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A phase decomposition approach and the Riemann problem for a model of two-phase flows

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A B S T R A C T

We present a phase decomposition approach to deal with the generalized Rankine–Hugoniot relations and then the Riemann problem for a model of two-phase flows. By investigating separately the jump relations for equations in conservative form in the solid phase, we show that the volume fractions can change only across contact discontinuities. Then, we prove that the generalized Rankine–Hugoniot relations are reduced to the usual form. It turns out that shock waves and rarefaction waves remain on one phase only, and the contact waves serve as a bridge between the two phases. By decomposing Riemann solutions into each phase, we show that Riemann solutions can be constructed for large initial data. Furthermore, the Riemann problem admits a unique solution for an appropriate choice of initial data.

1. Introduction

We consider in this paper the Riemann problem for the following model of two-phase flows, see [6],

$$
\begin{aligned}
&\partial_t(\alpha_g\rho_g) + \partial_x(\alpha_g\rho_g u_g) = 0,\\
&\partial_t(\alpha_g\rho_g u_g) + \partial_x\big(\alpha_g\big(\rho_g u_g^2 + p_g\big)\big) = p_g\partial_x\alpha_g,\\
&\partial_t(\alpha_s\rho_s) + \partial_x(\alpha_s\rho_s u_s) = 0,\\
&\partial_t(\alpha_s\rho_s u_s) + \partial_x\big(\alpha_s\big(\rho_s u_s^2 + p_s\big)\big) = -p_g\partial_x\alpha_g,\\
&\partial_t\rho_s + \partial_x(\rho_s u_s) = 0, \quad x \in \mathbb{R}, t > 0.
\end{aligned}
\tag{1.1}
$$

The system (1.1) is obtained from the full model of two-phase flows, see [4,6], namely the gas phase and the solid phase, by assuming that the flow is isentropic in both phases. The first and the third equation of (1.1) describe the conservation of mass in each phase; the second and the fourth equation of (1.1) describe

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the balance of momentum in each phase; the last equation of (1.1) is the so-called compaction dynamics equation. Throughout, we use the subscripts g and s to indicate the quantities in the gas phase and in the solid phase, respectively. The notations $\alpha_k, \rho_k, u_k, p_k, k = g, s$, respectively, stand for the volume fraction, density, velocity, and pressure in the k-phase, $k = g, s$. The volume fractions satisfy

$$\alpha_s + \alpha_g = 1. \tag{1.2}$$

The system (1.1) has the form of nonconservative systems of balance laws

$$U_t + A(U)U_x = 0, \tag{1.3}$$

for $U = (\rho_g, u_g, \rho_s, u_s, \alpha_g)$, and $A(U)$ is given at the beginning of Section 2. Weak solutions of such a system can be understood in the sense of *nonconservative products* – a concept introduced by Dal Maso, LeFloch and Murat [7]. In Section 2 we will brief this concept. Nonconservative systems can be used to model multi-phase flows. Multi-phase flow models have attracted attention of many scientists not only for the study theoretical problems such as existence, uniqueness, stability, and constructions of solutions, but also numerical approximations of the solutions. In the case of two-phase flow models, there may be two classes: the class of one-fluid models of two-phase flows and the class of two-fluid models of two-phase flows. The situation is similar to multi-phase flow models. Both classes are of nonconservative form, but there is a major difference between the two. That is, one-pressure models of two-phase flows are in general not hyperbolic (see [12]), but two-pressure models of two-phase flows such as (1.1) are hyperbolic and strictly hyperbolic except on a finite number of hyper-surfaces of the phase domain. Moreover, the characteristic fields of two-phase flow models such as (1.1) have explicit forms. This raises the hope for the study of the two-pressure models of two-phase flows: the theory of shock waves for hyperbolic systems of conservation laws may be developed to study these hyperbolic models. In the numerical approximations, numerical schemes employing an explicit form of characteristic fields, such as Roe-type schemes, can be implemented. Therefore, the model (1.1), having applications in the modeling of the deflagration-to-detonation transition (DDT) in granular explosives, is worth to study.

In this paper, we will present a method to construct solutions of the Riemann problem for the two-phase flow model (1.1) using a phase decomposition approach. Observe that the construction of solutions of the Riemann problem for hyperbolic systems when the solution vector has dimension larger than three is in general very complicated. The solution vector of (1.1) has dimension five, so how to work through? It is very interesting that in the system (1.1), four characteristic fields depend only on one phase, either gas or solid. This allows the waves associated with these characteristic fields to change only in one phase and remain constants in the other. This motivates us to propose a phase decomposition approach to build Riemann solutions of (1.1), where waves associated with the 5th characteristic field will then be used as a “bridge” to connect between the two phases. Precisely, we first investigate the impact of the jump conditions of the conservation equations in the solid phase. This leads us to a crucial conclusion that across a discontinuity, either the volume fractions remain constant, or the discontinuity is a contact wave. The first case yields the usual form of the jump relations for shock waves. In the later case we can also point out that the jump relations have the canonical form, by using a regularization method. Then, by a decomposition technique, we can separate the Riemann solutions in each phase, where the two phases are now constraint to each other via the solid contact. So, one can see that the solid contact in this method serves as a “bridge” to connect between the group of solid components and the group of gas components of the Riemann solutions. As usual, Riemann solutions are made from fundamental waves: shock waves, contact discontinuities, and rarefaction waves. Shock waves and contact discontinuities are weak solutions in the sense of nonconservative products and are of the usual form

$$U(x,t) = \begin{cases} U_-, & \text{for } x < \sigma t, \\ U_+, & \text{for } x > \sigma t, \end{cases} \tag{1.4}$$

for some constant states $U_\pm$ and a constant shock speed σ. We will show that shock waves and contact waves will be of path-independence. This allows us to determine uniquely these waves. The fact that the Riemann solutions in this work can be understood in the sense of nonconservative product makes us believe that this work will give a certain contribution to the theory of hyperbolic systems of balance laws in nonconservative forms. Since these Riemann solutions are constructed in a deterministic way, they would be effectively used in studying the front tracking method, the Glimm scheme, or Godunov-type schemes.

We observe that hyperbolic models in nonconservative forms have attracted many authors. The earlier works concerning nonconservative systems were carried out in [13,14,18,10]. The Riemann problem for the model of a fluid in a nozzle with discontinuous cross-section was solved in [15] for the isentropic case, and in [21] for the non-isentropic case. The Riemann problem for shallow water equations with discontinuous topography was solved in [16,17]. The Riemann problem for a general system in nonconservative form was studied by [8]. In [3,19], some Riemann solutions of the Baer–Nunziato model of two-phase flows were constructed. Two-fluid models of two-phase flows were studied in [12,20]. Numerical approximations for two-phase flows were considered in [5,2,9,23,22,24]. See also the references therein.

The organization of this paper is as follows. Section 2 provides us with basic properties of the system (1.1), and we recall the definition of weak solutions in the sense of nonconservative products. In Section 3 we study the jump relations by using phase decomposition. We will show that the generalized Rankine–Hugoniot relations can be reduced to the usual ones in the canonical form. Then, we define elementary waves that make up Riemann solutions. In Section 4 we use phase decomposition to construct all possible Riemann solutions, which can be done in each phase separately.

2. Preliminaries

2.1. Characteristic fields

When $\alpha_s \equiv 0$ or $\alpha_g \equiv 0$, the system (1.1) is reduced to the usual isentropic gas dynamics equations, and the Riemann problem has been well-treated. For simplicity, we assume in the sequel that

$$\alpha_s > 0, \qquad \alpha_g > 0. \tag{2.1}$$

This means that the flow always has exactly two phases. Furthermore, we will assume that the fluid in each phase is isentropic and ideal, where the equation of state is given by

$$p_k = \kappa_k \rho_k^{\gamma_k}, \quad \kappa_k > 0, \gamma_k > 1, k = s, g. \tag{2.2}$$

For smooth solutions, under the assumptions (2.1), the system (1.1) is equivalent to the following system

$$\begin{aligned}
&\partial_t \rho_g + u_g \partial_x \rho_g + \rho_g \partial_x u_g + \frac{\rho_g(u_g - u_s)}{\alpha_g} \partial_x \alpha_g = 0, \\
&\partial_t u_g + h_g'(\rho_g) \partial_x \rho_g + u_g \partial_x u_g = 0, \\
&\partial_t \rho_s + u_s \partial_x \rho_s + \rho_s \partial_x u_s = 0, \\
&\partial_t u_s + h_s'(\rho_s) \partial_x \rho_s + u_s \partial_x u_s + \frac{p_g - p_s}{\alpha_s \rho_s} \partial_x \alpha_g = 0, \\
&\partial_t \alpha_g + u_s \partial_x \alpha_g = 0, \quad x \in \mathbb{R}, t > 0,
\end{aligned} \tag{2.3}$$

where h_k is given by

$$h'_k(\rho) = \frac{p'_k(\rho)}{\rho}, \quad k = s, g.$$

From (2.3), choosing the dependent variable

$$U = (\rho_g, u_g, \rho_s, u_s, \alpha_g) \in \mathbb{R}^5,$$

we can re-write the system (1.1) as a system of balance laws in nonconservative form as

$$U_t + A(U)U_x = 0, \tag{2.4}$$

where

$$A(U) = \begin{pmatrix} u_g & \rho_g & 0 & 0 & \frac{\rho_g(u_g-u_s)}{\alpha_g} \\ h'_g(\rho_g) & u_g & 0 & 0 & 0 \\ 0 & 0 & u_s & \rho_s & 0 \\ 0 & 0 & h'_s(\rho_s) & u_s & \frac{p_g-p_s}{\alpha_s\rho_s} \\ 0 & 0 & 0 & 0 & u_s \end{pmatrix}.$$

Observe that if $\alpha_s = 0$, or $\alpha_g = 0$, then the matrix $A(U)$ is undefined.

