



# On $C^*$ -algebras of exponential solvable Lie groups and their real ranks <sup>☆</sup>



Ingrid Beltiță<sup>\*</sup>, Daniel Beltiță

*Institute of Mathematics “Simion Stoilow” of the Romanian Academy, P.O. Box 1-764, Bucharest, Romania*

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## ABSTRACT

For any solvable Lie group whose exponential map  $\exp_G: \mathfrak{g} \rightarrow G$  is bijective, we prove that the real rank of  $C^*(G)$  is equal to  $\dim(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$ . We also indicate a proof of a similar formula for the stable rank of  $C^*(G)$ , as well as some estimates on the ideal generated by the projections in  $C^*(G)$ .

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## 1. Introduction

The stable rank of  $C^*$ -algebras was introduced independently in [7] and [15], and subsequently, its self-adjoint version, called real rank, was introduced in [6], where it was proved among other things that the condition of positive real rank is equivalent to the fact that the linear span of projections is not dense in the  $C^*$ -algebra under consideration. Much information has been obtained on the real rank of  $C^*$ -algebras of locally compact groups, and yet some problems remained open so far. In the present note we answer some of these problems, based on the method of coadjoint orbits of exponential solvable Lie groups, particularly on the results of [8] (see also the monograph [12] for background information on that method).

First, there is the problem of investigating the set of projections in  $C^*(G)$  for any exponential Lie group  $G$ . We recall that by definition an exponential Lie group is any Lie group whose exponential map  $\exp_G: \mathfrak{g} \rightarrow G$  is bijective, and this condition implies that  $G$  is a solvable Lie group. Examples of exponential Lie groups include the connected, simply connected, nilpotent Lie groups, for which it was established in [16, Thm. 4] that their  $C^*$ -algebras contain no nonzero projections. The existence of projections has been left open so far

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<sup>\*</sup> Corresponding author.

*E-mail addresses:* [ingrid.beltita@gmail.com](mailto:ingrid.beltita@gmail.com), [Ingrid.Beltita@imar.ro](mailto:Ingrid.Beltita@imar.ro) (I. Beltiță), [beltita@gmail.com](mailto:beltita@gmail.com), [Daniel.Beltita@imar.ro](mailto:Daniel.Beltita@imar.ro) (D. Beltiță).

for general exponential Lie groups, whose  $C^*$ -algebras are not always liminary. It is well known that if  $G$  is the  $(ax + b)$ -group, then  $C^*(G)$  contains non-trivial projections. We provide more precise information on this phenomenon in [Theorem 4.4](#) and [Remark 4.7](#) below.

The next problem is motivated by the problem above and by related investigations in [\[17\]](#): Compute the real rank  $\text{RR}(C^*(G))$  for any exponential Lie group  $G$ . See also [\[1\]](#), where an answer was provided in particular for nilpotent Lie groups.

The answer to this problem is given in [Theorem 3.5](#), and the analogous result on stable ranks is given in [Theorem 5.4](#), which also fills some gaps in the literature (see [Remark 5.5](#)).

## 2. Preliminaries

**Definition 2.1.** Let  $\mathcal{A}$  be any unital  $C^*$ -algebra, and for any integer  $n \geq 1$  denote by  $\mathcal{L}_n(\mathcal{A})$  the set of all  $n$ -tuples  $(a_1, \dots, a_n) \in \mathcal{A}^n$  with  $\mathcal{A}a_1 + \dots + \mathcal{A}a_n = \mathcal{A}$ . We also denote  $\mathcal{A}^{\text{sa}} := \{a \in \mathcal{A} \mid a = a^*\}$ .

The *stable rank* of  $\mathcal{A}$  is defined by

$$\text{tsr}(\mathcal{A}) := \min\{n \geq 1 \mid \mathcal{L}_n(\mathcal{A}) \text{ is dense in } \mathcal{A}^n\}$$

with the usual convention  $\min \emptyset = \infty$ . The *real rank* of  $\mathcal{A}$  is similarly defined by

$$\text{RR}(\mathcal{A}) := \min\{n \geq 0 \mid \mathcal{L}_{n+1}(\mathcal{A}) \cap (\mathcal{A}^{\text{sa}})^{n+1} \text{ is dense in } (\mathcal{A}^{\text{sa}})^{n+1}\}.$$

For any non-unital  $C^*$ -algebra, its real rank and its stable rank are defined as the real rank, respectively the stable rank, of its unitization.

**Remark 2.2.** The real rank of a  $C^*$ -algebra was introduced in [\[6\]](#) as a noncommutative version of the covering dimension of a topological space. In this sense, it was established in [\[6, Prop. 1.1\]](#) that if  $X$  is any compact Hausdorff space, then the real rank of the commutative  $C^*$ -algebra  $\mathcal{C}(X)$  is equal to the covering dimension of  $X$ .

