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A maximum principle for fully coupled stochastic control systems of mean-field type ☆

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ABSTRACT

The present paper considers an optimal control problem for fully coupled forward–backward stochastic differential equations (FBSDEs) of mean-field type in the case of controlled diffusion coefficient. Moreover, the control domain is not assumed to be convex. By virtue of a reduction method, we establish the necessary optimality conditions of Pontryagin's type. As an application, a linear–quadratic stochastic control problem is studied.

1. Introduction

Study on the stochastic maximum principle (SMP) can be traced back to 1970s by Kushner [9], and later on by Haussmann [6,7]. From then on, a great deal of research has been devoted to different versions of SMP, see, e.g. the references Peng [13,14], Shi and Wu [16], Wu [17,18], Xu [19] and Yong [22]. Especially, for controlled FBSDEs, by introducing the reduction method to transform the original problem with endpoint constraint to another one called the reduced problem, which has no endpoint constraint, Wu [18] and Yong [22] obtained the optimality variational principle independently without convexity control domain. Their work makes great progress for general SMP, which extends Peng's SMP essentially.

In 2009, Buckdahn, Djehiche, Li and Peng [3] introduced a new kind of backward stochastic differential equations (BSDEs) called mean-field BSDEs, which were derived as a limit of some highly dimensional system of FBSDEs, corresponding to a large number of particles. Taking advantage of the dynamic programming, Buckdahn, Li and Peng [4] proved that this mean-field BSDE gave the viscosity solution of

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a nonlocal PDE. Since then, many authors studied the system of this kind of McKean–Vlasov type (Lasry and Lions [10]) adapted to different frameworks. Without being exhaustive, let us refer to Andersson and Djehiche [1], Buckdahn, Djehiche and Li [2], Li [11] as well as Meyer-Brandis, Øksendal and Zhou [12] and the references cited up there.

To the best of our knowledge, the topic about the SMP of mean-field FBSDEs is quite new in the literature. Up till now, there is only one paper dealing with this class of control problems. In the case of convex control domain, Xu and Zhang [20] studied the fully coupled mean-field FBSDEs, where well-posedness of these equations was presented under certain monotonic conditions assisted by the combination of classical methods (Hu and Peng [8]) with specific arguments in the mean-field theory. And also a SMP was concluded in view of spike variation techniques. Based on this, our aim of this paper is to investigate another kind of controlled FBSDEs of mean-field type under the condition of non-convex control domain. By making use of the monotonic conditions and reduction method developed in Yong [22], we not only prove the well-posedness for this kind of equations, but also acquire a series of necessary optimal conditions, which extend the classical results for fully coupled FBSDEs to the framework of mean-field theory. Meanwhile, our work is a great continuation of the result of Li [2] both in the mean-field context.

The paper is organized as follows: In Section 2, we state some preliminaries and obtain the well-posedness of the controlled mean-field FBSDEs. Section 3 is devoted to the main theorem and its detailed proof. In Section 4, an example of a linear–quadratic control problem is worked out to illustrate the theoretical applications.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ be a filtered probability space satisfying the usual condition, on which a one-dimensional standard Brownian motion $(W_t)_{t\geqslant 0}$ is defined, and $\mathcal{F} = \{\mathcal{F}_s,\ 0 \leqslant s \leqslant T\}$ be the natural filtration generated by $(W_t)_{t\geqslant 0}$, and augmented by all P-null sets, i.e.

$$\mathcal{F}_s = \sigma\{W_r,\ r \leqslant s\} \vee \mathcal{N}_p, \quad s \in [0,T],$$

where $\mathcal{N}_p$ is the set of all P-null subsets. We shall introduce some spaces to be used frequently in the sequel.

$$L^2_{\mathcal{F}}(\Omega; R) = \big\{X : \Omega \to R \bigm| X \text{ is } \mathcal{F}\text{-measurable},\ E|X|^2 < \infty\big\}.$$

$$\mathcal{S}^2_{\mathcal{F}}(0,T;R) = \Big\{\psi : [0,T]\times\Omega \to R \Bigm| \psi \text{ is } \mathcal{F}\text{-adapted and continuous},\ E\Big[\sup_{t\in[0,T]} |\psi_t|^2\Big] < \infty\Big\}.$$

$$\mathcal{H}^2_{\mathcal{F}}(0,T;R) = \left\{\psi : [0,T]\times\Omega \to R \,\middle|\, \psi \text{ is } \mathcal{F}\text{-adapted},\ E\left[\int_0^T |\psi_t|^2\, dt\right] < \infty\right\}.$$

$$M^2[0,T] := \mathcal{S}^2_{\mathcal{F}}(0,T;R) \times \mathcal{S}^2_{\mathcal{F}}(0,T;R) \times \mathcal{H}^2_{\mathcal{F}}(0,T;R).$$

Clearly, $M^2[0,T]$ is a Banach space. Any process in $M^2[0,T]$ is defined by $\Theta := (x,y,z)$ with the norm

$$\|\Theta\|_{M^2[0,T]} := \left\{E\left[\sup_{t\in[0,T]} |x_t|^2 + \sup_{t\in[0,T]} |y_t|^2 + \int_0^T |z_t|^2\, dt\right]\right\}^{\frac{1}{2}}.$$

2.1. Controlled mean-field FBSDEs

Consider the following fully coupled FBSDE of mean-field type,

$$\begin{cases} dx_t = b(t, x_t, y_t, z_t, Ex_t, Ey_t, Ez_t, u_t)\,dt + \sigma(t, x_t, y_t, z_t, Ex_t, Ey_t, Ez_t, u_t)\,dW_t, \\ -dy_t = f(t, x_t, y_t, z_t, Ex_t, Ey_t, Ez_t, u_t)\,dt - z_t\,dW_t, \\ x(0) = x_0, \quad y_T = h(x_T), \end{cases} \tag{2.1}$$

where $b, \sigma, f : [0,T] \times R \times R \times R \times R \times R \times R \times U \to R$; $h : R \to R$, and U is a non-empty subset of R.

The cost functional to be minimized over the space $\mathcal{U} = L^2_{\mathcal{F}}(0,T;U)$ of admissible controls takes the form

$$J(u) = E\big[g(x_T) + \gamma(y_0)\big], \tag{2.2}$$

where $g, \gamma : R \to R$. The optimal control problem under consideration in this paper is

Problem $\mathcal{A}$. Find $u \in \mathcal{U}$ such that

$$J(u) = \inf_{v \in \mathcal{U}} J(v).$$

Some notations and assumptions are presented before giving the well-posedness of system (2.1). We denote the scalar product by $\langle \cdot,\cdot \rangle$ and the norm by $|\cdot|$ of a Euclidean space. For $\Gamma := (x, y, z, \tilde{x}, \tilde{y}, \tilde{z})$, define $F(t, \Gamma, u) := (-f(t, \Gamma, u), b(t, \Gamma, u), \sigma(t, \Gamma, u))$.

(A_1) b, σ, f are continuously differentiable and Lipschitz continuous in Γ, g, γ are continuously differentiable in x and y respectively, and they are bounded by $C(1 + |x| + |y| + |z| + |\tilde{x}| + |\tilde{y}| + |\tilde{z}| + |u|)$, $C(1 + |x|)$ and $C(1 + |y|)$ respectively.

(A_2) All the derivatives in (A_1) are Lipschitz continuous and bounded.

(A_3) $\forall \Gamma$, $u \in \mathcal{U}$, $F(\cdot, \Gamma, u) \in \mathcal{H}^2_{\mathcal{F}}(0,T; R \times R \times R)$, and for each $x \in R$, $h(x) \in L^2_{\mathcal{F}}(\Omega; R)$; there exists a constant $C > 0$ such that

$$\begin{cases} \big|F(t, \Gamma_1, u) - F(t, \Gamma_2, u)\big| \leqslant C|\Gamma_1 - \Gamma_2|, \quad P\text{-a.s. a.e. } t \in [0,T], \\ \big|h(x_1) - h(x_2)\big| \leqslant C|x_1 - x_2|, \quad P\text{-a.s.}, \\ \forall \Gamma_j = (x_j, y_j, z_j, \tilde{x}_j, \tilde{y}_j, \tilde{z}_j), \quad j = 1, 2. \end{cases}$$

(A_4) (Monotonic conditions)

$$E\big\langle F(t, \Gamma, u) - F(t, \Gamma_1, u), \Theta - \Theta_1 \big\rangle \leqslant -\beta_1 E|\Theta - \Theta_1|^2, \quad P\text{-a.s.},$$
$$\big\langle h(x) - h(x_1), x - x_1 \big\rangle \geqslant \mu_1 |x - x_1|^2,$$

(A_4)′

$$E\big\langle F(t, \Gamma, u) - F(t, \Gamma_1, u), \Theta - \Theta_1 \big\rangle \geqslant \beta_2 E|\Theta - \Theta_1|^2, \quad P\text{-a.s.},$$
$$\big\langle h(x) - h(x_1), x - x_1 \big\rangle \leqslant -\mu_2 |x - x_1|^2, \quad \forall \Theta = (x, y, z),\ \Theta_1 = (x_1, y_1, z_1),$$

where β_1, μ_1 and β_2, μ_2 are given nonnegative constants.

Theorem 2.1. *Under the assumptions* (A_3) *and* (A_4)*, there exists a unique adapted solution* (x, y, z) *for the mean-field system* (2.1)*.*

We shall use the following two technical lemmas to give a proof to the existence part of Theorem 2.1, and the proof of these lemmas will be given in the sequel.

Lemma 2.1. *Suppose* $(r, \phi, \varphi) \in M^2[0, T]$*,* $\lambda \in L^2_{\mathcal{F}}(\Omega; R)$*, then*

$$\begin{cases} x_t = x_0 + \int_0^t (-y_s - Ey_s + r_s)\,ds + \int_0^t (-z_s - Ez_s + \phi_s)\,dW_s, \\ y_t = \lambda + x_T + \int_t^T (x_s + Ex_s - \varphi_s)\,ds - \int_t^T z_s\,dW_s \end{cases} \tag{2.3}$$

has a unique solution $(x, y, z) \in M^2[0, T]$*.*

Now, for any given $\alpha \in R$, we define

$$\begin{cases} b^\alpha(t, \Gamma, u) = \alpha b(t, \Gamma, u) + (1 - \alpha)(-y - Ey), \\ \sigma^\alpha(t, \Gamma, u) = \alpha\sigma(t, \Gamma, u) + (1 - \alpha)(-z - Ez), \\ f^\alpha(t, \Gamma, u) = \alpha f(t, \Gamma, u) + (\alpha - 1)(-x - Ex), \\ h^\alpha(x) = \alpha h(x) + (1 - \alpha)x, \\ \Gamma = (x, y, z, Ex, Ey, Ez). \end{cases}$$

Consider the following equations

$$\begin{cases} x_t = x_0 + \int_0^t \left[b^\alpha(s, \Gamma_s, u_s) + r_s\right] ds + \int_0^t \left[\sigma^\alpha(s, \Gamma_s, u_s) + \phi_s\right] dW_s, \\ y_t = \lambda + h^\alpha(x_T) + \int_t^T \left[f^\alpha(s, \Gamma_s, u_s) - \varphi_s\right] ds - \int_t^T z_s\,dW_s. \end{cases} \tag{2.4}$$

Lemma 2.2. *For a given* $\alpha_0 \in [0, 1)$ *and any* $(r, \phi, \varphi) \in M^2[0, T]$*,* $\lambda \in L^2_{\mathcal{F}}(\Omega; R)$*, Eqs.* (2.4) *have an adapted solution. Then there exists a* $\delta_0 \in (0, 1)$ *such that for all* $\alpha \in [\alpha_0, \alpha_0 + \delta_0]$ *and any* $(r, \phi, \varphi) \in M^2[0, T]$*,* $\lambda \in L^2_{\mathcal{F}}(\Omega; R)$*, Eqs.* (2.4) *have an adapted solution.*

Proof of Theorem 2.1. *Uniqueness.* If $\Theta := (x, y, z)$ and $\bar{\Theta} := (\bar{x}, \bar{y}, \bar{z})$ are two adapted solutions of (2.1), we set

$$\begin{cases} \hat{\Gamma} = (\hat{x}, \hat{y}, \hat{z}, E\hat{x}, E\hat{y}, E\hat{z}) = \Gamma - \bar{\Gamma} = (x - \bar{x}, y - \bar{y}, z - \bar{z}, Ex - E\bar{x}, Ey - E\bar{y}, Ez - E\bar{z}), \\ \hat{b}(t) = b(t, \Gamma, u) - b(t, \bar{\Gamma}, u), \quad \hat{\sigma}(t) = \sigma(t, \Gamma, u) - \sigma(t, \bar{\Gamma}, u), \\ \hat{f}(t) = f(t, \Gamma, u) - f(t, \bar{\Gamma}, u). \end{cases}$$

From (A_3), by standard estimates, it follows that $\{\hat{x}_t\}$ and $\{\hat{y}_t\}$ are continuous and

$$E\Big(\sup_{t\in[0,T]} |\hat{x}_t|^2\Big) + E\Big(\sup_{t\in[0,T]} |\hat{y}_t|^2\Big) < \infty.$$

Applying Itô's formula to $\hat{x}_t\hat{y}_t$ on $[0,T]$ yields

$$E\big[\hat{x}_T\big(h(x_T)-h(\bar{x}_T)\big)\big]=E\int_0^T\big\langle F(t,\Gamma_t,u_t)-F(t,\bar{\Gamma}_t,u_t),\Theta_t-\bar{\Theta}_t\big\rangle\,dt.$$

By assumption (A_4), we have

$$\mu_1E|x_T-\bar{x}_T|^2\leqslant E\int_0^T-\beta_1|\Theta_t-\bar{\Theta}_t|^2\,dt.$$

So

$$\Theta=\bar{\Theta}.$$

Existence. From Lemma 2.1, we can easily see that, when $\alpha=0$, for any $(r,\phi,\varphi)\in M^2[0,T]$, $\lambda\in L^2_{\mathcal{F}}(\Omega;R)$, (2.4) has an adapted solution. According to Lemma 2.2, for any $(r,\phi,\varphi)\in M^2[0,T]$, $\lambda\in L^2_{\mathcal{F}}(\Omega;R)$, (2.4) can be solved successively for the case $\alpha\in[0,\delta_0],[\delta_0,2\delta_0],\ldots.$ It turns out that when $\alpha=1$, for any $(r,\phi,\varphi)\in M^2[0,T]$, $\lambda\in L^2_{\mathcal{F}}(\Omega;R)$, the adapted solution of (2.1) exists. □

2.2. Proof of lemmas

Proof of Lemma 2.1. We consider the following BSDE:

$$\tilde{y}_t=\lambda+\int_t^T(-\tilde{y}_s-E\tilde{y}_s-\varphi_s+r_s)\,ds-\int_t^T(2\tilde{z}_s+E\tilde{z}_s-\phi_s)\,dW_s.$$

By the classical theory of mean-field BSDE (Theorem 3.1 in [4]), the above equation has a unique solution $(\tilde{y},\tilde{z})$. Then we solve the following forward equation

$$x_t=x_0+\int_0^t(-x_s-Ex_s-\tilde{y}_s-E\tilde{y}_s+r_s)\,ds+\int_0^t(-\tilde{z}_s-E\tilde{z}_s+\phi_s)\,dW_s.$$

Setting $y=\tilde{y}+x$, $z=\tilde{z}$, we can easily see that (x,y,z) is a solution of (2.3). Thus the existence is proved. As for the uniqueness, it suffices to use the method of the proof of uniqueness in Theorem 2.1, so we omit it. □

Proof of Lemma 2.2. Observe that

$$\begin{cases} b^{\alpha_0+\delta}(t,\Gamma,u)=b^{\alpha_0}(t,\Gamma,u)+\delta\big[b(t,\Gamma,u)+y+Ey\big],\\ \sigma^{\alpha_0+\delta}(t,\Gamma,u)=\sigma^{\alpha_0}(t,\Gamma,u)+\delta\big[\sigma(t,\Gamma,u)+z+Ez\big],\\ f^{\alpha_0+\delta}(t,\Gamma,u)=f^{\alpha_0}(t,\Gamma,u)+\delta\big[f(t,\Gamma,u)-x-Ex\big],\\ h^{\alpha_0+\delta}(x)=h^{\alpha_0}(x)+\delta\big[h(x)-x\big]. \end{cases}$$

We set $\Gamma^0 = (x^0, y^0, z^0, Ex^0, Ey^0, Ez^0) = 0$, and solve iteratively the following equations

$$\begin{cases} x_t^{i+1} = x_0 + \int_0^t \{ b^{\alpha_0}(s, \Gamma_s^{i+1}, u_s) + \delta[b(s, \Gamma_s^i, u_s) + y_s^i + Ey_s^i] + r_s \} \, ds \\ \qquad + \int_0^t \{ \sigma^{\alpha_0}(s, \Gamma_s^{i+1}, u_s) + \delta[\sigma(s, \Gamma_s^i, u_s) + z_s^i + Ez_s^i] + \phi_s \} \, dW_s, \\ y_t^{i+1} = h^{\alpha_0}(x_T^{i+1}) + \delta[h(x_T^i) - x_T^i] + \lambda \\ \qquad + \int_t^T \{ f^{\alpha_0}(s, \Gamma_s^{i+1}, u_s) + \delta[f(s, \Gamma_s^i, u_s) - x_s^i - Ex_s^i] - \varphi_s \} \, ds - \int_t^T z_s^{i+1} \, dW_s, \end{cases} \tag{2.5}$$

where $\Gamma^i = (x^i, y^i, z^i, Ex^i, Ey^i, Ez^i)$.