Now, let us recall the concept of weak solutions of a nonconservative system of the form (2.4) for the general case $U \in \mathbb{R}^N$ in the sense of nonconservative products. Given a *family of Lipschitz paths* $\phi : [0,1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying

$$\begin{aligned} &\phi(0; U, V) = U \qquad \phi(1; U, V) = V, \\ &\left|\partial_s\phi(s; U, V)\right| \le K|V - U|, \\ &\left|\partial_s\phi(s; U_1, V_1) - \partial_s\phi(s; U_2, V_2)\right| \le K\big(|V_1 - V_2| + |U_1 - U_2|\big), \end{aligned} \tag{2.5}$$

for some constant $K > 0$, and for all $s \in [0,1], U, V, U_1, U_2, V_1, V_2 \in \mathbb{R}^N$. Let U be a function with bounded variation in an interval $[a,b] \subset \mathbb{R}$ (in the following we will call it a BV-function for simplicity), then dU is a Borel measure which coincides with the distributional derivative of U, i.e.,

$$\int_a^b U\varphi' dx = -\int_a^b \varphi dU, \quad \forall \varphi \in C_0^\infty[a,b]$$

(see [1]). The nonconservative product of a locally Borel bounded function and a Borel measure is given as follows, see [7].

Definition 2.1. Let $U = U(x)$, $x \in [a,b]$, be a function with bounded variation. Then, the nonconservative product $\mu := [g(U) \cdot dU]_\phi$ of a locally Borel bounded function $g : \mathbb{R}^N \to \mathbb{R}^N$ by the vector-valued Borel measure dU is a real-valued bounded Borel measure μ with the following properties:

(i) For any Borel set B, s.t. U is continuous on B:

$$\mu(B) = \int_B g(U) dU \tag{2.6}$$

(ii) For any $x_0 \in [a, b]$:

$$\mu(x_0) = \int_0^1 g\big(\phi\big(s; U(x_0-), U(x_0+)\big)\big)\partial_s\phi\big(s; U(x_0-), U(x_0+)\big)\,ds \tag{2.7}$$

Weak solutions can be understood in the sense of nonconservative products as follows, see [14].

Definition 2.2. A function $U \in L^\infty \cap BV_{loc}(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}^N)$ is called a weak solution of the system in nonconservative form

$$U_t + A(U)U_x = 0, \quad U = U(x,t) \in \mathbb{R}^N$$

if the Borel measure

$$\partial_t U + \big[A\big(U(.,t)\big)\partial_x U(.,t)\big]_\phi \tag{2.8}$$

is equal to zero.

Let us consider now the hyperbolicity of the system (2.4). The characteristic equation of the matrix $A(U)$ in (2.4) is given by

$$(u_s - \lambda)\big((u_g - \lambda)^2 - p'_g\big)\big((u_s - \lambda)^2 - p'_s\big) = 0,$$

which admits five roots as

$$\begin{aligned} &\lambda_1(U) = u_g - \sqrt{p'_g}, \qquad \lambda_2(U) = u_g + \sqrt{p'_g}, \\ &\lambda_3(U) = u_s - \sqrt{p'_s}, \qquad \lambda_4(U) = u_s + \sqrt{p'_s}, \qquad \lambda_5(U) = u_s. \end{aligned} \tag{2.9}$$

The corresponding right eigenvectors can be chosen as

$$\begin{aligned} &r_1(U) = \mu \begin{pmatrix} \dfrac{-\rho_g}{\sqrt{p'_g(\rho_g)}} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad r_2(U) = \mu \begin{pmatrix} \dfrac{\rho_g}{\sqrt{p'_g(\rho_g)}} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\ &r_3(U) = \nu \begin{pmatrix} 0 \\ 0 \\ \dfrac{-\rho_s}{\sqrt{p'_s(\rho_s)}} \\ 0 \end{pmatrix}, \qquad r_4(U) = \nu \begin{pmatrix} 0 \\ 0 \\ \dfrac{\rho_s}{\sqrt{p'_s(\rho_s)}} \\ 0 \end{pmatrix}, \\ &r_5(U) = \begin{pmatrix} -(u_g - u_s)^2\rho_g\alpha_s p'_s(\rho_s) \\ (u_g - u_s)p'_g(\rho_g)p'_s(\rho_s)\alpha_s \\ (p_s(\rho_s) - p_g(\rho_g))((u_g - u_s)^2 - p'_g(\rho_g))\alpha_g \\ 0 \\ ((u_g - u_s)^2 - p'_g(\rho_g))\alpha_g\alpha_s p'_s(\rho_s) \end{pmatrix}, \end{aligned} \tag{2.10}$$

where

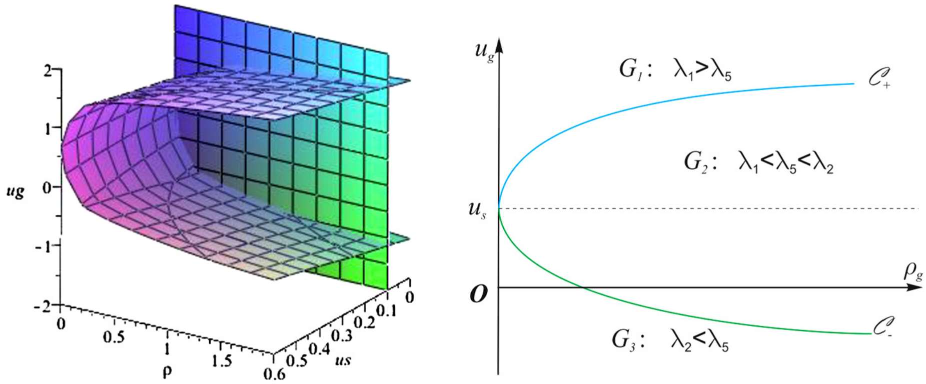

Fig. 1. The projection of the phase domain in the (ρ_g, u_g, u_s)-space and the projection of its intersection with the hyper-plane $u_s \equiv$ constant in the (ρ_g, u_g)-plane.

$$\mu = \frac{2\sqrt{p_g'(\rho_g)}}{p_g''(\rho_g)\rho_g + 2p_g'(\rho_g)},$$
$$\nu = \frac{2\sqrt{p_s'(\rho_s)}}{p_s''(\rho_s)\rho_s + 2p_s'(\rho_s)}.$$

It is not difficult to check that the eigenvectors r_i, $i = 1, 2, 3, 4, 5$ are linearly independent. Thus, the system is hyperbolic. Furthermore, it holds that

$$\lambda_3 < \lambda_5 < \lambda_4.$$

However, the eigenvalues λ_5 may coincide with either λ_1 or λ_2 on a certain hyper-surface of the phase domain, called the *resonant surface*. Due to the change of order of these eigenvalues, we set

$$\begin{aligned}
\Omega_1 &:= \{U \mid \lambda_1(U) > \lambda_5(U)\},\\
\Omega_2 &:= \{U \mid \lambda_1(U) < \lambda_5(U) < \lambda_2(U)\},\\
\Omega_3 &:= \{U \mid \lambda_2(U) < \lambda_5(U)\},\\
\Sigma_+ &:= \{U \mid \lambda_1(U) = \lambda_5(U)\},\\
\Sigma_- &:= \{U \mid \lambda_2(U) = \lambda_5(U)\}.
\end{aligned} \tag{2.11}$$

The system is thus strictly hyperbolic in each domain Ω_i, $i = 1, 2, 3$, but fails to be strictly hyperbolic on the resonant surface

$$\Sigma := \Sigma_+ \cup \Sigma_-. \tag{2.12}$$

Let us take an arbitrary and fixed value of the solid velocity u_{s0}. It will be useful to consider the phase domain in the hyper-plane $u_s \equiv u_{s0}$. We denote by $G_i, \mathcal{C}_\pm$ and $\mathcal{C}$ the projection in the (ρ_g, u_g)-plane of the intersection of the hyper-plane $u_s \equiv u_{s0}$ with the sets Ω_i, $\Sigma_\pm$ and Σ, $i = 1, 2, 3$, respectively, see Fig. 1.

On the other hand, it is not difficult to verify that

$$\begin{aligned}
&D\lambda_i(U) \cdot r_i(U) = 1, \quad i = 1, 2, 3, 4,\\
&D\lambda_5(U) \cdot r_5(U) = 0,
\end{aligned} \tag{2.13}$$

so that the first, second, third, fourth characteristic fields $(\lambda_i(U), r_i(U))$, $i = 1, 2, 3, 4$, are genuinely nonlinear, while the fifth characteristic field $(\lambda_5(U), r_5(U))$ is linearly degenerate.

2.2. Rarefaction waves

Next, let us look for rarefaction waves of the system (2.4). These waves are the continuous piecewise-smooth self-similar solutions of (1.1) associated with nonlinear characteristic fields, which have the form

$$U(x,t) = V(\xi), \quad \xi = \frac{x}{t}, t > 0, x \in \mathbb{R}.$$

Substituting this into (2.4), we can see that rarefaction waves are solutions of the following initial-value problem for ordinary differential equations

$$\begin{aligned} &\frac{dV(\xi)}{d\xi} = r_i\big(V(\xi)\big), \quad \xi \ge \lambda_i(U_0), i = 1,2,3,4, \\ &V\big(\lambda_i(U_0)\big) = U_0. \end{aligned} \tag{2.14}$$

Thus, the integral curve of the first characteristic field is given by

$$\begin{aligned} \frac{d\rho_g(\xi)}{d\xi} &= \frac{-2\sqrt{p_g'(\rho_g)}}{p_g''(\rho_g)\rho_g + 2p_g'(\rho_g)}\rho_g(\xi) < 0, \\ \frac{du_g(\xi)}{d\xi} &= \frac{2\sqrt{p_g'(\rho_g)}}{p_g''(\rho_g)\rho_g + 2p_g'(\rho_g)}\sqrt{p_g'(\xi)} > 0, \\ \frac{d\rho_s(\xi)}{d\xi} &= \frac{du_s(\xi)}{d\xi} = \frac{d\alpha_g(\xi)}{d\xi} = 0. \end{aligned} \tag{2.15}$$

This implies that ρ_s, u_s, α_g *are constant through* 1*-rarefaction waves*, ρ_g is strictly decreasing with respect to ξ, and u_g is strictly increasing with respect to ξ. Moreover, since ρ_g is strictly monotone though 1-rarefaction waves, we can use ρ_g as a parameter of the integral curve

$$\frac{du_g}{d\rho_g} = \frac{-\sqrt{p_g'(\rho_g)}}{\rho_g}. \tag{2.16}$$

The integral curve (2.16) determines the *forward* curve of 1-rarefaction wave $\mathcal{R}_1(U_0)$ consisting of all right-hand states that can be connected to the left-hand state U_0 using 1-rarefaction waves

$$\mathcal{R}_1(U_0): \quad u_g = \omega_1\big((\rho_{g0}, u_{g0}); \rho_g\big) := u_{g0} - \int_{\rho_{g0}}^{\rho_g} \frac{\sqrt{p_g'(y)}}{y} dy, \quad \rho_g \le \rho_{g0}, \tag{2.17}$$

where $\rho_g \le \rho_{g0}$ follows from the condition that the characteristic speed must be increasing through a rarefaction fan.