On the other hand, by [\[6, Thm. 2.6\]](#), the real rank of some  $C^*$ -algebra  $\mathcal{A}$  is zero if and only if every self-adjoint element in  $\mathcal{A}$  is the limit of a sequence of self-adjoint elements with finite spectra. Therefore, if  $\mathcal{A}$  contains no projections, then its real rank is  $\geq 1$ , and this points out the connection between the two problems in the Introduction.

The combination of [\[3, Thm. 4.11\]](#) ( $C^*$ -algebras of nilpotent Lie groups are special solvable) with the following proposition gives a short proof of one of the main results of [\[1\]](#) in the special case of Lie groups, namely that  $\text{RR}(C^*(G)) = \dim(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$  for any connected, simply connected, nilpotent Lie group  $G$ . This equality will be extended in [Theorem 3.5](#) to all exponential Lie groups, using however a different approach, because the  $C^*(G)$  in that case might not be liminary. We refer to [\[3, Defn. 2.9\]](#) for the definition of special solvable  $C^*$ -algebras.

**Proposition 2.3.** *Let  $\mathcal{A}$  be any liminary, solvable  $C^*$ -algebra with a special solving series*

$$\{0\} = \mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \dots \subseteq \mathcal{J}_n = \mathcal{A}$$

*with  $\mathcal{J}_j/\mathcal{J}_{j-1} \simeq \mathcal{C}_0(\Gamma_j, \mathcal{K}(\mathcal{H}_j))$  for  $j = 1, \dots, n$ . Then  $\text{RR}(\mathcal{A}) = \dim \Gamma_n$ .*

**Proof.** The  $C^*$ -algebra  $\mathcal{A}$  is liminary, hence it follows by [\[5, Cor. 3.7\]](#) that  $\text{RR}(\mathcal{A}) = \max\{\text{RR}(\mathcal{J}_j/\mathcal{J}_{j-1}) \mid j = 1, \dots, n\}$ . On the other hand, for  $j = 1, \dots, n-1$ , we have  $\dim \mathcal{H}_j = \infty$ , hence  $\text{RR}(\mathcal{C}_0(\Gamma_j, \mathcal{K}(\mathcal{H}_j))) \leq 1$  by [\[2, Prop. 3.3\]](#). However  $\dim \mathcal{H}_n = 1$ , hence  $\mathcal{J}_n/\mathcal{J}_{n-1} \simeq \mathcal{C}_0(\Gamma_n)$  and then it follows by [\[6, Prop. 1.1\]](#) that  $\text{RR}(\mathcal{J}_n/\mathcal{J}_{n-1}) = \dim \Gamma_n \geq 1$ , since  $\Gamma_n$  is homeomorphic to a vector space. Hence the assertion follows.  $\square$

### 3. Real rank for exponential Lie groups

In this section we obtain one of our main results ([Theorem 3.5](#)), which gives a formula for computing  $\text{RR}(C^*(G))$  in terms of the Lie algebra  $\mathfrak{g}$ , for any exponential Lie group  $G$ . We must mention that the spectra finite-dimensionality hypotheses in [Lemmas 3.1–3.2](#) are crucial for the conclusion of these lemmas, and we need [Lemma 3.4](#) in order to check that these hypotheses are satisfied in the setting of the proof of [Theorem 3.5](#).

**Lemma 3.1.** *Let  $\mathcal{A}$  be any separable  $C^*$ -algebra with continuous trace, for which all its irreducible representations are infinite-dimensional. If moreover  $\dim \widehat{\mathcal{A}} < \infty$ , then  $\text{RR}(\mathcal{A}) \leq 1$ .*

**Proof.** It follows by [[4, Cor. IV.1.7.22](#)] that  $\mathcal{A}$  is stable, hence there exists a  $*$ -isomorphism  $\mathcal{A} \simeq \mathcal{A} \otimes \mathcal{K}$ , where  $\mathcal{K}$  is the  $C^*$ -algebra of compact operators on some separable infinite-dimensional Hilbert space. On the other hand, by [[2, Prop. 3.3](#)], one has  $\text{RR}(\mathcal{A} \otimes \mathcal{K}) \leq 1$ , and then  $\text{RR}(\mathcal{A}) \leq 1$  as well.  $\square$

**Lemma 3.2.** *Let  $\mathcal{A}$  be any separable  $C^*$ -algebra with an ideal  $\mathcal{J}$  that is a continuous trace  $C^*$ -algebra with  $\dim(\widehat{\mathcal{J}}) < \infty$ , and such that all irreducible representations of  $\mathcal{J}$  are infinite dimensional. Then  $\text{RR}(\mathcal{A}) = \max\{\text{RR}(\mathcal{J}), \text{RR}(\mathcal{A}/\mathcal{J})\}$ .*