We set $\hat{\Theta}^{i+1} = \Theta^{i+1} - \Theta^i$, and apply Itô's formula to $\hat{x}_t^{i+1}\hat{y}_t^{i+1}$ yielding

$$\begin{aligned} E[(h^{\alpha_0}(x_T^{i+1}) - h^{\alpha_0}(x_T^i))\hat{x}_T^{i+1}] &= \delta E\langle \hat{x}_T^i - (h(x_T^i) - h(x_T^{i-1})), \hat{x}_T^{i+1} \rangle \\ &\quad + E\int_0^T \langle F^{\alpha_0}(t, \Gamma_t^{i+1}, u_t) - F^{\alpha_0}(t, \Gamma_t^i, u_t), \hat{\Theta}_t^{i+1} \rangle \, dt \\ &\quad + \delta E\int_0^T \langle \hat{\Theta}_t^i + E\hat{\Theta}_t^i + F(t, \Gamma_t^i, u_t) - F(t, \Gamma_t^{i-1}, u_t), \hat{\Theta}_t^{i+1} \rangle \, dt. \end{aligned}$$

From assumptions (A_3) and (A_4), we have

$$\begin{aligned} & E\langle F^{\alpha_0}(t, \Gamma_t^{i+1}, u_t) - F^{\alpha_0}(t, \Gamma_t^i, u_t), \hat{\Theta}_t^{i+1} \rangle \\ & \quad = \alpha_0 E\langle F(t, \Gamma_t^{i+1}, u_t) - F(t, \Gamma_t^i, u_t), \hat{\Theta}_t^{i+1} \rangle + (1-\alpha_0) E\langle -\hat{\Theta}_t^{i+1} - E\hat{\Theta}_t^{i+1}, \hat{\Theta}_t^{i+1} \rangle \\ & \quad \leqslant -\alpha_0 \beta_1 E|\hat{\Theta}_t^{i+1}|^2 - (1-\alpha_0) E\langle \hat{\Theta}_t^{i+1} + E\hat{\Theta}_t^{i+1}, \hat{\Theta}_t^{i+1} \rangle. \end{aligned}$$

Hence

$$\begin{aligned} & (\alpha_0\mu_1 + 1 - \alpha_0) E|\hat{x}_T^{i+1}|^2 + (\alpha_0\beta_1 + 1 - \alpha_0) E\int_0^T |\hat{\Theta}_t^{i+1}|^2 \, dt + (1-\alpha_0) \int_0^T |E\hat{\Theta}_t^{i+1}|^2 \, dt \\ & \quad \leqslant \delta(1+C) \left\{ E(|\hat{x}_T^i||\hat{x}_T^{i+1}|) + \int_0^T [E(|\hat{\Theta}_t^i||\hat{\Theta}_t^{i+1}|) + E|\hat{\Theta}_t^i| E|\hat{\Theta}_t^{i+1}|] \, dt \right\}. \end{aligned}$$

Set $C' = \min(\alpha_0\mu_1 + 1 - \alpha_0, \alpha_0\beta_1 + 1 - \alpha_0, 1 - \alpha_0, 1)$, then we obtain

$$E|\hat{x}_T^{i+1}|^2 + E\int_0^T |\hat{\Theta}_t^{i+1}|^2 \, dt \leqslant \frac{\delta(1+C)}{C'} \left\{ E(|\hat{x}_T^i||\hat{x}_T^{i+1}|) + \int_0^T [E(|\hat{\Theta}_t^i||\hat{\Theta}_t^{i+1}|) + E|\hat{\Theta}_t^i| E|\hat{\Theta}_t^{i+1}|] \, dt \right\}.$$

Letting $\varepsilon = \frac{C'}{\delta(1+C)}$, by virtue of $ab \leqslant \frac{a^2}{\varepsilon} + \frac{\varepsilon}{4}b^2$, we get

$$E|\hat{x}_T^{i+1}|^2 + E\int_0^T |\hat{\Theta}_t^{i+1}|^2 \, dt \leqslant \left(\frac{2\delta(1+C)}{C'} \right)^2 \left\{ E|\hat{x}_T^i|^2 + E\int_0^T |\hat{\Theta}_t^i|^2 \, dt \right\}. \tag{2.6}$$

Remember that $\forall i \geqslant 1$,

$$\hat{x}_T^i = \int_0^T \{b^{\alpha_0}(s, \Gamma_s^i, u_s) - b^{\alpha_0}(s, \Gamma_s^{i-1}, u_s) + \delta[b(s, \Gamma_s^{i-1}, u_s) - b(s, \Gamma_s^{i-2}, u_s) + \hat{y}_s^{i-1} + E\hat{y}_s^{i-1}]\}\, ds$$
$$+ \int_0^T \{\sigma^{\alpha_0}(s, \Gamma_s^i, u_s) - \sigma^{\alpha_0}(s, \Gamma_s^{i-1}, u_s) + \delta[\sigma(s, \Gamma_s^{i-1}, u_s) - \sigma(s, \Gamma_s^{i-2}, u_s) + \hat{z}_s^{i-1} + E\hat{z}_s^{i-1}]\}\, dW_s.$$

By a standard method of estimation, we derive

$$E|\hat{x}_T^i|^2 \leqslant KE\int_0^T (|\hat{\Theta}_t^i|^2 + |\hat{\Theta}_t^{i-1}|^2)\, dt, \tag{2.7}$$

where K is a constant only depending on C and T.

From (2.6) and (2.7), there exists a constant $K' > 0$ which only depends on C, C' and T such that

$$E\int_0^T |\hat{\Theta}_t^{i+1}|^2\, dt \leqslant K'\delta^2 E\int_0^T (|\hat{\Theta}_t^i|^2 + |\hat{\Theta}_t^{i-1}|^2)\, dt.$$

Hence there exists a $\delta_0 \in (0,1)$ depending on C, C' and T, such that when $0 < \delta \leqslant \delta_0$,

$$E\int_0^T |\hat{\Theta}_t^{i+1}|^2\, dt \leqslant \frac{1}{4}E\int_0^T |\hat{\Theta}_t^i|^2\, dt + \frac{1}{8}E\int_0^T |\hat{\Theta}_t^{i-1}|^2\, dt.$$

The following inequality is presented from this recursion formula,

$$E\int_0^T \left(|\hat{\Theta}_t^{i+1}|^2 + \frac{1}{4}|\hat{\Theta}_t^i|^2\right) dt \leqslant \left(\frac{1}{2}\right)^{i-1} E\int_0^T \left(|\hat{\Theta}_t^2|^2 + \frac{1}{4}|\hat{\Theta}_t^1|^2\right) dt, \quad \forall i \geqslant 1.$$

It turns out that Θ^i is a Cauchy sequence in $M^2[0,T]$. We denote its limit by $\Theta = (x,y,z)$. Passing to the limit in (2.5), we see that, when $0 < \delta \leqslant \delta_0$, $\Theta = (x,y,z)$ solves (2.4) for $\alpha = \alpha_0 + \delta$. The proof is completed. □

Theorem 2.2. *Under the assumptions (A_3) and $(A_4)'$, there exists a unique adapted solution (x,y,z) for the mean-field system (2.1).*

Actually, the method to prove the existence is similar to Theorem 2.1. We now consider the following system, for each $\alpha \in [0,1]$:

$$\begin{cases} dx_s^\alpha = [\alpha b(s, \Gamma_s, u_s) + r_s]\, ds + [\alpha\sigma(s, \Gamma_s, u_s) + \phi_s]\, dW_s, \\ -dy_s^\alpha = [(\alpha - 1)\beta_2 x_s + \alpha f(s, \Gamma_s, u_s) + \varphi_s]\, ds - z_s\, dW_s, \\ x^\alpha(0) = x_0, \quad y_T^\alpha = \alpha h(x_T) + (\alpha - 1)x_T + \lambda, \end{cases} \tag{2.8}$$

where $(r, \phi, \varphi) \in M^2[0,T]$, $\lambda \in L^2_{\mathcal{F}}(\Omega; R)$. Clearly, when $\alpha = 1$, the existence of (2.8) implies the existence of (2.1). Next, we give a lemma to provide a priori estimate for the existence interval of (2.8) with respect to $\alpha \in [0,1]$.

Lemma 2.3. *Let (A_3) and $(A_4)'$ hold, then there exists a constant $\delta_0 > 0$ such that if a priori, for an $\alpha_0 \in [0,1)$ there exists a solution $(x^{\alpha_0}, y^{\alpha_0}, z^{\alpha_0})$ of (2.8), then for each $\delta \in [0,\delta_0]$, there exists a solution $(x^{\alpha_0+\delta}, y^{\alpha_0+\delta}, z^{\alpha_0+\delta})$ of (2.8) for $\alpha = \alpha_0 + \delta$.*

Proof. For simplicity, we use the notations:

$$\begin{cases} \Upsilon = (X,Y,Z), \quad \bar{\Upsilon} = (\bar{X},\bar{Y},\bar{Z}), \\ \Lambda = (X,Y,Z,EX,EY,EZ), \\ \bar{\Lambda} = (\bar{X},\bar{Y},\bar{Z},E\bar{X},E\bar{Y},E\bar{Z}), \\ \hat{\Upsilon} = \Upsilon - \bar{\Upsilon}, \quad \hat{\Lambda} = \Lambda - \bar{\Lambda}. \end{cases}$$

The symbols Θ, $\bar{\Theta}$, $\hat{\Theta}$, Γ, $\bar{\Gamma}$, $\hat{\Gamma}$ are the same as used before.

Since $(r,\phi,\varphi) \in M^2[0,T]$, $\lambda \in L^2_{\mathcal{F}}(\Omega;R)$, $\alpha_0 \in [0,1)$, there exists a unique solution of (2.8), thus for each $X_T \in L^2_{\mathcal{F}}(\Omega;R)$ and $(X,Y,Z) \in M^2[0,T]$ there exists a unique solution $\Theta = (x,y,z) \in M^2[0,T]$ satisfying the following system

$$\begin{cases} dx_s = \left[\alpha_0 b(s,\Gamma_s,u_s) + \delta b(s,\Lambda_s,u_s) + r_s\right] ds + \left[\alpha_0 \sigma(s,\Gamma_s,u_s) + \delta\sigma(s,\Lambda_s,u_s) + \phi_s\right] dW_s, \\ -dy_s = \left[(\alpha_0 - 1)\beta_2 x_s + \alpha_0 f(s,\Gamma_s,u_s) + \delta\left(\beta_2 X_s + f(s,\Lambda_s,u_s)\right) + \varphi_s\right] ds - z_s\, dW_s, \\ x(0) = x_0, \quad y_T = \alpha_0 h(x_T) + (\alpha_0 - 1)x_T + \delta\left(h(X_T) + X_T\right) + \lambda. \end{cases}$$

We proceed to prove that, if δ is sufficiently small, the mapping defined by

$$I_{\alpha_0+\delta}(\Upsilon \times X_T) = \Theta \times x_T : M^2[0,T] \times L^2_{\mathcal{F}}(\Omega;R) \to M^2[0,T] \times L^2_{\mathcal{F}}(\Omega;R)$$

is a contraction.

Let $I_{\alpha_0+\delta}(\bar{\Upsilon} \times \bar{X}_T) = \bar{\Theta} \times \bar{x}_T$. Using Itô's formula to $\hat{x}_t\hat{y}_t$ fulfills

$$\begin{aligned} &\alpha_0 E\left[\hat{x}_T\left(h(x_T) - h(\bar{x}_T)\right)\right] + (\alpha_0 - 1)E|\hat{x}_T|^2 + \delta E\left[\hat{x}_T\left(\left(h(X_T) - h(\bar{X}_T)\right) + \hat{X}_T\right)\right] \\ &\quad = E\int_0^T \left\{\alpha_0\left\langle F(t,\Gamma_t,u_t) - F(t,\bar{\Gamma}_t,u_t), \hat{\Theta}_t\right\rangle + \delta\left\langle F(t,\Lambda_t,u_t) - F(t,\bar{\Lambda}_t,u_t), \hat{\Theta}_t\right\rangle \right. \\ &\qquad \left. + (1-\alpha_0)\beta_2|\hat{x}_t|^2 - \delta\beta_2\hat{x}_t\hat{X}_t\right\} dt. \end{aligned}$$

From (A_3) and $(A_4)'$, we can get

$$\begin{aligned} &(\alpha_0\mu_2 + 1 - \alpha_0)E|\hat{x}_T|^2 + \beta_2 E\int_0^T |\hat{x}_t|^2\, dt + \alpha_0\beta_2 E\int_0^T \left(|\hat{y}_t|^2 + |\hat{z}_t|^2\right) dt \\ &\quad \leqslant C'\delta E\left(|\hat{x}_T|^2 + |\hat{X}_T|^2\right) + C'\delta E\int_0^T \left(|\hat{\Upsilon}_t|^2 + |\hat{\Theta}_t|^2\right) dt. \end{aligned}$$

This implies that

$$\mu E|\hat{x}_T|^2 + \beta_2 E\int_0^T |\hat{x}_t|^2\, dt \leqslant C'\delta E\left(|\hat{x}_T|^2 + |\hat{X}_T|^2\right) + C'\delta E\int_0^T \left(|\hat{\Upsilon}_t|^2 + |\hat{\Theta}_t|^2\right) dt,$$

where $\alpha_0\mu_2 + 1 - \alpha_0 \geqslant \mu = \min(1,\mu_2)$.

For the difference of the solutions $(\hat{y}, \hat{z}) = (y - \bar{y}, z - \bar{z})$, applying the usual technique to the BSDE,

$$E\int_0^T \left(|\hat{y}_t|^2 + |\hat{z}_t|^2\right) dt \leqslant K\delta E\left\{\int_0^T |\hat{\Upsilon}_t|^2\, dt + |\hat{X}_T|^2\right\} + KE\left\{\int_0^T |\hat{x}_t|^2\, dt + |\hat{x}_T|^2\right\}.$$

Here the constant K depends on the Lipschitz constant C, C' and T.

Combining the above two estimates, we have

$$E|\hat{x}_T|^2 + E\int_0^T |\hat{\Theta}_t|^2\, dt \leqslant \delta K' E\left\{\int_0^T |\hat{\Upsilon}_t|^2\, dt + |\hat{X}_T|^2\right\}.$$

Here the constant K' depends on the μ, β_2, C' and T.

Choosing $\delta = \frac{1}{2K'}$. For each fixed $\delta \in [0, \delta_0]$, the mapping $I_{\alpha_0+\delta}$ is a contraction in the sense that

$$E|\hat{x}_T|^2 + E\int_0^T |\hat{\Theta}_t|^2\, dt \leqslant \frac{1}{2}E\left\{\int_0^T |\hat{\Upsilon}_t|^2\, dt + |\hat{X}_T|^2\right\}.$$

By the fixed point theorem, there exists a unique point $\Theta^{\alpha_0+\delta} = (x^{\alpha_0+\delta}, y^{\alpha_0+\delta}, z^{\alpha_0+\delta})$ which is the solution of (2.8) for $\alpha = \alpha_0 + \delta$. This completes the proof. □

Proof of Theorem 2.2. The uniqueness can be deduced from similar arguments as those in Theorem 2.1. When $\alpha = 0$, (2.8) has a unique solution, then by Lemma 2.3, there exists a $\delta_0 \in (0, 1)$ such that for any $\delta \in [0, \delta_0]$, (2.8) has a unique solution for $\alpha = \alpha_0 + \delta$. Repeat this process for N-times with $1 \leqslant N\delta_0 < 1 + \delta_0$. It then follows that for $\alpha = 1$, $\lambda = 0$, FBSDE (2.8) has a unique solution. This completes the proof. □

We now consider the following FBSDE:

$$\begin{cases} dx_t = b(t, x_t, y_t, z_t, Ex_t, Ey_t, Ez_t, u_t)\, dt + \sigma(t, x_t, y_t, z_t, Ex_t, Ey_t, Ez_t, u_t)\, dW_t, \\ -dy_t = f(t, x_t, y_t, z_t, Ex_t, Ey_t, Ez_t, u_t)\, dt - z_t\, dW_t, \\ x(0) = x_0, \quad y_T = h(x_T) + h_T, \end{cases} \tag{2.9}$$

with $h_T \in L^2_{\mathcal{F}}(\Omega; R)$. Clearly, when $h_T = 0$, the system (2.9) is reduced to the system (2.1).