Similarly, ρ_s, u_s, α_g *are constant through* 2*-rarefaction waves*. The *backward* curve of 2-rarefaction wave $\mathcal{R}_2(U_0)$ consisting of all left-hand states that can be connected to the right-hand state U_0 using 2-rarefaction waves is given by

$$\mathcal{R}_2(U_0): \quad u_g = \omega_2\big((\rho_{g0}, u_{g0}); \rho_g\big) := u_{g0} + \int_{\rho_{g0}}^{\rho_g} \frac{\sqrt{p_g'(y)}}{y} dy, \quad \rho_g \le \rho_{g0}. \tag{2.18}$$

In the same way, ρ_g, u_g, α_g *are constant through* 3- *and* 4-*rarefaction waves*. The *forward* curve of 3-rarefaction wave $\mathcal{R}_3(U_0)$ consisting of all right-hand states that can be connected to the left-hand state U_0 using 3-rarefaction waves is given by

$$\mathcal{R}_3(U_0)\colon\quad u_s = \omega_3\big((\rho_{s0}, u_{s0}); \rho_s\big) := u_{s0} - \int_{\rho_{s0}}^{\rho_s} \frac{\sqrt{p_s'(y)}}{y} dy, \quad \rho_s \le \rho_{s0}. \tag{2.19}$$

The *backward* curve of 4-rarefaction wave $\mathcal{R}_4(U_0)$ consisting of all left-hand states that can be connected to the right-hand state U_0 using 4-rarefaction waves is given by

$$\mathcal{R}_4(U_0)\colon\quad u_s = \omega_4\big((\rho_{s0}, u_{s0}); \rho_s\big) := u_{s0} + \int_{\rho_{s0}}^{\rho_s} \frac{\sqrt{p_s'(y)}}{y} dy, \quad \rho_s \le \rho_{s0}. \tag{2.20}$$

3. Phase decomposition for jump relations and elementary waves

Given a discontinuity of (1.1) of the form (1.4) in the sense of nonconservative product, this discontinuity satisfies the generalized Rankine–Hugoniot relations for a given family of Lipschitz paths. We will show that these generalized Rankine–Hugoniot relations will be reduced to the usual ones.

3.1. Phase decomposition for shock waves

Let us first consider the conservative equations in the solid phase of (1.1), which are the equation of conservation of mass and the compaction dynamics equation. Since these equations are of conservative form, the generalized Rankine–Hugoniot relations corresponding to any family of Lipschitz path must coincide with the usual ones. This means that the following jump relations hold

$$\begin{aligned} &-\sigma[\alpha_s\rho_s] + [\alpha_s\rho_s u_s] = 0,\\ &-\sigma[\rho_s] + [\rho_s u_s] = 0, \end{aligned} \tag{3.1}$$

where σ is the shock speed, $[A] = A_+ - A_-$, and $A_\pm$ denote the values on the right and left of the jump on the quantity A. Eq. (3.1) can be rewritten as

$$\begin{aligned} &\big[\alpha_s\rho_s(u_s - \sigma)\big] = 0,\\ &\big[\rho_s(u_s - \sigma)\big] = 0, \end{aligned}$$

or

$$\begin{aligned} &\rho_s(u_s - \sigma) = M = \text{constant},\\ &M[\alpha_s] = 0. \end{aligned} \tag{3.2}$$

The second equation of (3.2) implies that either $M = 0$ or $[\alpha_s] = 0$. Since $\rho_s > 0$, one obtains the following conclusion: across any discontinuity (1.4) of (1.1)

$$\text{– either } [\alpha_s] = 0, \quad \text{or} \quad u_s = \sigma = \text{constant}. \tag{3.3}$$

It is derived from (3.3) that if $[\alpha_s] = 0$, then the volume fractions remain constant across the discontinuity. The system (1.1) is therefore reduced to the two independent sets of isentropic gas dynamics equations in both phases

$$
\begin{aligned}
&\partial_t \rho_g + \partial_x(\rho_g u_g) = 0, \\
&\partial_t(\rho_g u_g) + \partial_x\big(\rho_g u_g^2 + p_g\big) = 0, \\
&\partial_t \rho_s + \partial_x(\rho_s u_s) = 0, \\
&\partial_t(\rho_s u_s) + \partial_x\big(\rho_s u_s^2 + p_s\big) = 0, \quad x \in \mathbb{R}, t > 0.
\end{aligned} \tag{3.4}
$$

This implies that ρ_s, u_s, α_g *are constant through* 1- *and* 2-*shock waves, while* ρ_g, u_g, α_g *are constant through* 3- *and* 4-*shock waves.*

Given a left-hand state U_0, let us denote by $\mathcal{S}_i(U_0)$, $i = 1, 3$ the *forward* shock curves consisting of all right-hand states U that can be connected to the left-hand state U_0 by an i-Lax shock, $i = 1, 3$, and by $\mathcal{S}_j(U_0)$, $j = 2, 4$ the *backward* shock curves consisting of all left-hand states U that can be connected to the right-hand state U_0 by a j-Lax shock, $j = 2, 4$. These curves are given by:

$$
\begin{aligned}
&\mathcal{S}_1(U_0)\colon \quad u_g = \omega_1\big((\rho_{g0}, u_{g0}); \rho_g\big) := u_{g0} - \left(\frac{(p_g - p_{g0})(\rho_g - \rho_{g0})}{\rho_{g0}\rho_g}\right)^{1/2}, \quad \rho_g > \rho_{g0}, \\
&\mathcal{S}_2(U_0)\colon \quad u_g = \omega_2\big((\rho_{g0}, u_{g0}); \rho_g\big) := u_{g0} + \left(\frac{(p_g - p_{g0})(\rho_g - \rho_{g0})}{\rho_{g0}\rho_g}\right)^{1/2}, \quad \rho_g > \rho_{g0}, \\
&\mathcal{S}_3(U_0)\colon \quad u_s = \omega_3\big((\rho_{s0}, u_{s0}); \rho_s\big) := u_{s0} - \left(\frac{(p_s - p_{s0})(\rho_s - \rho_{s0})}{\rho_{s0}\rho_s}\right)^{1/2}, \quad \rho_s > \rho_{s0}, \\
&\mathcal{S}_4(U_0)\colon \quad u_s = \omega_4\big((\rho_{s0}, u_{s0}); \rho_s\big) := u_{s0} + \left(\frac{(p_s - p_{s0})(\rho_s - \rho_{s0})}{\rho_{s0}\rho_s}\right)^{1/2}, \quad \rho_s > \rho_{s0}.
\end{aligned} \tag{3.5}
$$

From (2.17)–(2.20) and (3.5), we can now define the *forward wave curves* issuing from U_0 as

$$
\begin{aligned}
&\mathcal{W}_1(U_0) = \mathcal{R}_1(U_0) \cup \mathcal{S}_1(U_0), \\
&\mathcal{W}_3(U_0) = \mathcal{R}_3(U_0) \cup \mathcal{S}_3(U_0),
\end{aligned} \tag{3.6}
$$

and the *backward wave curves* issuing from U_0 by

$$
\begin{aligned}
&\mathcal{W}_2(U_0) = \mathcal{R}_2(U_0) \cup \mathcal{S}_2(U_0), \\
&\mathcal{W}_4(U_0) = \mathcal{R}_4(U_0) \cup \mathcal{S}_4(U_0).
\end{aligned} \tag{3.7}
$$

These curves are parameterized in such a way that the velocity is given as a function of the density in each phase, under the form $u = \omega_i(U_0; \rho)$, $\rho > 0$, $i = 1, 2, 3, 4$. It is not difficult to check that ω_1, ω_3 are strictly decreasing; and that ω_2, ω_4 are strictly increasing.

Summarizing the above argument, we get the following result.

Lemma 3.1. *Through an i-wave (shock or rarefaction), $i = 1, 2$, the quantities ρ_s, u_s, α_g are constant. Through a j-wave (shock or rarefaction), $j = 3, 4$, the quantities ρ_g, u_g, α_g are constant. The wave curves $\mathcal{W}_i(U_0)$, $i = 1, 2, 3, 4$, associated with the genuinely nonlinear characteristic fields issuing from a given state U_0 are given by (3.6) and (3.7).*

The following result shows that the shock speeds of shock waves associated with nonlinear characteristic fields may alter the order with the speed of contact waves.

Proposition 3.2. *Consider the projection of the hyper-plane $u_s \equiv u_{s0}$, for an arbitrarily fixed u_{s0}, in the (ρ_g, u_g)-plane. The following conclusions hold.*

(a) *For any state* $U_0 = (\rho_{g0}, u_{g0}) \in G_1$*, there exists a unique state denoted by* $U_0^{\#} = (\rho_{g0}^{\#}, u_{g0}^{\#}) \in \mathcal{S}_1(U_0) \cap G_2$, $u_{g0}^{\#} > u_{s0}$ *such that the 1-shock speed* $\sigma_1(U_0, U_0^{\#})$ *coincides with the characteristic speed* $\lambda_5(U_0^{\#})$*. More precisely,*

$$\begin{aligned}
&\sigma_1\big(U_0, U_0^{\#}\big) = \lambda_5\big(U_0^{\#}\big),\\
&\sigma_1(U_0, U) > \lambda_5\big(U_0^{\#}\big), \quad \rho \in \big(\rho_0, \rho_0^{\#}\big),\\
&\sigma_1(U_0, U) < \lambda_5\big(U_0^{\#}\big), \quad \rho \in \big(\rho_0^{\#}, +\infty\big).
\end{aligned} \tag{3.8}$$

(b) *For any state* $U_0 = (\rho_{g0}, u_{g0}) \in G_3$*, there exists a unique state denoted by* $U_0^{\natural} = (\rho_{g0}^{\natural}, u_{g0}^{\natural}) \in \mathcal{S}_2(U_0) \cap G_2$, $u_{g0}^{\natural} < u_{s0}$ *such that the 2-shock speed* $\sigma_2(U_0, U_0^{\natural})$ *coincides with the characteristic speed* $\lambda_5(U_0^{\natural})$*. More precisely,*

$$\begin{aligned}
&\sigma_2\big(U_0, U_0^{\natural}\big) = \lambda_5\big(U_0^{\natural}\big),\\
&\sigma_2(U_0, U) < \lambda_5\big(U_0^{\natural}\big), \quad \rho \in \big(\rho_0, \rho_0^{\natural}\big),\\
&\sigma_2(U_0, U) > \lambda_5\big(U_0^{\natural}\big), \quad \rho \in \big(\rho_0^{\natural}, +\infty\big).
\end{aligned} \tag{3.9}$$

We omit the proof, since it is similar to the one of Proposition 2.4 [15].

