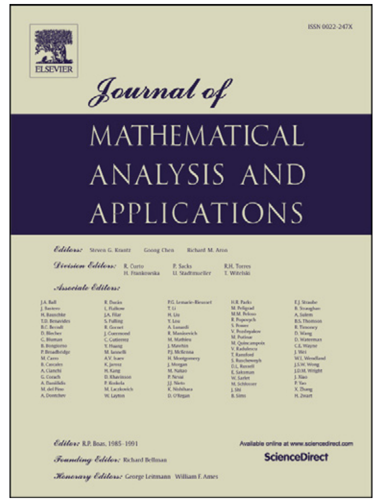
Accepted Manuscript



Strong solutions to the equations of electrically conductive magnetic fluids

Youcef Amirat, Kamel Hamdache

PII: S0022-247X(14)00620-9
DOI: 10.1016/j.jmaa.2014.06.073
Reference: YJMAA 18664

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 16 December 2013

Strong solutions to the equations of electrically conductive magnetic fluids

Youcef Amirat* and Kamel Hamdache†

Abstract

We study the equations of flow of an electrically conductive magnetic fluid, when the fluid is subjected to the action of an external applied magnetic field. The system is formed by the incompressible Navier-Stokes equations, the magnetization relaxation equation of Bloch type and the magnetic induction equation. The system takes into account the Kelvin and Lorentz force densities. We prove the local-in-time existence of the unique strong solution to the system equipped with initial and boundary conditions. We also establish a blow-up criterion for the local strong solution.

2010 Mathematics Subject Classification: 35Q35, 76D05.
Key words and phrases: magnetic fluid, Navier-Stokes equations, magnetization relaxation equation, induction equation, strong solution, blow-up criterion.

1 Introduction

Electrically conductive magnetic fluid models describe the dynamics of electromagnetic fine particles having internal rotations in a carrier fluid. Many applications, with different modeling, are used in engineering sciences in view of their potential applications in clutches, valves, actuators and also in bioengineering and medicine sciences. See [4, 20, 27] for example.

The model we are concerned in this work is described by the fluid velocity U, the magnetic induction B, the electric induction D and the magnetization field M; the electric polarization is assumed to be 0. The magnetic induction satisfies the state law $B = \mu_0(H + M)$ where H is the magnetic field and μ_0 is the magnetic permeability constant. The electromagnetic fields satisfy the Maxwell equations while the magnetization obeys the Bloch equation. The fluid velocity satisfies the incompressible Navier-Stokes equation with volume forces as the Kelvin and Lorentz force densities.

Consider a laminar incompressible flow of a Newtonian and electrically conducting magnetic fluid under the influence of an applied external magnetic field. The fluid flows in a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$. Let $T > 0$ be a fixed time and $\Omega_T = (0, T) \times \Omega$. The electromagnetic fields satisfy the Maxwell equations where the displacement current is neglected ([13, 14, 18, 27]):

$$\partial_t B + \operatorname{curl} E = 0, \quad \operatorname{div} B = 0, \tag{1}$$

$$J = \sigma 1_\Omega (E + U \wedge B), \tag{2}$$

$$\operatorname{curl} H = J. \tag{3}$$

*Laboratoire de Mathématiques, CNRS UMR 6620, Université Blaise Pascal (Clermont-Ferrand 2) 63177 Aubière Cedex, France (Email: amirat@math.univ-bpclermont.fr)

†Centre de Mathématiques Appliquées, CNRS, Ecole Polytechnique, 91128 Palaiseau Cedex, France (Email: hamdache@cmapx.polytechnique.fr)

Equations (1)–(3) are considered in $(0,T)\times\mathbb{R}^3$, J is the electric current density, E is the electric field, σ is the electric conductivity and 1_Ω is the characteristic function of Ω. The magnetization M obeys the equation

$$\partial_t M + (U\cdot\nabla)M = \frac{1}{2}\operatorname{curl} U\wedge M - \frac{1}{t_m}(M-\chi_m H) - \frac{\mu_0}{4\xi_r}M\wedge(M\wedge H) \quad \text{in } \Omega_T, \tag{4}$$

where t_m is the Brownian relaxation time, ξ_r is the vortex (rotational) viscosity and χ_m denotes the total magnetic susceptibility. Equation (4), proposed by M.I. Shliomis [23], is the magnetization relaxation equation of Bloch type, which takes account of rotational Brownian motion. It follows from (2) and (3) that the electric field E satisfies

$$E + U\wedge B = \frac{1}{\sigma}\operatorname{curl} H \quad \text{in } \Omega_T, \tag{5}$$

then taking the curl of the above equation and using (1) we deduce that B satisfies the equation

$$\partial_t B + \operatorname{curl}\left(\frac{1}{\sigma}\operatorname{curl} H\right) = \operatorname{curl}(U\wedge B) \quad \text{in } \Omega_T.$$

The fluid motion is governed by the incompressible Navier-Stokes equations

$$\begin{aligned} &\operatorname{div} U = 0 \quad \text{in } \Omega_T,\\ &\rho(\partial_t U + (U\cdot\nabla)U) - \mu\Delta U + \nabla p = \mu_0 M\cdot\nabla H + \frac{\mu_0}{2}\operatorname{curl}(M\wedge H) + J\wedge B \quad \text{in } \Omega_T, \end{aligned}$$

where p is the pressure, ρ is the fluid density and μ is the dynamical viscosity. The term $\mu_0(M\cdot\nabla)H$ represents the Kelvin body force due to magnetization, $\mu_0 M\wedge H$ is the body torque density which causes the magnetic nanoparticles and surrounding fluid to spin and the term $J\wedge B$ represents the Lorentz force due to the induced electric current of magneto-hydrodynamics (MHD).

We require the functions U, B and E to satisfy the following boundary conditions. For the velocity we impose the no-slip boundary condition, namely

$$U = 0 \quad \text{on } (0,T)\times\partial\Omega.$$

We also impose the perfect conductor boundary condition

$$E\wedge n = 0, \quad B\cdot n = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{6}$$

n being the unit outward normal vector to $\partial\Omega$. It results from (5) and (6) that

$$\operatorname{curl} H\wedge n = 0 \quad \text{on } (0,T)\times\partial\Omega.$$

We thus obtain a boundary-value problem set in Ω_T for the velocity U, the pressure p, the magnetic field H and the magnetization M formed by the equations

$$\operatorname{div} B = 0,\ \ \operatorname{div} U = 0, \tag{7}$$

$$\partial_t M + (U\cdot\nabla)M = \frac{1}{2}\operatorname{curl} U\wedge M - \frac{1}{t_m}(M-\chi_m H) - \frac{\mu_0}{4\xi_r}M\wedge(M\wedge H), \tag{8}$$

$$\partial_t B + \operatorname{curl}\left(\frac{1}{\sigma}\operatorname{curl} H\right) = \operatorname{curl}(U\wedge B), \tag{9}$$

$$\rho(\partial_t U + (U\cdot\nabla)U) - \mu\Delta U + \nabla p = \mu_0 M\cdot\nabla H + \frac{\mu_0}{2}\operatorname{curl}(M\wedge H) + J\wedge B, \tag{10}$$

where

$$B = \mu_0(H + M) \quad \text{and} \quad J = \operatorname{curl} H. \tag{11}$$

System (7)–(11) is equipped with the boundary and initial conditions

$$B \cdot n = 0, \quad \operatorname{curl} H \wedge n = 0, \quad U = 0 \quad \text{on } (0,T) \times \partial\Omega, \tag{12}$$

$$U|_{t=0} = U_0,\ M|_{t=0} = M_0,\ B|_{t=0} = B_0 \quad \text{in } \Omega. \tag{13}$$

Before we can formulate our main result we need to introduce some notations. We assume that Ω is a simply-connected bounded domain in $\mathbb{R}^3$, with smooth boundary $\partial\Omega$. Let $L^q(\Omega)$ and $W^{s,q}(\Omega)$ $(1 \le q \le \infty,\ s \in \mathbb{R})$ be the usual Lebesgue and Sobolev spaces of scalar-valued functions, respectively. When $q = 2$, $W^{s,q}(\Omega)$ is denoted by $H^s(\Omega)$. By $\|\cdot\|$ and $(\cdot,\cdot)$ we denote the L^2-norm and its scalar product, respectively. The Hölder spaces $C^{k,\alpha}(\overline{\Omega})$ $(k \in \mathbb{N},\ 0 < \alpha < 1)$ are defined as the subspaces of $C^k(\overline{\Omega})$ consisting of functions whose k-th order partial derivatives are Hölder continuous with exponent α. We set $\mathbb{L}^q(\Omega) = (L^q(\Omega))^3$, $\mathbb{W}^{s,q}(\Omega) = (W^{s,q}(\Omega))^3$, $\mathbb{H}^s(\Omega) = (H^s(\Omega))^3$ and $C^{k,\alpha}(\overline{\Omega},\mathbb{R}^3) = (C^{k,\alpha}(\overline{\Omega}))^3$. We denote by $\mathscr{D}(\Omega,\mathbb{R}^3)$ (resp. $\mathscr{D}(\overline{\Omega},\mathbb{R}^3)$) the space of infinitely differentiable functions with compact support in Ω (resp. $\overline{\Omega}$) and valued in $\mathbb{R}^3$. We introduce the classical function spaces in the theory of the Navier-Stokes equations (see [9, 10, 15, 16, 17, 25, 26]):

$$\begin{aligned}
&\mathscr{D}_s(\Omega) = \left\{ v \in \mathscr{D}(\Omega,\mathbb{R}^3) : \operatorname{div} v = 0 \ \text{ in } \Omega \right\}, \\
&\mathcal{U} \ = \ \text{closure of } \mathscr{D}_s(\Omega) \text{ in } \mathbb{H}^1(\Omega), \\
&\mathcal{U}_0 \ = \ \text{closure of } \mathscr{D}_s(\Omega) \text{ in } \mathbb{L}^2(\Omega).
\end{aligned}$$

As is well known,

$$\begin{aligned}
&\mathcal{U} = \left\{ v \in \mathbb{H}^1_0(\Omega) : \ \operatorname{div} v = 0 \ \text{ in } \Omega \right\}, \\
&\mathcal{U}_0 = \left\{ v \in \mathbb{L}^2(\Omega) : \operatorname{div} v = 0 \ \text{ in } \Omega,\ v \cdot n = 0 \ \text{ on } \partial\Omega \right\}, \\
&\mathcal{U} \subset \mathcal{U}_0 \subset \mathcal{U}' \ = \ \text{dual space of } \mathcal{U} \text{ when } \mathcal{U}_0 \text{ is identified with its dual.}
\end{aligned}$$

We also introduce the spaces

$$\begin{aligned}
&\mathscr{D}_s(\overline{\Omega}) = \left\{ C \in \mathscr{D}(\overline{\Omega},\mathbb{R}^3) : \operatorname{div} C = 0 \ \text{ in } \Omega,\ C \cdot n = 0 \ \text{ on } \partial\Omega \right\}, \\
&\mathcal{B} \ = \ \text{closure of } \mathscr{D}_s(\overline{\Omega}) \text{ in } \mathbb{H}^1(\Omega), \\
&\mathcal{B}_0 \ = \ \text{closure of } \mathscr{D}_s(\overline{\Omega}) \text{ in } \mathbb{L}^2(\Omega).
\end{aligned}$$

We have

$$\begin{aligned}
&\mathcal{B} = \left\{ C \in \mathbb{H}^1(\Omega) : \ \operatorname{div} C = 0 \ \text{ in } \Omega,\ C \cdot n = 0 \ \text{ on } \partial\Omega \right\}, \\
&\mathcal{B}_0 = \mathcal{U}_0, \\
&\mathcal{B} \subset \mathcal{B}_0 \subset \mathcal{B}' \ = \ \text{dual space of } \mathcal{B}.
\end{aligned}$$

Note that $\left(\int_\Omega |\operatorname{curl} \boldsymbol{C}|^2\, dx\right)^{\frac{1}{2}}$ defines a norm on $\mathcal{B}$ which is equivalent to that induced by $\mathbb{H}^1(\Omega)$ on W, see [7, Chap. 7, Theorem 6.1]. Recall that $v \cdot n$ makes sense in $H^{-\frac{1}{2}}(\partial\Omega)$ when v belongs to the space

$$H(\operatorname{div},\Omega) = \{ v \in \mathbb{L}^2(\Omega) : \operatorname{div} v \in L^2(\Omega) \}$$

and we have the Stokes formula: $\forall v \in H(\mathrm{div}, \Omega)$, $\forall \varphi \in H^1(\Omega)$,

$$\int_\Omega v \cdot \nabla \varphi \, dx = -\int_\Omega \varphi \, \mathrm{div}\, v \, dx + \langle v \cdot n, \varphi \rangle_{\partial\Omega},$$

where $\langle \cdot, \cdot \rangle_{\partial\Omega}$ is the duality pairing between $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$. Similarly, if v belongs to the space

$$H(\mathrm{curl}, \Omega) = \{v \in \mathbb{L}^2(\Omega) : \mathrm{curl}\, v \in \mathbb{L}^2(\Omega)\},$$

then v has a tangential component $v \wedge n \in \mathbb{H}^{-\frac{1}{2}}(\partial\Omega)$ and the following Green's formula holds:

$$\forall w \in \mathbb{H}^1(\Omega), \quad \int_\Omega \mathrm{curl}\, v \cdot w \, dx = \int_\Omega v \cdot \mathrm{curl}\, w \, dx + \langle v \wedge n, w \rangle_{\partial\Omega}.$$

Without loss of generality, in the sequel we will suppose that $\rho = c_p = \sigma = \mu_0 = 1$. For notational convenience, we refer to Problem (7)–(13) to as problem $(\mathcal{P})$. We assume that

$$U_0 \in \mathbb{H}^2(\Omega) \cap \mathcal{U}, \tag{14}$$

$$M_0 \in \mathbb{H}^2(\Omega), \tag{15}$$

$$B_0 \in \mathbb{H}^2(\Omega) \cap \mathcal{B}. \tag{16}$$

Definition 1. *We say that (U, M, H) is a strong solution of problem $(\mathcal{P})$ if:*

(i)

$$\begin{aligned}
&U \in C([0,T]; \mathcal{U} \cap \mathbb{H}^2(\Omega)) \cap L^2(0,T; \mathbb{H}^3(\Omega)) \cap W^{1,\infty}(0,T; \mathbb{L}^2(\Omega)) \cap H^1(0,T; \mathcal{U}),\\
&M \in L^\infty(0,T; \mathbb{H}^2(\Omega)) \cap W^{1,\infty}(0,T; \mathbb{L}^2(\Omega)) \cap H^1(0,T; \mathbb{H}^1(\Omega)),\\
&H \in L^\infty(0,T; \mathbb{H}^2(\Omega)) \cap W^{1,\infty}(0,T; \mathbb{L}^2(\Omega)) \cap H^1(0,T; \mathbb{H}^1(\Omega)).
\end{aligned}$$

(ii) The function M satisfies the problem

$$\begin{aligned}
&\partial_t M + (U \cdot \nabla) M = \frac{1}{2} \mathrm{curl}\, U \wedge M - \frac{1}{t_m}(M - \chi_m H) - \frac{1}{4\xi_r} M \wedge (M \wedge H) \quad \text{in } \Omega_T,\\
&M|_{t=0} = M_0.
\end{aligned}$$

(iii) The function H solves the problem

$$\begin{aligned}
&\partial_t H + \mathrm{curl}^2 H = \mathrm{curl}\,(U \wedge B) - \partial_t M \quad \text{in } \Omega_T,\\
&\mathrm{div}\, B = 0 \quad \text{in } \Omega_T,\\
&B \cdot n = 0, \quad \mathrm{curl}\, H \wedge n = 0 \quad \text{on } (0,T) \times \partial\Omega,\\
&H|_{t=0} = H_0 \equiv B_0 - M_0 \quad \text{in } \Omega,
\end{aligned}$$

where $B = H + M$.