Lemma 2.4. *Let (A_3), (A_4) (or $(A_4)'$) hold, for any $u \in \mathcal{U}$, the fully coupled FBSDE (2.9) admits a unique solution $(x, y, z) \in M^2[0, T]$. Moreover, the following estimate holds*:

$$\begin{aligned} \|\Theta\|^2_{M^2[0,T]} &= E\left\{\sup_{t\in[0,T]} |x_t|^2 + \sup_{t\in[0,T]} |y_t|^2 + \int_0^T |z_t|^2\, dt\right\} \\ &\leqslant CE\left\{|x_0|^2 + \left|h(0) + h_T\right|^2 + \int_0^T \left[\left|b(t, 0, u_t)\right|^2 + \left|\sigma(t, 0, u_t)\right|^2 + \left|f(t, 0, u_t)\right|^2\right] dt\right\}, \end{aligned}$$

where C is a multiplicative constant which can be different from line to line. Furthermore, if $\tilde{\Theta} = (\tilde{x}, \tilde{y}, \tilde{z}) \in M^2[0, T]$ is the unique adapted solution of (2.9) controlled by $(\tilde{x}_0, \tilde{h}_T) \in R \times L^2_{\mathcal{F}}(\Omega; R)$ and $\tilde{u} \in \mathcal{U}$, then we have the following estimate:

$$\|\tilde{\Theta}-\Theta\|^2_{M^2[0,T]} = E\left\{\sup_{t\in[0,T]}|\tilde{x}_t-x_t|^2+\sup_{t\in[0,T]}|\tilde{y}_t-y_t|^2+\int_0^T|\tilde{z}_t-z_t|^2\,dt\right\}$$

$$\leqslant CE\left\{|\tilde{x}_0-x_0|^2+|\tilde{h}_T-h_T|^2+\int_0^T\Big[\big|b(t,\Gamma_t,\tilde{u}_t)-b(t,\Gamma_t,u_t)\big|^2\right.$$

$$\left.+\big|\sigma(t,\Gamma_t,\tilde{u}_t)-\sigma(t,\Gamma_t,u_t)\big|^2+\big|f(t,\Gamma_t,\tilde{u}_t)-f(t,\Gamma_t,u_t)\big|^2\Big]\,dt\right\},$$

where $\Gamma_t := (x_t, y_t, z_t, Ex_t, Ey_t, Ez_t)$.

Proof. The well-posedness of (2.9) can be proved by using the same techniques as in Theorem 2.1 and Theorem 2.2. Next, we will use the method of continuation to gain the above two estimates.

Using Itô's formula to $|x_t|^2$, we have

$$|x_t|^2 = |x_0|^2+\int_0^t\Big[2x_sb(s,\Gamma_s,u_s)+\big|\sigma(s,\Gamma_s,u_s)\big|^2\Big]\,ds+\int_0^t 2x_s\sigma(s,\Gamma_s,u_s)\,dW_s$$

$$\leqslant |x_0|^2+C\int_0^t\Big[|x_s|^2+|y_s|^2+|z_s|^2+|Ex_s|^2+|Ey_s|^2+|Ez_s|^2$$

$$+b^2(s,0,u_s)+\sigma^2(s,0,u_s)\Big]\,ds+2\int_0^t x_s\sigma(s,\Gamma_s,u_s)\,dW_s. \tag{2.10}$$

By taking expectations and using Gronwall's inequality, one has that

$$E|x_t|^2 \leqslant CE\left\{|x_0|^2+\int_0^T\big(|y_s|^2+|z_s|^2+b^2(s,0,u_s)+\sigma^2(s,0,u_s)\big)\,ds\right\}. \tag{2.11}$$

Through the use of Burkholder–Davis–Gundy's inequality to (2.10) (note (2.11)) yields

$$E\Big(\sup_{t\in[0,T]}|x_t|^2\Big) \leqslant CE\left\{|x_0|^2+\int_0^T\big(|y_s|^2+|z_s|^2+b^2(s,0,u_s)+\sigma^2(s,0,u_s)\big)\,ds\right\}. \tag{2.12}$$

On the other hand, by applying Itô's formula to $|y_t|^2$ fulfills

$$|y_t|^2+\int_t^T|z_s|^2\,ds = |y_T|^2+2\int_t^T y_sf(s,\Gamma_s,u_s)\,ds-2\int_t^T y_sz_s\,dW_s$$

$$\leqslant C\left\{|x_T|^2+\big|h(0)+h_T\big|^2+\int_t^T\big[|x_s|^2+|y_s|^2+E|x_s|^2+E|y_s|^2+f^2(s,0,u_s)\big]\,ds\right\}$$

$$+\frac{1}{4}\int_t^T|z_s|^2\,ds+\frac{1}{4}\int_t^T E|z_s|^2\,ds-2\int_t^T y_sz_s\,dW_s. \tag{2.13}$$

Similar to the argument of getting (2.12), we have

$$\begin{aligned} &E\Big(\sup_{t\in[0,T]}|y_t|^2\Big)+E\int_0^T|z_t|^2\,dt \\ &\quad\leqslant CE\bigg\{|x_T|^2+\big|h(0)+h_T\big|^2+\int_0^T\big(|x_t|^2+\big|f(t,0,u_t)\big|^2\big)\,dt\bigg\}. \end{aligned}\tag{2.14}$$

Let a C^1 function $\Phi=\left(\begin{smallmatrix}A&B\\B&D\end{smallmatrix}\right):[0,T]\to R^{2\times 2}$ be a bridge (definition 2.5 in [21]) extending from (b,σ,f,h), with some constant K, and $A,B,D:[0,T]\to R$, satisfying

$$\begin{cases} D_T\leqslant 0,\quad A_t\geqslant 0,\quad \forall t\in[0,T],\\ \Phi_0\leqslant K\begin{pmatrix}1&0\\0&0\end{pmatrix}. \end{cases}$$

Applying Itô's formula to $\langle\Phi_t\left(\begin{smallmatrix}x_t\\y_t\end{smallmatrix}\right),\left(\begin{smallmatrix}x_t\\y_t\end{smallmatrix}\right)\rangle$, it follows that

$$\begin{aligned} &E\Big\langle\Phi_T\begin{pmatrix}x_T\\y_T\end{pmatrix},\begin{pmatrix}x_T\\y_T\end{pmatrix}\Big\rangle-E\Big\langle\Phi_0\begin{pmatrix}x_0\\y_0\end{pmatrix},\begin{pmatrix}x_0\\y_0\end{pmatrix}\Big\rangle \\ &\quad=E\int_0^T\bigg\{2\Big\langle\Phi_t\begin{pmatrix}x_t\\y_t\end{pmatrix},\begin{pmatrix}b(t,\Gamma_t,u_t)\\-f(t,\Gamma_t,u_t)\end{pmatrix}\Big\rangle+\Big\langle\Phi_t\begin{pmatrix}\sigma(t,\Gamma_t,u_t)\\z_t\end{pmatrix},\begin{pmatrix}\sigma(t,\Gamma_t,u_t)\\z_t\end{pmatrix}\Big\rangle \\ &\qquad+\Big\langle\dot\Phi_t\begin{pmatrix}x_t\\y_t\end{pmatrix},\begin{pmatrix}x_t\\y_t\end{pmatrix}\Big\rangle\bigg\}\,dt. \end{aligned}\tag{2.15}$$

Case 1. If the following inequalities hold,

$$E\Big\langle\Phi_T\begin{pmatrix}x\\h(x)-h(0)\end{pmatrix},\begin{pmatrix}x\\h(x)-h(0)\end{pmatrix}\Big\rangle\geqslant\delta E|x|^2,\quad\forall x\in R,$$

and

$$\begin{aligned} &E\bigg\{\Big\langle\dot\Phi_t\begin{pmatrix}x\\y\end{pmatrix},\begin{pmatrix}x\\y\end{pmatrix}\Big\rangle+2\Big\langle\Phi_t\begin{pmatrix}x\\y\end{pmatrix},\begin{pmatrix}b(t,\Gamma_t,u_t)-b(t,0,u_t)\\-[f(t,\Gamma_t,u_t)-f(t,0,u_t)]\end{pmatrix}\Big\rangle \\ &\qquad+\Big\langle\Phi_t\begin{pmatrix}\sigma(t,\Gamma_t,u_t)-\sigma(t,0,u_t)\\z\end{pmatrix},\begin{pmatrix}\sigma(t,\Gamma_t,u_t)-\sigma(t,0,u_t)\\z\end{pmatrix}\Big\rangle\bigg\} \\ &\quad\leqslant-\delta E|x|^2,\quad\forall\Gamma\in R^6,\ \text{a.e. } t\in[0,T],\ \text{a.s.} \end{aligned}$$

where $\delta>0$, then

$$\begin{aligned} E\Big\langle\Phi_T\begin{pmatrix}x_T\\y_T\end{pmatrix},\begin{pmatrix}x_T\\y_T\end{pmatrix}\Big\rangle&=E\Big\langle\Phi_T\begin{pmatrix}x_T\\h(x_T)+h_T\end{pmatrix},\begin{pmatrix}x_T\\h(x_T)+h_T\end{pmatrix}\Big\rangle \\ &=E\Big\langle\Phi_T\begin{pmatrix}x_T\\h(x_T)-h(0)\end{pmatrix},\begin{pmatrix}x_T\\h(x_T)-h(0)\end{pmatrix}\Big\rangle \\ &\quad+E\Big\langle\Phi_T\begin{pmatrix}x_T\\h(x_T)-h(0)\end{pmatrix},\begin{pmatrix}0\\h(0)+h_T\end{pmatrix}\Big\rangle \\ &\quad+E\Big\langle\Phi_T\begin{pmatrix}0\\h(0)+h_T\end{pmatrix},\begin{pmatrix}x_T\\h(x_T)+h_T\end{pmatrix}\Big\rangle \end{aligned}$$

$$\begin{aligned}
&\geqslant E\big\{\delta|x_T|^2 - 2|B_T||x_T|\big|h(0)+h_T\big| - 2C|D_T|\big|h(0)+h_T\big| \\
&\quad - 2C|D_T||x_T|\big|h(0)+h_T\big| - |D_T|\big|h(0)+h_T\big|^2\big\} \\
&\geqslant E\left\{\frac{\delta}{2}|x_T|^2 - L\big|h(0)+h_T\big|^2\right\}. \qquad (2.16)
\end{aligned}$$

Here the constant $L > 0$ only depends on C, δ, $|B_T|$ and $|D_T|$. Similarly,

$$\begin{aligned}
\text{the right side of (2.15)} &\leqslant E\int_0^T \left\{-\delta|x_t|^2 + 2\left\langle \Phi_t\begin{pmatrix} x_t \\ y_t\end{pmatrix}, \begin{pmatrix} b(t,0,u_t) \\ -f(t,0,u_t)\end{pmatrix}\right\rangle\right. \\
&\quad + 2\left\langle \Phi_t\begin{pmatrix} \sigma(t,0,u_t) \\ 0\end{pmatrix}, \begin{pmatrix} \sigma(t,\Gamma_t,u_t)-\sigma(t,0,u_t) \\ z_t\end{pmatrix}\right\rangle \\
&\quad \left. + \left\langle \Phi_t\begin{pmatrix} \sigma(t,0,u_t) \\ 0\end{pmatrix}, \begin{pmatrix} \sigma(t,0,u_t) \\ 0\end{pmatrix}\right\rangle\right\} dt \\
&\leqslant E\int_0^T -\frac{\delta}{2}|x_t|^2\,dt + C_\varepsilon E\int_0^T \big[\big|b(t,0,u_t)\big|^2 + \big|\sigma(t,0,u_t)\big|^2 \\
&\quad + \big|f(t,0,u_t)\big|^2\big]\,dt + \varepsilon E\int_0^T \big(|y_t|^2+|z_t|^2\big)\,dt, \qquad (2.17)
\end{aligned}$$

where $C_\varepsilon > 0$ only depends on the bound of $|\Phi_t|$ as well as δ, ε.

Combining (2.15)–(2.17) and noting (2.14), we have proved that

$$\begin{aligned}
E|x_T|^2 + E\int_0^T |x_t|^2\,dt &\leqslant C_\varepsilon\left\{E\big[|x_0|^2 + \big|h(0)+h_T\big|^2\big] + E\int_0^T \big[\big|b(t,0,u_t)\big|^2 + \big|\sigma(t,0,u_t)\big|^2\right. \\
&\quad \left. + \big|f(t,0,u_t)\big|^2\big]\,dt\right\} + \frac{2\varepsilon}{\delta}E\int_0^T \big(|y_t|^2+|z_t|^2\big)\,dt \\
&\leqslant C_\varepsilon\left\{E\big[|x_0|^2 + \big|h(0)+h_T\big|^2\big] + E\int_0^T \big[\big|b(t,0,u_t)\big|^2 + \big|\sigma(t,0,u_t)\big|^2\right. \\
&\quad \left. + \big|f(t,0,u_t)\big|^2\big]\,dt\right\} + \varepsilon C E\left\{|x_T|^2 + \big|h(0)+h_T\big|^2 + \int_0^T \big(|x_t|^2 + \big|f(t,0,u_t)\big|^2\big)\,dt\right\},
\end{aligned}$$

with the constant C independent of ε. Choose suitable ε such that

$$\begin{aligned}
E|x_T|^2 + E\int_0^T |x_t|^2\,dt &\leqslant CE\left\{\big[|x_0|^2 + \big|h(0)+h_T\big|^2\big] + \int_0^T \big[\big|b(t,0,u_t)\big|^2\right. \\
&\quad \left. + \big|\sigma(t,0,u_t)\big|^2 + \big|f(t,0,u_t)\big|^2\big]\,dt\right\}.
\end{aligned}$$

Then returning to (2.14), we deduce

$$E\Big(\sup_{t\in[0,T]}|y_t|^2\Big)+E\int_0^T|z_t|^2\,dt\leqslant CE\bigg\{\big[|x_0|^2+\big|h(0)+h_T\big|^2\big]+\int_0^T\big[\big|b(t,0,u_t)\big|^2+\big|\sigma(t,0,u_t)\big|^2+\big|f(t,0,u_t)\big|^2\big]\,dt\bigg\}.\tag{2.18}$$

Finally, by (2.12), we have

$$E\Big(\sup_{t\in[0,T]}|x_t|^2\Big)\leqslant CE\bigg\{\big[|x_0|^2+\big|h(0)+h_T\big|^2\big]+\int_0^T\big[\big|b(t,0,u_t)\big|^2+\big|\sigma(t,0,u_t)\big|^2+\big|f(t,0,u_t)\big|^2\big]\,dt\bigg\}.\tag{2.19}$$

Hence, the first estimation follows immediately from (2.18) and (2.19).