**Proof.** This is a special case of [[5, Thm. 3.12\(ii\)](#)].  $\square$

**Proposition 3.3.** *Let  $\mathcal{A}$  be any separable  $C^*$ -algebra with a family of closed two-sided ideals*

$$\{0\} = \mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \dots \subseteq \mathcal{J}_n = \mathcal{A}$$

where for each  $j = 1, \dots, n$ ,  $\mathcal{J}_j/\mathcal{J}_{j-1}$  has continuous trace, with its spectrum of finite covering dimension, and all its irreducible representations are infinite dimensional. Then

$$\text{RR}(\mathcal{A}) = \max\{\text{RR}(\mathcal{J}_j/\mathcal{J}_{j-1}) \mid j = 1, \dots, n\}.$$

**Proof.** We proceed by induction on  $n$ . The case  $n = 1$  is obvious. If we assume  $n \geq 2$  and the assertion already proved for  $n - 1$ , then, using the family of closed two-sided ideals

$$\{0\} = \mathcal{J}_1/\mathcal{J}_1 \subseteq \mathcal{J}_2/\mathcal{J}_1 \subseteq \dots \subseteq \mathcal{J}_n/\mathcal{J}_1 = \mathcal{A}/\mathcal{J}_1$$

of  $\mathcal{A}/\mathcal{J}_1$  for which  $(\mathcal{J}_j/\mathcal{J}_1)/(\mathcal{J}_{j-1}/\mathcal{J}_1) \simeq \mathcal{J}_j/\mathcal{J}_{j-1}$  has Hausdorff spectrum, for  $j = 2, \dots, n$ , then the induction hypothesis implies

$$\text{RR}(\mathcal{A}/\mathcal{J}_1) = \max\{\text{RR}(\mathcal{J}_j/\mathcal{J}_{j-1}) \mid j = 2, \dots, n\}.$$

On the other hand, by [Lemma 3.2](#) we have  $\text{RR}(\mathcal{A}) = \max\{\text{RR}(\mathcal{J}_1), \text{RR}(\mathcal{A}/\mathcal{J}_1)\}$ , hence we directly obtain the assertion for  $n$ , and this completes the proof.  $\square$

**Lemma 3.4.** *Let  $X$  be a metric space and  $A$  a locally closed subset of  $X$ . Then  $\dim A \leq \dim X$ .*

**Proof.** The set  $A$  is locally closed, hence there are sets  $D \subseteq X$ ,  $F \subseteq X$ , with  $D$  open and  $F$  closed such that  $A = D \cap F$ .

On the other hand  $D$  is an  $F_\sigma$  set in the metric space  $X$ , that is, there is a countable family of closed subsets  $F_n$ ,  $n \geq 0$ , such that  $D = \cup_{n \geq 1} F_n$ . Hence  $A = \cup_{n \geq 1} (F_n \cap F)$ , and from [[14, Prop. 3.1.5, Thm. 3.2.5](#)] it follows that  $\dim A \leq \sup_{n \geq 1} \dim(F_n \cap F) \leq \dim X$ .  $\square$

**Theorem 3.5.** For every exponential Lie group  $G$  with its Lie algebra  $\mathfrak{g}$ , we have

$$\text{RR}(C^*(G)) = \dim(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]).$$

**Proof.** Denoting  $\mathcal{A} := C^*(G)$ , it follows by [8, Cor. 3.2] that there exists a family of closed two-sided ideals  $\{0\} = \mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \dots \subseteq \mathcal{J}_n = \mathcal{A}$  where  $\mathcal{J}_j/\mathcal{J}_{j-1}$  is a separable continuous trace  $C^*$ -algebra whose irreducible representations are infinite-dimensional and for which  $\Sigma_j := \widehat{\mathcal{J}_j/\mathcal{J}_{j-1}}$  is homeomorphic to a semi-algebraic subset of  $\mathfrak{g}^*$  for  $j = 1, \dots, n-1$ . Moreover  $\mathcal{A}/\mathcal{J}_{n-1} \simeq \mathcal{C}_0([\mathfrak{g}, \mathfrak{g}]^\perp)$ , hence  $\Sigma_n := \widehat{\mathcal{A}/\mathcal{J}_{n-1}}$  is homeomorphic to the vector space  $[\mathfrak{g}, \mathfrak{g}]^\perp \simeq (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^* \simeq \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ . Then  $\Sigma_j$  is a locally closed subset of  $\mathfrak{g}^*$ , hence it follows by Lemma 3.4 that  $\dim \Sigma_j < \infty$ . Thus by Lemma 3.1 we obtain  $\text{RR}(\mathcal{J}_j/\mathcal{J}_{j-1}) \leq 1$  for  $j = 1, \dots, n-1$ .