3.2. Jump relations for contact waves

Let us now consider the second equation of (3.3) where $[\alpha_s] \neq 0$. We will show that this is the case of a contact discontinuity associated with the fifth characteristic field.

Theorem 3.3. *Let* U *be a contact discontinuity of the form (1.4) associated with the linearly degenerate characteristic field* (λ_5, r_5)*, that is,* $[\alpha_s] \neq 0$ *and* $U_\pm$ *belongs to the same trajectory of the integral field of the 5th characteristic field. Then,* U *is a weak solution of (1.1) in the sense of nonconservative products and independent of paths. Moreover, this contact discontinuity* U *satisfies the jump relations in the usual form*

$$\begin{aligned}
&u_{s\pm} = \sigma,\\
&\big[\alpha_g \rho_g (u_g - u_s)\big] = 0,\\
&\big[(u_g - u_s)^2 + 2h_g\big] = 0,\\
&[m u_g + \alpha_g p_g + \alpha_s p_s] = 0,
\end{aligned} \tag{3.10}$$

where m *is a constant given by*

$$m = \alpha_g \rho_g (u_g - u_s).$$

Proof. Consider the integral curve corresponding to the 5th family passing through U_-

$$\begin{aligned}
&\frac{dV(\xi)}{d\xi} = r_5\big(V(\xi)\big), \quad \xi \geq \xi_0 := \lambda_5(U_-),\\
&V(\xi_0) = U_-.
\end{aligned} \tag{3.11}$$

A (unique) solution $V = V(\xi)$ of (3.11) always exists on a certain interval $[\xi_0, \xi_1]$, where $\xi_1 > \xi_0$. Let

$$U_+ = V(\xi_1).$$

Since $[\alpha_s] \neq 0$, it follows from (3.2) that the first equation of (3.10) holds. Moreover, it is derived from (2.13) that

$$\frac{d\lambda_5(V(\xi))}{d\xi} = D\lambda_5\big(V(\xi)\big) \cdot \frac{dV(\xi)}{d\xi} = D\lambda_5\big(V(\xi)\big) \cdot r_5\big(V(\xi)\big) \equiv 0.$$

This means that the 5th characteristic speed remains constant through trajectory of (3.11). Next, let $\eta : [\xi_0, \xi_1] \to \mathbb{R}$ be any smooth function. We define a function

$$U(x,t) = V\big(\eta(x-\sigma t)\big) = W(x-\sigma t). \tag{3.12}$$

Then U is the classical solution of (2.4) with the smooth initial data

$$U(x,0) = V \circ \eta(x) = W(x), \quad x \in [\xi_0, \xi_1].$$

Actually, it holds that, for $\xi = x - \sigma t$

$$\begin{aligned}
U_t + A(U)U_x &= -V_\xi(\xi)\eta'\sigma + A\big(V(\xi)\big)V_\xi(\xi)\eta'(\xi) \\
&= \big(-I\sigma + A\big(V(\xi)\big)\big)V_\xi(\xi)\eta'(\xi) \\
&= \big(-I\lambda_5\big(V(\xi)\big) + A\big(V(\xi)\big)\big)r_5\big(V(\xi)\big)\eta'(\xi) = 0,
\end{aligned}$$

where the last equation is deduced from the fact that λ_5 remains constant across the trajectory (3.11). For such a smooth solution U as in (3.12), the system (1.1) becomes

$$\begin{aligned}
&-\sigma(\alpha_g\rho_g)' + (\alpha_g\rho_g u_g)' = 0, \\
&-\sigma(\alpha_g\rho_g u_g)' + \big(\alpha_g\big(\rho_g u_g^2 + p_g\big)\big)' = p_g\alpha_g', \\
&-\sigma(\alpha_s\rho_s)' + (\alpha_s\rho_s u_s)' = 0, \\
&-\sigma(\alpha_s\rho_s u_s)' + \big(\alpha_s\big(\rho_s u_s^2 + p_s\big)\big)' = -p_g\alpha_g', \\
&-\sigma\rho_s' + (\rho_s u_s)' = 0.
\end{aligned} \tag{3.13}$$

The first equation of (3.13) can be re-written in the divergence form as

$$\big(\alpha_g\rho_g(u_g - \sigma)\big)' = 0, \tag{3.14}$$

which implies

$$\big(\alpha_g\rho_g(u_g - \sigma)\big) \equiv m = \text{constant}. \tag{3.15}$$

The second equation of (3.13) is expressed as

$$\big(\alpha_g\rho_g u_g(u_g - \sigma)\big)' + \alpha_g p_g' + p_g\alpha_g' = p_g\alpha_g'.$$

Using (3.14) and (3.15), and canceling the terms, we get from the last equation

$$(mu_g)' + \alpha_g p_g' = 0$$

or, since m is constant,

$$mu_g' + \alpha_g p_g' = 0.$$

Using (3.15) in a reverse way, we obtain from the last equation

$$\big(\alpha_g \rho_g (u_g - \sigma)\big) u_g' + \alpha_g p_g' = 0.$$

Canceling $\alpha_g > 0$ from the last equation to get

$$\rho_g (u_g - \sigma) u_g' + p_g' = 0,$$

or

$$(u_g - \sigma) u_g' + p_g'/\rho_g = 0.$$

The last equation yields the following equation in divergence form

$$\big((u_g - \sigma)^2 + 2h_g\big)' = 0, \qquad h_g' = p_g'/\rho_g. \tag{3.16}$$

The third and the fifth equations of (3.13) are trivial, since $\sigma = u_s \equiv$ constant, we discard them. Consider the fourth equation of (3.13). Adding up the second equation to the fourth equation of (3.13) we obtain the conservation of momentum of the mixture

$$-\sigma(\alpha_g \rho_g u_g)' + \big(\alpha_g\big(\rho_g u_g^2 + p_g\big)\big)' - \sigma(\alpha_s \rho_s u_s)' + \big(\alpha_s\big(\rho_s u_s^2 + p_s\big)\big)' = 0.$$

The last equation can be simplified using (3.14) and that $u_s = \sigma$ as follows

$$\big(\alpha_g \rho_g u_g (u_g - \sigma)\big)' + (\alpha_g p_g)' + \big(\alpha_s \rho_s (u_s - \sigma)\big)' + (\alpha_s p_s)' = 0,$$

or

$$(mu_g + \alpha_g p_g + \alpha_s p_s)' = 0. \tag{3.17}$$

We have established that the system (1.1) for the traveling wave solution is reduced to the system (3.14)–(3.17):

$$\begin{aligned}
&\big(\alpha_g \rho_g (u_g - \sigma)\big)' = 0,\\
&\big((u_g - \sigma)^2 + 2h_g\big)' = 0,\\
&(mu_g + \alpha_g p_g + \alpha_s p_s)' = 0,
\end{aligned} \tag{3.18}$$

where m is given by (3.15). Since (3.18) has a divergence form, it is independent of paths, if U is considered as a weak solution in the sense of nonconservative products.

Now, let η_ε be a sequence of smooth functions such that

$$\eta_\varepsilon(x) \to \begin{cases} \xi_0, & \text{for } x < 0,\\ \xi_1, & \text{for } x > 0, \end{cases} \quad \text{as } \varepsilon \to 0. \tag{3.19}$$

Then, it holds that

$$W_\varepsilon(x) = V \circ \eta_\varepsilon(x) \to W_0(x) = \begin{cases} V(\xi_0) = U_-, & \text{for } x < 0,\\ V(\xi_1) = U_+, & \text{for } x > 0. \end{cases} \tag{3.20}$$

As seen in the above argument, we obtain the sequence of corresponding smooth solutions $U_\varepsilon(x,t) = W_\varepsilon(x-\sigma t) = V \circ \eta_\varepsilon(x-\sigma t)$ of (1.1) that satisfy Eqs. (3.18). Let U still stand for $U_\varepsilon(x,t)$ in (3.18). Then, passing to the limit of (3.18) for $U = U_\varepsilon(x,t)$ as $\varepsilon \to 0$, we obtain the jump relations (3.10). Due to the stability result of weak solutions in the sense of nonconservative products (see Theorem 2.2 [7]), the function of the form (1.4), where the jump relations (3.10) hold, is a weak solution of the system (1.1) and independent of paths. □

In the sequel, we fix one state U_0 and look for any state U that can be connected with U_0 by a contact discontinuity. As seen above, the state U satisfies the equations

$$\begin{aligned} &\alpha_g\rho_g(u_g-u_s) = \alpha_{g0}\rho_{g0}(u_{g0}-u_s) := m, \\ &(u_g-u_s)^2 + 2h_g = (u_{g0}-u_{s0})^2 + 2h_{g0}, \end{aligned} \tag{3.21}$$

and

$$p_s = \frac{\alpha_{s0}p_{s0} - [mu_g + \alpha_g p_g]}{\alpha_s}. \tag{3.22}$$

The quantities in the gas phase can be found using (3.21). Then, the solid pressure is given by (3.22), and therefore the solid density can be calculated by using the equation of state (2.2).

3.3. Contact waves and admissibility criterion

Not all the jumps satisfying (3.10) are admissible. In the rest of this section we will investigate properties of contact waves as well as the admissibility criterion for contact discontinuities.

