we will state a result concerning the surjectivity of the asymptotic Borel map $\tilde{\mathcal{B}}$ in the classes $\mathcal{A}_{\mathbb{M}}(S_\gamma)$, and the existence of suitably defined linear continuous right inverses for $\tilde{\mathcal{B}}$.

Theorem 6.2. *Let $\mathbb{M}$ be strongly regular and such that $d(r)$ is a proximate order, and let $\gamma > 0$ be given.*

(a) *Each of the following assertions implies the next one:*
    (i) $\gamma < \omega(\mathbb{M})$.

(ii) *There exists $d \geqslant 1$ such that for every $A > 0$ there is a linear continuous operator*

$$T_{\mathbb{M},A,\gamma} : \Lambda_{\mathbb{M},A} \to \mathcal{A}_{\mathbb{M},dA}(S_\gamma)$$

   such that $\tilde{\mathcal{B}} \circ T_{\mathbb{M},A,\gamma} = Id_{\Lambda_{\mathbb{M},A}}$, the identity map in $\Lambda_{\mathbb{M},A}$.
(iii) *The Borel map $\tilde{\mathcal{B}} : \mathcal{A}_{\mathbb{M}}(S_\gamma) \to \Lambda_{\mathbb{M}}$ is surjective.*
(iv) *There exists a function $f \in \mathcal{A}_{\mathbb{M}}(S_\gamma)$ such that for every $m \in \mathbb{N}_0$ we have $f^{(m)}(0) = \delta_{1,m}$ (where $\delta_{1,m}$ stands for Kronecker's delta).*

(b) *If one has*

$$\sum_{n=0}^{\infty} \left(\frac{1}{(n+1)m_n} \right)^{1/(\omega(\mathbb{M})+1)} = \infty, \tag{23}$$

   then (i) *is equivalent to*:
   (v) *The Borel map $\tilde{\mathcal{B}} : \mathcal{A}_{\mathbb{M}}(S_\gamma) \to \Lambda_{\mathbb{M}}$ is not injective, i.e., $\mathcal{A}_{\mathbb{M}}(S_\gamma)$ is not quasianalytic.*
(c) *If one has*

$$\sum_{n=0}^{\infty} \left(\frac{1}{m_n} \right)^{1/\omega(\mathbb{M})} = \infty, \tag{24}$$

   then all the conditions (i)–(v) *are equivalent to each other.*

Proof. (a) (i) $\Longrightarrow$ (ii) Fix $A > 0$. For every $\boldsymbol{a} = (a_p)_{p \in \mathbb{N}_0} \in \Lambda_{\mathbb{M},A}$, the series $\hat{g}$ given in (18) converges in a disc $D(0, R)$ not depending on $\boldsymbol{a}$. We define $T_{\mathbb{M},A,\gamma}(\boldsymbol{a})$ as the restriction to S_γ of the function defined in (19), which was shown to belong to $\tilde{\mathcal{A}}_{\mathbb{M}}(S_{\omega(\mathbb{M})})$. By combining the information in Proposition 2.3 with that in Remark 2.5, we conclude that there exists $d \geqslant 1$ such that $T_{\mathbb{M},A,\gamma}$ sends $\Lambda_{\mathbb{M},A}$ into $\mathcal{A}_{\mathbb{M},dA}(S_\gamma)$ and solves the problem.

(ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (iv) are immediate.

(b) By the definition of $\omega(\mathbb{M})$, we always have that (i) implies (v), and that (v) implies $\gamma \leqslant \omega(\mathbb{M})$. But condition (23) excludes equality by Theorem 3.2.

(c) Under condition (24), the fact that (iv) implies (i) may be obtained in the same way as Proposition 3.3 in [12]. So, (i)–(iv) are all equivalent to each other. According to (b), in order to conclude it suffices to prove that condition (24) implies condition (23), but this was obtained in Proposition 4.8(i) in [13]. $\square$

Remark 6.3.

(i) Of course, all the results in this paper are valid for general unbounded sectors $S(d, \gamma)$. We have considered the case $d = 0$ in the previous arguments only for convenience.

(ii) V. Thilliez [27] obtained (i) $\Longrightarrow$ (ii) in the previous result for $\gamma < \gamma(\mathbb{M})$, where $\gamma(\mathbb{M})$ is the growth index described in Definition 2.17. Since $\gamma(\mathbb{M}) \leqslant \omega(\mathbb{M})$ in general, our result would mean an improvement for those $\mathbb{M}$ (if any) such that $\gamma(\mathbb{M}) < \omega(\mathbb{M})$. Also, note that in our present construction of right inverses for $\tilde{\mathcal{B}}$ we need to consider just a "global" kernel e_V in $S_{\omega(\mathbb{M})}$, while in V. Thilliez's and A. Lastra, S. Malek and the author's previous approaches (see [27,15]) the kernel had to be chosen depending on the sector S_γ on which the class was defined.

(iii) As commented before, the integral expression for the operators $T_{\mathbb{M},A,\gamma}$ is well suited for their extension to the several variable case. The interested reader may compare this and other approaches in [24,14,15].

(iv) For Gevrey sequences, condition (24) holds, since it amounts to the divergence of the harmonic series. In general, condition (23) does not imply (24). For instance, as stated in Example 3.10, the sequence $\mathbb{M}_{\alpha,\beta}$ satisfies (23) if, and only if, $\alpha \geqslant \beta - 1$. One easily checks that it satisfies (24) if, and only if,

$\alpha \geqslant \beta$. So, if $\beta - 1 \leqslant \alpha < \beta$ we have that $\mathbb{M}_{\alpha,\beta}$ satisfies (23) and not (24). Whenever this is the case, it is an open problem to decide whether (iv) in the previous theorem implies (i).

Acknowledgments

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