(iv) Equation (10)*, together with the incompressibility condition* $\mathrm{div}\, U = 0$ *in Ω_T, holds weakly, that is, for every $v \in \mathcal{U}$,*

$$\begin{aligned}
&\frac{d}{dt} \int_\Omega U \cdot v \, dx + \int_\Omega (U \cdot \nabla) U \cdot v \, dx + \mu \int_\Omega \nabla U \cdot \nabla v \, dx\\
&= \int_\Omega (M \cdot \nabla) H \cdot v \, dx + \frac{1}{2} \int_\Omega (\mathrm{curl}\,(M \wedge H)) \cdot v \, dx + \int_\Omega (J \wedge B) \cdot v \, dx \quad \text{in } \mathscr{D}'(]0,T[),\\
&U|_{t=0} = U_0.
\end{aligned}$$

(v) There exists $p \in L^2(0,T;H^2(\Omega))$ such that equation (10) holds a.e. in Ω_T.

Definition 2. *A positive number $T^\star$ is called a finite blow-up time of the strong solution (U,M,H) of problem $(\mathcal{P})$ if*

$$J(t) < \infty \quad \text{for } 0 \le t < T^\star \text{ and } \lim_{t\to T^\star} J(t) = +\infty,$$

where the functional $J(t)$ is defined by

$$\begin{aligned} J(t) = \sup_{0\le s\le t} \Big(&\|U(s)\|_{\mathbb{H}^2(\Omega)} + \|U_t(s)\| + \|M(s)\|_{\mathbb{H}^2(\Omega)} + \|H(s)\|_{\mathbb{H}^2(\Omega)} + \|M_t(s)\| + \|H_t(s)\|\Big) \\ &+ \int_0^t \Big(\|U(s)\|^2_{\mathbb{H}^3(\Omega)} + \|U_t(s)\|^2_{\mathbb{H}^1(\Omega)} + \|M_t(s)\|^2_{\mathbb{H}^1(\Omega)} + \|H_t(s)\|^2_{\mathbb{H}^1(\Omega)}\Big)\, ds, \quad t \ge 0. \end{aligned} \tag{17}$$

Our main result is:

Theorem 1. *Under assumptions (14)–(16), there is a time $T_* > 0$ such that problem $(\mathcal{P})$ admits a unique strong solution (U,M,H) in Ω_{T_*}. Moreover, if $T^\star$ is a finite blow-up time of (U,M,H), we have*

$$\int_0^{T^\star} \left(\|\nabla U(s)\| + \|M(s)\|_{\mathbb{H}^2(\Omega)} + \|H(s)\|_{\mathbb{H}^2(\Omega)}\right)^{12} ds = +\infty. \tag{18}$$

To prove the local-in-time existence of a strong solution, we use a classical linearization and iteration method, see for instance [6].

Assume that $(U^\sharp, M^\sharp, H^\sharp)$ is given, $U^\sharp$ belongs to $L^\infty(0,T;\mathcal{U}\cap\mathbb{H}^2(\Omega)) \cap L^2(0,T;\mathbb{H}^3(\Omega))$, $\partial_t U^\sharp$ belongs to $L^\infty(0,T;\mathbb{L}^2(\Omega)) \cap L^2(0,T;\mathcal{U})$, $M^\sharp \in L^\infty(0,T;\mathbb{H}^2(\Omega))$, $\partial_t M^\sharp \in L^\infty(0,T;\mathbb{L}^2(\Omega))$, $H^\sharp \in L^\infty(0,T;\mathbb{H}^2(\Omega))$ and $\partial_t H^\sharp \in L^\infty(0,T;\mathbb{L}^2(\Omega))$. We define the function M as the solution of the linearized hyperbolic equation

$$\partial_t M + (U^\sharp\cdot\nabla)M - \frac{1}{2}\mathrm{curl}\, U^\sharp \wedge M + \frac{1}{t_m}M + \frac{1}{4\xi_r} M \wedge (M^\sharp \wedge H^\sharp) = \frac{\chi_m}{t_m} H^\sharp \quad \text{in } \Omega_T, \tag{19}$$

$$M|_{t=0} = M_0 \quad \text{in } \Omega. \tag{20}$$

Then we define H as the solution of

$$\partial_t H + \mathrm{curl}^2 H = \mathrm{curl}\,(U^\sharp \wedge B) - \partial_t M \quad \text{in } \Omega_T, \tag{21}$$

$$\mathrm{div}\, B = 0 \quad \text{in } \Omega_T, \tag{22}$$

$$B\cdot n = 0, \quad \mathrm{curl}\, H \wedge n = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{23}$$

$$H|_{t=0} = H_0 \equiv B_0 - M_0 \quad \text{in } \Omega, \tag{24}$$

where $B = H + M$. Setting $J = \mathrm{curl}\, H$, we define U as the solution of the linearized problem

$$\partial_t U + (U^\sharp\cdot\nabla)U - \mu\Delta U + \nabla p = M\cdot\nabla H + \frac{1}{2}\mathrm{curl}\,(M\wedge H) + J\wedge B \quad \text{in } \Omega_T, \tag{25}$$

$$\mathrm{div}\, U = 0 \quad \text{in } \Omega_T, \tag{26}$$

$$U = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{27}$$

$$U|_{t=0} = U_0 \quad \text{in } \Omega. \tag{28}$$

For convenience, in the sequel, we refer to problem (19), (20) to as problem $(\mathcal{P}_M)$, problem (21)–(24) to as problem $(\mathcal{P}_H)$ and problem (25)–(28) to as problem $(\mathcal{P}_U)$.

Using this approach by linearization we will construct a sequence (U^n, M^n, H^n) of approximate solutions to problem $(\mathcal{P})$. Then we will derive some uniform bounds of the sequence (U^n, M^n, H^n) which allow to prove the convergence of the sequence to a strong solution of problem $(\mathcal{P})$. Our proof of the existence of a strong solution also provides a priori estimates which allow to obtain the blow-up criterion (18). See Section 3.

Let us now mention some previous studies [1, 2, 3, 24]. These studies deal with the flow of electrically *nonconducting* magnetic fluids. A differential system describing the motion of an isothermal electrically nonconducting ferrofluid driven by an external magnetic field is considered in [1]. The system reads

$$\operatorname{div} U = 0 \quad \text{in } \Omega_T, \tag{29}$$

$$\partial_t M + (U \cdot \nabla) M = \frac{1}{2} \operatorname{curl} U \wedge M - \frac{1}{t_m}(M - \chi_m H) - \frac{\mu_0}{4\xi_r} M \wedge (M \wedge H) \quad \text{in } \Omega_T, \tag{30}$$

$$\operatorname{curl} H = 0, \ \ \operatorname{div} B = F \ \ \text{in } \Omega_T, \tag{31}$$

$$\rho(\partial_t U + (U \cdot \nabla) U) - \mu \Delta U + \nabla p = \mu_0 M \cdot \nabla H + \frac{\mu_0}{2} \operatorname{curl}(M \wedge H) \quad \text{in } \Omega_T, \tag{32}$$

where F is a given function in Ω_T such that $\int_\Omega F\, dx = 0$, for all $t \in [0, T]$, and the boundary and initial conditions are taken as

$$U = 0 \quad \text{on } (0,T) \times \partial\Omega, \tag{33}$$

$$B \cdot n = 0 \ \ \text{on } (0,T) \times \partial\Omega, \tag{34}$$

$$U|_{t=0} = U_0, \ \ M|_{t=0} = M_0 \ \ \text{in } \Omega. \tag{35}$$

Note that system (29)–(32) can be deduced from system (7)–(11) by taking $J = 0$ and replacing the magnetic induction equation (9) by the magnetostatic equations (31) with $F = 0$. We prove the local-in-time existence of the unique strong solution to problem (29)–(35).

The paper [24] deals with the differential system formed by the equations (29), (30), (32) posed in the whole domain $\mathbb{R}^3$ and coupled with the equations

$$\operatorname{curl} H = 0, \ \ \operatorname{div} B = -\operatorname{div} H_{ext} \ \ \text{in } (0,T) \times \mathbb{R}^3. \tag{36}$$

The authors study the Cauchy problem, they obtain a local-in-time existence result of a strong solution, establish a blow-up criterion and a global-in-time existence result under smallness assumptions on the data.

In [2] we are concerned with the model proposed by R.E. Rosensweig [19] to describe the motion of a ferrofluid under the action of an external applied magnetic field. The state variables are the fluid velocity U, the angular velocity ω, the magnetization field M and the magnetic field H. The differential system is formed by the Navier-Stokes equations, the angular momentum equation, the magnetization equation and the magnetostatic equations. The magnetization relaxation equation has the form

$$\partial_t M + (U \cdot \nabla) M = \omega \wedge M - \frac{1}{t_m}(M - \chi_m H).$$

We prove the local-in-time existence of the unique strong solution to the differential system equipped with initial and boundary conditions.

In the paper [3] we study the equations of flow and heat transfer in an electrically non-conducting magnetic fluid, when the fluid is subjected to the action of an external applied magnetic field. The system of equations is formed by the Navier-Stokes equations, the magnetization relaxation equation of Bloch type, the magnetostatic equations and the temperature equation. We prove the local-in-time existence of the unique strong solution to the system equipped with initial and boundary conditions and establish a blow-up criterion for the strong solution. We also prove the global-in-time existence of strong solutions, under smallness assumptions on the initial data and the external magnetic field.

Regarding involved mathematical methods, we note that the mathematical analysis of problem $(\mathcal{P})$ requires more refined estimates, particularly for the magnetization M and the magnetic field H, to that in [1, 2, 3, 24]. In [1, 2, 3, 24], the magnetic field H is the solution of (31) (or (36)), and the estimates on H follow straightforwardly from that on M. While for problem $(\mathcal{P})$ we need additional estimates of M to derive suitable estimates of H. Namely, we establish estimates of M in $L^\infty(0,T;\mathbb{H}^2(\Omega))\cap W^{1,\infty}(0,T;\mathbb{L}^2(\Omega))\cap H^1(0,T;\mathbb{H}^1(\Omega))$, see Lemmas 2 and 3. These estimates allow to derive from the magnetic induction equation estimates of H in the spaces $L^\infty(0,T;\mathbb{H}^2(\Omega))$, $W^{1,\infty}(0,T;\mathbb{L}^2(\Omega))$ and $H^1(0,T;\mathbb{H}^1(\Omega))$, see Lemma 4. Lemma 5 is used to estimate the electromagnetic forces. Then we obtain estimates of U in the spaces $L^\infty(0,T;\mathbb{H}^2(\Omega))\cap L^2(0,T;\mathbb{H}^3(\Omega))$ and $W^{1,\infty}(0,T;\mathbb{L}^2(\Omega))\cap H^1(0,T;\mathbb{H}^1(\Omega))$, see Lemma 6. Finally, note that our uniform estimates on the approximate solutions of problem $(\mathcal{P})$ provide estimates which allow to obtain the blow-up criterion (18), see Section 3.

There have been extensive mathematical studies on the solutions of the equations of MHD viscous and resistive incompressible fluids. Global weak solutions and local strong solutions have been constructed in [8]. Properties of weak and strong solutions have been examined in [22]. Some sufficient conditions for regularity of weak solutions to the MHD equations were obtained in [12]. Blow up criteria for smooth solutions of the incompressible MHD equations were obtained, see for example [5].

Throughout the paper, C indicates a generic constant that depends only on some bounds of the physical data.

2 Study of problems $(\mathcal{P}_M)$, $(\mathcal{P}_H)$ and $(\mathcal{P}_U)$

2.1 Problem $(\mathcal{P}_M)$

Throughout the paper we will make frequent use of the Sobolev embedding:

$$\begin{cases} \text{if } 1\le r<3, \text{ then } W^{1,r}(\Omega)\hookrightarrow L^{r^\star}(\Omega), \text{ with } r^\star=\dfrac{3r}{3-r}, \\ \text{if } r=3, \text{ then } W^{1,r}(\Omega)\hookrightarrow L^{\gamma}(\Omega) \text{ for any real number } \gamma\ge 1, \\ \text{if } r>3, \text{ then } W^{1,r}(\Omega)\hookrightarrow L^{\infty}(\Omega). \end{cases} \tag{37}$$

In particular, we will use estimates in the space $\mathbb{L}^r(\Omega)$ of functions of the type $A\wedge B$ and $A\wedge(B\wedge D)$. The Hölder inequality gives

$$\|A\wedge B\|_{\mathbb{L}^r(\Omega)}\le \|A\|_{\mathbb{L}^6(\Omega)}\|B\|_{\mathbb{L}^{q(r)}(\Omega)},$$

with $q(r)=\frac{6r}{6-r}$ if $2\le r<6$, and $q(r)=+\infty$ if $r=6$. Observe that, for any $2\le r\le 6$,

$$W^{1,r}(\Omega)\hookrightarrow L^{q(r)}(\Omega). \tag{38}$$

Indeed, if $2 \le r < 3$ we have $L^{r^\star}(\Omega) \hookrightarrow L^{q(r)}(\Omega)$ (with $r^\star = \frac{3r}{3-r}$) and in view of $(37)_1$ we have $W^{1,r}(\Omega) \hookrightarrow L^{r^\star}(\Omega)$. If $3 \le r \le 6$, according to $(37)_2$ and $(37)_3$, we have $W^{1,r}(\Omega) \hookrightarrow L^{q(r)}(\Omega)$. The claim is proved. We deduce that

$$\|A \wedge B\|_{\mathbb{L}^r(\Omega)} \le C\|A\|_{\mathbb{L}^6(\Omega)}\|B\|_{\mathbb{W}^{1,r}(\Omega)}.$$

Employing the Hölder inequality, $(37)_3$ and (38), we have

$$\|A \wedge (B \wedge D)\|_{\mathbb{L}^r(\Omega)} \le C\|A\|_{\mathbb{W}^{1,r}(\Omega)}\|B\|_{\mathbb{L}^6(\Omega)}\|D\|_{\mathbb{W}^{1,6}(\Omega)}. \tag{39}$$