For simplicity, in the rest of this section we will drop the subindex "g" for the quantities in the gas phase. It follows from (3.21) that the gas density is a root of the nonlinear algebraic equation

$$F(U_0,\rho,\alpha) := \operatorname{sgn}(u_0-u_s)\left((u_0-u_s)^2 - \frac{2\kappa\gamma}{\gamma-1}\left(\rho^{\gamma-1}-\rho_0^{\gamma-1}\right)\right)^{1/2}\rho - \frac{\alpha_0(u_0-u_s)\rho_0}{\alpha} = 0. \tag{3.23}$$

Let us investigate properties of the function $F(U_0,\rho,\alpha)$. First, the domain of this function is given by

$$(u_0-u_s)^2 - \frac{2\kappa\gamma}{\gamma-1}\left(\rho^{\gamma-1}-\rho_0^{\gamma-1}\right) \geq 0.$$

That is

$$\rho \leq \bar\rho(U_0) := \left(\frac{\gamma-1}{2\kappa\gamma}(u_0-u_s)^2 + \rho_0^{\gamma-1}\right)^{\frac{1}{\gamma-1}}.$$

A straightforward calculation shows that

$$\frac{\partial F(U_0,\rho;\alpha)}{\partial\rho} = \frac{(u_0-u_s)^2 - \frac{2\kappa\gamma}{\gamma-1}(\rho^{\gamma-1}-\rho_0^{\gamma-1}) - \kappa\gamma\rho^{\gamma-1}}{((u_0-u_s)^2 - \frac{2\kappa\gamma}{\gamma-1}(\rho^{\gamma-1}-\rho_0^{\gamma-1}))^{1/2}}.$$

Consider first the case $u_0 - u_s > 0$. Then, the last expression yields

$$\begin{aligned} &\frac{\partial F(U_0,\rho;\alpha)}{\partial\rho} > 0, \quad \rho < \rho_{\max}(\rho_0,u_0), \\ &\frac{\partial F(U_0,\rho;\alpha)}{\partial\rho} < 0, \quad \rho > \rho_{\max}(\rho_0,u_0), \end{aligned}$$

where

$$\rho_{\max}(\rho_0, u_0) := \left(\frac{\gamma - 1}{\kappa\gamma(\gamma + 1)}(u_0 - u_s)^2 + \frac{2}{\gamma + 1}\rho_0^{\gamma-1}\right)^{\frac{1}{\gamma-1}}. \tag{3.24}$$

The values of F at the limits are given by

$$F(U_0, \rho = 0, \alpha) = F(U_0, \rho = \bar{\rho}, \alpha) = -\frac{\alpha_0(u_0 - u_s)\rho_0}{\alpha} < 0.$$

Thus, the function $\rho \mapsto F(U_0, \rho; \alpha)$ admits a root if and only if the maximum value is non-negative:

$$F(U_0, \rho = \rho_{\max}, \alpha) \geq 0.$$

In other words,

$$\alpha \geq \alpha_{\min}(U_0) := \frac{\alpha_0\rho_0|u_0 - u_s|}{\sqrt{\kappa\gamma}\rho_{\max}^{\frac{\gamma+1}{2}}(\rho_0, u_0)}. \tag{3.25}$$

Similar argument can be made for $u_0 - u_s < 0$.

The following lemma characterizes the properties of roots of Eq. (3.23).

Lemma 3.4. *(i) The function $F(U_0, \rho, \alpha)$ in (3.23) admits a zero if and only if $a \geq \alpha_{\min}(U_0)$. In this case, $F(U_0, \rho, \alpha)$ admits two distinct zeros, denoted by $\rho = \varphi_1(U_0, \alpha)$, $\rho = \varphi_2(U_0, \alpha)$ such that*

$$\varphi_1(U_0, \alpha) \leq \rho_{\max}(U_0) \leq \varphi_2(U_0, \alpha) \tag{3.26}$$

the equality in (3.26) holds only if $\alpha = \alpha_{\min}(U_0)$.

(ii) The state $(\varphi_1(U_0, \alpha), u = u_s + m/\alpha\varphi_1(U_0, \alpha)) \in G_1$ if $u_0 > u_s$, and the state $(\varphi_1(U_0, \alpha), u = u_s + m/\alpha\varphi_1(U_0, \alpha)) \in G_3$ if $u_0 < u_s$; the state $(\varphi_2(U_0, \alpha), u = u_s + m/\alpha\varphi_2(U_0, \alpha)) \in G_2$.

(iii)

- *If $\alpha > \alpha_0$, then*

$$\varphi_1(U_0, \alpha) < \rho_0 < \varphi_2(U_0, \alpha).$$

- *If $\alpha < \alpha_0$, then*

$$\begin{aligned} &\rho_0 < \varphi_1(U_0, \alpha) \quad \text{for } (\rho_0, u_0) \in G_1 \cup G_3, \\ &\rho_0 > \varphi_2(U_0, \alpha) \quad \text{for } (\rho_0, u_0) \in G_2. \end{aligned}$$

(iv) Given $u_s\alpha$, and let $\alpha_{\min}(\rho, u)$ be defined as in (3.25). The following conclusions hold

$$\begin{aligned} &\alpha_{\min}(\rho, u) < \alpha, \quad (\rho, u) \in G_i, i = 1, 2, 3, \\ &\alpha_{\min}(\rho, u) = \alpha, \quad (\rho, u) \in \mathcal{C}, \\ &\alpha_{\min}(\rho, u) = 0, \quad u = 0. \end{aligned}$$

The proof of Lemma 3.4 is omitted, since it is similar to the ones in [15].

Arguing similarly as in [15], one can construct a two-parameter family of solutions using shock waves, rarefaction waves and contact discontinuities. This is because there are two possible choices of contact discontinuities, as seen by Lemma 3.4. Therefore, to select a unique stationary wave, we need the following admissibility criterion. Observe that the equations of (3.21) also define a curve

$$\rho \mapsto \alpha = \alpha(U_-, \rho), \tag{3.27}$$

where $U_- = U_0$ and ρ varies between $\rho_\pm$. We will postulate the following admissibility criterion for contact waves.

(MC) *Through any contact wave connecting $U_\pm$, the volume fraction α given by (3.27) as a function of ρ must vary monotonically in ρ between the two values ρ_- and ρ_+. Furthermore, the total variation of the volume fraction α of any Riemann solution must not exceed $|\alpha_R - \alpha_L|$.*

A similar criterion was used in [15,21,10,11]. Given $u_s, U_0 = (\rho_0, u_0), \alpha_0, \alpha$, we denote by

$$\Upsilon(U_0, \alpha_0, \alpha) = (\rho, u, \alpha)$$

a state on the other side of the 5-contact wave from (ρ_0, u_0, α_0) satisfying the (MC) criterion. In view of Lemma 3.4, $\Upsilon(U_0, \alpha_0, \alpha)$ is a single point except on the resonant surface, where it consists of two points. In discussing about the location of point sin the (ρ, u)-plane, we still refer to $\Upsilon(U_0, \alpha_0, \alpha)$ as its projection onto this plane. The location of $\Upsilon(U_0, \alpha_0, \alpha)$ is given in the following lemma.

Lemma 3.5. *Given $u_s, U_0 = (\rho_0, u_0), \alpha_0, \alpha$, let*

$$\Upsilon(U_0, \alpha_0, \alpha) = (\rho, u, \alpha)$$

be a state on the other side of the 5-contact wave from (ρ_0, u_0, α_0) satisfying the (MC) criterion. It holds that

(i) If $U_0 \in G_1 \cup G_3$, then the admissible 5-contact satisfying the (MC) criterion corresponds to the zero $\varphi(U_0, \alpha) = \varphi_1(U_0, \alpha)$ defined by Lemma 3.4. This means that $\Upsilon(U_0, \alpha_0, \alpha) \in G_1 \cup G_3$.
(ii) If $U_0 \in G_2$, then the admissible 5-contact satisfying the (MC) criterion corresponds to the zero $\varphi(U_0, \alpha) = \varphi_2(U_0, \alpha)$ defined by Lemma 3.4. This means that $\Upsilon(U_0, \alpha_0, \alpha) \in G_2$.
(iii) It holds that

$$\Upsilon\big(U_0, \alpha_0, \alpha_{\min}(U_0)\big) \in \mathcal{C}_+, \quad u_0 > u_s,$$
$$\Upsilon\big(U_0, \alpha_0, \alpha_{\min}(U_0)\big) \in \mathcal{C}_-, \quad u_0 < u_s.$$

Proof. Since the proofs of (i) and (ii) are similar to the ones of Lemma 4.1 in [15], they are omitted.

For any given $U_0 = (\rho_0, u_0)$ and α_0, according to the part (i) of Lemma 3.4, it holds that

$$\varphi_1(U_0, \alpha) = \rho_{\max}(U_0) = \varphi_2(U_0, \alpha).$$

This means that the states on the other side of the 5-contacts from U_0 with gas volume fraction α_0 to a state with gas volume fraction $\alpha_{\min}(U_0)$ coincide and so admissible. In our notations, it is denoted by $\Upsilon(U_0, \alpha_0, \alpha_{\min}(U_0))$. According to the part (ii), the state $\Upsilon(U_0, \alpha_0, \alpha_{\min}(U_0))$ must be on the curve $\mathcal{C}$. Furthermore, since the state $\Upsilon(U_0, \alpha_0, \alpha_{\min}(U_0))$ and U_0 must lie on the same side with respect to the line $u = u_s$ in the (ρ, u)-plane, the conclusion follows. □

The above argument leads to defining elementary waves of the system (1.1), which make up solutions of the Riemann problem.

Definition 3.6. (a) (Elementary waves) The elementary waves of the system (1.1) are

- i-Lax shock waves, $i = 1, 2, 3, 4$;
- i-rarefaction waves, $i = 1, 2, 3, 4$;
- admissible 5-contact waves satisfying the criterion (MC).

(b) (Riemann solutions) Riemann solutions of (1.1) are made of elementary waves and are constraint with the criterion (MC).

4. Solving the Riemann problem via phase decomposition

4.1. Notations

Throughout this section, we will use the following notations:

(i) $W_k(U_i, U_j)$ ($S_k(U_i, U_j), R_k(U_i, U_j)$) denotes a k-wave (k-shock, or k-rarefaction wave, respectively) connecting the left-hand state U_i to the right-hand state U_j;
(ii) $W_m(U_i, U_j) \oplus W_n(U_j, U_k)$ indicates that there is an m-wave from the left-hand state U_i to the right-hand state U_j, followed by an n-wave from the left-hand state U_j to the right-hand state U_k.