Case 2. Let the following inequalities hold,

$$E\left\langle \Phi_T\begin{pmatrix} x\\ h(x)-h(0)\end{pmatrix},\begin{pmatrix} x\\ h(x)-h(0)\end{pmatrix}\right\rangle\geqslant 0,\quad \forall x\in R,$$

and

$$\begin{aligned}&E\bigg\{\left\langle\dot{\Phi}_t\begin{pmatrix}x\\ y\end{pmatrix},\begin{pmatrix}x\\ y\end{pmatrix}\right\rangle+2\left\langle\Phi_t\begin{pmatrix}x\\ y\end{pmatrix},\begin{pmatrix}b(t,\Gamma_t,u_t)-b(t,0,u_t)\\ -[f(t,\Gamma_t,u_t)-f(t,0,u_t)]\end{pmatrix}\right\rangle\\&\quad+\left\langle\Phi_t\begin{pmatrix}\sigma(t,\Gamma_t,u_t)-\sigma(t,0,u_t)\\ z\end{pmatrix},\begin{pmatrix}\sigma(t,\Gamma_t,u_t)-\sigma(t,0,u_t)\\ z\end{pmatrix}\right\rangle\bigg\}\\&\leqslant-\delta E\big[|y|^2+|z|^2\big],\quad\forall\Gamma\in R^6,\ \text{a.e. } t\in[0,T],\ \text{a.s.}\end{aligned}$$

where $\delta>0$. In this case, we still have (2.12), (2.14) and (2.15). Further, by applying the similar arguments as in Case 1, it follows that

$$\begin{aligned}&\text{Left side of (2.15)}\geqslant-\varepsilon E|x_T|^2-C_\varepsilon E\big[|x_0|^2+\big|h(0)+h_T\big|^2\big],\\&\text{Right side of (2.15)}\leqslant E\int_0^T-\frac{\delta}{2}\big(|y_t|^2+|z_t|^2\big)\,dt+C_\varepsilon E\int_0^T\big[\big|b(t,0,u_t)\big|^2+\big|\sigma(t,0,u_t)\big|^2\\&\qquad+\big|f(t,0,u_t)\big|^2\big]\,dt+\varepsilon E\int_0^T|x_t|^2\,dt,\end{aligned}$$

with the constant $C_\varepsilon>0$ depending on C, δ, $|B_T|$ and $|D_T|$.

Combining the above two inequalities (note (2.12)) and choosing suitable $\varepsilon>0$ yield

$$E\int_0^T\big(|y_t|^2+|z_t|^2\big)\,dt\leqslant CE\bigg\{\big[|x_0|^2+\big|h(0)+h_T\big|^2\big]+\int_0^T\big[\big|b(t,0,u_t)\big|^2+\big|\sigma(t,0,u_t)\big|^2+\big|f(t,0,u_t)\big|^2\big]\,dt\bigg\}.$$

Again by (2.12) and (2.14), we can obtain the first estimation. As for the estimate of the difference of solutions, we refer to the above arguments. □

It is remarkable that, in the fully coupled FBSDE (2.1) both z and u appear in the diffusion coefficient of the forward equation, and the regularity of the process z (as a part of state process) seems to be not enough to obtain a second-order expansion with respect to the control u. So a reduction method is adopted to overcome this difficulty. First, we pose the following problem.

Problem $\mathcal{B}$. Minimize $J(x_0, y_0, z, u) = E[g(x_T) + \gamma(y_0)]$ over $(x_0, y_0, z, u) \in \mathcal{R} := R \times R \times \mathcal{H}^2_{\mathcal{F}}(0,T;R) \times \mathcal{U}$ subject to the forward control system

$$\begin{cases} dx_t = b(t, x_t, y_t, z_t, Ex_t, Ey_t, Ez_t, u_t)\,dt + \sigma(t, x_t, y_t, z_t, Ex_t, Ey_t, Ez_t, u_t)\,dW_t, \\ -dy_t = f(t, x_t, y_t, z_t, Ex_t, Ey_t, Ez_t, u_t)\,dt - z_t\,dW_t, \\ x(0) = x_0, \quad y(0) = y_0, \end{cases} \tag{2.20}$$

with an optimal state constraint

$$E\big|y_T - h(x_T)\big|^2 = 0.$$

Clearly, the original problem $\mathcal{A}$ is embedded into problem $\mathcal{B}$, but the reverse is not true in general. So based on the optimal control $(\bar{x}_0, \bar{y}_0, \bar{z}, \bar{u})$ of $\mathcal{B}$, we know that $\bar{u}$ is optimal for $\mathcal{A}$. In the following section, the classical second-order variational technique is used to solve $\mathcal{B}$.

3. Stochastic maximum principle

This section is devoted to the main theorem and its corresponding proof. Now, we assume that $(\bar{x}, \bar{y}, \bar{z}, \bar{u})$ is an optimal 4-tuple of problem $\mathcal{B}$. For simplicity, we denote

$$\begin{cases} \bar{\rho}_j(t) := \rho_j(t, \bar{x}_t, \bar{y}_t, \bar{z}_t, E\bar{x}_t, E\bar{y}_t, E\bar{z}_t, \bar{u}_t), \\ \bar{\rho}_{ij}(t) := \rho_{ij}(t, \bar{x}_t, \bar{y}_t, \bar{z}_t, E\bar{x}_t, E\bar{y}_t, E\bar{z}_t, \bar{u}_t), \\ \bar{B}_X(t) := \begin{pmatrix} \bar{b}_x(t) & \bar{b}_y(t) \\ -\bar{f}_x(t) & -\bar{f}_y(t) \end{pmatrix}, \quad \bar{B}_{\tilde{X}}(t) := \begin{pmatrix} \bar{b}_{\tilde{x}}(t) & \bar{b}_{\tilde{y}}(t) \\ -\bar{f}_{\tilde{x}}(t) & -\bar{f}_{\tilde{y}}(t) \end{pmatrix}, \\ \bar{\Sigma}_X(t) := \begin{pmatrix} \bar{\sigma}_x(t) & \bar{\sigma}_y(t) \\ 0 & 0 \end{pmatrix}, \quad \bar{\Sigma}_{\tilde{X}}(t) := \begin{pmatrix} \bar{\sigma}_{\tilde{x}}(t) & \bar{\sigma}_{\tilde{y}}(t) \\ 0 & 0 \end{pmatrix}, \\ \rho = b, \sigma, f; \quad i, j = x, y, z, \tilde{x}, \tilde{y}, \tilde{z}. \end{cases}$$

Theorem 3.1. *Let (A_1)–(A_4) or (A_1)–$(A_4)'$ hold, and $(\bar{x}, \bar{y}, \bar{z}, \bar{u})$ be an optimal 4-tuple of problem $\mathcal{B}$. Then there exists an adapted solution $(\xi, \eta, \zeta) \in M^2[0,T]$ of the following FBSDE:*

$$\begin{cases} d\xi_t = \big\{\bar{f}_y(t)\xi_t - \bar{b}_y(t)\eta_t - \bar{\sigma}_y(t)\zeta_t + E\big[\bar{f}_{\tilde{y}}(t)\xi_t - \bar{b}_{\tilde{y}}(t)\eta_t - \bar{\sigma}_{\tilde{y}}(t)\zeta_t\big]\big\}\,dt \\ \qquad\quad + \big\{\bar{f}_z(t)\xi_t - \bar{b}_z(t)\eta_t - \bar{\sigma}_z(t)\zeta_t + E\big[\bar{f}_{\tilde{z}}(t)\xi_t - \bar{b}_{\tilde{z}}(t)\eta_t - \bar{\sigma}_{\tilde{z}}(t)\zeta_t\big]\big\}\,dW_t, \\ -d\eta_t = \big\{-\bar{f}_x(t)\xi_t + \bar{b}_x(t)\eta_t + \bar{\sigma}_x(t)\zeta_t + E\big[-\bar{f}_{\tilde{x}}(t)\xi_t + \bar{b}_{\tilde{x}}(t)\eta_t + \bar{\sigma}_{\tilde{x}}(t)\zeta_t\big]\big\}\,dt - \zeta_t\,dW_t, \\ \xi_0 = \bar{\gamma}_{y_0}, \quad \eta_T = -\bar{g}_{x_T} - \bar{h}_{x_T}\xi_T, \end{cases}$$

and let $(P, Q) \in \mathcal{S}^2_{\mathcal{F}}(0,T;R^{2\times 2}) \times \mathcal{H}^2_{\mathcal{F}}(0,T;R^{2\times 2})$ be the unique adapted solution of the following BSDE:

$$\begin{cases} dP_t = -\big[\bar{B}_X(t)^\top P_t + P_t\bar{B}_X(t) + \bar{\Sigma}_X(t)^\top P_t\bar{\Sigma}_X(t) + \bar{\Sigma}_X(t)^\top Q_t + Q_t\bar{\Sigma}_X(t) + \bar{H}_{XX}(t)\big]\,dt + Q_t\,dW_t, \\ P_T = -\begin{pmatrix} \bar{g}_{x_Tx_T} + \xi_T\bar{h}_{x_Tx_T} & 0 \\ 0 & 0 \end{pmatrix}, \end{cases}$$

where $\bar{H}_{XX}(t)$ *is defined by*

$$\bar{H}_{XX}(t) = \begin{pmatrix} -\xi_t \bar{f}_{xx}(t) + \eta_t \bar{b}_{xx}(t) + \zeta_t \bar{\sigma}_{xx}(t) & -\xi_t \bar{f}_{xy}(t) + \eta_t \bar{b}_{xy}(t) + \zeta_t \bar{\sigma}_{xy}(t) \\ -\xi_t \bar{f}_{yx}(t) + \eta_t \bar{b}_{yx}(t) + \zeta_t \bar{\sigma}_{yx}(t) & -\xi_t \bar{f}_{yy}(t) + \eta_t \bar{b}_{yy}(t) + \zeta_t \bar{\sigma}_{yy}(t) \end{pmatrix}.$$

Then the following inequalities hold:

$$P_0 \leqslant \begin{pmatrix} 0 & 0 \\ 0 & \bar{\gamma}_{y_0 y_0} \end{pmatrix}, \tag{3.1}$$

and

$$\begin{aligned} &H(t, \bar{x}_t, \bar{y}_t, \bar{z}_t, u, \xi_t, \eta_t, \zeta_t) - H(t, \bar{x}_t, \bar{y}_t, \bar{z}_t, \bar{u}_t, \xi_t, \eta_t, \zeta_t) + \frac{1}{2} P_1(t) \big[\sigma(t, \bar{\Gamma}_t, u) - \sigma(t, \bar{\Gamma}_t, \bar{u}_t)\big]^2 \\ &\quad \leqslant 0, \quad \forall u \in \mathcal{U}, \ \text{a.e. } t \in [0, T], \ P\text{-a.s.} \end{aligned} \tag{3.2}$$

where $\bar{\Gamma}_t = (\bar{x}_t, \bar{y}_t, \bar{z}_t, E\bar{x}_t, E\bar{y}_t, E\bar{z}_t)$, *with* $P = \left(\begin{smallmatrix} P_1 & P_2 \\ P_2 & P_3 \end{smallmatrix}\right)$ *and* $H(t, x, y, z, u, \xi, \eta, \zeta)$ *being the Hamiltonian defined by*

$$H(t, x, y, z, u, \xi, \eta, \zeta) = -\xi f(t, \Gamma, u) + \eta b(t, \Gamma, u) + \zeta \sigma(t, \Gamma, u).$$

Proof. Since the proof is lengthy, we divide it into a few steps to make the process clear.

Step 1: Introduction of the penalty functional and application of Ekeland's variational principle. Let $(\bar{x}_0, \bar{y}_0, \bar{z}, \bar{u})$ be an optimal control of problem $\mathcal{B}$, with the corresponding optimal state process $(\bar{x}, \bar{y})$. Without loss of generality, we assume that

$$J(\bar{x}_0, \bar{y}_0, \bar{z}, \bar{u}) = 0.$$

For any $\delta > 0$ and $(x_0, y_0, z, u) \in \mathcal{R}$, we define the penalty functional

$$J^\delta(x_0, y_0, z, u) = \big\{\big[\big(J(x_0, y_0, z, u) + \delta\big)^+\big]^2 + E\big|y_T - h(x_T)\big|^2\big\}^{\frac{1}{2}}. \tag{3.3}$$

To apply Ekeland's variational principle, we endow the set $\mathcal{U}$ with the distance

$$d(u, \tilde{u}) = \tilde{P}\big\{(t, w) \in [0, T] \times \Omega \ \big| \ u_t(w) \neq \tilde{u}_t(w)\big\}, \quad \forall u, \tilde{u} \in \mathcal{U},$$

where $\tilde{P}$ is the product measure of the Lebesgue measure and P. Then $\mathcal{H}^2_{\mathcal{F}}(0, T; R) \times \mathcal{U}$ is a complete metric space under the metric

$$\tilde{d}\big((z, u), (\tilde{z}, \tilde{u})\big) = \big[\|z - \tilde{z}\|_2^2 + d(u, \tilde{u})^2\big]^{\frac{1}{2}}, \quad \forall (z, u), (\tilde{z}, \tilde{u}) \in \mathcal{H}^2_{\mathcal{F}}(0, T; R) \times \mathcal{U},$$

where $\|z\|_2^2 = E \int_0^T |z_t|^2 \, dt$. Indeed, let $\{(z_n, u_n)\}$ be a Cauchy sequence in $\mathcal{H}^2_{\mathcal{F}}(0, T; R) \times \mathcal{U}$ under $\tilde{d}$. Then for any $k \geqslant 2$,

$$\begin{aligned} \bar{P}\big\{(t, w) \in [0, T] \times \Omega \ \big| \ |z_n - z_m| > 2^{-k}\big\} &\leqslant 2^k \Bigg\{ \int_0^T \int_{\Omega(|z_n - z_m| > 2^{-k})} |z_n - z_m|^2 \, dP \, dt \Bigg\}^{\frac{1}{2}} \\ &\leqslant 2^k \|z_n - z_m\|_2, \end{aligned}$$

where $\bar{P}$ denotes the product measure of the Lebesgue measure and P. Hence, $\{z_n\}$ is a Cauchy sequence convergent in measure $\bar{P}$. By the Riesz Theorem, there exists a subsequence $\{z_{n_k}\}$, which converges almost everywhere. We denote the limit by $\bar{z}$. Then for any $n \geqslant 1$,

$$\lim_{k\to\infty} |z_{n_k} - z_n| = |\bar{z} - z_n|.$$

On the other hand, since $\{z_n\}$ is a Cauchy sequence in the norm $\|\cdot\|_2$, for any $\varepsilon > 0$, there is an N, such that

$$\|z_n - z_m\|_2 < \varepsilon, \quad n, m \geqslant N.$$

For any fixed n, we set $m = n_k$, by Fatou's lemma,

$$\begin{aligned}
\varepsilon &\geqslant \lim_{k\to\infty} \|z_{n_k} - z_n\|_2 \\
&= \lim_{k\to\infty} \left\{ \int_0^T \int_\Omega |z_{n_k} - z_n|^2 \, dP \, dt \right\}^{\frac{1}{2}} \\
&= \left\{ \lim_{k\to\infty} \int_0^T \int_\Omega |z_{n_k} - z_n|^2 \, dP \, dt \right\}^{\frac{1}{2}} \\
&\geqslant \left\{ \int_0^T \int_\Omega |z_n - \bar{z}|^2 \, dP \, dt \right\}^{\frac{1}{2}}.
\end{aligned}$$

Thus, $z_n - \bar{z} \in \mathcal{H}^2_{\mathcal{F}}(0, T; R)$, which suggests $\bar{z} = (\bar{z} - z_n) + z_n \in \mathcal{H}^2_{\mathcal{F}}(0, T; R)$. Besides,

$$\|\bar{z} - z_n\|_2 \leqslant \varepsilon, \quad \forall n \geqslant N.$$

This implies that $\{z_n\}$ converges to $\bar{z}$ in the norm $\|\cdot\|_2$ in $\mathcal{H}^2_{\mathcal{F}}(0, T; R)$. So far, we have proved the completeness of $\mathcal{H}^2_{\mathcal{F}}(0, T; R)$. As for the completeness of $\mathcal{U}$, the argument is the same as Lemma 6.4 in Yong and Zhou [23], we omit the details here, but we still have $d(u_n, \bar{u}) \to 0$ in $\mathcal{U}$ as $n \to \infty$.

Hence

$$\tilde{d}\big((z_n, u_n), (\bar{z}, \bar{u})\big) = \big[\|z_n - \bar{z}\|_2^2 + d(u_n, \bar{u})^2\big]^{\frac{1}{2}} \to 0, \quad n \to \infty.$$

This yields the completeness of $\mathcal{H}^2_{\mathcal{F}}(0, T; R) \times \mathcal{U}$.