Lemma 1. *Problem* $(\mathcal{P}_M)$ *has a unique global-in-time strong solution* $M \in L^\infty(0,T;\mathbb{H}^2(\Omega)) \cap H^1(0,T;\mathbb{H}^1(\Omega)) \cap H^2(0,T;\mathbb{L}^2(\Omega))$. *Moreover, the following estimates hold:*

$$\|M(t)\|^r_{\mathbb{L}^r(\Omega)} + \int_0^t \|M(s)\|^r_{\mathbb{L}^r(\Omega)}\,ds \le C\left(\|M_0\|^r_{\mathbb{L}^r(\Omega)} + \int_0^t \|H^\sharp(s)\|^r_{\mathbb{L}^r(\Omega)}\,ds\right), \tag{40}$$

$$\|\nabla M(t)\|^r_{(\mathbb{L}^r(\Omega))^3} + \int_0^t \|\nabla M(s)\|^r_{(\mathbb{L}^r(\Omega))^3}\,ds \le Ca_1(t), \tag{41}$$

$$\|M_t(t)\| \le Ca_2(t), \tag{42}$$

for any $2 \le r \le 6$ *and* $t \in (0,T)$. *The functions* a_1 *and* a_2 *are given by*

$$a_1(t) = a_1^1(t)\exp\left(C\int_0^t a_1^2(s)\,ds\right),$$

with

$$\begin{aligned}
a_1^1(t) = {}& \|\nabla M_0\|^r_{(\mathbb{L}^r(\Omega))^3} + \int_0^t \left(\|M(s)\|^r_{\mathbb{L}^r(\Omega)}\|U^\sharp(s)\|_{\mathbb{W}^{2,6}(\Omega)} + \|\nabla H^\sharp(s)\|^r_{(\mathbb{L}^r(\Omega))^3}\right)ds \\
&+ \int_0^t \|M(s)\|^r_{\mathbb{L}^r(\Omega)}\|H^\sharp(s)\|_{\mathbb{W}^{1,6}(\Omega)}\|\nabla M^\sharp(s)\|_{(\mathbb{L}^6(\Omega))^3}\,ds \\
&+ \int_0^t \|M(s)\|^r_{\mathbb{L}^r(\Omega)}\|M^\sharp(s)\|_{\mathbb{W}^{1,6}(\Omega)}\|\nabla H^\sharp(s)\|_{(\mathbb{L}^6(\Omega))^3}\,ds, \\
a_1^2(t) = {}& \|H^\sharp(t)\|_{\mathbb{W}^{1,6}(\Omega)}\|\nabla M^\sharp(t)\|_{(\mathbb{L}^6(\Omega))^3} + \|M^\sharp(t)\|_{\mathbb{W}^{1,6}(\Omega)}\|\nabla H^\sharp(t)\|_{(\mathbb{L}^6(\Omega))^3} \\
&+ \|U^\sharp(t)\|_{\mathbb{W}^{2,6}(\Omega)} + 1,
\end{aligned}$$

and

$$\begin{aligned}
a_2(t) = {}& \|U^\sharp(t)\|_{\mathbb{H}^1(\Omega)}\|M(t)\|_{\mathbb{W}^{1,6}(\Omega)} + \|H^\sharp(t)\| + \|M(t)\| \\
&+ \|M^\sharp(t)\|_{\mathbb{W}^{1,6}(\Omega)}\|M(t)\|_{\mathbb{W}^{1,6}(\Omega)}\|H^\sharp(t)\|.
\end{aligned}$$

Since $\nabla U^\sharp$ belongs to the space $L^2(0,T;C^{0,\frac{1}{2}}(\overline{\Omega},\mathbb{R}^9))$ and $M^\sharp \wedge H^\sharp$ belongs to $L^\infty(0,T;\mathbb{H}^2(\Omega))$, the existence, regularity and uniqueness of a solution to (19), (20) is classical. For the proof of (40)–(42) we refer to [1] (Lemma 2).

Lemma 2. *We have*

$$\|\partial^2_{x_ix_j}M(t)\|^2 + \int_0^t \|\partial^2_{x_ix_j}M(s)\|^2\,ds \le Ca_3(t), \tag{43}$$

for any $i,j = 1,2,3$ *and* $t \in (0,T)$. *Here* a_3 *is a function from* $L^\infty(0,T)$, *depending only on* $\|U^\sharp\|_{L^2(0,T;\mathbb{H}^3(\Omega))}$, $\|M_0\|_{\mathbb{H}^2(\Omega)}$, $\|M\|_{L^\infty(0,T;\mathbb{W}^{1,6}(\Omega))}$, $\|M^\sharp\|_{L^\infty(0,T;\mathbb{H}^2(\Omega))}$, $\|H^\sharp\|_{L^\infty(0,T;\mathbb{H}^2(\Omega))}$.

Remark 1. *The explicit formula of the function a_3 is given in the following proof.*

Proof. Differentiating equation (19) with respect to x_i $(1 \leq i \leq 3)$ yields

$$\partial_t N + (U^\sharp \cdot \nabla)N - \frac{1}{2}\mathrm{curl}\, U^\sharp \wedge N + \frac{1}{4\xi_r} N \wedge (M^\sharp \wedge H^\sharp) + \frac{1}{t_m} N = Z, \tag{44}$$

with $N = \partial_{x_i} M$, $K^\sharp = \partial_{x_i} H^\sharp$, $N^\sharp = \partial_{x_i} M^\sharp$, $V^\sharp = \partial_{x_i} U^\sharp$, and

$$Z = \frac{\chi_m}{t_m} K^\sharp - (V^\sharp \cdot \nabla)M + \frac{1}{2}\mathrm{curl}\, V^\sharp \wedge M - \frac{1}{4\xi_r} M \wedge (N^\sharp \wedge H^\sharp) - \frac{1}{4\xi_r} M \wedge (M^\sharp \wedge K^\sharp). \tag{45}$$

Then, differentiating equation (44) with respect to x_j $(1 \leq j \leq 3)$ we obtain

$$\partial_t \widetilde{N} + (U^\sharp \cdot \nabla)\widetilde{N} - \frac{1}{2}\mathrm{curl}\, U^\sharp \wedge \widetilde{N} + \frac{1}{4\xi_r} \widetilde{N} \wedge (M^\sharp \wedge H^\sharp) + \frac{1}{t_m} \widetilde{N} = Z^{(1)} + Z^{(2)}. \tag{46}$$

Here the tilde sign denotes the derivative with respect to x_j, thus $\widetilde{N} = \partial_{x_j} N$, $\widetilde{K}^\sharp = \partial_{x_j} K^\sharp$, $\widetilde{V}^\sharp = \partial_{x_j} V^\sharp$, $\cdots$, and

$$Z^{(1)} = -(\widetilde{U}^\sharp \cdot \nabla)N + \frac{1}{2}\mathrm{curl}\, \widetilde{U}^\sharp \wedge N - \frac{1}{4\xi_r} N \wedge (\widetilde{M}^\sharp \wedge H^\sharp) - \frac{1}{4\xi_r} N \wedge (M^\sharp \wedge \widetilde{H}^\sharp)$$

and $Z^{(2)} = \widetilde{Z}$ where Z is given by (45), that is

$$\begin{aligned} Z^{(2)} =& \frac{\chi_m}{t_m} \widetilde{K}^\sharp - (\widetilde{V}^\sharp \cdot \nabla)M - (V^\sharp \cdot \nabla)\widetilde{M} + \frac{1}{2}\mathrm{curl}\, \widetilde{V}^\sharp \wedge M + \frac{1}{2}\mathrm{curl}\, V^\sharp \wedge \widetilde{M} \\ & - \frac{1}{4\xi_r} \widetilde{M} \wedge (N^\sharp \wedge H^\sharp) - \frac{1}{4\xi_r} M \wedge (\widetilde{N}^\sharp \wedge H^\sharp) - \frac{1}{4\xi_r} M \wedge (N^\sharp \wedge \widetilde{H}^\sharp) \\ & - \frac{1}{4\xi_r} \widetilde{M} \wedge (M^\sharp \wedge K^\sharp) - \frac{1}{4\xi_r} M \wedge (\widetilde{M}^\sharp \wedge K^\sharp) - \frac{1}{4\xi_r} M \wedge (M^\sharp \wedge \widetilde{K}^\sharp). \end{aligned}$$

Multiplying equation (46) by $\widetilde{N}$ and integrating over Ω yields

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{N}\|^2 + \frac{1}{t_m}\|\widetilde{N}\|^2 = \int_\Omega (Z^{(1)} + Z^{(2)}) \cdot \widetilde{N}\, dx. \tag{47}$$

The right-hand side is estimated as follows:

$$\begin{aligned} \left| \int_\Omega Z^{(1)} \cdot \widetilde{N}\, dx \right| \leq& \int_\Omega \left(|(\widetilde{U}^\sharp \cdot \nabla)N| + \frac{1}{2}|\mathrm{curl}\, \widetilde{U}^\sharp \wedge N| \right) |\widetilde{N}|\, dx \\ & + \frac{1}{4\xi_r} \int_\Omega \left(|N \wedge (\widetilde{M}^\sharp \wedge H^\sharp)| + |N \wedge (M^\sharp \wedge \widetilde{H}^\sharp)| \right) |\widetilde{N}|\, dx \\ \equiv& J_1 + J_2. \end{aligned} \tag{48}$$

Cauchy-Schwarz's inequality yields

$$J_1 \leq C\left(\|(\widetilde{U}^\sharp \cdot \nabla)N\| + \|\mathrm{curl}\, \widetilde{U}^\sharp \wedge N\| \right) \|\nabla N\|.$$

Using the Sobolev embedding we have

$$\begin{aligned} &\|(\widetilde{U}^\sharp \cdot \nabla)N\| \leq C\|\widetilde{U}^\sharp\|_{\mathbb{W}^{1,6}(\Omega)}\|\nabla N\| \leq C\|U^\sharp\|_{\mathbb{W}^{2,6}(\Omega)}\|\nabla N\|, \\ &\|\mathrm{curl}\, \widetilde{U}^\sharp \wedge N\| \leq \|\mathrm{curl}\, \widetilde{U}^\sharp\|_{\mathbb{L}^6(\Omega)}\|N\|_{\mathbb{L}^3(\Omega)} \leq C\|U^\sharp\|_{\mathbb{W}^{2,6}(\Omega)} \left(\|N\| + \|\nabla N\| \right), \end{aligned}$$

and applying the Young inequality we obtain

$$J_1 \le C\|U^\sharp\|_{\mathbb{W}^{2,6}(\Omega)} \left(\|N\|^2 + \|\nabla N\|^2\right).$$

Employing (39) we have

$$\begin{aligned}
&\|N \wedge (\widetilde{M}^\sharp \wedge H^\sharp)\| \le C\|H^\sharp\|_{\mathbb{W}^{1,6}(\Omega)}\|\nabla M^\sharp\|_{(\mathbb{L}^6(\Omega))^3}\| \left(\|N\| + \|\nabla N\|\right),\\
&\|N \wedge (M^\sharp \wedge \widetilde{H}^\sharp)\| \le C\|M^\sharp\|_{\mathbb{W}^{1,6}(\Omega)}\|\nabla H^\sharp\|_{(\mathbb{L}^6(\Omega))^3} \left(\|N\| + \|\nabla N\|\right),
\end{aligned}$$

hence

$$J_2 \le C\left(\|H^\sharp\|_{\mathbb{W}^{1,6}(\Omega)}\|\nabla M^\sharp\|_{(\mathbb{L}^6(\Omega))^3} + \|M^\sharp\|_{\mathbb{W}^{1,6}(\Omega)}\|\nabla H^\sharp\|_{(\mathbb{L}^6(\Omega))^3}\right)\left(\|N\|^2 + \|\nabla N\|^2\right).$$

We thus have

$$J_1 + J_2 \le Ca_3^1\left(\|N\|^2 + \|\nabla N\|^2\right), \tag{49}$$

where a_3^1 is defined by

$$\begin{aligned}
a_3^1(t) = \|U^\sharp(t)\|_{\mathbb{W}^{2,6}(\Omega)} &+ \|H^\sharp(t)\|_{\mathbb{W}^{1,6}(\Omega)}\|\nabla M^\sharp(t)\|_{(\mathbb{L}^6(\Omega))^3}\\
&+ \|M^\sharp(t)\|_{\mathbb{W}^{1,6}(\Omega)}\|\nabla H^\sharp(t)\|_{(\mathbb{L}^6(\Omega))^3}.
\end{aligned}$$

Employing the Hölder inequality and the Sobolev embedding we also have

$$\begin{aligned}
&\|(\widetilde{V}^\sharp \cdot \nabla)M\| \le C\|U^\sharp\|_{\mathbb{W}^{2,6}(\Omega)}\|M\|_{\mathbb{W}^{1,6}(\Omega)},\\
&\|(V^\sharp \cdot \nabla)\widetilde{M}\| \le C\|U^\sharp\|_{\mathbb{W}^{2,6}(\Omega)}\|\nabla\widetilde{M}\|,\\
&\|\mathrm{curl}\,\widetilde{V}^\sharp \wedge M\| \le C\|U^\sharp\|_{\mathbb{H}^3(\Omega)}\|M\|_{\mathbb{W}^{1,6}(\Omega)},\\
&\|\mathrm{curl}\,V^\sharp \wedge \widetilde{M}\| \le C\|U^\sharp\|_{\mathbb{W}^{2,6}(\Omega)}\|M\|_{\mathbb{W}^{1,6}(\Omega)},
\end{aligned}$$

and

$$\begin{aligned}
&\|\widetilde{M} \wedge (N^\sharp \wedge H^\sharp)\| \le C\|\nabla M\|_{(\mathbb{L}^6(\Omega))^3}\|\nabla M^\sharp\|_{(\mathbb{L}^6(\Omega))^3}\|H^\sharp\|_{\mathbb{L}^6(\Omega)},\\
&\|M \wedge (\widetilde{N}^\sharp \wedge H^\sharp)\| \le C\|M\|_{\mathbb{W}^{1,6}(\Omega)}\|M^\sharp\|_{\mathbb{H}^2(\Omega)}\|H^\sharp\|_{\mathbb{W}^{1,6}(\Omega)},\\
&\|M \wedge (N^\sharp \wedge \widetilde{H}^\sharp)\| \le C\|M\|_{\mathbb{L}^6(\Omega)}\|M^\sharp\|_{\mathbb{W}^{1,6}(\Omega)}\|H^\sharp\|_{\mathbb{W}^{1,6}(\Omega)},\\
&\|\widetilde{M} \wedge (M^\sharp \wedge K^\sharp)\| \le C\|M\|_{\mathbb{W}^{1,6}(\Omega)}\|M^\sharp\|_{\mathbb{L}^6(\Omega)}\|H^\sharp\|_{\mathbb{W}^{1,6}(\Omega)},\\
&\|M \wedge (\widetilde{M}^\sharp \wedge K^\sharp)\| \le C\|M\|_{\mathbb{L}^6(\Omega)}\|M^\sharp\|_{\mathbb{W}^{1,6}(\Omega)}\|H^\sharp\|_{\mathbb{W}^{1,6}(\Omega)},\\
&\|M \wedge (M^\sharp \wedge \widetilde{K}^\sharp)\| \le C\|M\|_{\mathbb{W}^{1,6}(\Omega)}\|M^\sharp\|_{\mathbb{W}^{1,6}(\Omega)}\|H^\sharp\|_{\mathbb{H}^2(\Omega)}.
\end{aligned}$$