4.2. Phase decomposition

Recall from Lemma 3.1 that

– Through i-Lax shock waves and i-rarefaction waves, $i = 1, 2$, the quantities in the solid phase ρ_s, u_s and α_s remain constant;
– Through j-Lax shock waves and j-rarefaction waves, $j = 3, 4$, the quantities in the gas phase ρ_g, u_g and α_g remain constant.

Clearly, the fact that the gas volume fraction remains constant is the same as that the solid volume fraction remains constant. Only 5-contacts involve quantities in both phases, and so the 5-contacts conserve as a "bridge" to provide a connection between the two phases. This suggests us that we can consider the wave structure in each phase separately. In other words, we use a phase decomposition approach to treat the Riemann problem. Without any confusion, when considering separately a phase, we still use the letter U to denote a state in that phase.

Precisely, quantities in the solid phase involve only in the 3rd, 4th and 5th characteristic fields, where

$$\lambda_3(U) < \lambda_5(U) < \lambda_4(U),$$

for all U in the phase domain. Thus, solid components in any Riemann solution involve only in these 3 families of waves which are separated by four constant states $U_L, V_\pm = (\rho_{s\pm}, u_{s\pm}), U_R$, as illustrated by Fig. 2. One has in the solid phase

$$V_- = (\rho_{s-}, u_{s-}) \in \mathcal{W}_3(U_L), \qquad V_+ = (\rho_{s+}, u_{s+}) \in \mathcal{W}_4(U_R). \tag{4.1}$$

Therefore, since u_s remains constant across the 5-contact, it holds that

$$\begin{aligned} u_{s-} &= \omega_3\big((\rho_{sL}, u_{sL}); \rho_{s-}\big) = u_s, \\ u_{s+} &= \omega_4\big((\rho_{sR}, u_{sR}); \rho_{s+}\big) = u_s, \end{aligned} \tag{4.2}$$

where ω_3, ω_4 are given by (2.19)–(2.20) and (3.5).

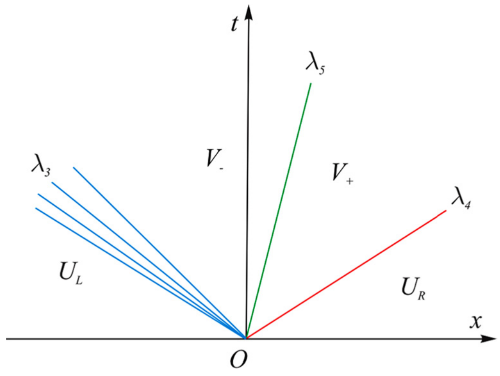

Fig. 2. Solid components of a Riemann solution in the (x,t)-plane.

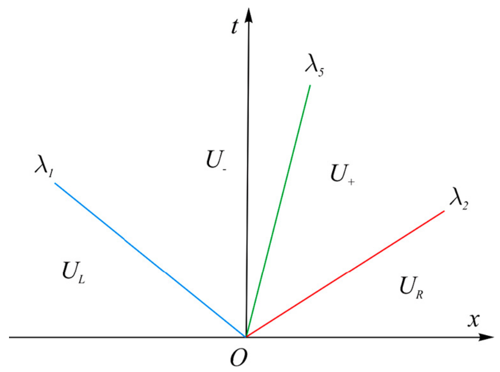

Fig. 3. Case 1: gas components of a Riemann solution in the (x,t)-plane.

Quantities in the gas phase also involve only in the 3 families of waves from the 1st, 2nd, and 5th characteristic fields. However, although

$$\lambda_1(U) < \lambda_2(U),$$

for all U in the phase domain, the values $\lambda_5(U)$ can be in any order with $\lambda_1(U)$ and $\lambda_2(U)$. This requires us to consider several cases as in the following.

Case 1: the contact wave lies between the two nonlinear waves This case represents a situation where a 5-contact wave lies in the middle region between a 1-wave and a 2-wave, see Fig. 3. Since

$$U_- = (\rho_{g-}, u_{g-}) \in \mathcal{W}_1(U_L), \qquad U_+ = (\rho_{g+}, u_{g+}) \in \mathcal{W}_2(U_R), \tag{4.3}$$

it holds that

$$\begin{aligned} u_{g-} &= \omega_1\big((\rho_{gL}, u_{gL}); \rho_{g-}\big), \\ u_{g+} &= \omega_2\big((\rho_{gR}, u_{gR}); \rho_{g+}\big), \end{aligned} \tag{4.4}$$

where ω_1, ω_2 are given by (2.17)–(2.18) and (3.5). From two equations in (4.2), two equations in (4.4), and two equations in (3.21) determine the 5 quantities

$$\rho_{g\pm}, \quad u_{g\pm}, \quad u_s.$$

Observe that this is an over-determined system, since $u_{s\pm} = u_s$.

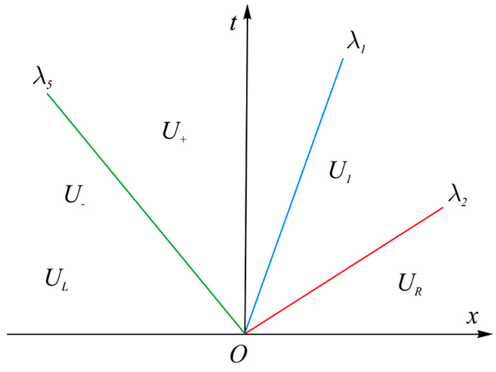

Fig. 4. Case 2: gas components of a Riemann solution in the (x, t)-plane.

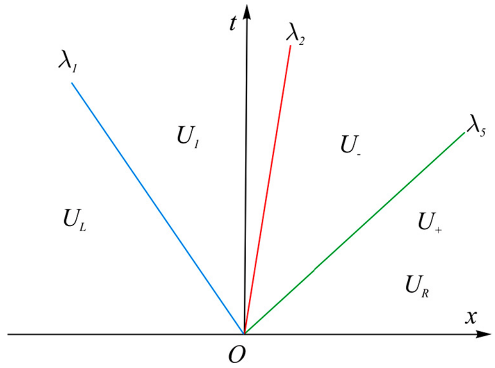

Fig. 5. Case 3: gas components of a Riemann solution in the (x, t)-plane.

Case 2: the contact wave lies on the left of the two nonlinear waves This case represents a situation where a 5-contact wave lies on the left of both waves in nonlinear families, see Fig. 4. One has

$$\begin{aligned} &U_- = (\rho_{g-}, u_{g-}) = (\rho_{gL}, u_{gL}),\\ &\{U_1\} = \mathcal{W}_1(U_+) \cap \mathcal{W}_2(U_R). \end{aligned} \tag{4.5}$$

From two equations in (4.2) and two equations in (3.21) determine the 3 quantities

$$\rho_{g+}, \quad u_{g+}, \quad u_s.$$

Again, this is also an over-determined system, since $u_{s\pm} = u_s$. The state U_1 can be determined easily.

Case 3: the contact wave lies on the right of the two nonlinear waves This case represents a situation where a 5-contact wave lies on the right of both waves in nonlinear families, see Fig. 5. One has

$$\begin{aligned} &U_+ = (\rho_{g+}, u_{g+}) = (\rho_{gL}, u_{gL}),\\ &\{U_1\} = \mathcal{W}_1(U_L) \cap \mathcal{W}_2(U_-). \end{aligned} \tag{4.6}$$

From two equations in (4.2) and two equations in (3.21) determine the 3 quantities

$$\rho_{g-}, \quad u_{g-}, \quad u_s.$$

Again, this is also an over-determined system, since $u_{s\pm} = u_s$. The state U_1 can be determined easily.

Case 4: collision of waves This case represents a situation where a there are three waves of the same speed, where two 5-contacts enclose a shock wave. We just consider the case where a 5-contact from $U_L \in G_1$ to a state $U_1 \in G_1$ with an intermediate value of volume fraction α between α_L and α_R, followed by a 1-wave from U_1 to $U_2 \in G_2$ with shock speed $\sigma_1(U_1, U_2) = \lambda_5(U_1)$, then followed by another 5-contact from U_2 to a state $U_3 \in G_2$, and finally arrive at U_R by a 2-wave, see Construction N3 below.

One has

$$(\rho_{g-}, u_{g-}) = (\rho_{gL}, u_{gL}),$$

so U_1 depends on the intermediate value of the gas volume fraction α only: $U_1 = U_1(a)$. Then U_2 is determined immediately using (3.8), and U_2 also depends only on α, i.e., $U_2 = U_2(\alpha)$. From two equations in (4.2), two equations in (3.21) for the second 5-contact between U_2 and U_3, and one equation from

$$U_3 \in \mathcal{W}_2(U_R)$$

one can determines the 4 quantities

$$\rho_{g3}, \quad u_{g3}, \quad u_s, \quad \alpha.$$

Then, other quantities can be calculated.

Other similar cases will be treated in the same way.

4.3. Solid components of Riemann solutions

Since this subsection deal with mainly the solid phase, for simplicity, in this subsection, we will refer to the components of Riemann solutions in the solid phase as Riemann solutions. From any $U \in \mathcal{W}_3(U_L)$, one can use an admissible contact wave to jump to a state $\bar{U}$. These states $\bar{U}$ form a curve in the (ρ_s, u_s)-plane, denoted by $W_{3+}(U_L)$. Let

$$U_+ \in W_{3+}(U_L) \cap W_4(U_R). \tag{4.7}$$

By definition of $W_{3+}(U_L)$, there exists a state $U_- \in \mathcal{W}_3(U_L)$ such that the state U_- is connected to U_+ be a contact wave. The Riemann solution will be a 3-wave from U_L to U_-, followed by a contact from U_- to U_+, and then followed by a 4-wave from U_+ to U_R. Using the notations, one has

$$W_3(U_L, U_-) \oplus W_5(U_-, U_+) \oplus W_4(U_+, U_R). \tag{4.8}$$

See Fig. 6. Whenever the contact wave is well-determined, solid components of Riemann solutions can be constructed for any initial data, by allowing that a vacuum can occur.