Moreover, $J^\delta(x_0, y_0, z, u)$ is continuous from $\mathcal{R}$ into R. Obviously,

$$\begin{cases} J^\delta(x_0, y_0, z, u) > 0, \\ J^\delta(\bar{x}_0, \bar{y}_0, \bar{z}, \bar{u}) = \delta \leqslant \inf\limits_{(x_0, y_0, z, u)\in\mathcal{R}} J^\delta(x_0, y_0, z, u) + \delta. \end{cases}$$

By Ekeland's variational principle [5], there exists a 4-tuple $(x_0^\delta, y_0^\delta, z^\delta, u^\delta) \in \mathcal{R}$ such that

$$\begin{cases} J^\delta\big(x_0^\delta, y_0^\delta, z^\delta, u^\delta\big) \leqslant J^\delta(\bar{x}_0, \bar{y}_0, \bar{z}, \bar{u}) = \delta, \\ \big|x_0^\delta - \bar{x}_0\big|^2 + \big|y_0^\delta - \bar{y}_0\big|^2 + \big\|z^\delta - \bar{z}\big\|_2^2 + d\big(u^\delta, \bar{u}\big)^2 \leqslant \delta, \\ -\sqrt{\delta}\big[\big|x_0^\delta - x_0\big|^2 + \big|y_0^\delta - y_0\big|^2 + \big\|z^\delta - z\big\|_2^2 + d\big(u^\delta, u\big)^2\big]^{\frac{1}{2}} \\ \quad \leqslant J^\delta(x_0, y_0, z, u) - J^\delta\big(x_0^\delta, y_0^\delta, z^\delta, u^\delta\big), \quad \forall (x_0, y_0, z, u) \in \mathcal{R}. \end{cases} \tag{3.4}$$

Therefore, $(x_0^\delta, y_0^\delta, z^\delta, u^\delta)$ is optimal for the system (2.20) with the new cost functional

$$J^\delta(x_0, y_0, z, u) + \sqrt{\delta}\big[\big|x_0^\delta - x_0\big|^2 + \big|y_0^\delta - y_0\big|^2 + \big\|z^\delta - z\big\|_2^2 + d\big(u^\delta, u\big)^2\big]^{\frac{1}{2}}.$$

Up to this point, we have transformed the original problem with endpoint constraints to the penalized optimal control problem with no endpoint constraints, and the optimal 4-tuple $(x_0^\delta, y_0^\delta, z^\delta, u^\delta)$ approaches $(\bar{x}_0, \bar{y}_0, \bar{z}, \bar{u})$ as $\delta \to 0$.

Step 2: Nontriviality of the multiplier. Let $\Theta^\delta := (x^\delta, y^\delta, z^\delta)$ be the unique adapted solution of the following FBSDE:

$$\begin{cases} dx_t^\delta = b\big(t, x_t^\delta, y_t^\delta, z_t^\delta, Ex_t^\delta, Ey_t^\delta, Ez_t^\delta, u_t^\delta\big)\,dt + \sigma\big(t, x_t^\delta, y_t^\delta, z_t^\delta, Ex_t^\delta, Ey_t^\delta, Ez_t^\delta, u_t^\delta\big)\,dW_t, \\ -dy_t^\delta = f\big(t, x_t^\delta, y_t^\delta, z_t^\delta, Ex_t^\delta, Ey_t^\delta, Ez_t^\delta, u_t^\delta\big)\,dt - z_t^\delta\,dW_t, \\ x^\delta(0) = x_0^\delta, \quad y_T^\delta = h\big(x_T^\delta\big) + h_T^\delta, \end{cases}$$

with

$$h_T^\delta = y_T^\delta - h\big(x_T^\delta\big).$$

Make some small disturbance to the above initial-terminal value to get

$$\begin{cases} dx_t^{\delta,\varepsilon} = b\big(t, x_t^{\delta,\varepsilon}, y_t^{\delta,\varepsilon}, z_t^{\delta,\varepsilon}, Ex_t^{\delta,\varepsilon}, Ey_t^{\delta,\varepsilon}, Ez_t^{\delta,\varepsilon}, u_t^\delta\big)\,dt + \sigma\big(t, x_t^{\delta,\varepsilon}, y_t^{\delta,\varepsilon}, z_t^{\delta,\varepsilon}, Ex_t^{\delta,\varepsilon}, Ey_t^{\delta,\varepsilon}, Ez_t^{\delta,\varepsilon}, u_t^\delta\big)\,dW_t, \\ -dy_t^{\delta,\varepsilon} = f\big(t, x_t^{\delta,\varepsilon}, y_t^{\delta,\varepsilon}, z_t^{\delta,\varepsilon}, Ex_t^{\delta,\varepsilon}, Ey_t^{\delta,\varepsilon}, Ez_t^{\delta,\varepsilon}, u_t^\delta\big)\,dt - z_t^{\delta,\varepsilon}\,dW_t, \\ x^{\delta,\varepsilon}(0) = x_0^\delta + \varepsilon\varsigma_0, \quad y_T^{\delta,\varepsilon} = h\big(x_T^{\delta,\varepsilon}\big) + h_T^\delta + \varepsilon\vartheta_T, \end{cases}$$

with $(\varsigma_0, \vartheta_T) \in R \times L^2_{\mathcal{F}}(\Omega; R)$ satisfying $|\varsigma_0|^2 + E|\vartheta_T|^2 \leqslant 1$. By virtue of Lemma 2.4,

$$\lim_{\varepsilon\to 0}\big\|\Theta^{\delta,\varepsilon} - \Theta^\delta\big\|^2_{M^2[0,T]} = 0,$$

which implies

$$\lim_{\varepsilon\to 0} E\left\{\big|x^{\delta,\varepsilon}(0) - x^\delta(0)\big|^2 + \big|y_0^{\delta,\varepsilon} - y_0^\delta\big|^2 + \int_0^T \big|z_t^{\delta,\varepsilon} - z_t^\delta\big|^2\,dt\right\} = 0.$$

By using the same argument as in Yong [22], we obtain

$$\begin{aligned} -C\sqrt{\delta}\varepsilon &\leqslant J^\delta\big(x_0^{\delta,\varepsilon}, y_0^{\delta,\varepsilon}, z^{\delta,\varepsilon}, u^\delta\big) - J^\delta\big(x_0^\delta, y_0^\delta, z^\delta, u^\delta\big) \\ &= I_0^{\delta,\varepsilon}\big[J\big(x_0^{\delta,\varepsilon}, y_0^{\delta,\varepsilon}, z^{\delta,\varepsilon}, u^\delta\big) - J\big(x_0^\delta, y_0^\delta, z^\delta, u^\delta\big)\big] + E\big[\bar{I}_T^{\delta,\varepsilon}\big(y_T^{\delta,\varepsilon} - h\big(x_T^{\delta,\varepsilon}\big) - h_T^\delta\big)\big] \\ &= \big(I_0^\delta + o(1)\big)\big[J\big(x_0^{\delta,\varepsilon}, y_0^{\delta,\varepsilon}, z^{\delta,\varepsilon}, u^\delta\big) - J\big(x_0^\delta, y_0^\delta, z^\delta, u^\delta\big)\big] + \varepsilon E\big[\big(\bar{I}_T^\delta + o(1)\big)\vartheta_T\big], \end{aligned}$$

where $o(1)$ stands for certain scalars that go to 0 as $\varepsilon \to 0$,

$$\begin{cases} I_0^{\delta,\varepsilon} = \dfrac{2\int_0^1[\beta J(x_0^{\delta,\varepsilon}, y_0^{\delta,\varepsilon}, z^{\delta,\varepsilon}, u^\delta) + (1-\beta)J(x_0^\delta, y_0^\delta, z^\delta, u^\delta) + \delta]^+\,d\beta}{J^\delta(x_0^{\delta,\varepsilon}, y_0^{\delta,\varepsilon}, z^{\delta,\varepsilon}, u^\delta) + J^\delta(x_0^\delta, y_0^\delta, z^\delta, u^\delta)}, \\ \bar{I}_T^{\delta,\varepsilon} = \dfrac{y_T^{\delta,\varepsilon} - h(x_T^{\delta,\varepsilon}) + y_T^\delta - h(x_T^\delta)}{J^\delta(x_0^{\delta,\varepsilon}, y_0^{\delta,\varepsilon}, z^{\delta,\varepsilon}, u^\delta) + J^\delta(x_0^\delta, y_0^\delta, z^\delta, u^\delta)}, \end{cases}$$

and

$$\begin{cases} I_0^\delta = \dfrac{(J(x_0^\delta, y_0^\delta, z^\delta, u^\delta) + \delta)^+}{J^\delta(x_0^\delta, y_0^\delta, z^\delta, u^\delta)}, \\ \bar{I}_T^\delta = \dfrac{y_T^\delta - h(x_T^\delta)}{J^\delta(x_0^\delta, y_0^\delta, z^\delta, u^\delta)}. \end{cases}$$

We point out that $(I_0^\delta, \bar{I}_T^\delta)$ is independent of $(\varsigma_0, \vartheta_T)$ and

$$I_0^\delta \geqslant 0, \qquad \left|I_0^\delta\right|^2 + E\left|\bar{I}_T^\delta\right|^2 = 1.$$

Thus, there exists a subsequence still denoted by $(I_0^\delta, \bar{I}_T^\delta)$ convergent, i.e.

$$\lim_{\delta\to 0}\left(I_0^\delta, \bar{I}_T^\delta\right) = (I_0, \bar{I}_T), \quad \text{with } |I_0|^2 + E|\bar{I}_T|^2 = 1.$$

We claim that $I_0 \neq 0$. The detailed illustration of this point refers to Shi [15]. Here $(I_0, \bar{I}_T)$ is called the Lagrange multiplier of the corresponding optimal 4-tuple $(\bar{x}_0, \bar{y}_0, \bar{z}, \bar{u})$.

Step 3: Variations. For the sake of convenience, we denote

$$\begin{cases} X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad v = \begin{pmatrix} Ez \\ z \\ u \end{pmatrix}, \quad X_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad X_T = \begin{pmatrix} x_T \\ y_T \end{pmatrix}, \\ B(t, X, EX, v) = \begin{pmatrix} b(t, x, y, z, Ex, Ey, Ez, u) \\ -f(t, x, y, z, Ex, Ey, Ez, u) \end{pmatrix}, \\ \Sigma(t, X, EX, v) = \begin{pmatrix} \sigma(t, x, y, z, Ex, Ey, Ez, u) \\ z \end{pmatrix}, \\ G(X_T) = y_T - h(x_T), \quad F(X_0, X_T) = g(x_T) + \gamma(y_0), \\ F_j = F_j(X_0, X_T), \quad F_{ij} = F_{ij}(X_0, X_T), \quad i, j = X_0, X_T. \end{cases}$$

Consequently,

$$J(x_0, y_0, z, u) = J(X_0, v), \qquad J^\delta(x_0, y_0, z, u) = J^\delta(X_0, v).$$

We denote the gradient DF and the Hessian D^2F as follows:

$$DF(X_0, X_T) = (F_{X_0}, F_{X_T}), \qquad D^2F(X_0, X_T) = \begin{pmatrix} F_{X_0X_0} & F_{X_0X_T} \\ F_{X_TX_0} & F_{X_TX_T} \end{pmatrix}.$$

Clearly,

$$\begin{cases} F_{X_0} = (0, \gamma_{y_0}), \quad F_{X_T} = (g_{x_T}, 0), \quad F_{X_0X_0} = \begin{pmatrix} 0 & 0 \\ 0 & \gamma_{y_0y_0} \end{pmatrix}, \\ F_{X_0X_T} = F_{X_TX_0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad F_{X_TX_T} = \begin{pmatrix} g_{x_Tx_T} & 0 \\ 0 & 0 \end{pmatrix}. \end{cases}$$

For G, we have

$$DG(X_T) = G_{X_T}(X_T), \qquad D^2G(X_T) = G_{X_TX_T}(X_T).$$

To make the above more specific, let us take any $\hat{I}_T \in L^2_{\mathcal{F}}(\Omega; R)$. Then

$$\langle G(X_T), \hat{I}_T\rangle = \big(y_T - h(x_T)\big)\hat{I}_T.$$

Thus

$$\begin{cases} \big[DG(X_T)\hat{I}_T\big] \equiv D\big[\langle G(X_T), \hat{I}_T\rangle\big] = \langle G(X_T), \hat{I}_T\rangle_{X_T} = G_{X_T}(X_T)\hat{I}_T, \\ \big[D^2G(X_T)\hat{I}_T\big] = D^2\big[\langle G(X_T), \hat{I}_T\rangle\big] = \langle G(X_T), \hat{I}_T\rangle_{X_T X_T} = G_{X_T X_T}(X_T)\hat{I}_T, \end{cases}$$

with

$$\begin{cases} G_{X_T}(X_T)\hat{I}_T = (-\hat{I}_T h_{x_T}, \hat{I}_T), \\ G_{X_T X_T}(X_T)\hat{I}_T = \begin{pmatrix} -\hat{I}_T h_{x_T x_T} & 0 \\ 0 & 0 \end{pmatrix}. \end{cases}$$

Fixing $X_0 \in R^2$ and $v \in \mathcal{H}^2_{\mathcal{F}}(0,T;R) \times \mathcal{H}^2_{\mathcal{F}}(0,T;R) \times \mathcal{U}$, for any $\varepsilon \in (0,1)$, set

$$X_0^{\delta,\varepsilon} = X_0^\delta + \sqrt{\varepsilon} X_0,$$

and

$$v_t^{\delta,\varepsilon} = \begin{cases} v_t^\delta, & t \in [0,T]\backslash S_\varepsilon, \\ v_t, & t \in S_\varepsilon, \end{cases}$$

for some $S_\varepsilon \subseteq [0,T]$ with $|S_\varepsilon| = \varepsilon T$, where $|\cdot|$ denotes the Lebesgue measure. Let $X^{\delta,\varepsilon}$ be the state process corresponding to $(X_0^{\delta,\varepsilon}, v^{\delta,\varepsilon})$, and $X_1^{\delta,\varepsilon}$, $X_2^{\delta,\varepsilon}$ be the solutions of the following SDEs:

$$\begin{cases} dX_1^{\delta,\varepsilon}(t) = \big\{B_X^\delta(t) X_1^{\delta,\varepsilon}(t) + B_{\tilde{X}}^\delta(t) E\big[X_1^{\delta,\varepsilon}(t)\big]\big\}\,dt \\ \qquad\qquad + \big\{\Sigma_X^\delta(t) X_1^{\delta,\varepsilon}(t) + \Sigma_{\tilde{X}}^\delta(t) E\big[X_1^{\delta,\varepsilon}(t)\big] + \Delta\Sigma^\delta(t) I_{S_\varepsilon}(t)\big\}\,dW_t, \\ X_1^{\delta,\varepsilon}(0) = \sqrt{\varepsilon}\, X_0, \end{cases}$$

$$\begin{cases} dX_2^{\delta,\varepsilon}(t) = \big\{B_X^\delta(t) X_2^{\delta,\varepsilon}(t) + B_{\tilde{X}}^\delta(t) E\big[X_2^{\delta,\varepsilon}(t)\big] + \big[\Delta B^\delta(t) \\ \qquad\qquad + \Delta B_X^\delta(t) X_1^{\delta,\varepsilon}(t)\big] I_{S_\varepsilon}(t) + \frac{1}{2} B_{XX}^\delta(t) X_1^{\delta,\varepsilon}(t)^2\big\}\,dt \\ \qquad\qquad + \big\{\Sigma_X^\delta(t) X_2^{\delta,\varepsilon}(t) + \Sigma_{\tilde{X}}^\delta(t) E\big[X_2^{\delta,\varepsilon}(t)\big] + \Delta\Sigma_X^\delta(t) X_1^{\delta,\varepsilon}(t) I_{S_\varepsilon}(t) \\ \qquad\qquad + \frac{1}{2}\Sigma_{XX}^\delta(t) X_1^{\delta,\varepsilon}(t)^2\big\}\,dW_t, \\ X_2^{\delta,\varepsilon}(0) = 0, \end{cases}$$

where

$$\begin{cases} \phi_X^\delta(t) = \phi_X\big(t, X_t^\delta, EX_t^\delta, v_t^\delta\big), \\ \Delta\phi^\delta(t) = \phi\big(t, X_t^\delta, EX_t^\delta, v_t\big) - \phi\big(t, X_t^\delta, EX_t^\delta, v_t^\delta\big), \\ \Delta\phi_X^\delta(t) = \phi_X\big(t, X_t^\delta, EX_t^\delta, v_t\big) - \phi_X\big(t, X_t^\delta, EX_t^\delta, v_t^\delta\big), \\ \phi_{XX}^\delta(t) X^2 = \begin{pmatrix} \langle \phi_{XX}^{1,\delta}(t) X, X\rangle \\ \langle \phi_{XX}^{2,\delta}(t) X, X\rangle \end{pmatrix}, \\ \phi_{XX}^{i,\delta}(t) = \phi_{XX}^i\big(t, X_t^\delta, EX_t^\delta, v_t^\delta\big), \quad \phi = B, \Sigma, \quad i = 1,2. \end{cases}$$

Then, we have the following estimates, whose proofs are similar as those given in Buckdahn, Djehiche and Li [2].

$$\begin{cases} E\Big[\sup\limits_{t\in[0,T]}\big|X_1^{\delta,\varepsilon}(t)\big|^{2k}\Big]+E\Big[\sup\limits_{t\in[0,T]}\big|X_t^{\delta,\varepsilon}-X_t^{\delta}\big|^{2k}\Big]\leqslant C\varepsilon^k, \\ E\Big[\sup\limits_{t\in[0,T]}\big|X_t^{\delta,\varepsilon}-X_t^{\delta}-X_1^{\delta,\varepsilon}(t)\big|^{2k}\Big]+E\Big[\sup\limits_{t\in[0,T]}\big|X_2^{\delta,\varepsilon}(t)\big|^{2k}\Big]\leqslant C\varepsilon^{2k}, \\ E\Big[\sup\limits_{t\in[0,T]}\big|X_t^{\delta,\varepsilon}-X_t^{\delta}-X_1^{\delta,\varepsilon}(t)-X_2^{\delta,\varepsilon}(t)\big|^{2k}\Big]\leqslant \varepsilon^{2k}\rho_k(\varepsilon), \end{cases} \tag{3.5}$$

where $\rho_k:(0,\infty)\to(0,\infty)$ such that $\lim_{\varepsilon\to 0}\rho_k(\varepsilon)=0$.