Using the Young inequality we deduce that

$$\left|\int_\Omega Z^{(2)} \cdot \widetilde{N}\,dx\right| \le C\left(\|\nabla N\|^2 + a_3^2\right), \tag{50}$$

where a_3^2 is defined by

$$\begin{aligned}
a_3^2(t) = \|H^\sharp(t)\|^2_{\mathbb{H}^2(\Omega)} &+ \|U^\sharp(t)\|^2_{\mathbb{H}^3(\Omega)}\|M(t)\|^2_{\mathbb{W}^{1,6}(\Omega)}\\
&+ \|M(t)\|^2_{\mathbb{W}^{1,6}(\Omega)}\|M^\sharp(t)\|^2_{\mathbb{H}^2(\Omega)}\|H^\sharp(t)\|^2_{\mathbb{W}^{1,6}(\Omega)}\\
&+ \|M(t)\|^2_{\mathbb{W}^{1,6}(\Omega)}\|M^\sharp(t)\|^2_{\mathbb{W}^{1,6}(\Omega)}\|H^\sharp(t)\|^2_{\mathbb{H}^2(\Omega)}.
\end{aligned}$$

We deduce from (47)–(50) that

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{N}\|^2 + \frac{1}{t_m}\|\widetilde{N}\|^2 \le C\Big(a_3^3\left(\|N\|^2 + \|\nabla N\|^2\right) + a_3^2\Big). \tag{51}$$

where $a_3^3(t) = a_3^1(t) + 1$. Let us introduce the quantities

$$R(t) = \sum_{i,j=1}^{3} \|\partial^2_{x_i x_j} M(t)\|^2, \quad R_0 = \sum_{i,j=1}^{3} \|\partial^2_{x_i x_j} M_0\|^2.$$

Summing (51) over $i, j = 1, 2, 3$ and using the inequality $\|N\| \le \|\nabla M\|$ we obtain

$$\frac{1}{2}\frac{dR}{dt} + \frac{R}{t_m} \le C\left(a_3^3 R + a_3^3\|\nabla M\|^2 + a_3^2\right)$$

and Gronwall's inequality implies that

$$R(t) + \int_0^t R(s)\,ds \le C\left(R_0 + \int_0^t \left(a_3^3(s)\|\nabla M(s)\|^2 + a_3^2(s)\right) ds\right)\exp\left(C\int_0^t a_3^3(s)\,ds\right),$$

hence (43) with

$$a_3(t) = \left(\|M_0\|^2_{\mathbb{H}^2(\Omega)} + \int_0^t \left(a_3^3(s)\|\nabla M(s)\|^2 + a_3^2(s)\right) ds\right)\exp\left(C\int_0^t a_3^3(s)\,ds\right).$$

Lemma 2 is proved. □

Lemma 3. *We have*

$$\|\partial^2_{t x_i} M(t)\| \le C a_4(t), \tag{52}$$

$$\|M_{tt}(t)\| \le C a_5(t), \tag{53}$$

for any $i = 1, 2, 3$ and $t \in (0, T)$. Here a_4 is a function from $L^\infty(0,T)$, depending only on $\|U^\sharp\|_{L^\infty(0,T;\mathbb{H}^2(\Omega))}$, $\|M\|_{L^\infty(0,T;\mathbb{H}^2(\Omega))}$, $\|M^\sharp\|_{L^\infty(0,T;\mathbb{W}^{1,6}(\Omega))}$, $\|H^\sharp\|_{L^\infty(0,T;\mathbb{W}^{1,6}(\Omega))}$, while a_5 belongs to $L^2(0,T)$, depending only on $\|U^\sharp\|_{L^2(0,T;\mathbb{W}^{2,6}(\Omega))}$, $\|U_t^\sharp\|_{L^2(0,T;\mathbb{H}^1(\Omega))}$, $\|M\|_{L^\infty(0,T;\mathbb{W}^{1,6}(\Omega))}$, $\|M_t\|_{L^\infty(0,T;\mathbb{L}^2(\Omega))}$, $\|M^\sharp\|_{L^\infty(0,T;\mathbb{W}^{1,6}(\Omega))}$, $\|H^\sharp\|_{L^\infty(0,T;\mathbb{W}^{1,6}(\Omega))}$, $\|M_t^\sharp\|_{L^\infty(0,T;\mathbb{L}^2(\Omega))}$, $\|H_t^\sharp\|_{L^\infty(0,T;\mathbb{L}^2(\Omega))}$.

Remark 2. *The explicit formulae of the functions a_4 and a_5 are given in the following proof.*

Proof. Writing

$$\partial_t N = -(U^\sharp\cdot\nabla)N + \frac{1}{2}\mathrm{curl}\, U^\sharp \wedge N - \frac{1}{4\xi_r} N \wedge (M^\sharp \wedge H^\sharp) - \frac{1}{t_m} N + Z$$

with $N = \partial_{x_i} M$, $K^\sharp = \partial_{x_i} H^\sharp$, $N^\sharp = \partial_{x_i} M^\sharp$, $V^\sharp = \partial_{x_i} U^\sharp$, and Z given by (45), we easily deduce that (52) holds with

$$\begin{aligned} a_4(t) = {} & \|U^\sharp(t)\|_{\mathbb{H}^2(\Omega)}\|M(t)\|_{\mathbb{H}^2(\Omega)} + \|\nabla M(t)\| + \|\nabla H^\sharp(t)\| \\ & + \|\nabla M(t)\|\|M^\sharp(t)\|_{\mathbb{W}^{1,6}(\Omega)}\|H^\sharp(t)\|_{\mathbb{W}^{1,6}(\Omega)} \\ & + \|M(t)\|_{\mathbb{W}^{1,6}(\Omega)}\|\nabla M^\sharp(t)\|\|H^\sharp(t)\|_{\mathbb{W}^{1,6}(\Omega)} \\ & + \|M(t)\|_{\mathbb{W}^{1,6}(\Omega)}\|M^\sharp(t)\|_{\mathbb{W}^{1,6}(\Omega)}\|\nabla H^\sharp(t)\|. \end{aligned}$$

Differentiating equation (19) with respect to t gives

$$\begin{aligned}M_{tt} = &-(U_t^\sharp \cdot \nabla)M - (U^\sharp \cdot \nabla)M_t + \frac{1}{2}\operatorname{curl} U_t^\sharp \wedge M + \frac{1}{2}\operatorname{curl} U^\sharp \wedge M_t \\ &- \frac{1}{t_m}(M_t - \chi_m H_t^\sharp) - \frac{1}{4\xi_r} M_t \wedge (M^\sharp \wedge H^\sharp) - \frac{1}{4\xi_r} M \wedge (M_t^\sharp \wedge H^\sharp) \\ &- \frac{1}{4\xi_r} M \wedge (M^\sharp \wedge H_t^\sharp)\end{aligned}$$

from which we deduce (53) with

$$\begin{aligned}a_5(t) = &\|U_t^\sharp(t)\|_{\mathbb{H}^1(\Omega)}\|M(t)\|_{\mathbb{W}^{1,6}(\Omega)} + \|U^\sharp(t)\|_{\mathbb{W}^{1,6}(\Omega)}\|M_t(t)\|_{\mathbb{H}^1(\Omega)} \\ &+ \|M_t(t)\| + \|H_t^\sharp(t)\| + \|M_t(t)\|\|M^\sharp(t)\|_{\mathbb{W}^{1,6}(\Omega)}\|H^\sharp(t)\|_{\mathbb{W}^{1,6}(\Omega)} \\ &+ \|M(t)\|_{\mathbb{W}^{1,6}(\Omega)}\|M_t^\sharp(t)\|\|H^\sharp(t)\|_{\mathbb{W}^{1,6}(\Omega)} \\ &+ \|M(t)\|_{\mathbb{W}^{1,6}(\Omega)}\|M^\sharp(t)\|_{\mathbb{W}^{1,6}(\Omega)}\|H_t^\sharp(t)\|.\end{aligned}$$

Lemma 3 is proved. □

2.2 Problem ($\mathcal{P}_H$)

Lemma 4. *Problem ($\mathcal{P}_H$) has a unique global-in-time strong solution $H \in L^\infty(0,T;\mathbb{H}^2(\Omega)) \cap W^{1,\infty}(0,T;\mathbb{L}^2(\Omega)) \cap H^1(0,T;\mathbb{H}^1(\Omega))$. Moreover, the following estimates hold for $t \in (0,T)$:*

(i)

$$\|H(t)\|^2 + \int_0^t \|\operatorname{curl} H(s)\|^2\,ds \le b_1(t), \tag{54}$$

$$\|H\|_{L^2(0,T;\mathbb{H}^1(\Omega))} \le C\left(\|\operatorname{curl} H\| + \|M\|_{L^2(0,T;\mathbb{H}^1(\Omega))}\right), \tag{55}$$

where

$$\begin{aligned}b_1(t) = &\left(\|H_0\|^2 + \int_0^t \left(\|U^\sharp(s)\|^2_{\mathbb{L}^\infty(\Omega)}\|M(s)\|^2 + \|M_t(s)\|^2\right)ds\right) \times \\ &\times \exp\left(t + \int_0^t \|U^\sharp(s)\|^2_{\mathbb{L}^\infty(\Omega)}\,ds\right);\end{aligned}$$

(ii)

$$\|\operatorname{curl} H(t)\|^2 + \int_0^t \|H_t(s)\|^2\,ds \le b_2(t), \tag{56}$$

$$\|H\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} \le C\left(\|\operatorname{curl} H\|_{L^\infty(0,T;\mathbb{L}^2(\Omega))} + \|M\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))}\right), \tag{57}$$

where

$$\begin{aligned}b_2(t) = &2\int_0^t \left(\|\nabla U^\sharp(s)\|^2_{(\mathbb{L}^\infty(\Omega))^3}\|B(s)\|^2 + \|U^\sharp(s)\|^2_{\mathbb{L}^\infty(\Omega)}\|B(s)\|^2_{\mathbb{H}1(\Omega)} + \|M_t(s)\|^2\right)ds \\ &+ \|\operatorname{curl} H_0\|^2;\end{aligned}$$

(iii)

$$\|\mathrm{curl}\, H(t)\|^2 + \int_0^t \|\mathrm{curl}^{\,2} H(s)\|^2 \, ds \le b_3(t), \tag{58}$$

$$\|H\|_{L^2(0,T;\mathbb{H}^2(\Omega))} \le C \left(\|\mathrm{curl}^{\,2} H\| + \|M\|_{L^2(0,T;\mathbb{H}^2(\Omega))} \right), \tag{59}$$

where

$$b_3(t) = \left(\|\mathrm{curl}\, H_0\|^2 + \int_0^t \left(C\|U^\sharp(s)\|^2_{\mathbb{W}^{2,6}(\Omega)} \|\mathrm{curl}\, M(s)\|^2 + \|\mathrm{curl}\, M_t(s)\|^2 \right) ds \right) \times$$
$$\times \exp\left(t + C \int_0^t \|U^\sharp(s)\|^2_{\mathbb{W}^{2,6}(\Omega)} \, ds \right);$$

(iv)

$$\|H_t(t)\|^2 + \frac{1}{4} \int_0^t \|\mathrm{curl}\, H_t(s)\|^2 \, ds \le b_4(t), \tag{60}$$

$$\|H_t\|_{L^2(0,T;\mathbb{H}^1(\Omega))} \le C \left(\|\mathrm{curl}\, H_t\| + \|M_t\|_{L^2(0,T;\mathbb{H}^1(\Omega))} \right), \tag{61}$$

where

$$b_4(t) = \left(\|H_t(0)\|^2 + \int_0^t \left(\left(\|U_t^\sharp(s)\|_{\mathbb{L}^6(\Omega)} \|B(s)\|_{\mathbb{L}^3(\Omega)} + \|U^\sharp(s)\|_{\mathbb{L}^\infty(\Omega)} \|B_t(s)\| \right)^2 \right.\right.$$
$$\left.\left. + \frac{1}{2} \|M_{tt}(s)\|^2 \right) ds \right) \exp\left(\frac{t}{2} \right),$$

with

$$H_t(0) = -\mathrm{curl}^{\,2} H_0 + \mathrm{curl}\,(U^\sharp(0) \wedge B_0) - M_t(0),$$
$$M_t(0) = -(U^\sharp(0)\nabla) M_0 + \frac{1}{2} \mathrm{curl}\,(U^\sharp(0) \wedge M_0) - \frac{1}{t_m} M_0 - \frac{1}{4\xi_r} M_0 \wedge (M^\sharp(0) \wedge H^\sharp(0))$$
$$+ \frac{\chi_m}{t_m} H^\sharp(0);$$

(v)

$$\|H(t)\|_{\mathbb{H}^2(\Omega)} \le C \left(\|\nabla U^\sharp(t)\| \|B(t)\|_{\mathbb{W}^{1,4}(\Omega)} + \|B_t(t)\| + \|M(t)\|_{\mathbb{H}^2(\Omega)} \right). \tag{62}$$

Proof. We only need to prove the estimates.

(i) Multiplying equation (21) by H and integrating by parts yields

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |H|^2 \, dx + \int_\Omega |\mathrm{curl}\, H|^2 \, dx = \int_\Omega \left(U^\sharp \wedge (H + M) \right) \cdot \mathrm{curl}\, H \, dx - \int_\Omega M_t H \, dx.$$
$$\equiv I_1 + I_2.$$

Using Cauchy-Schwarz and Young inequalities it holds that

$$|I_1| \le \|U^\sharp\|_{\mathbb{L}^\infty(\Omega)} \|H\| \|\mathrm{curl}\, H\| + \|U^\sharp\|_{\mathbb{L}^\infty(\Omega)} \|M\| \|\mathrm{curl}\, H\|$$
$$\le \frac{1}{2} \|\mathrm{curl}\, H\|^2 + \frac{1}{2} \|U^\sharp\|^2_{\mathbb{L}^\infty(\Omega)} \left(\|H\|^2 + \|M\|^2 \right)$$

and

$$|I_2| \leq \frac{1}{2}(\|H\|^2 + \|M_t\|^2).$$

We thus have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |H|^2\,dx + \int_\Omega |\mathrm{curl}\,H|^2\,dx \\ &\leq \left(\|U^\sharp\|^2_{\mathbb{L}^\infty(\Omega)} + 1\right)\|H\|^2 + \|U^\sharp\|^2_{\mathbb{L}^\infty(\Omega)}\|M\|^2 + \|M_t\|^2. \end{aligned}$$

Integrating over $(0,t)$ and using the Gronwall lemma we get (54). From the equality $B = H + M$ it holds that

$$\|\mathrm{curl}\,B\| \leq \|\mathrm{curl}\,H\| + \|\mathrm{curl}\,M\|,$$

since $\mathrm{div}\,B = 0$ in Ω_T and $B \cdot n = 0$ on $(0,T)\times\partial\Omega$, it results that

$$\|B\|_{L^2(0,T;\mathbb{H}^1(\Omega))} \leq C\|\mathrm{curl}\,B\|,$$

hence (55).