4.4. Gas components of Riemann solutions

By phase decomposition, more precisely, by considering now the solutions in the gas phase, we can see that this situation is quite similar to the one in the case of the model of fluid flows in a nozzle with discontinuous cross-section [15,21]. Thus, the constructions of Riemann solutions in [15,21], which were mainly relied on geometry of the space, can be applied. However, we will present out here an innovative analysis of the system that leads to results on existence and uniqueness of the Riemann problem for large data. For simplicity, we will consider only the case $U_L \in G_1 \cup \mathcal{C}_+$ and $\alpha_L \le \alpha_R$, since similar argument can be made for other cases.

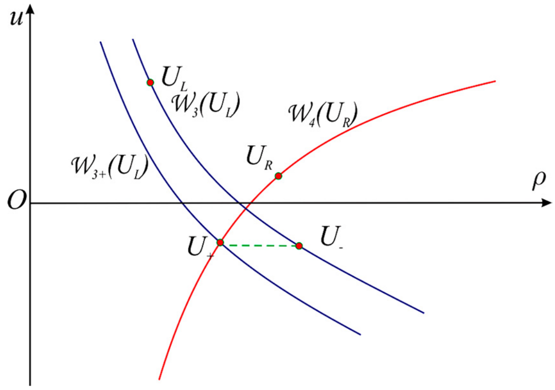

Fig. 6. Riemann solutions in the solid phase (4.8).

Notations First, let us introduce and recall some more notations. For $U_0 = (\rho_0, u_0)$, $U = (\rho, u)$-plane we set

$$\Phi_i(U_0, U) = u - \omega_i(U_0; \rho), \tag{4.9}$$

where ω_i, $i = 1, 2$ are defined by (2.17), (2.18) and (3.5). The curve $\mathcal{W}_i(U_0)$ is thus corresponding to the equation

$$\Phi(U_0, .) = 0.$$

It is not difficult to verify that the function $\Phi_i(U_0, .)$ for any fixed U_0 is smooth in the right-half plane $\rho > 0$, and continuous until the boundary $\rho = 0$. The region above $\mathcal{W}_i(U_0)$ is determined by the algebraic inequality

$$\Phi(U_0, .) > 0,$$

while region below $\mathcal{W}_i(U_0)$ is determined by the inequality

$$\Phi(U_0, .) < 0,$$

$i = 1, 2$. The last two inequalities can therefore be used to determine whether a point is in on one side or in the other side of $\mathcal{W}_i(U_0)$, $i = 1, 2$.

Fix an arbitrary u_s. As in Lemma 3.5, for any $U_0 = (\rho_0, u_0)$, let us denote by

$$\Upsilon(U_0, \alpha_0, \alpha) = (\rho, u, \alpha) \tag{4.10}$$

a state on the other side of the admissible 5-contact wave from (ρ_0, u_0, α_0).

Constructions of Riemann solutions for the case $U_L \in G_1 \cup \mathcal{C}_+, \alpha_L \le \alpha_R$ We will build the curve of composite waves of the 1- and 5-waves. Then, the existence of the Riemann problem can be obtained by letting the backward wave curve $\mathcal{W}_2(U_R)$ intersect with this curve of composite waves. As seen below, this construction makes sense for U_R in a large neighborhood of U_L, which contains parts of the regions G_1, G_2 and G_3.

There are three possibilities for the starting waves as follows. The first possibility is that the solution can start by an admissible 5-contact wave from U_L to some state

$$U_L^{@} = \Upsilon(U_L, \alpha_L, \alpha_R) \in G_1, \tag{4.11}$$

where Υ is defined as in (4.10). The solution then continues with a 1-wave along the wave curve $\mathcal{W}_1(U_L^{@})$.

The second possibility is that the solution may begin with a 5-contact from $U_L \in G_1$ to some intermediate state $U_1 = \Upsilon(U_L, \alpha_L, \alpha) \in G_1$ with an intermediate value of volume fraction $\alpha \in [\alpha_L, \alpha_R]$. Then, the solution continues with a 1-shock to $U_2 = U_1^{\#} \in G_2$ with speed

$$\sigma_1\big(U_1, U_1^{\#}\big) = \lambda_5\big(U_1^{\#}\big),$$

see Proposition 3.2. This 1-shock could be followed by 5-contact from $U_2 \in G_2$ to a state $U_3 = \Upsilon(U_2, \alpha, \alpha_R) \in G_2$ to shift the gas volume fraction until α_R. It is not difficult to check that the mapping

$$[\alpha_L, \alpha_R] \ni \alpha \mapsto U_3 = \Pi(\alpha) := \Upsilon\big(\big(\Upsilon(U_L, \alpha_L, \alpha)\big)^{\#}, \alpha, \alpha_R\big) \tag{4.12}$$

is locally Lipschitz. We thus can define a (continuous) path

$$\mathcal{L} = \big\{\Pi(\alpha) \bigm| \alpha \in [\alpha_L, \alpha_R]\big\}, \tag{4.13}$$

where $\Pi(\alpha), \alpha \in [\alpha_L, \alpha_R]$, is defined by (4.12).

Next, it follows from Proposition 3.2 that there exists a point $U_L^{\#}$ such that

$$U_L^{\#} = \big(\rho_L^{\#}, u_L^{\#}\big) \in \mathcal{W}_1(U_L), \qquad \sigma_1\big(U_L, U_L^{\#}\big) = \lambda_5(U_L). \tag{4.14}$$

This gives a third possibility where the solution can start with a strong 1-shock from U_L to some state $U = (\rho, u) \in \mathcal{W}_1(U_L) \cap (G_2 \cup \mathcal{C}_-)$ with

$$\sigma_1(U_L, U) \le \lambda_5(U_L),$$

and then continues with a 5-contact wave from U to a state $U^{\#} \in G_2$ by an admissible 5-contact wave. Let $U_L^{\#} = (\rho_L^{\#}, u_L^{\#})$ and let

$$U_{L*} = (\rho_{L*}, u_{L*}) \in \mathcal{W}_1(U_L) \cap \mathcal{C}_-. \tag{4.15}$$

The condition $U = (\rho, u) \in \mathcal{W}_1(U_L) \cap (G_2 \cup \mathcal{C}_-), \sigma_1(U_L, U) \le \lambda_5(U_L)$ becomes

$$\rho_L^{\#} \le \rho \le \rho_{L*}.$$

The solution then can continue with a 5-contact from any point in $\mathcal{W}_1(U_L) \cap (G_2 \cup \mathcal{C}_-)$. Let $U_L^{\#@}$ and $U_{L*}^{@}$ denote the states on the other side of the admissible 5-contacts from $U_L^{\#}$ defined by (4.14) and from U_{L*} defined by (4.15), respectively:

$$\begin{aligned} U_L^{\#@} &= \big(\rho_L^{\#@}, u_L^{\#@}\big) = \Upsilon\big(U_L^{\#}, \alpha_L, \alpha_R\big), \\ U_{L*}^{@} &= \big(\rho_{L*}^{@}, u_{L*}^{@}\big) = \Upsilon(U_{L*}, \alpha_L, \alpha_R). \end{aligned} \tag{4.16}$$

The states on the other side of the admissible 5-contact waves from $U \in \mathcal{W}_1(U_L)$, U between the state $U_L^{\#}$ defined by (4.14) and the state U_{L*} defined by (4.15) from a curve, denoted by $\mathcal{W}_{1@}(U_L)$. Precisely,

$$\mathcal{W}_{1@}(U_L) = \big\{U^{@} := \Upsilon(U, a_L, a_R) \bigm| U = (\rho, u) \in \mathcal{W}_1(U_L), \rho_L^{\#} \le \rho \le \rho_{L*}\big\}. \tag{4.17}$$

The composite curve $\mathcal{W}_{1@}(U_L)$ thus has the two end-points $U_L^{\#@}$ and $U_{L*}^{@}$ defined by (4.16). See Fig. 7.

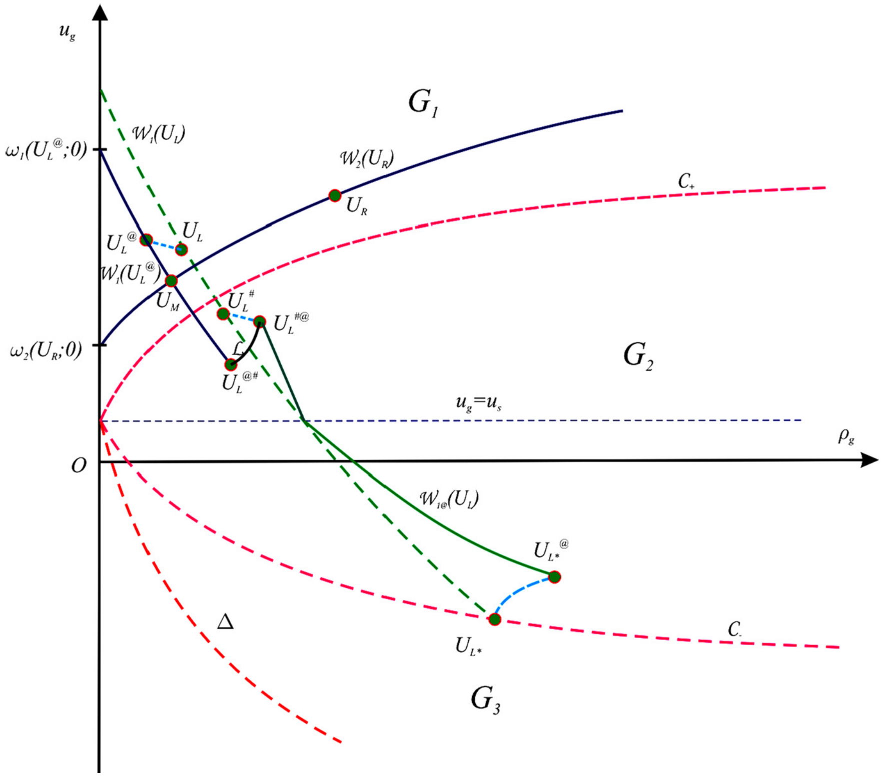

Fig. 7. The curve of composite waves $\Gamma = \mathcal{W}_1(U_L^@) \cup \mathcal{L} \cup \mathcal{W}_{1@}(U_L)$ defined by (4.18), and an intersection of Γ and the curve $\mathcal{W}_2(U_R)$ on the path $\mathcal{W}_1(U_L^@)$ that leads to a Riemann solution.