From (3.4), we obtain

$$\begin{aligned} -\sqrt{\delta}\big(\sqrt{\varepsilon}\,|X_0|+\varepsilon T\big) &\leqslant J^\delta\big(X_0^{\delta,\varepsilon},v^{\delta,\varepsilon}\big)-J^\delta\big(X_0^\delta,v^\delta\big) \\ &= I_0^{\delta,\varepsilon}\big[J\big(X_0^{\delta,\varepsilon},v^{\delta,\varepsilon}\big)-J\big(X_0^\delta,v^\delta\big)\big]+E\big\langle \bar I_T^{\delta,\varepsilon},G\big(X_T^{\delta,\varepsilon}\big)-G\big(X_T^\delta\big)\big\rangle \\ &= \big(I_0^\delta+o(1)\big)\big[J\big(X_0^{\delta,\varepsilon},v^{\delta,\varepsilon}\big)-J\big(X_0^\delta,v^\delta\big)\big]+E\big\langle \bar I_T^{\delta}+o(1),G\big(X_T^{\delta,\varepsilon}\big)-G\big(X_T^\delta\big)\big\rangle, \end{aligned}$$

with

$$\begin{cases} I_0^{\delta,\varepsilon} = \dfrac{2\int_0^1[\beta J(X_0^{\delta,\varepsilon},v^{\delta,\varepsilon})+(1-\beta)J(X_0^\delta,v^\delta)+\delta]^+\,d\beta}{J^\delta(X_0^{\delta,\varepsilon},v^{\delta,\varepsilon})+J^\delta(X_0^\delta,v^\delta)}, \\ \bar I_T^{\delta,\varepsilon} = \dfrac{G(X_T^{\delta,\varepsilon})+G(X_T^\delta)}{J^\delta(X_0^{\delta,\varepsilon},v^{\delta,\varepsilon})+J^\delta(X_0^\delta,v^\delta)}, \end{cases}$$

and

$$\begin{cases} I_0^\delta = \dfrac{(J(X_0^\delta,v^\delta)+\delta)^+}{J^\delta(X_0^\delta,v^\delta)}, \\ \bar I_T^\delta = \dfrac{y_T^\delta-h(x_T^\delta)}{J^\delta(X_0^\delta,v^\delta)}. \end{cases}$$

In the following, we shall prove the conclusion that $(I_0^\delta,\bar I_T^\delta)\to(I_0,\bar I_T)$ and $I_0\neq 0$. By using Taylor's expansion,

$$\begin{aligned} &J\big(X_0^{\delta,\varepsilon},v^{\delta,\varepsilon}\big)-J\big(X_0^\delta,v^\delta\big) \\ &\quad= E\big[F\big(X_0^{\delta,\varepsilon},X_T^{\delta,\varepsilon}\big)-F\big(X_0^\delta,X_T^\delta\big)\big] \\ &\quad= E\bigg\{DF^\delta\begin{pmatrix} X_0^{\delta,\varepsilon}-X_0^\delta \\ X_T^{\delta,\varepsilon}-X_T^\delta\end{pmatrix}+\frac12\bigg\langle D^2F^\delta\begin{pmatrix} X_0^{\delta,\varepsilon}-X_0^\delta \\ X_T^{\delta,\varepsilon}-X_T^\delta\end{pmatrix},\begin{pmatrix} X_0^{\delta,\varepsilon}-X_0^\delta \\ X_T^{\delta,\varepsilon}-X_T^\delta\end{pmatrix}\bigg\rangle \\ &\qquad+\bigg\langle D^2F^{\delta,\varepsilon}\begin{pmatrix} X_0^{\delta,\varepsilon}-X_0^\delta \\ X_T^{\delta,\varepsilon}-X_T^\delta\end{pmatrix},\begin{pmatrix} X_0^{\delta,\varepsilon}-X_0^\delta \\ X_T^{\delta,\varepsilon}-X_T^\delta\end{pmatrix}\bigg\rangle\bigg\} \\ &\quad= E\Big\{\big\langle F_{X_0}^\delta,X_0^{\delta,\varepsilon}-X_0^\delta\big\rangle+\big\langle F_{X_T}^\delta,X_T^{\delta,\varepsilon}-X_T^\delta\big\rangle \\ &\qquad+\frac12\Big[\big\langle F_{X_0X_0}^\delta\big(X_0^{\delta,\varepsilon}-X_0^\delta\big),X_0^{\delta,\varepsilon}-X_0^\delta\big\rangle+\big\langle F_{X_TX_T}^\delta\big(X_T^{\delta,\varepsilon}-X_T^\delta\big),X_T^{\delta,\varepsilon}-X_T^\delta\big\rangle \\ &\qquad+\bigg\langle D^2F^{\delta,\varepsilon}\begin{pmatrix} X_0^{\delta,\varepsilon}-X_0^\delta \\ X_T^{\delta,\varepsilon}-X_T^\delta\end{pmatrix},\begin{pmatrix} X_0^{\delta,\varepsilon}-X_0^\delta \\ X_T^{\delta,\varepsilon}-X_T^\delta\end{pmatrix}\bigg\rangle\Big]\Big\}, \end{aligned}$$

where

$$\begin{cases} DF^\delta = \big(F^\delta_{X_0}, F^\delta_{X_T}\big), \quad D^2F^\delta = \begin{pmatrix} F^\delta_{X_0X_0} & F^\delta_{X_0X_T} \\ F^\delta_{X_TX_0} & F^\delta_{X_TX_T} \end{pmatrix}, \\ D^2F^{\delta,\varepsilon} = \int_0^1 \beta\big[D^2F\big(\beta X_0^\delta + (1-\beta)X_0^{\delta,\varepsilon}, \beta X_T^\delta + (1-\beta)X_T^{\delta,\varepsilon}\big) - D^2F\big(X_0^\delta, X_T^\delta\big)\big]\, d\beta, \\ F_j^\delta = F_j\big(X_0^\delta, X_T^\delta\big), \quad F_{ij}^\delta = F_{ij}\big(X_0^\delta, X_T^\delta\big), \quad i,j = X_0, X_T. \end{cases}$$

Similarly,

$$\begin{aligned} E\big\langle \bar{I}_T^{\delta,\varepsilon}, G\big(X_T^{\delta,\varepsilon}\big) - G\big(X_T^\delta\big)\big\rangle &= E\Big\{\big\langle\big[DG^\delta \bar{I}_T^{\delta,\varepsilon}\big], X_T^{\delta,\varepsilon} - X_T^\delta\big\rangle + \frac{1}{2}\big[\big\langle\big[D^2G^\delta \bar{I}_T^{\delta,\varepsilon}\big]\big(X_T^{\delta,\varepsilon} - X_T^\delta\big), X_T^{\delta,\varepsilon} - X_T^\delta\big\rangle \\ &\quad + \big\langle\big[D^2G^{\delta,\varepsilon}\bar{I}_T^{\delta,\varepsilon}\big]\big(X_T^{\delta,\varepsilon} - X_T^\delta\big), X_T^{\delta,\varepsilon} - X_T^\delta\big\rangle\big]\Big\} \\ &= E\Big\{\big\langle\big[G^\delta_{X_T} \bar{I}_T^{\delta,\varepsilon}\big], X_T^{\delta,\varepsilon} - X_T^\delta\big\rangle + \frac{1}{2}\big[\big\langle\big[G^\delta_{X_TX_T} \bar{I}_T^{\delta,\varepsilon}\big]\big(X_T^{\delta,\varepsilon} - X_T^\delta\big), X_T^{\delta,\varepsilon} - X_T^\delta\big\rangle \\ &\quad + \big\langle\big[D^2G^{\delta,\varepsilon}\bar{I}_T^{\delta,\varepsilon}\big]\big(X_T^{\delta,\varepsilon} - X_T^\delta\big), X_T^{\delta,\varepsilon} - X_T^\delta\big\rangle\big]\Big\}, \end{aligned}$$

with

$$\begin{cases} \big[DG^\delta \bar{I}_T^{\delta,\varepsilon}\big] = DG\big(X_T^\delta\big)\bar{I}_T^{\delta,\varepsilon}, \quad \big[D^2G^\delta \bar{I}_T^{\delta,\varepsilon}\big] = D^2G\big(X_T^\delta\big)\bar{I}_T^{\delta,\varepsilon}, \\ \big[D^2G^{\delta,\varepsilon} \bar{I}_T^{\delta,\varepsilon}\big] = \int_0^1 \beta\big\{\big[D^2G\big(\beta X_T^\delta + (1-\beta)X_T^{\delta,\varepsilon}\big)\bar{I}_T^{\delta,\varepsilon}\big] - \big[D^2G\big(X_T^\delta\big)\bar{I}_T^{\delta,\varepsilon}\big]\big\}\, d\beta, \\ \big[G^\delta_{X_T} \bar{I}_T^{\delta,\varepsilon}\big] = G_{X_T}\big(X_T^\delta\big)\bar{I}_T^{\delta,\varepsilon}, \quad \big[G^\delta_{X_TX_T} \bar{I}_T^{\delta,\varepsilon}\big] = G_{X_TX_T}\big(X_T^\delta\big)\bar{I}_T^{\delta,\varepsilon}. \end{cases}$$

By use of the assumption (A_2) and (3.5),

$$\begin{aligned} &E\left|\left\langle I_0^{\delta,\varepsilon} D^2F^{\delta,\varepsilon}\begin{pmatrix} X_0^{\delta,\varepsilon} - X_0^\delta \\ X_T^{\delta,\varepsilon} - X_T^\delta \end{pmatrix}, \begin{pmatrix} X_0^{\delta,\varepsilon} - X_0^\delta \\ X_T^{\delta,\varepsilon} - X_T^\delta \end{pmatrix}\right\rangle\right| + E\big|\big\langle\big[D^2G^{\delta,\varepsilon}\bar{I}_T^{\delta,\varepsilon}\big]\big(X_T^{\delta,\varepsilon} - X_T^\delta\big), X_T^{\delta,\varepsilon} - X_T^\delta\big\rangle\big| \\ &\quad \leqslant CE\big[\big|X_0^{\delta,\varepsilon} - X_0^\delta\big|^3 + \big|X_T^{\delta,\varepsilon} - X_T^\delta\big|^3\big] \leqslant C\varepsilon^{\frac{3}{2}}. \end{aligned}$$

Consequently,

$$\begin{aligned} -\sqrt{\delta}\big(\sqrt{\varepsilon}\,|X_0| + \varepsilon T\big) &\leqslant I_0^{\delta,\varepsilon}\big[J\big(X_0^{\delta,\varepsilon}, v^{\delta,\varepsilon}\big) - J\big(X_0^\delta, v^\delta\big)\big] + E\big\langle \bar{I}_T^{\delta,\varepsilon}, G\big(X_T^{\delta,\varepsilon}\big) - G\big(X_T^\delta\big)\big\rangle \\ &= E\Big\{\sqrt{\varepsilon}\,\big\langle I_0^{\delta,\varepsilon}\big(F^\delta_{X_0}\big)^\top, X_0\big\rangle + \frac{\varepsilon}{2}\big\langle I_0^{\delta,\varepsilon}F^\delta_{X_0X_0}X_0, X_0\big\rangle \\ &\quad + \big\langle I_0^{\delta,\varepsilon}\big(F^\delta_{X_T}\big)^\top + \big[G^\delta_{X_T}\bar{I}_T^{\delta,\varepsilon}\big], X_1^{\delta,\varepsilon}(T) + X_2^{\delta,\varepsilon}(T)\big\rangle \\ &\quad + \frac{1}{2}\big\langle\big(I_0^{\delta,\varepsilon}F^\delta_{X_TX_T} + \big[G^\delta_{X_TX_T}\bar{I}_T^{\delta,\varepsilon}\big]\big)X_1^{\delta,\varepsilon}(T), X_1^{\delta,\varepsilon}(T)\big\rangle\Big\} + o\big(\varepsilon^{\frac{3}{2}}\big). \end{aligned}$$

Step 4: Duality. Let $(\Phi^{\delta,\varepsilon}, \Psi^{\delta,\varepsilon})$ be the adapted solution of the following BSDE:

$$\begin{cases} d\Phi_t^{\delta,\varepsilon} = -\big\{B_X^\delta(t)^\top\Phi_t^{\delta,\varepsilon} + E\big[B_{\tilde{X}}^\delta(t)^\top\Phi_t^{\delta,\varepsilon}\big] + \Sigma_X^\delta(t)^\top\Psi_t^{\delta,\varepsilon} + E\big[\Sigma_{\tilde{X}}^\delta(t)^\top\Psi_t^{\delta,\varepsilon}\big]\big\}\, dt + \Psi_t^{\delta,\varepsilon}\, dW_t, \\ \Phi_T^{\delta,\varepsilon} = -\big\{I_0^{\delta,\varepsilon}\big(F^\delta_{X_T}\big)^\top + \big[G^\delta_{X_T}\bar{I}_T^{\delta,\varepsilon}\big]\big\}. \end{cases}$$

Then $(\Phi^{\delta,\varepsilon}, \Psi^{\delta,\varepsilon})$ tends to (Φ, Ψ) as $(\delta, \varepsilon) \to (0,0)$, which is the adapted solution of the following BSDE:

$$\begin{cases} d\Phi_t = -\big\{\bar{B}_X(t)^\top \Phi_t + E\big[\bar{B}_{\tilde{X}}(t)^\top \Phi_t\big] + \bar{\Sigma}_X(t)^\top \Psi_t + E\big[\bar{\Sigma}_{\tilde{X}}(t)^\top \Psi_t\big]\big\}\,dt + \Psi_t\,dW_t, \\ \Phi_T = -\big\{I_0(\bar{F}_{X_T})^\top + [\bar{G}_{X_T}\bar{I}_T]\big\}. \end{cases}$$