(ii) We multiply equation (21) by H_t and integrate by parts to obtain

$$\begin{aligned} \int_\Omega |H_t|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\mathrm{curl}\,H|^2\,dx &= \int_\Omega \mathrm{curl}\left(U^\sharp \wedge B\right)\cdot H_t\,dx - \int_\Omega M_t H_t\,dx. \\ &\equiv I_3 + I_4. \end{aligned}$$

In view of the vector identity

$$\mathrm{curl}\,(U^\sharp \wedge B) = (\mathrm{div}\,B)U^\sharp - (\mathrm{div}\,U^\sharp)B + (B\cdot\nabla)U^\sharp - (U^\sharp\cdot\nabla)B,$$

and the conditions $\mathrm{div}\,B = \mathrm{div}\,U = 0$, we have

$$\mathrm{curl}\,(U^\sharp \wedge B) = (B\cdot\nabla)U^\sharp - (U^\sharp\cdot\nabla)B. \tag{63}$$

We deduce that

$$|I_3| \leq \frac{1}{4}\|H_t\|^2 + \left(\|\nabla U^\sharp\|^2_{(\mathbb{L}^\infty(\Omega))^3}\|B\|^2 + \|U^\sharp\|^2_{\mathbb{L}^\infty(\Omega)}\|B\|^2_{\mathbb{H}1(\Omega)}\right).$$

We also have

$$|I_4| \leq \frac{1}{4}\|H_t\|^2 + \|M_t\|^2.$$

We thus have

$$\begin{aligned} &\frac{1}{2}\int_\Omega |H_t|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\mathrm{curl}\,H|^2\,dx \\ &\leq \|\nabla U^\sharp\|^2_{(\mathbb{L}^\infty(\Omega))^3}\|B\|^2 + \|U^\sharp\|^2_{\mathbb{L}^\infty(\Omega)}\|B\|^2_{\mathbb{H}^1(\Omega)} + \|M_t\|^2. \end{aligned}$$

Integrating over $(0,t)$ we deduce (56). Arguing as above we show that

$$\|B\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} \leq C\left(\|\mathrm{curl}\,H\|_{L^\infty(0,T;\mathbb{L}^2(\Omega))} + \|M\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))}\right),$$

hence (57).

(iii) Multiplying equation (21) by $\operatorname{curl}^2 H$ and integrating by parts yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\operatorname{curl} H|^2\,dx + \int_\Omega |\operatorname{curl}^2 H|^2\,dx &= \int_\Omega \operatorname{curl}\left(U^\sharp \wedge B\right)\cdot \operatorname{curl}^2 H\,dx \\ &\quad - \int_\Omega \operatorname{curl} H \cdot \operatorname{curl} M_t\,dx. \\ &\equiv I_5 + I_6.\end{aligned}$$

Using (63), the Cauchy-Schwarz inequality, the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, the equality $B = H + M$ and the equivalence of norms in $\mathcal{B}$ we have

$$\begin{aligned}|I_5| &\le \left(\|\nabla U^\sharp\|_{(\mathbb{L}^\infty(\Omega))^3}\|B\| + \|U^\sharp\|_{\mathbb{L}^\infty(\Omega)}\|\nabla B\|\right)\|\operatorname{curl}^2 H\| \\ &\le C\|U^\sharp\|_{\mathbb{W}^{2,6}(\Omega)}\|\operatorname{curl} B\|\|\operatorname{curl}^2 H\| \\ &\le C\|U^\sharp\|_{\mathbb{W}^{2,6}(\Omega)}\left(\|\operatorname{curl} H\| + \|\operatorname{curl} M\|\right)\|\operatorname{curl}^2 H\| \\ &\le \frac{1}{2}\|\operatorname{curl}^2 H\|^2 + C\|U^\sharp\|^2_{\mathbb{W}^{2,6}(\Omega)}\left(\|\operatorname{curl} H\|^2 + \|\operatorname{curl} M\|^2\right).\end{aligned}$$

We also have

$$|I_6| \le \frac{1}{2}\|\operatorname{curl} H\|^2 + \frac{1}{2}\|\operatorname{curl} M_t\|^2.$$

It results that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega |\operatorname{curl} H|^2\,dx + \int_\Omega |\operatorname{curl}^2 H|^2\,dx \\ &\le \|\operatorname{curl} H\|^2\left(C\|U^\sharp\|^2_{\mathbb{W}^{2,6}(\Omega)} + 1\right) + C\|U^\sharp\|^2_{\mathbb{W}^{2,6}(\Omega)}\|\operatorname{curl} M\|^2 + \|\operatorname{curl} M_t\|^2.\end{aligned}$$

Integrating over $(0,t)$ we deduce (58). Writing $\operatorname{curl}^2 B = \operatorname{curl}^2 H + \operatorname{curl}^2 M$, using Lemma 2, the conditions $\operatorname{div} B = 0$ in Ω_T and $B\cdot n = 0$ in $(0,T)\times\partial\Omega$, applying a classical regularity result for curl^2 equations with a nonhomogeneous boundary condition $\operatorname{curl} B \times n = \operatorname{curl} M \times n$ with $\operatorname{curl} M \times n \in L^2(0,T;\mathbb{H}^{-\frac{1}{2}}(\partial\Omega))$, see [11] and [21, Proposition 2.1], gives $B \in L^2(0,T;\mathbb{H}^2(\Omega))$ and the following estimate

$$\|B\|_{L^2(0,T;\mathbb{H}^2(\Omega))} \le C\left(\|\operatorname{curl}^2 H\| + \|\operatorname{curl}^2 M\| + \|\operatorname{curl} M\|_{L^2(0,T;\mathbb{H}^{-\frac{1}{2}}(\partial\Omega))}\right)$$

holds. We deduce that $H \in L^2(0,T;\mathbb{H}^2(\Omega))$ and satisfies (59).

(iv) Differentiating equation (21) with respect to t, multiplying the resulting equation by H_t and integrating by parts yields

$$\begin{aligned}&\frac{d}{dt}\int_\Omega |H_t|^2\,dx + \frac{1}{2}\int_\Omega |\operatorname{curl} H_t|^2\,dx \\ &= \int_\Omega \left(U^\sharp_t \wedge B + U^\sharp \wedge B_t\right)\cdot \operatorname{curl} H_t\,dx - \int_\Omega M_{tt}\cdot H_t\,dx \\ &\equiv I_7 + I_8.\end{aligned}$$

We have

$$|I_7| \le \frac{1}{4}\|\operatorname{curl} H_t\|^2 + \left(\|U^\sharp_t\|_{\mathbb{L}^6(\Omega)}\|B\|_{\mathbb{L}^3(\Omega)} + \|U^\sharp\|_{\mathbb{L}^\infty(\Omega)}\|B_t\|\right)^2.$$

We also have

$$|I_8| \le \frac{1}{2}\left(\|M_{tt}\|^2 + \|H_t\|^2\right).$$

It results that

$$\frac{d}{dt}\int_\Omega |H_t|^2\,dx + \frac{1}{4}\int_\Omega |\mathrm{curl}\,H_t|^2\,dx$$
$$\leq \Big(\|U_t^\sharp\|_{\mathbb{L}^6(\Omega)}\|B\|_{\mathbb{L}^3(\Omega)} + \|U^\sharp\|_{\mathbb{L}^\infty(\Omega)}\|B_t\|\Big)^2 + \frac{1}{2}\left(\|M_{tt}\|^2 + \|H_t\|^2\right).$$

Integrating over $(0,t)$ and using the Gronwall Lemma we deduce (60). Writing $\mathrm{curl}\,B_t = \mathrm{curl}\,H_t + \mathrm{curl}\,M_t$, we have $\|\mathrm{curl}\,B_t\| \leq \|\mathrm{curl}\,H_t\| + \|\mathrm{curl}\,M_t\|$ and using the conditions $\mathrm{div}\,B_t = 0$ in Ω_T and $B_t \cdot n = 0$ in $(0,T)\times\partial\Omega$ we get that $B_t \in L^2(0,T;\mathbb{H}^1(\Omega))$ and we have the estimate

$$\|B_t\|_{L^2(0,T;\mathbb{H}^1(\Omega))} \leq C\left(\|\mathrm{curl}\,H_t\| + \|\mathrm{curl}\,M_t\|\right).$$

We deduce that $H_t \in L^2(0,T;\mathbb{H}^1(\Omega))$ and we have (61).

(v) Using (63) and the Hölder inequality we deduce from equation (21) that

$$\|\mathrm{curl}^2 H(t)\| \leq \|U^\sharp(t)\|_{\mathbb{L}^6(\Omega)}\|\nabla B(t)\|_{(\mathbb{L}^3(\Omega))^3} + \|B(t)\|_{\mathbb{L}^\infty(\Omega)}\|\nabla U^\sharp(t)\| + \|B_t(t)\|.$$

Using the Poincaré inequality and the Sobolev embeddings $(37)_1$ and $(37)_3$, we obtain

$$\|\mathrm{curl}^2 H(t)\| \leq C\|\nabla U^\sharp(t)\|\|B(t)\|_{\mathbb{W}^{1,4}(\Omega)} + \|B_t(t)\|, \ \ t \in (0,T).$$

Since the right-hand side belongs to $L^\infty(0,T)$, the function $\mathrm{curl}^2 H$ belongs to $L^\infty(0,T;\mathbb{L}^2(\Omega))$ which implies that $\mathrm{curl}^2 B \in L^\infty(0,T;\mathbb{L}^2(\Omega))$. Therefore H and B belong to $L^\infty(0,T;\mathbb{H}^2(\Omega))$ and satisfy (62). The proof of Lemma 4 is complete. □

Lemma 5. *Let M be the solution of problem $(\mathcal{P}_M)$, H the solution of problem $(\mathcal{P}_H)$ and $J = \mathrm{curl}\,H$. Then:*

(i) $(M\cdot\nabla)H$, $M \wedge H$, $\mathrm{curl}\,(M\wedge H)$ and $J \times B$ satisfy, for $t \in (0,T)$,

$$\|(M\cdot\nabla)H(t)\|_{\mathbb{H}^1(\Omega)} \leq C\|M(t)\|_{\mathbb{H}^2(\Omega)}\|H(t)\|_{\mathbb{H}^2(\Omega)}, \tag{64}$$
$$\|M\wedge H(t)\|_{\mathbb{H}^1(\Omega)} \leq C\|M(t)\|_{\mathbb{H}^2(\Omega)}\|H(t)\|_{\mathbb{H}^2(\Omega)}, \tag{65}$$
$$\|\mathrm{curl}\,(M\wedge H)(t)\|_{\mathbb{H}^1(\Omega)} \leq C\|M(t)\|_{\mathbb{H}^2(\Omega)}\|H(t)\|_{\mathbb{H}^2(\Omega)}, \tag{66}$$
$$\|J\wedge B(t)\|_{\mathbb{H}^1(\Omega)} \leq C\|B(t)\|_{\mathbb{H}^2(\Omega)}\|H(t)\|_{\mathbb{H}^2(\Omega)}; \tag{67}$$

(ii) $[M\wedge H]_t$, $\mathrm{curl}\,[M\wedge H]_t$, $[(M\cdot\nabla)H]_t$, $[J\times B]_t$ satisfy, for $t\in(0,T)$,

$$\|[M\wedge H]_t(t)\| \leq C\left(\|H(t)\|_{\mathbb{W}^{1,6}(\Omega)}\|M_t(t)\|_{\mathbb{L}^6(\Omega)} + \|M(t)\|_{\mathbb{W}^{1,6}(\Omega)}\|H_t(t)\|\right), \tag{68}$$
$$\|\mathrm{curl}\,[M\wedge H]_t(t)\|_{\mathbb{H}^{-1}(\Omega)} \leq C\|[M\wedge H]_t(t)\|, \tag{69}$$
$$\begin{aligned}\|[(M\cdot\nabla)H(t)]_t\|_{\mathbb{H}^{-1}(\Omega)} &\leq C\big(\|[M\wedge H]_t(t)\| + \|\mathrm{div}\,M_t(t)\|\|B(t)\|_{\mathbb{W}^{1,6}(\Omega)}\big)\\ &\quad + C\big(\|H_t(t)\|\|M(t)\|_{\mathbb{W}^{1,6}(\Omega)} + \|H(t)\|_{\mathbb{H}^1(\Omega)}\|M(t)|_{\mathbb{H}^1(\Omega)}\big),\end{aligned} \tag{70}$$
$$\begin{aligned}\|[J\wedge B]_t(t)\|_{\mathbb{H}^{-1}(\Omega)} &\leq C\left(\|H_t(t)\|\|B(t)\|_{\mathbb{L}^\infty(\Omega)} + \|H_t(t)\|\|\nabla B(t)\|_{(\mathbb{L}^3(\Omega))^3}\right)\\ &\quad + C\|B_t(t)\|\|J(t)\|_{\mathbb{L}^3(\Omega)}.\end{aligned} \tag{71}$$

Proof. The proofs of *(i)*, (68) and (69) are easy, we use the Hölder inequality and the Sobolev embedding. To prove (70) we use the vector identity

$$(M\cdot\nabla)H = -\mathrm{curl}\,(M\wedge H) + (\mathrm{div}\,M)B + (H\cdot\nabla)M,$$

which by differentiation with respect to t gives

$$[(M\cdot\nabla)H]_t = -\operatorname{curl}[M\wedge H]_t + (\operatorname{div} M_t)B + (\operatorname{div} M)B_t + (H_t\cdot\nabla)M + (H\cdot\nabla)M_t.$$

Estimating each term in the right-hand side gives (70). To show (71) we write, for any $w\in\mathcal{U}$,

$$\begin{aligned}\int_\Omega [J\wedge B]_t\cdot w\,dx &= \int_\Omega [J_t\wedge B]\cdot w\,dx + \int_\Omega [J\wedge B_t]\cdot w\,dx\\ &= \int_\Omega H_t\cdot\Big((B\nabla)w-(w\nabla)B\Big)\,dx + \int_\Omega \operatorname{curl} H\cdot[B_t\wedge w]\,dx,\end{aligned}$$

from which follows

$$\begin{aligned}\left|\int_\Omega [J\wedge B]_t\cdot w\,dx\right| &\le \|H_t\|\|B\|_{\mathbb{L}^\infty(\Omega)}\|\nabla w\| + \|H_t\|\|\nabla B\|_{(\mathbb{L}^3(\Omega))^3}\|w\|_{\mathbb{L}^6(\Omega)}\\ &\quad + \|B_t\|\|w\|_{\mathbb{L}^6(\Omega)}\|\operatorname{curl} H\|_{\mathbb{L}^3(\Omega)},\end{aligned}$$

hence (71). The lemma is proved. □

2.3 Problem $(\mathcal{P}_U)$

Arguing as in [1] (Lemma 6) we prove the following result.