We now define

$$\Gamma = \mathcal{W}_1\big(U_L^@\big) \cup \mathcal{L} \cup \mathcal{W}_{1@}(U_L), \tag{4.18}$$

where $U_L^@, \mathcal{L}$ and $\mathcal{W}_{1@}(U_L)$ are defined by (4.11), (4.13) and (4.17), respectively. Since the two end-points of $\mathcal{L}$ corresponding to $\alpha = \alpha_R$ and $\alpha = \alpha_L$ coincide with the an end-point of $\mathcal{W}_1(U_L^@)$ and $\mathcal{W}_{1@}(U_L)$, respectively, Γ is a (continuous) curve.

Any intersection point U_M between the curve of the composite waves Γ and the backward curve of 2-waves $\mathcal{W}_2(U_R)$ could complete the Riemann solution by a 2-wave from U_M to U_R. The curves Γ and $\mathcal{W}_2(U_R)$ in fact intersect whenever

$$\omega_1\big(U_L^@;0\big) > \omega_2(U_R;0), \qquad \Phi_2\big(U_R, U_{L*}^@\big) < 0.$$

However, when $U_R \in G_3$ and the intersection point U_M is located in the region $u < u_s$, the compatibility condition

$$\sigma_2(U_M, U_R) \geq \lambda_5(U_M) \tag{4.19}$$

must hold, in order for the 2-wave $W_2(U_M, U_R)$ to precede the 5-contact wave arriving at U_M. As seen from Proposition 3.2, there corresponds to any point $U \in \mathcal{W}_{1@}(U_L), u < u_s$, a point $U' \in \mathcal{W}_2(U) \cap G_3$ such that

$$\sigma_2\big(U, U'\big) = \lambda_5(U).$$

Let Δ denote the set of these points $U^\#$, which form a curve in G_3:

$$\Delta = \{U \in G_3 : \mathcal{W}_2(U) \cap \mathcal{W}_{1@}(U_L) = \{U^\natural\}\}, \tag{4.20}$$

where $U^\natural$ is defined in Proposition 3.2 (part (b)), which satisfies

$$\sigma_2(U, U^\natural) = \lambda_5(U).$$

Geometrically, the condition (4.19) means that U_R is located above or on the curve Δ.

Interestingly, whenever $U_L^{@\#}$ is located below the curve $\mathcal{W}_2(U_R)$ and $U_L^{\#@}$ is located above it, the curve $\mathcal{W}_2(U_R)$ intersects with all the three parts $\mathcal{W}_1(U_L^@)$, $\mathcal{L}$, and $\mathcal{W}_{1@}(U_L)$. The Riemann problem then admits three distinct solutions with different configurations. The above argument leads us to the following result on the existence of Riemann solutions.

Theorem 4.1 *(Existence for large data). Consider the case $U_L \in G_1 \cup \mathcal{C}_+$, $\alpha_L \le \alpha_R$. Assume that the right-hand state $U_R \in G_1 \cup \mathcal{C} \cup G_2$ such that*

$$\begin{aligned} u_R - u_L^@ &< \left(\int_0^{\rho_L^@} + \int_0^{\rho_R}\right) \frac{\sqrt{p'(y)}}{y} dy, \\ u_R - u_{L*}^@ &> \int_{\rho_{L*}^@}^{\rho_R} \frac{\sqrt{p'(y)}}{y} dy, \end{aligned} \tag{4.21}$$

where $U_L^@ = (\rho_L^@, u_L^@)$ and $U_{L*}^@ = (\rho_{L*}^@, u_{L*}^@)$ are defined by (4.11) and (4.16), respectively. The Riemann problem for (1.1) admits a solution made of Lax shocks, rarefaction waves, and admissible contacts.

Remark 4.1. As indicated in the above argument, the result of Theorem 4.1 is still valid if the condition U_R is above G_3 is replaced by a weaker condition that U_R is above Δ defined by (4.20), see Fig. 7.

Proof. It is easy to see that the curve $\mathcal{W}_{1@}(U_L)$ defined by (4.17) can be parameterized by a function $u = u_{1@}(\rho)$. Furthermore, let us show that this function $u_{1@}$ is decreasing. Indeed, if there were two states $U_1 = (\rho_1, u_1)$ and $U_2 = (\rho_2, u_2)$ on the curve $\mathcal{W}_1(U_L)$, U_1 is on a higher position than U_2 such that

$$u_{1@}(\rho_1) < u_{1@}(\rho_2),$$

i.e., the state $\Upsilon(U_1, a_L, a_R)$ is on a lower position than the state $\Upsilon(U_2, \alpha_L, \alpha_R)$ on the curve $\mathcal{W}_{1@}(U_L)$. By continuity, there would exist two states $V_1, V_2 \in \mathcal{W}_1(U_L) \cap G_2$, V_1 is on a higher position than V_2 on $\mathcal{W}_1(U_L)$ such that

$$\Upsilon(V_2, \alpha_L, \alpha_R) = \Upsilon(V_1, \alpha_L, \alpha_R) := V_0,$$

which is a contradiction, since there is a unique admissible 5-contact, which does not cross the resonant surface, from V_0 with the gas volume fraction α_R to another state with the gas volume fraction α_L.

Next, the first inequality in (4.21) can be re-written as

$$\omega_1(U_L^@; 0) > \omega_2(U_R; 0), \tag{4.22}$$

where ω_1 and ω_2 are defined by (2.17), (2.18) and (3.5). The condition (4.22) means that the u-intercept of the "increasing" curve $\mathcal{W}_2(U_R)$ (curve parameterized as an increasing function of the specific volume as a function of the density) is below the one of the "decreasing" curve $\mathcal{W}_1(U_L^@)$ in the (ρ, u)-plane. The second inequality of (4.21) can be re-written as

$$u_{L*}^@ < \omega_2\big(U_R; \rho_{L*}^@\big),$$

or

$$\Phi_2\big(U_R, U_{L*}^@\big) < 0, \tag{4.23}$$

where Φ_2 is defined by (4.9). The condition (4.23) means that the end-point $U_{L*}^@$ of Γ is located below the curve $\mathcal{W}_2(U_R)$ in the (ρ, u)-plane. The above argument implies that the two curves Γ and $\mathcal{W}_2(U_R)$ intersect. This leads us to a Riemann solution whose configuration can be described as follows. First, let the two curves Γ and $\mathcal{W}_2(U_R)$ intersect on the path $\mathcal{W}_L(U_L^@)$, where $U_L^@ = \Upsilon(U_L, a_L, a_R)$. Let

$$U_M \in \mathcal{W}_1\big(U_L^@\big) \cap \mathcal{W}_3(U_R).$$

The solution is described by

$$W_5\big(U_L, U_L^@\big) \oplus W_1\big(U_L^@, U_M\big) \oplus W_2(U_M, U_R). \tag{4.24}$$

Second, let the two curves Γ and $\mathcal{W}_2(U_R)$ intersect on the path $\mathcal{L}$. Let

$$U_M \in \mathcal{L} \cap \mathcal{W}_3(U_R).$$

The solution has the form

$$W_5(U_L, U_1) \oplus S_1(U_1, U_2) \oplus W_5(U_2, U_3) \oplus W_2(U_3, U_R), \tag{4.25}$$

where $U_1 = \Upsilon(U_L, \alpha_L, \alpha), U_2 = U_1^\#, U_3 = \Upsilon(U_2, \alpha, \alpha_R)$ for some $\alpha \in [\alpha_L, \alpha_R]$. Third, let the two curves Γ and $\mathcal{W}_2(U_R)$ intersect on the path $\mathcal{W}_{1@}(U_L)$ and let

$$U_M \in \mathcal{W}_{1@}(U_L) \cap \mathcal{W}_3(U_R).$$

The solution is described by

$$W_1(U_L, U_N) \oplus W_1(U_N, U_M) \oplus W_2(U_M, U_R), \tag{4.26}$$

where $U_N = \Upsilon(U_M, \alpha_R, \alpha_L)$. □

Theorem 4.2 *(Uniqueness). Under the assumptions of Theorem 4.1, the Riemann problem for (1.1) admits a unique solution if*

$$\min_{U \in \mathcal{L}} \Phi_2(U_R, U) > 0, \quad \text{or} \quad \max_{U \in \mathcal{L}} \Phi_2(U_R, U) < 0, \tag{4.27}$$

where Φ_2 is defined by (4.9). In particular, if $U_R, U_L \in G_1$ such that

$$u_s < u_R - \int_0^{\rho_R} \frac{\sqrt{p'(y)}}{y} dy < u_L^@ + \int_0^{\rho_L^@} \frac{\sqrt{p'(y)}}{y} dy, \tag{4.28}$$

where $U_L^@ = (\rho_L^@, u_L^@)$ is defined by (4.11), then the Riemann problem for (1.1) admits a unique solution of the form (4.24).

Proof. The condition (4.27) implies that the curve $\mathcal{W}_2(U_R)$ may intersect with Γ exactly either on the path $\mathcal{W}_1(U_L^@)$, or on the path $\mathcal{W}_{1@}(U_L)$. Note that the path $\mathcal{L}$ is always on one side of $\mathcal{W}_2(U_R)$. It is clear that the curve $\mathcal{W}_1(U_L^@)$ can be parameterized in such a way that the velocity is given as decreasing function of the density. This is also the case for the curve $\mathcal{W}_{1@}(U_L)$, see the proof of Theorem 4.1. The curve $\mathcal{W}_2(U_R)$ is given by an increasing function $u = \omega_2(U_R; \rho)$. Thus, it follows from Theorem 4.1 that the intersection of $\mathcal{W}_2(U_R)$ and Γ contains exactly one point. The Riemann problem therefore admit a unique solution. This establishes the first conclusion.

Next, the inequality on the left of (4.28) means that

$$\omega_2(U_R; 0) > u_s,$$

where ω_2 is defined by (2.18). Since the slope of ω_2 is larger than the one of $\mathcal{C}_+$, the last inequality implies that the curve $\mathcal{W}_2(U_R)$ remains in G_1. Therefore, the path $\mathcal{L}$ is located below $\mathcal{W}_2(U_R)$. In this case, $\mathcal{W}_2(U_R)$ intersects with Γ only on the path $\mathcal{W}_1(U_L^@)$. This yields the second conclusion. Theorem 4.2 is completely proved. $\square$

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