Applying Itô's formula to $\langle \Phi_t^{\delta,\varepsilon}, X_1^{\delta,\varepsilon}(t) + X_2^{\delta,\varepsilon}(t)\rangle$ fulfills

$$\begin{aligned} &-E\big\langle I_0^{\delta,\varepsilon}\big(F_{X_T}^\delta\big)^\top + \big[G_{X_T}^\delta \bar{I}_T^{\delta,\varepsilon}\big], X_1^{\delta,\varepsilon}(T) + X_2^{\delta,\varepsilon}(T)\big\rangle \\ &\quad = E\Bigg\{\big\langle \Phi_0^{\delta,\varepsilon}, \sqrt{\varepsilon}X_0\big\rangle + \int_0^T \Big\langle \Phi_t^{\delta,\varepsilon}, \big(\triangle B^\delta(t) + \triangle B_X^\delta(t)X_1^{\delta,\varepsilon}(t)\big)I_{S_\varepsilon}(t) + \frac{1}{2}B_{XX}^\delta(t)X_1^{\delta,\varepsilon}(t)^2\Big\rangle \\ &\qquad + \Big\langle \Psi_t^{\delta,\varepsilon}, \big(\triangle\Sigma^\delta(t) + \triangle\Sigma_X^\delta(t)X_1^{\delta,\varepsilon}(t)\big)I_{S_\varepsilon}(t) + \frac{1}{2}\Sigma_{XX}^\delta(t)X_1^{\delta,\varepsilon}(t)^2\Big\rangle\,dt\Bigg\} \\ &\quad = E\Bigg\{\big\langle \Phi_0^{\delta,\varepsilon}, \sqrt{\varepsilon}X_0\big\rangle + \int_0^T \big[\big\langle \Phi_t^{\delta,\varepsilon}, \triangle B^\delta(t)\big\rangle + \big\langle \Psi_t^{\delta,\varepsilon}, \triangle\Sigma^\delta(t)\big\rangle\big]I_{S_\varepsilon}(t) \\ &\qquad + \frac{1}{2}\big\langle H_{XX}^{\delta,\varepsilon}(t)X_1^{\delta,\varepsilon}(t), X_1^{\delta,\varepsilon}(t)\big\rangle\,dt\Bigg\} + o\big(\varepsilon^{\frac{3}{2}}\big), \end{aligned}$$

with $H_{XX}^{\delta,\varepsilon}(t) = H_{XX}(t, X_t^\delta, v_t^\delta, \Phi_t^{\delta,\varepsilon}, \Psi_t^{\delta,\varepsilon})$ being the Hessian of

$$H(t, X, v, \Phi, \Psi) = \big\langle \Phi, B(t, X, EX, v)\big\rangle + \big\langle \Psi, \Sigma(t, X, EX, v)\big\rangle.$$

Hence,

$$\begin{aligned} -\sqrt{\delta}\big(\sqrt{\varepsilon}\,|X_0| + \varepsilon T\big) \leqslant E\Bigg\{&\sqrt{\varepsilon}\,\big\langle I_0^{\delta,\varepsilon}\big(F_{X_0}^\delta\big)^\top - \Phi_0^{\delta,\varepsilon}, X_0\big\rangle + \frac{\varepsilon}{2}\big\langle I_0^{\delta,\varepsilon}F_{X_0X_0}^\delta X_0, X_0\big\rangle \\ &+ \frac{1}{2}\big\langle\big(I_0^{\delta,\varepsilon}F_{X_TX_T}^\delta + \big[G_{X_TX_T}^\delta \bar{I}_T^{\delta,\varepsilon}\big]\big)X_1^{\delta,\varepsilon}(T), X_1^{\delta,\varepsilon}(T)\big\rangle \\ &- \int_0^T \big[\big\langle \Phi_t^{\delta,\varepsilon}, \triangle B^\delta(t)\big\rangle + \big\langle \Psi_t^{\delta,\varepsilon}, \triangle\Sigma^\delta(t)\big\rangle\big]I_{S_\varepsilon}(t) \\ &+ \frac{1}{2}\big\langle H_{XX}^{\delta,\varepsilon}(t)X_1^{\delta,\varepsilon}(t), X_1^{\delta,\varepsilon}(t)\big\rangle\,dt\Bigg\} + o\big(\varepsilon^{\frac{3}{2}}\big). \end{aligned}$$

Note that $Y_t^{\delta,\varepsilon} = X_1^{\delta,\varepsilon}(t)X_1^{\delta,\varepsilon}(t)^\top$ satisfies

$$\begin{cases} dY_t^{\delta,\varepsilon} = \big\{B_X^\delta(t)Y_t^{\delta,\varepsilon} + Y_t^{\delta,\varepsilon}B_X^\delta(t)^\top + \Sigma_X^\delta(t)Y_t^{\delta,\varepsilon}\Sigma_X^\delta(t)^\top \\ \qquad + \Sigma_{\tilde{X}}^\delta(t)\big(EX_1^{\delta,\varepsilon}(t)\big)\big(EX_1^{\delta,\varepsilon}(t)\big)^\top\Sigma_{\tilde{X}}^\delta(t)^\top + \Lambda_1^{\delta,\varepsilon}(t) \\ \qquad + \big[\Sigma_X^\delta(t)X_1^{\delta,\varepsilon}(t)\triangle\Sigma^\delta(t)^\top + \triangle\Sigma^\delta(t)X_1^{\delta,\varepsilon}(t)^\top\triangle\Sigma^\delta(t)^\top \\ \qquad + \triangle\Sigma^\delta(t)\triangle\Sigma^\delta(t)^\top + \Lambda_2^{\delta,\varepsilon}(t)\big]I_{S_\varepsilon}(t)\big\}\,dt \\ \qquad + \big\{\Sigma_X^\delta(t)Y_t^{\delta,\varepsilon} + Y_t^{\delta,\varepsilon}\Sigma_X^\delta(t)^\top + \Lambda_3^{\delta,\varepsilon}(t)\big\}\,dW_t, \\ Y_0^{\delta,\varepsilon} = \varepsilon X_0X_0^\top, \end{cases}$$

where

$$\begin{cases} \Lambda_1^{\delta,\varepsilon}(t) = X_1^{\delta,\varepsilon}(t)\big(EX_1^{\delta,\varepsilon}(t)\big)^\top B_{\tilde{X}}^\delta(t)^\top + B_{\tilde{X}}^\delta(t)\big(EX_1^{\delta,\varepsilon}(t)\big)X_1^{\delta,\varepsilon}(t)^\top \\ \qquad + \Sigma_X^\delta(t)X_1^{\delta,\varepsilon}(t)\big(EX_1^{\delta,\varepsilon}(t)\big)^\top \Sigma_{\tilde{X}}^\delta(t)^\top + \Sigma_{\tilde{X}}^\delta(t)\big(EX_1^{\delta,\varepsilon}(t)\big)X_1^{\delta,\varepsilon}(t)^\top \Sigma_X^\delta(t)^\top, \\ \Lambda_2^{\delta,\varepsilon}(t) = \Sigma_{\tilde{X}}^\delta(t)\big(EX_1^{\delta,\varepsilon}(t)\big)\triangle\Sigma^\delta(t)^\top + \triangle\Sigma^\delta(t)\big(EX_1^{\delta,\varepsilon}(t)\big)^\top \Sigma_{\tilde{X}}^\delta(t)^\top, \\ \Lambda_3^{\delta,\varepsilon}(t) = X_1^{\delta,\varepsilon}(t)\big(EX_1^{\delta,\varepsilon}(t)\big)^\top \Sigma_{\tilde{X}}^\delta(t)^\top + \Sigma_{\tilde{X}}^\delta(t)\big(EX_1^{\delta,\varepsilon}(t)\big)X_1^{\delta,\varepsilon}(t)^\top \\ \qquad + \big[X_1^{\delta,\varepsilon}(t)\triangle\Sigma^\delta(t)^\top + \triangle\Sigma^\delta(t)X_1^{\delta,\varepsilon}(t)^\top\big] I_{S_\varepsilon}(t). \end{cases}$$

Now, let $(P^{\delta,\varepsilon}, Q^{\delta,\varepsilon})$ be the adapted solution of the following BSDE:

$$\begin{cases} dP_t^{\delta,\varepsilon} = -\big[B_X^\delta(t)^\top P_t^{\delta,\varepsilon} + P_t^{\delta,\varepsilon} B_X^\delta(t) + \Sigma_X^\delta(t)^\top P_t^{\delta,\varepsilon}\Sigma_X^\delta(t) \\ \qquad + \Sigma_X^\delta(t)^\top Q_t^{\delta,\varepsilon} + Q_t^{\delta,\varepsilon}\Sigma_X^\delta(t) + H_{XX}^{\delta,\varepsilon}(t)\big]\,dt + Q_t^{\delta,\varepsilon}\,dW_t, \\ P_T^{\delta,\varepsilon} = -\big\{I_0^{\delta,\varepsilon} F_{X_T X_T}^\delta + \big[G_{X_T X_T}^\delta \bar{I}_T^{\delta,\varepsilon}\big]\big\}. \end{cases}$$

Then, by (3.5) and applying Itô's formula to $\langle P_t^{\delta,\varepsilon}, Y_t^{\delta,\varepsilon}\rangle$ yields

$$\begin{aligned} &-E\big[\big\langle\big(I_0^{\delta,\varepsilon} F_{X_T X_T}^\delta + \big[G_{X_T X_T}^\delta \bar{I}_T^{\delta,\varepsilon}\big]\big)X_1^{\delta,\varepsilon}(T), X_1^{\delta,\varepsilon}(T)\big\rangle + \varepsilon\big\langle P_0^{\delta,\varepsilon}X_0, X_0\big\rangle\big] \\ &\quad = E\int_0^T \big\{\triangle\Sigma^\delta(t)^\top P_t^{\delta,\varepsilon}\triangle\Sigma^\delta(t) I_{S_\varepsilon}(t) - \big\langle H_{XX}^{\delta,\varepsilon}(t)X_1^{\delta,\varepsilon}(t), X_1^{\delta,\varepsilon}(t)\big\rangle\big\}\,dt + o\big(\varepsilon^{\frac{3}{2}}\big). \end{aligned}$$

Consequently,

$$\begin{aligned} -\sqrt{\delta}\big(\sqrt{\varepsilon}\,|X_0| + \varepsilon T\big) \leqslant{} & E\Big\{\sqrt{\varepsilon}\,\big\langle I_0^{\delta,\varepsilon}\big(F_{X_0}^\delta\big)^\top - \Phi_0^{\delta,\varepsilon}, X_0\big\rangle \\ & + \frac{\varepsilon}{2}\big\langle\big(I_0^{\delta,\varepsilon} F_{X_0 X_0}^\delta - P_0^{\delta,\varepsilon}\big)X_0, X_0\big\rangle\Big\} \\ & - E\int_0^T \Big[\big\langle\Phi_t^{\delta,\varepsilon}, \triangle B^\delta(t)\big\rangle + \big\langle\Psi_t^{\delta,\varepsilon}, \triangle\Sigma^\delta(t)\big\rangle \\ & + \frac{1}{2}\big\langle P_t^{\delta,\varepsilon}\triangle\Sigma^\delta(t), \triangle\Sigma^\delta(t)\big\rangle\Big] I_{S_\varepsilon}(t)\,dt + o\big(\varepsilon^{\frac{3}{2}}\big). \end{aligned} \tag{3.6}$$

Step 5: Derivation of the inequality. Dividing $\sqrt{\varepsilon}$ in (3.6) and then sending $\varepsilon \to 0$, $\delta \to 0$, we see that

$$E\big\langle I_0(\bar{F}_{X_0})^\top - \Phi_0, X_0\big\rangle \geqslant 0, \quad \forall X_0 \in R^2,$$

which implies

$$\Phi_0 = I_0(\bar{F}_{X_0})^\top. \tag{3.7}$$

Also, dividing $\varepsilon|X_0|^2$ in (3.6) and sending $|X_0| \to \infty$, $\varepsilon \to 0$, $\delta \to 0$, we get

$$E[I_0\bar{F}_{X_0X_0} - P_0] \geqslant 0. \tag{3.8}$$

Finally, by taking $X_0 = 0$ in (3.6), using a standard argument [23], the variational inequality follows

$$\begin{aligned} &\big\langle\Phi_t, B(t,\bar{X}_t,E\bar{X}_t,v) - B(t,\bar{X}_t,E\bar{X}_t,\bar{v}_t)\big\rangle + \big\langle\Psi_t, \Sigma(t,\bar{X}_t,E\bar{X}_t,v) - \Sigma(t,\bar{X}_t,E\bar{X}_t,\bar{v}_t)\big\rangle \\ &\quad + \frac{1}{2}\big(\Sigma(t,\bar{X}_t,E\bar{X}_t,v) - \Sigma(t,\bar{X}_t,E\bar{X}_t,\bar{v}_t)\big)^\top P_t\big(\Sigma(t,\bar{X}_t,E\bar{X}_t,v) - \Sigma(t,\bar{X}_t,E\bar{X}_t,\bar{v}_t)\big) \leqslant 0. \end{aligned} \tag{3.9}$$

Step 6: Completing the proof. Since $I_0 \neq 0$, we might as well set $I_0 = 1$. Then

$$\begin{cases} d\Phi_t = -\{\bar{B}_X(t)^\top \Phi_t + E[\bar{B}_{\tilde{X}}(t)^\top \Phi_t] + \bar{\Sigma}_X(t)^\top \Psi_t + E[\bar{\Sigma}_{\tilde{X}}(t)^\top \Psi_t]\}\, dt + \Psi_t\, dW_t, \\ \Phi_T = -\{(\bar{F}_{X_T})^\top + [\bar{G}_{X_T} \bar{I}_T]\}, \end{cases} \tag{3.10}$$

and

$$\begin{cases} dP_t = -[\bar{B}_X(t)^\top P_t + P_t \bar{B}_X(t) + \bar{\Sigma}_X(t)^\top P_t \bar{\Sigma}_X(t) \\ \qquad + \bar{\Sigma}_X(t)^\top Q_t + Q_t \bar{\Sigma}_X(t) + \bar{H}_{XX}(t)]\, dt + Q_t\, dW_t, \\ P_T = -\{\bar{F}_{X_T X_T} + [\bar{G}_{X_T X_T} \bar{I}_T]\}. \end{cases} \tag{3.11}$$

Also, from (3.7) and (3.8),

$$\Phi_0 = (\bar{F}_{X_0})^\top, \qquad E[\bar{F}_{X_0 X_0} - P_0] \geqslant 0. \tag{3.12}$$

Note that

$$\begin{cases} (\bar{F}_{X_0})^\top = F_{X_0}(\bar{X}_0, \bar{X}_T)^\top = \begin{pmatrix} 0 \\ \bar{\gamma}_{y_0} \end{pmatrix}, \\ (\bar{F}_{X_T})^\top = F_{X_T}(\bar{X}_0, \bar{X}_T)^\top = \begin{pmatrix} \bar{g}_{x_T} \\ 0 \end{pmatrix}, \\ \bar{G}_{X_T} \bar{I}_T = G_{X_T}(\bar{X}_T) \bar{I}_T = \begin{pmatrix} -\bar{h}_{x_T} \bar{I}_T \\ \bar{I}_T \end{pmatrix}. \end{cases}$$

Let

$$\Phi = \begin{pmatrix} \eta \\ \xi \end{pmatrix}, \qquad \Psi = \begin{pmatrix} \zeta \\ \tilde{\zeta} \end{pmatrix}. \tag{3.13}$$

Then it follows from (3.10), (3.12) and (3.13) that

$$\begin{pmatrix} \eta_0 \\ \xi_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{\gamma}_{y_0} \end{pmatrix}, \qquad \begin{pmatrix} \eta_T \\ \xi_T \end{pmatrix} = \begin{pmatrix} -\bar{g}_{x_T} + \bar{h}_{x_T} \bar{I}_T \\ -\bar{I}_T \end{pmatrix}.$$

So

$$\begin{cases} \xi_0 = \bar{\gamma}_{y_0}, \\ \eta_T = -\bar{g}_{x_T} - \bar{h}_{x_T} \xi_T. \end{cases}$$

By use of (3.13), we may rewrite (3.10) as

$$\begin{cases} d\begin{pmatrix} \eta_t \\ \xi_t \end{pmatrix} = -\Big\{ \begin{pmatrix} \bar{b}_x(t) & -\bar{f}_x(t) \\ \bar{b}_y(t) & -\bar{f}_y(t) \end{pmatrix} \begin{pmatrix} \eta_t \\ \xi_t \end{pmatrix} + E\Big[\begin{pmatrix} \bar{b}_{\tilde{x}}(t) & -\bar{f}_{\tilde{x}}(t) \\ \bar{b}_{\tilde{y}}(t) & -\bar{f}_{\tilde{y}}(t) \end{pmatrix} \begin{pmatrix} \eta_t \\ \xi_t \end{pmatrix} \Big] \\ \qquad + \begin{pmatrix} \bar{\sigma}_x(t) & 0 \\ \bar{\sigma}_y(t) & 0 \end{pmatrix} \begin{pmatrix} \zeta_t \\ \tilde{\zeta}_t \end{pmatrix} + E\Big[\begin{pmatrix} \bar{\sigma}_{\tilde{x}}(t) & 0 \\ \bar{\sigma}_{\tilde{y}}(t) & 0 \end{pmatrix} \begin{pmatrix} \zeta_t \\ \tilde{\zeta}_t \end{pmatrix} \Big] \Big\}\, dt \\ \qquad + \begin{pmatrix} \zeta_t \\ \tilde{\zeta}_t \end{pmatrix} dW_t, \\ \xi_0 = \bar{\gamma}_{y_0}, \quad \eta_T = -\bar{g}_{x_T} - \bar{h}_{x_T} \xi_T. \end{cases} \tag{3.14}$$