Lemma 6. *Problem $(\mathcal{P}_U)$ admits a unique global-in-time strong solution U satisfying:*

(i)

$$U\in C([0,T];\mathcal{U}\cap\mathbb{H}^2(\Omega))\cap C^1([0,T];\mathbb{H}^1(\Omega))\cap L^2(0,T;\mathbb{H}^3(\Omega));$$

(ii)

$$\begin{aligned}&\|U(t)\|^2 + \int_0^t \|\nabla U(s)\|^2\,ds\\ &\le C\left(\|U_0\|^2 + \int_0^t \left(\|(M\cdot\nabla)H(s)\|^2 + \|M\wedge H(s)\|^2 + \|J\wedge B(s)\|^2\right)\,ds\right),\quad t\in(0,T);\end{aligned}$$

(iii)

$$\|\nabla U(t)\|^2 + \int_0^t \|U_t(s)\|^2\,ds \le Cd_1(t),\quad t\in(0,T),$$

$$\int_0^t \|U(s)\|^2_{\mathbb{H}^2(\Omega))}\,ds \le Cd_2(t),\quad t\in(0,T),$$

where

$$\begin{aligned}d_1(t) &= \left(\|\nabla U_0\|^2 + \int_0^t \left(\|(M\cdot\nabla)H(s)\|^2 + \|\operatorname{curl}(M\wedge H)(s)\|^2\right)ds\right.\\ &\quad\left. + \int_0^t \|J\wedge B(s)\|^2\,ds\right)\exp\left(C\int_0^t \|U^\sharp(s)\|^2_{\mathbb{L}^\infty(\Omega)}\,ds\right),\\ d_2(t) &= \int_0^t \|U_t(s)\|^2\,ds + \|U^\sharp(s)\|^2_{\mathbb{L}^\infty(0,t;\mathbb{H}^2(\Omega))}\int_0^t \|\nabla U(s)\|^2\,ds\\ &\quad + \int_0^t \left(\|(M\cdot\nabla)H(s)\|^2 + \|\operatorname{curl}(M\wedge H)(s)\|^2 + \|J\wedge B(s)\|^2\right)ds;\end{aligned}$$

(iv)

$$\|U_t(t)\|^2 + \int_0^t \|\nabla U_t(s)\|^2\,ds \le C(d_3^1 + d_3^2)(t), \quad t \in (0,T),$$
$$\|U(t)\|^2_{\mathbb{H}^2(\Omega)} \le C d_4(t), \quad t \in (0,T),$$

where d_3^1, d_3^2 and d_4 are given by

$$\begin{aligned} d_3^1 &= \|\Delta U_0\|^2_{\mathbb{L}^2(\Omega)} + \|U^\sharp(0)\|^2_{\mathbb{H}^2(\Omega)}\|\nabla U_0\|^2 + \|(M_0\cdot\nabla)H_0\|^2 + \|(M_0 \wedge H_0)\|^2 \\ &\quad + \|\mathrm{curl}\, H_0 \wedge B_0\|^2, \\ d_3^2(t) &= \int_0^t \big(\|[(M\cdot\nabla)H]_t(s)\|^2_{\mathbb{H}^{-1}(\Omega)} + \|[J\wedge B]_t(s)\|^2_{\mathbb{H}^{-1}(\Omega)}\big)\,ds \\ &\quad + \int_0^t \|[M\wedge H]_t(s)\|^2_{\mathbb{H}^{-1}(\Omega)}\,ds + \|U_t^\sharp\|^2_{L^\infty(0,t;\mathbb{L}^2(\Omega))}\int_0^t \|U\|^2_{\mathbb{H}^2(\Omega))}\,ds, \\ d_4(t) &= \|U_t(t)\|^2 + \|U^\sharp(t)\|^2_{\mathbb{H}^2(\Omega)}\|\nabla U_t(t)\|^2 + \|(M\cdot\nabla)H(t)\|^2 \\ &\quad + \|\mathrm{curl}\,(M\wedge H)(t)\|^2 + \|J\wedge B(t)\|^2; \end{aligned}$$

(v)

$$\begin{aligned} \int_0^T \|U(s)\|^2_{\mathbb{H}^3(\Omega)}\,ds &\le C\int_0^T \|U_t(s)\|^2_{\mathbb{H}^1(\Omega)}\,ds + C\int_0^T \|U^\sharp(s)\|^2_{\mathbb{H}^2(\Omega)}\|U(s)\|^2_{\mathbb{H}^2(\Omega)}\,ds \\ &\quad + C\int_0^T \|(M\cdot\nabla)H(s)\|^2_{\mathbb{H}^1(\Omega)}\,ds + C\int_0^T \|\mathrm{curl}\,(M\wedge H)(s)\|^2_{\mathbb{H}^1(\Omega)}\,ds \\ &\quad + C\int_0^T \|J\wedge B(s)\|^2_{\mathbb{H}^1(\Omega)}\,ds. \end{aligned}$$

3 Proof of Theorem 1

3.1 Approximate solutions

Set $(U^0, M^0, H^0) = (0,0,0)$. Assuming that the triplet (U^n, M^n, H^n) is defined, let $(U^{n+1}, M^{n+1}, H^{n+1})$ be the solution of problem (19)–(28) with $(U^\sharp, M^\sharp, H^\sharp)$ replaced by (U^n, M^n, H^n). Thus M^{n+1} satisfies

$$\partial_t M^{n+1} + (U^n\cdot\nabla)M^{n+1} - \frac{1}{2}\mathrm{curl}\, U^n \wedge M^{n+1} + \frac{1}{t_m}M^{n+1} + \frac{1}{4\xi_r}M^{n+1}\wedge(M^n\wedge H^n)$$
$$= \frac{\chi_m}{t_m}H^n \ \text{ in } \Omega_T, \tag{72}$$
$$M^{n+1}(0) = M_0 \ \text{ in } \Omega. \tag{73}$$

The function H^{n+1} satisfies

$$\partial_t H^{n+1} + \mathrm{curl}^{\,2} H^{n+1} = \mathrm{curl}\,(U^n\wedge B^{n+1}) - \partial_t M^{n+1} \quad \text{in } \Omega_T, \tag{74}$$
$$\mathrm{div}\, B^{n+1} = 0 \quad \text{in } \Omega_T, \tag{75}$$
$$B^{n+1}\cdot n = 0, \quad \mathrm{curl}\, H^{n+1}\wedge n = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{76}$$
$$H^{n+1}|_{t=0} = H_0 \equiv B_0 - M_0 \quad \text{in } \Omega, \tag{77}$$

where $B^{n+1} = H^{n+1} + M^{n+1}$. The function U^{n+1} satisfies

$$
\begin{aligned}
&\partial_t U^{n+1} + (U^n \cdot \nabla)U^{n+1}) - \mu\Delta U^{n+1} + \nabla p^{n+1} \\
&= (M^{n+1} \cdot \nabla)H^{n+1} + \frac{1}{2}\operatorname{curl}(M^{n+1} \wedge H^{n+1}) + J^{n+1} \wedge B^{n+1} \quad \text{in } \Omega_T, \qquad (78)\\
&\operatorname{div} U^{n+1} = 0 \quad \text{in } \Omega_T, \qquad (79)\\
&U^{n+1} = 0 \quad \text{on } (0,T) \times \partial\Omega, \quad U^{n+1}(0) = U_0 \quad \text{in } \Omega. \qquad (80)
\end{aligned}
$$

The previous study of problems $(\mathcal{P}_M)$, $(\mathcal{P}_H)$ and $(\mathcal{P}_U)$ shows that $(U^{n+1}, M^{n+1}, H^{n+1})$ is well-defined by (72)–(80).

3.2 Uniform bounds

Let N be a large fixed integer and let us introduce an auxiliary function Φ_N defined on $(0,T)$ by

$$
\Phi_N(t) = \max_{0\le n\le N}\left(\sup_{0\le s\le t}\left(1 + \|\nabla U^{n+1}(s)\| + \|M^{n+1}(s)\|_{\mathbb{H}^2(\Omega)} + \|H^{n+1}(s)\|_{\mathbb{H}^2(\Omega)}\right)\right).
$$

Arguing as in [1] (Lemmas 7, 8) and using Lemma 5 we prove the following result.

Lemma 7. *We have*

$$
\|\nabla U^{n+1}(t)\|^2 + \int_0^t \|U_t^{n+1}(s)\|^2\, ds \le C + C\int_0^t \Phi_N^6(s)\, ds, \qquad (81)
$$

$$
\|U_t^{n+1}(t)\|^2 + \int_0^t \|\nabla U_t^{n+1}(s)\|^2\, ds \le C\exp\left(C\int_0^t \Phi_N^8(s)\, ds\right), \qquad (82)
$$

for any $0 \le n \le N$ and $t \in (0,T)$.

Consider the Stokes system (for fixed $t \in (0,T)$)

$$
-\mu\Delta U^{n+1} + \nabla p^{n+1} = G^{n+1}, \quad \operatorname{div} U^{n+1} = 0 \quad \text{in } \Omega, \quad U^{n+1} = 0 \quad \text{on } \partial\Omega, \qquad (83)
$$

where

$$
G^{n+1} = -(U_t^{n+1} + (U^n \cdot \nabla)U^{n+1}) + S^{n+1},
$$

and S^{n+1} is given by

$$
S^{n+1} = (M^{n+1} \cdot \nabla)H^{n+1} + \frac{1}{2}\operatorname{curl}(M^{n+1} \wedge H^{n+1}) + J^{n+1} \wedge B^{n+1}.
$$

We have

$$
\|(M^{n+1} \cdot \nabla)H^{n+1}\| \le C\|M^{n+1}\|_{\mathbb{L}^\infty(\Omega)}\|\nabla H^{n+1}\|, \qquad (84)
$$

$$
\|J^{n+1} \wedge B^{n+1}\| \le C\left(\|M^{n+1}\|_{\mathbb{L}^\infty(\Omega)} + \|H^{n+1}\|_{\mathbb{L}^\infty(\Omega)}\right)\|\nabla H^{n+1}\|, \qquad (85)
$$

and

$$
\|\operatorname{curl}(M^{n+1} \wedge H^{n+1})\| \le C\left(\|M^{n+1}\|_{\mathbb{L}^\infty(\Omega)}\|\nabla H^{n+1}\| + \|H^{n+1}\|_{\mathbb{L}^\infty(\Omega)}\|\nabla M^{n+1}\|\right). \qquad (86)
$$

For the last inequality we used the identity

$$\begin{aligned}\operatorname{curl}(M^{n+1}\wedge H^{n+1}) =&(\operatorname{div} H^{n+1})M^{n+1} - (\operatorname{div} M^{n+1})H^{n+1}\\ &+(H^{n+1}\cdot\nabla)M^{n+1} - (M^{n+1}\cdot\nabla)H^{n+1}.\end{aligned}$$

Using the Sobolev embedding $H^2(\Omega)\hookrightarrow L^\infty(\Omega)$ we deduce from (84)–(86) that

$$\|S^{n+1}\| \le C\|H^{n+1}\|_{\mathbb{H}^2(\Omega)}\|\nabla M^{n+1}\| + C\left(\|M^{n+1}\|_{\mathbb{H}^2(\Omega)} + \|H^{n+1}\|_{\mathbb{H}^2(\Omega)}\right)\|\nabla H^{n+1}\|.$$

Similarly, using (64)–(67) and the Sobolev embedding $H^1(\Omega)\hookrightarrow L^6(\Omega)$ we show that

$$\|S^{n+1}\|_{\mathbb{L}^6(\Omega)} \le C\left(\|M^{n+1}\|_{\mathbb{H}^2(\Omega)} + \|H^{n+1}\|_{\mathbb{H}^2(\Omega)}\right)\|H^{n+1}\|_{\mathbb{H}^2(\Omega)}.$$

Now applying the elliptic regularity results for the Stokes system (83), by using the techniques in [1], we derive the estimates

$$\|U^{n+1}(t)\|_{\mathbb{H}^2(\Omega)} \le C\left(\|U_t^{n+1}(t)\| + \|\nabla U^n(t)\|^2\ \|\nabla U^{n+1}(t)\| + \|S^{n+1}(t)\|\right), \tag{87}$$

$$\|U^{n+1}(t)\|^2_{\mathbb{H}^2(\Omega)} \le C\left(\|U_t^{n+1}(t)\|^2 + \Phi_N^6(t)\right), \tag{88}$$

$$\|U^{n+1}(t)\|^2_{\mathbb{W}^{2,6}(\Omega)} \le C\left(\|\nabla U_t^{n+1}(t)\|^2 + \|U_t^{n+1}(t)\|^4 + \Phi_N^{12}(t)\right), \tag{89}$$

$$\int_0^t \|U^{n+1}(s)\|^2_{\mathbb{W}^{2,6}(\Omega)}\,ds \le C\exp\left(C\int_0^t \Phi_N^{12}(s)\,ds\right), \tag{90}$$

for any $0\le n\le N$ and $t\in(0,T)$.

Lemma 8. *We have*

$$\|M^{n+1}(t)\|^r_{\mathbb{L}^r(\Omega)} + \int_0^t \|M^{n+1}(s)\|^r_{\mathbb{L}^r(\Omega)}\,ds \le C + C\int_0^t \Phi_N^r(s)\,ds, \quad 2\le r\le 6, \tag{91}$$

$$\|M^{n+1}(t)\|_{\mathbb{W}^{1,r}(\Omega)} \le C\exp\left(C\exp\left(C\int_0^t \Phi_N^{12}(s)\,ds\right)\right), \quad 2\le r\le 6, \tag{92}$$

$$\int_0^t \|\nabla M_t^{n+1}(s)\|^2\,ds \le C\exp\left(C\int_0^t \Phi_N^8(s)\,ds\right), \tag{93}$$

for any $0\le n\le N$ and $t\in(0,T)$. Moreover,

$$\|\partial^2_{x_ix_j}M^{n+1}(t)\|^2 + \int_0^t \|\partial^2_{x_ix_j}M^{n+1}(s)\|^2\,ds \le C\exp\left(C\exp\left(C\int_0^t \Phi_N^{12}(s)\,ds\right)\right), \tag{94}$$

for any $i,j=1,2,3$, for any $0\le n\le N$ and $t\in(0,T)$.