Variational inequality (3.9) takes the form:

$$\begin{aligned} &\eta_t [b(v_t) - b(\bar{v}_t)] - \xi_t [f(v_t) - f(\bar{v}_t)] + \zeta_t [\sigma(v_t) - \sigma(\bar{v}_t)] \\ &\quad + \tilde{\zeta}_t (z_t - \bar{z}_t) + \frac{1}{2} \Big\langle P_t \begin{pmatrix} \sigma(v_t) - \sigma(\bar{v}_t) \\ z_t - \bar{z}_t \end{pmatrix}, \begin{pmatrix} \sigma(v_t) - \sigma(\bar{v}_t) \\ z_t - \bar{z}_t \end{pmatrix} \Big\rangle \leqslant 0, \end{aligned} \tag{3.15}$$

where

$$\begin{cases} \rho(v_t) = \rho(t, \bar{x}_t, \bar{y}_t, z_t, E\bar{x}_t, E\bar{y}_t, Ez_t, u_t), \\ \rho(\bar{v}_t) = \phi(t, \bar{x}_t, \bar{y}_t, \bar{z}_t, E\bar{x}_t, E\bar{y}_t, E\bar{z}_t, \bar{u}_t), \quad \rho = b, \sigma, f. \end{cases}$$

Then taking $u = \bar{u}$, $z = \bar{z} + \varepsilon z_0$, $Ez = E\bar{z} + \varepsilon Ez_0$, dividing by ε and sending $\varepsilon \to 0$, we have

$$\eta_t\big[\bar{b}_z(t)z_0 + \bar{b}_{\bar{z}}(t)Ez_0\big] - \xi_t\big[\bar{f}_z(t)z_0 + \bar{f}_{\bar{z}}(t)Ez_0\big] + \zeta_t\big[\bar{\sigma}_z(t)z_0 + \bar{\sigma}_{\bar{z}}(t)Ez_0\big] + \tilde{\zeta}_t z_0 \leqslant 0, \quad \forall z_0 \in R.$$

Hence

$$\tilde{\zeta}_t = -\big\{\bar{b}_z(t)\eta_t - \bar{f}_z(t)\xi_t + \bar{\sigma}_z(t)\zeta_t + E\big[\bar{b}_{\bar{z}}(t)\eta_t - \bar{f}_{\bar{z}}(t)\xi_t + \bar{\sigma}_{\bar{z}}(t)\zeta_t\big]\big\}. \tag{3.16}$$

Combining (3.14) and (3.16) gives the first-order adjoint equation. Furthermore,

$$\bar{F}_{X_T X_T} + [\bar{G}_{X_T X_T} \bar{I}_T] = \begin{pmatrix} \bar{g}_{x_T x_T} - \bar{I}_T \bar{h}_{x_T x_T} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \bar{g}_{x_T x_T} + \xi_T \bar{h}_{x_T x_T} & 0 \\ 0 & 0 \end{pmatrix}.$$

This provides the terminal value of P_T. Similarly, we have

$$\bar{F}_{X_0 X_0} = \begin{pmatrix} 0 & 0 \\ 0 & \bar{\gamma}_{y_0 y_0} \end{pmatrix}.$$

Hence, inequality (3.1) follows from (3.12). Next, taking $z = \bar{z}$, $Ez = E\bar{z}$ in the variational inequality (3.15) leads to

$$\begin{aligned} &\eta_t\big[b(t, \bar{\Gamma}_t, u_t) - b(t, \bar{\Gamma}_t, \bar{u}_t)\big] - \xi_t\big[f(t, \bar{\Gamma}_t, u_t) - f(t, \bar{\Gamma}_t, \bar{u}_t)\big] + \zeta_t\big[\sigma(t, \bar{\Gamma}_t, u_t) - \sigma(t, \bar{\Gamma}_t, \bar{u}_t)\big] \\ &\quad + \frac{1}{2}\left\langle P_t \begin{pmatrix} \sigma(t, \bar{\Gamma}_t, u_t) - \sigma(t, \bar{\Gamma}_t, \bar{u}_t) \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma(t, \bar{\Gamma}_t, u_t) - \sigma(t, \bar{\Gamma}_t, \bar{u}_t) \\ 0 \end{pmatrix} \right\rangle \\ &= H(t, \bar{x}_t, \bar{y}_t, \bar{z}_t, u_t, \xi_t, \eta_t, \zeta_t) - H(t, \bar{x}_t, \bar{y}_t, \bar{z}_t, \bar{u}_t, \xi_t, \eta_t, \zeta_t) \\ &\quad + \frac{1}{2}P_1(t)\big[\sigma(t, \bar{\Gamma}_t, u_t) - \sigma(t, \bar{\Gamma}_t, \bar{u}_t)\big]^2 \leqslant 0. \end{aligned}$$

This gives the optimal variational inequality (3.2), thus we have completed the total proof. □

Remark 3.1. Compared with the optimality variational principle of the classical fully coupled FBSDE (Yong [22]), a conclusion similar to (3.6) in Yong [22] cannot be reached for the reason that the second-order expansions of $b(t, \bar{x}, \bar{y}, z, E\bar{x}, E\bar{y}, Ez, \bar{u})$, $\sigma(t, \bar{x}, \bar{y}, z, E\bar{x}, E\bar{y}, Ez, \bar{u})$, $f(t, \bar{x}, \bar{y}, z, E\bar{x}, E\bar{y}, Ez, \bar{u})$ with respect to z and Ez not only contain $(z - \bar{z})^2$, but also contain $(z - \bar{z})(Ez - E\bar{z})$ and $(Ez - E\bar{z})^2$.

Remark 3.2. If the coefficients of the system (2.1) do not depend on the expected values of the states, Theorem 3.1 reduces to the general SMP for fully coupled FBSDE. However, if the system (2.20) has the state constraint that $y_T = h(x_T, Ex_T)$, duality analysis of deriving the variational inequality is rather complicated. We will leave it for a future study.

4. A linear–quadratic problem

Consider the optimal control problem:

Problem $\mathcal{D}$. Minimize $J(u) = \frac{1}{2}E\{g_1 x_T^2 + \gamma_1 y_0^2\}$ over $\mathcal{U}$, subject to

$$\begin{cases} dx_t = \{b_1 x_t + b_2 y_t + b_3 z_t + b_4 Ex_t + b_5 Ey_t + b_6 Ez_t + b_7 u_t\}\,dt \\ \qquad\quad + \{\sigma_1 x_t + \sigma_2 y_t + \sigma_3 z_t + \sigma_4 Ex_t + \sigma_5 Ey_t + \sigma_6 Ez_t + \sigma_7 u_t\}\,dW_t, \\ -dy_t = \{f_1 x_t + f_2 y_t + f_3 z_t + f_4 Ex_t + f_5 Ey_t + f_6 Ez_t + f_7 u_t\}\,dt - z_t\,dW_t, \\ x(0) = x_0, \quad y_T = h_1 x_T, \end{cases} \tag{4.1}$$

where b_i, σ_i, f_i, $i = 1, \ldots, 7$ and g_1, γ_1, h_1 are real constants of R, $\mathcal{U} = \{u \in L^2_{\mathcal{F}}(0,T;U) \mid u \in U\}$ and $U \subseteq R$ could be arbitrary.

Noting the notations we have used before and the monotonic conditions (A_4), we make assumptions for coefficients of the state equation (4.1).

$$\begin{cases} E\langle F(t,\Gamma,u) - F(t,\Gamma_1,u), \Theta - \Theta_1\rangle \leqslant -\beta_1 E|\Theta - \Theta_1|^2, \quad P\text{-a.s.}, \\ h_1 \geqslant \mu_1 > 0, \quad \Gamma = (x,y,z,Ex,Ey,Ez), \quad \Theta = (x,y,z), \end{cases} \tag{4.2}$$

where β_1, μ_1 are given nonnegative constants. Then, by virtue of Theorem 2.1, for any $u \in \mathcal{U}$, the system (4.1) has a unique solution (x,y,z). Now, we suppose Problem $\mathcal{D}$ admits an optimal 4-tuple $(\bar{x},\bar{y},\bar{z},\bar{u})$. Then the first-order adjoint equation reads

$$\begin{cases} d\xi_t = \big\{f_2\xi_t - b_2\eta_t - \sigma_2\zeta_t + E[f_5\xi_t - b_5\eta_t - \sigma_5\zeta_t]\big\}\,dt \\ \qquad\quad + \big\{f_3\xi_t - b_3\eta_t - \sigma_3\zeta_t + E[f_6\xi_t - b_6\eta_t - \sigma_6\zeta_t]\big\}\,dW_t, \\ -d\eta_t = \big\{-f_1\xi_t + b_1\eta_t + \sigma_1\zeta_t + E[-f_4\xi_t + b_4\eta_t + \sigma_4\zeta_t]\big\}\,dt - \zeta_t\,dW_t, \\ \xi_0 = \gamma_1\bar{y}_0, \quad \eta_T = -g_1\bar{x}_T - h_1\xi_T. \end{cases} \tag{4.3}$$

Similarly, by (4.2), it is easy to verify that the mean-field FBSDE (4.3) satisfies the monotonic conditions $(A_4)'$. Then by Theorem 2.2, (4.3) admits a unique solution (ξ,η,ζ). And the BSDE for (P,Q) becomes

$$\begin{cases} dP_t = -\big\{B^\top P_t + P_t B + \Sigma^\top P_t \Sigma + \Sigma^\top Q_t + Q_t \Sigma\big\}\,dt + Q_t\,dW_t, \\ P_T = -\begin{pmatrix} g_1 & 0 \\ 0 & 0 \end{pmatrix}, \end{cases} \tag{4.4}$$

with

$$B = \begin{pmatrix} b_1 & b_2 \\ -f_1 & -f_2 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \sigma_1 & \sigma_2 \\ 0 & 0 \end{pmatrix}.$$

Since all the coefficients in (4.4) are constants and the terminal value of P_T is deterministic, we deduce $Q = 0$. Note $P = \left(\begin{smallmatrix} P_1 & P_2 \\ P_2 & P_3 \end{smallmatrix}\right)$, then (4.4) can be written as

$$\dot{P}_1(t) = -\big(2b_1 + \sigma_1^2\big)P_1(t) + 2f_1 P_2(t), \qquad P_1(T) = -g_1, \tag{4.5}$$

$$\dot{P}_2(t) = -(b_2 + \sigma_1\sigma_2)P_1(t) + (f_2 - b_1)P_2(t) + f_1 P_3(t), \qquad P_2(T) = 0, \tag{4.6}$$

$$\dot{P}_3(t) = -\sigma_2^2 P_1(t) - 2b_2 P_2(t) + 2f_2 P_3(t), \qquad P_3(T) = 0. \tag{4.7}$$

Clearly, linear ODEs (4.5)–(4.7) are independent of the optimal 4-tuple $(\bar{x},\bar{y},\bar{z},\bar{u})$ and only depend on the coefficients of the state equation and the cost functional. Hence, condition (3.1), which takes the form

$$\begin{pmatrix} 0 & 0 \\ 0 & \gamma_1 \end{pmatrix} - P_0 \geqslant 0, \tag{4.8}$$

presents necessary conditions satisfied by these coefficients. Meanwhile, the optimality variational inequality (3.2) takes the form

$$(-f_7\xi_t + b_7\eta_t + \sigma_7\zeta_t)(u - \bar{u}_t) + \frac{1}{2}\sigma_7^2 P_1(t)(u - \bar{u}_t)^2 \leqslant 0, \quad \forall u \in \mathcal{U}.$$

This implies that

$$\begin{cases} -f_7\xi_t + b_7\eta_t + \sigma_7\zeta_t \leqslant -\dfrac{1}{2}\sigma_7^2 P_1(t)(u - \bar{u}_t), & \forall u \in \mathcal{U},\ u > \bar{u}_t, \\ -f_7\xi_t + b_7\eta_t + \sigma_7\zeta_t \geqslant -\dfrac{1}{2}\sigma_7^2 P_1(t)(u - \bar{u}_t), & \forall u \in \mathcal{U},\ u < \bar{u}_t. \end{cases} \tag{4.9}$$

Let us now look at a special case. Suppose that

$$\begin{array}{cccccc} f_1 > 0, & b_2 \leqslant 0, & \sigma_3 \leqslant 0, & f_4 > 0, & b_5 \leqslant 0, & \sigma_6 \leqslant 0, \\ b_1 = f_2, & \sigma_1 = f_3, & \sigma_2 = -b_3, & b_4 = f_5, & \sigma_4 = f_6, & \sigma_5 = -b_6. \end{array}$$

It is easy to check that monotonic conditions (A_4) and $(A_4)'$ hold, this assures the well-posedness of (4.1) and (4.3). Furthermore, if $b_2 = \sigma_2 = 0$, systems (4.6) and (4.7) can be changed as

$$\dot{P}_2(t) = f_1 P_3(t), \qquad P_2(T) = 0, \tag{4.10}$$

$$\dot{P}_3(t) = 2f_2 P_3(t), \qquad P_3(T) = 0. \tag{4.11}$$

From (4.5), (4.10) and (4.11), we see that

$$P_1(t) = -g_1 e^{(2b_1+\sigma_1^2)(T-t)}, \qquad P_2(\cdot) = 0, \qquad P_3(\cdot) = 0.$$

Consequently, (4.8) becomes

$$\begin{pmatrix} g_1 e^{(2b_1+\sigma_1^2)T} & 0 \\ 0 & \gamma_1 \end{pmatrix} \geqslant 0.$$

Thus, it is necessary that

$$g_1 \geqslant 0, \qquad \gamma_1 \geqslant 0.$$

On the other hand, the optimality variational inequality (4.9) becomes

$$\begin{cases} -f_7\xi_t + b_7\eta_t + \sigma_7\zeta_t \leqslant \dfrac{1}{2}\sigma_7^2 g_1 e^{(2b_1+\sigma_1^2)(T-t)}(u - \bar{u}_t), & \forall u \in \mathcal{U},\ u > \bar{u}_t, \\ -f_7\xi_t + b_7\eta_t + \sigma_7\zeta_t \geqslant \dfrac{1}{2}\sigma_7^2 g_1 e^{(2b_1+\sigma_1^2)(T-t)}(u - \bar{u}_t), & \forall u \in \mathcal{U},\ u < \bar{u}_t. \end{cases} \tag{4.12}$$

If in such a condition that $\sigma_7 = 0$, we have

$$-f_7\xi_t + b_7\eta_t + \sigma_7\zeta_t = 0, \quad \forall u \in \mathcal{U}, \tag{4.13}$$

but in the condition that $\sigma_7 \neq 0$, $g_1 \neq 0$, relations (4.12) imply

$$\sup_{u\in\mathcal{U},\, u<\bar{u}_t} u - \bar{u}_t \leqslant \frac{2(-f_7\xi_t + b_7\eta_t + \sigma_7\zeta_t)}{\sigma_7^2 g_1 e^{(2b_1+\sigma_1^2)(T-t)}} \leqslant \inf_{u\in\mathcal{U},\, u>\bar{u}_t} u - \bar{u}_t. \tag{4.14}$$

When U is discrete and not a singleton, the two sides of (4.14) are different. While, when $U = R$, we always have (4.13) as the necessary optimality condition regardless of the value σ_7. Besides, it is necessary to point out that we cannot obtain the explicit optimal control from the optimality variational inequality (4.9) in general due to its heavy dependence on adjoint processes (ξ, η, ζ), which is rather difficult to work out. However, this can be achieved in some special cases mentioned below.

Remark 4.1. If we only focus on the forward control system, i.e. $f_i = 0$, $h_1 = 0$, $b_2 = b_3 = b_5 = b_6 = 0$, $\sigma_2 = \sigma_3 = \sigma_5 = \sigma_6 = 0$, $i = 1, \dots, 7$, then (4.1) reduces to the system investigated in Li [11]. There, under the assumption of convex control domain, an optimal control for the linear quadratic control problem is given in the feedback form.

Remark 4.2. If the coefficients of (4.1) do not depend on the expected values of the states, i.e. $b_4 = b_5 = b_6 = 0$, $\sigma_4 = \sigma_5 = \sigma_6 = 0$, $f_4 = f_5 = f_6 = 0$, then it is just the classical fully coupled FBSDEs. The corresponding linear quadratic problem with mixed initial-terminal conditions is studied in Yong [22], where an optimal control in its explicit form is not presented. However, if in addition, $b_2 = b_3 = 0$, $\sigma_2 = \sigma_3 = 0$, an explicit optimal control in its state feedback form is obtained in Wu [18].

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