Proof. We show inequalities (91) and (92) by similar arguments to that used in [1] (Lemma 9). Inequality (93) follows directly from (52), (82) and (88). Let us prove (94). Arguing as in the proof of (43) we have

$$\|\partial^2_{x_ix_j}M^{n+1}(t)\|^2 + \int_0^t \|\partial^2_{x_ix_j}M^{n+1}(s)\|^2\,ds \le Ca_3(t), \tag{95}$$

with

$$a_3(t) = \left(\|M_0\|^2_{\mathbb{H}^2(\Omega)} + \int_0^t \left(a_3^3(s)\|\nabla M^{n+1}(s)\|^2 + a_3^2(s)\right)ds\right)\exp\left(C\int_0^t a_3^3(s)\,ds\right)$$

and

$$\begin{aligned}
a_3^2(t) &= \|H^n(t)\|^2_{\mathbb{H}^2(\Omega)} + \|U^n(t)\|^2_{\mathbb{H}^3(\Omega)}\|M^{n+1}(t)\|^2_{\mathbb{W}^{1,6}(\Omega)} \\
&\quad + \|M^{n+1}(t)\|^2_{\mathbb{W}^{1,6}(\Omega)}\|M^n(t)\|^2_{\mathbb{H}^2(\Omega)}\|H^n(t)\|^2_{\mathbb{W}^{1,6}(\Omega)} \\
&\quad + \|M^{n+1}(t)\|^2_{\mathbb{W}^{1,6}(\Omega)}\|M^n(t)\|^2_{\mathbb{W}^{1,6}(\Omega)}\|H^n(t)\|^2_{\mathbb{H}^2(\Omega)}, \\
a_3^3(t) &= \|U^n(t)\|_{\mathbb{W}^{2,6}(\Omega)} + \|H^n(t)\|_{\mathbb{W}^{1,6}(\Omega)}\|\nabla M^n(t)\|_{(\mathbb{L}^6(\Omega))^3} \\
&\quad + \|M^n(t)\|_{\mathbb{W}^{1,6}(\Omega)}\|\nabla H^n(t)\|_{(\mathbb{L}^6(\Omega))^3} + 1.
\end{aligned}$$

Using Young's inequality we have

$$\begin{aligned}
a_3(t) \le &\left(\|M_0\|^2_{\mathbb{H}^2(\Omega)} + \int_0^t \left(\frac{1}{2}(a_3^3)^2(s) + \frac{1}{2}\|\nabla M^{n+1}(s)\|^4 + a_3^2(s)\right) ds\right) \times \\
&\times \exp\left(C\int_0^t a_3^3(s)\, ds\right).
\end{aligned} \tag{96}$$

Using (82) and (89) we find that

$$\int_0^t (a_3^3)^2(s)\, ds \le C\exp\left(C\int_0^t \Phi_N^{12}(s)\, ds\right). \tag{97}$$

According to Lemma 6 *(v)* we have

$$\begin{aligned}
&\int_0^t \|U^{n+1}(s)\|^2_{\mathbb{H}^3(\Omega)}\, ds \\
&\le C\int_0^t \|U_t^{n+1}(s)\|^2_{\mathbb{H}^1(\Omega)}\, ds + C\int_0^t \|U^n(s)\|^2_{\mathbb{H}^2(\Omega)}\|U^{n+1}(s)\|^2_{\mathbb{H}^2(\Omega)}\, ds \\
&\quad + C\int_0^t \|(M^{n+1}\cdot\nabla)H^{n+1}(s)\|^2_{\mathbb{H}^1(\Omega)}\, ds + C\int_0^t \|\operatorname{curl}(M^{n+1}\wedge H^{n+1})(s)\|^2_{\mathbb{H}^1(\Omega)}\, ds \\
&\quad + C\int_0^t \|J^{n+1}\wedge B^{n+1}(s)\|^2_{\mathbb{H}^1(\Omega)}\, ds.
\end{aligned}$$

Then, using (64)–(67), (82) and (88) it holds that

$$\int_0^t \|U^{n+1}(s)\|^2_{\mathbb{H}^3(\Omega)}\, ds \le C\exp\left(C\int_0^t \Phi_N^{12}(s)\, ds\right). \tag{98}$$

Using (92) and (98) we easily see that

$$\int_0^t a_3^2(s)\, ds \le C\exp\left(C\exp\left(C\int_0^t \Phi_N^{12}(s)\, ds\right)\right). \tag{99}$$

It results from (96), (97) and (99) that

$$a_3(t) \le C\exp\left(C\exp\left(C\int_0^t \Phi_N^{12}(s)\, ds\right)\right).$$

This inequality, together with (95), implies (94). The lemma is proved. □

Lemma 9. *We have*

$$\|H^{n+1}(t)\|^2 + \int_0^t \|\operatorname{curl} H^{n+1}(s)\|^2\,ds \le C \exp\left(C \exp\left(C \int_0^t \Phi_N^8(s)\,ds\right)\right), \tag{100}$$

$$\|\operatorname{curl} H^{n+1}(t)\|^2 + \int_0^t \|H_t^{n+1}(s)\|^2\,ds \le C \exp\left(C \exp\left(C \int_0^t \Phi_N^{12}(s)\,ds\right)\right), \tag{101}$$

$$\|\operatorname{curl} H^{n+1}(t)\|^2 + \int_0^t \|\operatorname{curl}^2 H^{n+1}(s)\|^2\,ds \le C \exp\left(C \exp\left(C \int_0^t \Phi_N^{12}(s)\,ds\right)\right), \tag{102}$$

$$\|H_t^{n+1}(t)\|^2 + \int_0^t \|\operatorname{curl} H_t^{n+1}(s)\|^2\,ds \le C \exp\left(C \exp\left(C \int_0^t \Phi_N^{12}(s)\,ds\right)\right), \tag{103}$$

for any $0 \le n \le N$ *and* $t \in (0,T)$.

Proof. Multiplying equation (74) by H^{n+1} yields the analogue of (54):

$$\|H^{n+1}(t)\|^2 + \int_0^t \|\operatorname{curl} H^{n+1}(s)\|^2\,ds \le b_1(t),$$

with

$$\begin{aligned} b_1(t) = \left(\|H_0\|^2 + \int_0^t \left(\|U^n(s)\|^2_{\mathbb{L}^\infty(\Omega)}\|M^{n+1}(s)\|^2 + \|M_t^{n+1}(s)\|^2\right) ds\right) \times \\ \times \exp\left(t + \int_0^t \|U^n(s)\|^2_{\mathbb{L}^\infty(\Omega)}\,ds\right). \end{aligned}$$

Using the Sobolev embedding, (82) and (88), we obtain

$$\int_0^t \|U^n(s)\|^2_{\mathbb{L}^\infty(\Omega)}\,ds \le C \int_0^t \|U^n(s)\|^2_{\mathbb{H}^2(\Omega)}\,ds \le C \exp\left(C \int_0^t \Phi_N^8(s)\,ds\right). \tag{104}$$

Similar arguments, together with the inequality $\|M^{n+1}(s)\| \le \Phi_N(s)$, imply that

$$\int_0^t \|U^n(s)\|^2_{\mathbb{L}^\infty(\Omega)}\|M^{n+1}(s)\|^2\,ds \le C \exp\left(C \int_0^t \Phi_N^8(s)\,ds\right). \tag{105}$$

We deduce from equation (72) that

$$\begin{aligned} \|M_t^{n+1}\| &\le C\left(\|\nabla U^n\|\|M^{n+1}\|_{\mathbb{W}^{1,6}(\Omega)} + \|M^{n+1}\| + \|H^n\|\right) \\ &\quad + C\|M^{n+1}\|_{\mathbb{W}^{1,6}(\Omega)}\|M^n\|_{\mathbb{W}^{1,6}(\Omega)}\|H^n\| \\ &\le C\Phi_N^3, \end{aligned} \tag{106}$$

Estimates (104)–(106) imply that

$$b_1(t) \le C \exp\left(C \exp\left(\int_0^t \Phi_N^8(s)\,ds\right)\right)$$

and (100) follows.

Arguing as in the proof of (56) we have

$$\|\operatorname{curl} H^{n+1}(t)\|^2 + \int_0^t \|H_t^{n+1}(s)\|^2\,ds \le b_2(t),$$

with

$$b_2(t) = 2\int_0^t \Big(\|\nabla U^n(s)\|^2_{(\mathbb{L}^\infty(\Omega))^3} \|B^{n+1}(s)\|^2 + \|U^n(s)\|^2_{\mathbb{L}^\infty(\Omega)} \|B^{n+1}(s)\|^2_{\mathbb{H}1(\Omega)} + \|M_t^{n+1}(s)\|^2 \Big) ds + \|\mathrm{curl}\, H_0\|^2.$$

Using the Sobolev embedding $W^{1,6}(\Omega) \hookrightarrow L^\infty(\Omega)$ we have

$$\begin{aligned} \int_0^t \|\nabla U^n(s)\|^2_{(\mathbb{L}^\infty(\Omega))^3} \|B^{n+1}(s)\|^2\, ds &\le C \max_{0\le s\le t} \|B^{n+1}(s)\|^2 \int_0^t \|U^n(s)\|^2_{\mathbb{W}^{2,6}(\Omega)}\, ds \\ &\le C \max_{0\le s\le t} \big(\|M^{n+1}(s)\|^2 + \|H^{n+1}(s)\|^2 \big) \times \\ &\quad \times \int_0^t \|U^n(s)\|^2_{\mathbb{W}^{2,6}(\Omega)}\, ds. \end{aligned} \tag{107}$$

Using inequalities (89), (91) and (100), we deduce from (107) that

$$\int_0^t \|\nabla U^n(s)\|^2_{(\mathbb{L}^\infty(\Omega))^3} \|B^{n+1}(s)\|^2\, ds \le C \exp\left(C \exp\left(C \int_0^t \Phi_N^{12}(s)\, ds \right)\right).$$

Similarly to (105) we show that

$$\int_0^t \|U^n(s)\|^2_{\mathbb{L}^\infty(\Omega)} \|B^{n+1}(s)\|^2_{\mathbb{H}^1(\Omega)}\, ds \le C \exp\left(C \int_0^t \Phi_N^{8}(s)\, ds \right).$$

We conclude, together with (106), that

$$b_2(t) \le C \exp\left(C \exp\left(C \int_0^t \Phi_N^{12}(s)\, ds \right)\right),$$

hence (101).

As for (58) we have

$$\|\mathrm{curl}\, H^{n+1}(t)\|^2 + \int_0^t \|\mathrm{curl}^{\,2} H^{n+1}(s)\|^2\, ds \le b_3(t), \tag{108}$$

with

$$\begin{aligned} b_3(t) = &\Big(\|\mathrm{curl}\, H_0\|^2 + \int_0^t \big(C\|U^n(s)\|^2_{\mathbb{W}^{2,6}(\Omega)} \|\mathrm{curl}\, M^{n+1}(s)\|^2 + \|\mathrm{curl}\, M_t^{n+1}(s)\|^2 \big) ds \Big) \times \\ &\times \exp\left(t + C \int_0^t \|U^n(s)\|^2_{\mathbb{W}^{2,6}(\Omega)}\, ds \right). \end{aligned}$$

We have

$$\int_0^t \|U^n(s)\|^2_{\mathbb{W}^{2,6}(\Omega)} \|\mathrm{curl}\, M^{n+1}(s)\|^2\, ds \le C \sup_{0\le s\le t} \|M^{n+1}(s)\|^2_{\mathbb{H}^1(\Omega)} \int_0^t \|U^n(s)\|^2_{\mathbb{W}^{2,6}(\Omega)}\, ds,$$

then, using (90) and (92) we deduce that

$$\int_0^t \|U^n(s)\|^2_{\mathbb{W}^{2,6}(\Omega)} \|\mathrm{curl}\, M^{n+1}(s)\|^2\, ds \le C \exp\left(C \exp\left(C \int_0^t \Phi_N^{12}(s)\, ds \right)\right).$$

It results, together with (93), that

$$b_3(t) \le C \exp\left(C \exp\left(C \int_0^t \Phi_N^{12}(s)\, ds \right)\right),$$

hence (102).

Arguing as for (60) we derive the inequality

$$\|H_t^{n+1}(t)\|^2 + \frac{1}{4}\int_0^t \|\mathrm{curl}\, H_t^{n+1}(s)\|^2\, ds \le b_4(t),$$

with

$$b_4(t) = \Bigg(\|H_t^{n+1}(0)\|^2 + \int_0^t \bigg(\Big(\|U_t^n(s)\|_{\mathbb{L}^6(\Omega)} \|B^{n+1}(s)\|_{\mathbb{L}^3(\Omega)} + \|U^n(s)\|_{\mathbb{L}^\infty(\Omega)} \|B_t^{n+1}(s)\|\Big)^2 + \frac{1}{2}\|M_{tt}^{n+1}(s)\|^2 \bigg) ds \Bigg) \exp\left(\frac{t}{2}\right)$$

and

$$\begin{aligned} H_t^{n+1}(0) &= -\mathrm{curl}^{\,2} H_0 + \mathrm{curl}\,(U_0 \wedge B_0) - M_t^{n+1}(0),\\ M_t^{n+1}(0) &= -(U_0\nabla)M_0 + \frac{1}{2}\mathrm{curl}\,(U_0\wedge M_0) - \frac{1}{t_m}M_0 - \frac{1}{4\xi_r} M_0 \wedge (M_0 \wedge H_0) + \frac{\chi_m}{t_m} H_0. \end{aligned}$$

We deduce from equation (74) and inequality (106) that

$$\begin{aligned} \|H_t^{n+1}\| &\le C\left(\|H^{n+1}\|_{\mathbb{H}^2(\Omega)} + \|\nabla U^n\| \|B^{n+1}\|_{\mathbb{W}^{1,6}(\Omega)} + \|M_t^{n+1}\|\right)\\ &\le C\Phi_N^3. \end{aligned} \tag{109}$$

Using the Sobolev embedding, (82), (88), (106) and (109), we deduce that

$$\begin{aligned} \int_0^t \|U^n(s)\|^2_{\mathbb{L}^\infty(\Omega)} \|B_t^{n+1}(s)\|^2\, ds &\le C \int_0^t \|U^n(s)\|^2_{\mathbb{H}^2(\Omega)} \left(\|M_t^{n+1}(s)\|^2 + \|H_t^{n+1}(s)\|^2\right) ds\\ &\le C\exp\left(C\int_0^t \Phi_N^{12}(s)\, ds\right). \end{aligned}$$

We also have

$$\begin{aligned} &\int_0^t \|U_t^n(s)\|^2_{\mathbb{L}^6(\Omega)} \|B^{n+1}(s)\|^2_{\mathbb{L}^3(\Omega)}\, ds\\ &\le C\left(\max_{0\le s\le t} \|M^{n+1}(s)\|^2_{\mathbb{H}^1(\Omega)} + \max_{0\le s\le t}\|H^{n+1}(s)\|^2_{\mathbb{H}^1(\Omega)}\right) \int_0^t \|\nabla U_t^n(s)\|^2\, ds, \end{aligned}$$

then, using the inequality (analogue of (57))

$$\|H^{n+1}\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} \le C\left(\|\mathrm{curl}\, H^{n+1}\|_{L^\infty(0,T;\mathbb{L}^2(\Omega))} + \|M^{n+1}\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))}\right) \tag{110}$$

and estimates (82), (92) and (101) we obtain

$$\int_0^t \|U_t^n(s)\|^2_{\mathbb{L}^6(\Omega)} \|B^{n+1}(s)\|^2_{\mathbb{L}^3(\Omega)}\, ds \le C\exp\left(C\exp\left(C\int_0^t \Phi_N^{12}(s)\, ds\right)\right).$$

Similar calculations (see (53)) show that

$$\int_0^t \|M_{tt}^{n+1}(s)\|^2\, ds \le C \exp\left(\int_0^t \Phi_N^8\, ds \right).$$

Since $\|H_t^{n+1}(0)\|^2 \le C$, we conclude that $b_4(t) \le C \exp\left(C \exp\left(C \int_0^t \Phi_N^{12}(s)\, ds \right)\right)$, hence (103). The proof of the lemma is finished. □

Lemma 10. *We have*

$$\|M_t^{n+1}(t)\| \le C \exp\left(C \exp\left(C \int_0^t \Phi_N^{12}(s)\, ds \right)\right), \tag{111}$$

$$\|H^{n+1}(t)\|_{\mathbb{H}^2(\Omega)} + \int_0^t \|\nabla H_t^{n+1}(s)\|^2\, ds \le C \exp\left(C \exp\left(C \int_0^t \Phi_N^{12}(s)\, ds \right)\right), \tag{112}$$

$$\|U^{n+1}(t)\|_{\mathbb{H}^2(\Omega)} \le C \exp\left(C \exp\left(C \int_0^t \Phi_N^{12}(s)\, ds \right)\right), \tag{113}$$

for any $0 \le n \le N$ *and* $t \in (0, T)$.

Proof. Since (see (106))

$$\begin{aligned}\|M_t^{n+1}(t)\| \le C \left(\|\nabla U^n(t)\| \|M^{n+1}(t)\|_{\mathbb{W}^{1,6}(\Omega)} + \|M^{n+1}(t)\| + \|H^n(t)\| \right) \\ + C \|M^{n+1}(t)\|_{\mathbb{W}^{1,6}(\Omega)} \|M^n(t)\|_{\mathbb{W}^{1,6}(\Omega)} \|H^n(t)\|,\end{aligned}$$

inequality (111) follows from (81), (92), (100).

In accordance with (61) we have

$$\|H_t^{n+1}\|_{L^2(0,T;\mathbb{H}^1(\Omega))} \le C \left(\|\mathrm{curl}\, H_t^{n+1}\| + \|M_t^{n+1}\|_{L^2(0,T;\mathbb{H}^1(\Omega))} \right).$$

Using (91), (93) and (103) we deduce that

$$\int_0^t \|\nabla H_t^{n+1}(s)\|^2\, ds \le C \exp\left(C \exp\left(C \int_0^t \Phi_N^{12}(s)\, ds \right)\right).$$

According to (62) we have

$$\|H^{n+1}(t)\|_{\mathbb{H}^2(\Omega)} \le C \left(\|\nabla U^n(t)\| \|B^{n+1}(t)\|_{\mathbb{W}^{1,4}(\Omega)} + \|B_t^{n+1}(t)\| + \|M^{n+1}(t)\|_{\mathbb{H}^2(\Omega)} \right). \tag{114}$$

Using the Gagliardo-Nirenberg-Sobolev inequality

$$\|\nabla v\|_{\mathbb{L}^4(\Omega)} \le C \|\nabla v\|^{\frac14} \|v\|_{H^2(\Omega)}^{\frac34}, \quad \forall v \in H^2(\Omega),$$

we have

$$\|B^{n+1}(t)\|_{\mathbb{W}^{1,4}(\Omega)} \le C \|\nabla B^{n+1}(t)\|^{\frac14} \|B^{n+1}(t)\|_{\mathbb{H}^2(\Omega)}^{\frac34} + C \|B^{n+1}(t)\|_{\mathbb{L}^4(\Omega)}. \tag{115}$$

Inserting (115) in (114) and applying the Young inequality we obtain

$$\begin{aligned}\|H^{n+1}(t)\|_{\mathbb{H}^2(\Omega)} \le & C \left(\|\nabla U^n(t)\|^4 \|\nabla B^{n+1}(t)\| + \|\nabla U^n(t)\| \|B^{n+1}(t)\|_{\mathbb{L}^4(\Omega)} + \|B_t^{n+1}(t)\| \right) \\ & + C \|M^{n+1}(t)\|_{\mathbb{H}^2(\Omega)} + \frac{1}{2} \|B^{n+1}(t)\|_{\mathbb{H}^2(\Omega)}.\end{aligned}$$

Since $\|B^{n+1}(t)\|_{\mathbb{H}^2(\Omega)} \leq \|H^{n+1}(t)\|_{\mathbb{H}^2(\Omega)} + \|M^{n+1}(t)\|_{\mathbb{H}^2(\Omega)}$ and $H^1(\Omega) \hookrightarrow L^4(\Omega)$ we obtain

$$\begin{aligned}\|H^{n+1}(t)\|_{\mathbb{H}^2(\Omega)} \leq & C\left(\|\nabla U^n(t)\|^4\|B^{n+1}(t)\|_{\mathbb{H}^1(\Omega)} + \|\nabla U^n(t)\|\|B^{n+1}(t)\|_{\mathbb{H}^1(\Omega)} + \|B_t^{n+1}(t)\|\right)\\ &+ C\|M^{n+1}(t)\|_{\mathbb{H}^2(\Omega)}.\end{aligned}$$

Using (81), (92), (94), (101), (103), (110) and (111) we obtain

$$\|H^{n+1}(t)\|_{\mathbb{H}^2(\Omega)} \leq C\exp\left(C\exp\left(C\int_0^t \Phi_N^{12}(s)\,ds\right)\right),$$

Since (see (87))

$$\|U^{n+1}(t)\|_{\mathbb{H}^2(\Omega)} \leq C\left(\|U_t^{n+1}(t)\| + \|\nabla U^n(t)\|^2\ \|\nabla U^{n+1}(t)\| + \|S^{n+1}(t)\|\right)$$

and

$$\begin{aligned}\|S^{n+1}(t)\| \leq & C\|H^{n+1}(t)\|_{\mathbb{H}^2(\Omega)}\|\nabla M^{n+1}(t)\|\\ &+ C\left(\|M^{n+1}(t)\|_{\mathbb{H}^2(\Omega)} + \|H^{n+1}(t)\|_{\mathbb{H}^2(\Omega)}\right)\|\nabla H^{n+1}(t)\|,\end{aligned}$$

inequality (113) follows from (81), (82), (92), (94) and (112). The lemma is proved. □

Lemma 11. *There is a time $T_* > 0$ such that*

$$\begin{aligned}&\sup_{0\leq t\leq T_*}\left(\|U^{n+1}(t)\|_{\mathbb{H}^2(\Omega)} + \|M^{n+1}(t)\|_{\mathbb{H}^2(\Omega)} + \|H^{n+1}(t)\|_{\mathbb{H}^2(\Omega)}\right)\\ &+\int_0^{T_*}\left(\|U^{n+1}(s)\|^2_{\mathbb{H}^3(\Omega)} + \|U_t^{n+1}(s)\|^2_{\mathbb{H}^1(\Omega)} + \|M_t^{n+1}(s)\|^2_{\mathbb{H}^1(\Omega)} + \|H_t^{n+1}(s)\|^2_{\mathbb{H}^1(\Omega)}\right)ds \leq C,\end{aligned} \tag{116}$$

for any $n \geq 0$.

Proof. It results from estimates (81), (92), (94) and (112) that the function Φ_N satisfies the integral inequality

$$\Phi_N(t) \leq C\exp\left(C\exp\left(C\int_0^t \Phi_N^{12}(s)\,ds\right)\right).$$

We deduce as in [1] that there is a time $T_* > 0$ such that $\Phi_N(t) \leq C$, for all $t \in (0, T_*)$. Then, using (82), (93), (98), (103), (111) and (113), we easily derive (116). The prof of Lemma 11 is complete. □

3.3 End of the proof of Theorem 1

(i) Existence and uniqueness. With the bound (116) one can easily show, following the technique in [1], that the whole sequence (U^n, M^n, H^n) converges to a limit (U, M, H) which is the unique strong solution of problem $(\mathcal{P})$ in $\Omega_{T_\star}$.
(ii) Blow-up criterion. Suppose that $T^\star < T$ and let us introduce the function

$$\Phi(t) = 1 + \|\nabla U(t)\| + \|M(t)\|_{\mathbb{H}^2(\Omega)} + \|H(t)\|_{\mathbb{H}^2(\Omega)}.$$

defined for $0 < t < T^\star$. Following the same arguments as in Section 3.2, one can establish the following estimates (for $t \in (0, T^\star)$):

$$
\begin{aligned}
\|U(t)\|_{\mathbb{H}^2(\Omega)} &\le C \exp\left(C \exp\left(C \int_0^t \Phi^{12}(s)\, ds\right)\right),\\
\int_0^t \|U(s)\|^2_{\mathbb{H}^3(\Omega)}\, ds &\le C \exp\left(C \int_0^t \Phi^{12}(s)\, ds\right),\\
\int_0^t \|U_t(s)\|^2_{\mathbb{H}^1(\Omega)}\, ds &\le C \exp\left(C \int_0^t \Phi^{8}(s)\, ds\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\|M(t)\|_{\mathbb{H}^2(\Omega)} &\le C \exp\left(C \exp\left(C \int_0^t \Phi^{12}(s)\, ds\right)\right),\\
\|M_t(t)\| &\le C \exp\left(C \exp\left(C \int_0^t \Phi^{12}(s)\, ds\right)\right),\\
\int_0^t \|M_t(s)\|^2_{\mathbb{H}^1(\Omega)}\, ds &\le C \exp\left(C \int_0^t \Phi^{8}(s)\, ds\right),\\
\|H(t)\|_{\mathbb{H}^2(\Omega)} &\le C \exp\left(C \exp\left(C \int_0^t \Phi^{12}(s)\, ds\right)\right),\\
\|H_t(t)\| &\le C \exp\left(C \exp\left(C \int_0^t \Phi^{12}(s)\, ds\right)\right),\\
\int_0^t \|H_t(s)\|^2_{\mathbb{H}^1(\Omega)}\, ds &\le C \exp\left(C \exp\left(C \int_0^t \Phi^{12}(s)\, ds\right)\right).
\end{aligned}
$$

Combining these estimates we conclude that

$$
J(t) \le C \exp\left(C \exp\left(C \int_0^t \Phi^{12}(s)\, ds\right)\right),
$$

where J is the functional defined by (17). The later estimate allows to conclude. The proof of Theorem 1 is complete.

References

[1] Y. Amirat and K. Hamdache, *Strong solutions to the equations of a ferrofluid flow model,* J. Math. Anal. Appl., **353** (2009), 271–294.

[2] Y. Amirat and K. Hamdache, *Unique solvability of equations of motion for ferrofluids,* Nonlinear Analysis, Series A: Theory, Methods & Applications, **73** (2010) 471–494.

[3] Y. Amirat and K. Hamdache, *Strong solutions to the equations of flow and heat transfer in magnetic fluids with internal rotations,* Disc. Cont. Dyn. Syst. (A), **33** (2013), 3289–3320.

[4] I.A. Brigadnov and A. Dorfmann, *Mathematical modeling of magnetorheological fluids,* Continuum Mech. Thermodyn., **17** (2005), 29–42.

[5] Q. Chen, Z. Tan and Y. Wang, *Strong solutions to the incompressible magnetohydrodynamic equations,* Math. Meth. Appl. Sci., **34**, (2011), 94–107.

[6] Y. Cho, H.J. Choe and H. Kim, *Unique solvability of the initial boundary value problems for compressible viscous fluids,* J. Math. Pures Appl., **83** (2004), 243–275.

[7] R. Dautray, J.L. Lions, *Analyse Mathématique et Calcul Numérique pour les Sciences et les Techniques,* T2, Masson, 1985.

[8] G. Duvaut and J.L. Lions, *Inéquations en thermoélasticité et magnétohydrodynamique,* Archive Rational Mech. Anal., **46** (1972), 241–279.

[9] G.P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations. I. Linearized steady problems,* Springer tracts in Natural Philosophy, 38, Springer Verlag, New-York, 1994.

[10] G.P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations. II. Nonlinear steady problems,* Springer tracts in Natural Philosophy, 39, Springer Verlag, 1994.

[11] V. Georgescu, *Some boundary value problems for differential forms on compact Riemannian manifolds,* Annali di Matematica Pura ed Applicata, **4** (1979), 159–198.

[12] C. He and Z. Xin, *On the regularity of weak solutions to the magnetohydrodynamic equations,* J. Differential Equations, **213** (2005), 235–254.

[13] J.D. Jackson, *Classical Electrodynamics,* Second Edition, John Wiley & Son, New York, 1975, (Third Edition, 1999).

[14] A. Kovetz, *The Principles of Electromagnetic Theory,* Cambridge University Press, 1990.

[15] O.A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow,* Gordon and Breach, Rev. second edition, 1969.

[16] J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires,* Dunod-Gauthier-Villars, 1969.

[17] P.L. Lions, *Mathematical topics in fluid mechanics. Volume 1. Incompressible models,* Oxford Science Publications, 1996.

[18] R.J. Moreau, *Magnetohydrodynamics,* Kluwer Academic Publishers, 1990.

[19] R.E. Rosensweig, *Ferrohydrodynamics,* Dover Publications, Inc., 1997.

[20] R.E. Rosensweig, *Continuum equations for magnetic and dielectric fluids with internal rotations,* J. Chem. Phys., **121** (2004), 1228–1242.

[21] B. Saramito, *Stabilité d'un Plasma: Modélisation mathématique et simulation numérique,* Masson, 1994.

[22] M. Sermange and R. Temam, *Some mathematical questions related to the MHD equations,* Communications on Pure and Applied Mathematics, **36** (1983), 635–664.

[23] M.I. Shliomis, *Effective viscosity of magnetic suspension,* Sov. Phys. JETP, **44** (1972), 1291–1294.

[24] Z. Tan and Y. Wang, *Global analysis for strong solutions to the equations of a ferrofluid flow model,* J. Math. Anal. Appl., **364** (2010), 424–436.

[25] L. Tartar, *Topics in nonlinear analysis,* 78.13, Publications Mathématiques d'Orsay, 1978.

[26] R. Temam, *Navier-Stokes equations,* 3rd (revised) edition, Elsevier Science Publishers B. V., Amsterdam, 1984.

[27] E.E. Tzirtzilakis, *A mathematical model for blood flow in magnetic field,* Physics of Fluids, **17** (2005), 1-15.