On the other hand, denoting  $r := \dim(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$ , it follows that the one-point compactification of  $\Sigma_n$  is homeomorphic to the  $r$ -dimensional sphere  $S^r$ , hence the unitization of the  $C^*$ -algebra  $\mathcal{A}/\mathcal{J}_{n-1} \simeq \mathcal{C}_0(\Sigma_n)$  is  $*$ -isomorphic to  $\mathcal{C}(S^r)$ . Using [6, Prop. 1.1], we then obtain  $\text{RR}(\mathcal{A}/\mathcal{J}_{n-1}) = r = \dim(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$ . Now, as an application of Proposition 3.3, we obtain  $\text{RR}(C^*(G)) = \max\{r, 1\} = r$ , and this completes the proof.  $\square$

**Corollary 3.6.** Let  $G$  be any connected Lie group with its Lie algebra  $\mathfrak{g}$ . If the universal covering group of  $G$  is an exponential Lie group, then  $\text{RR}(C^*(G)) \leq \dim(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$ .

**Proof.** Let  $p: \widetilde{G} \rightarrow G$  be the universal covering map of  $G$ , so that we have a short exact sequence of Lie groups

$$\mathbf{1} \rightarrow N \rightarrow \widetilde{G} \rightarrow G \rightarrow \mathbf{1}$$

where  $N := \text{Ker } p$  is a discrete subgroup of the center of the exponential Lie group  $\widetilde{G}$ . Then all the groups involved in the above short exact sequence are amenable even as discrete groups, and we then obtain a short exact sequence of  $C^*$ -algebras

$$0 \rightarrow \mathcal{J} \rightarrow C^*(\widetilde{G}) \rightarrow C^*(G) \rightarrow 0$$

for the ideal  $\widehat{\mathcal{J}} = \{[\pi] \in \widehat{C^*(\widetilde{G})} \simeq \widehat{\widetilde{G}} \mid N \not\subseteq \text{Ker } \pi\}$  of  $C^*(\widetilde{G})$ . We then obtain

$$\text{RR}(C^*(G)) = \text{RR}(C^*(\widetilde{G})/\mathcal{J}) \leq \text{RR}(C^*(\widetilde{G})) = \dim(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]).$$

The above inequality follows by [11, Thm. 1.4] (as in the proof of Lemma 3.2 above) and the final equality follows by Theorem 3.5 applied for  $\widetilde{G}$ , taking also into account that the Lie algebra of  $\widetilde{G}$  is isomorphic to  $\mathfrak{g}$ . This concludes the proof.  $\square$

**Remark 3.7.** The inequality in Corollary 3.6 can be strict if  $G$  is not simply connected. For instance, if  $G = \mathbb{T} := \mathbb{R}/\mathbb{Z}$ , then  $C^*(G) = c_0(\mathbb{Z})$  has real rank zero by [6, Thm. 2.6((i)  $\Leftrightarrow$  (ii))], so  $\text{RR}(C^*(G)) = 0 < 1 = \dim(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$ .

*Some remarks on abelianization.* Let  $\mathcal{A}$  be any  $C^*$ -algebra and denote by  $\mathcal{J}(\mathcal{A})$  its closed two-sided ideal generated by its subset of commutators  $\text{span}\{[a, b] \mid a, b \in \mathcal{A}\}$ . Then  $\mathcal{A}/\mathcal{J}(\mathcal{A})$  is a commutative  $C^*$ -algebra, hence there exists a locally compact space  $\Gamma_{\mathcal{A}}$  (uniquely determined by  $\mathcal{A}$  up to a homeomorphism) and a  $*$ -isomorphism  $\mathcal{A}/\mathcal{J}(\mathcal{A}) \simeq \mathcal{C}_0(\Gamma_{\mathcal{A}})$ , and thus we obtain the short exact sequence

$$0 \rightarrow \mathcal{J}(\mathcal{A}) \rightarrow \mathcal{A} \rightarrow \mathcal{C}_0(\Gamma_{\mathcal{A}}) \rightarrow 0. \quad (3.1)$$

A natural question is to estimate the real rank of  $\mathcal{A}$  in terms of the covering dimension  $\dim(\Gamma_{\mathcal{A}}^*)$ , where  $\Gamma_{\mathcal{A}}^*$  denotes the one-point compactification of  $\Gamma_{\mathcal{A}}$ . We always have that  $\text{RR}(\mathcal{A}) \geq \dim(\Gamma_{\mathcal{A}}^*)$ , by (3.1) and [11, Thm. 1.4]. The interesting equality  $\text{RR}(\mathcal{A}) = \dim(\Gamma_{\mathcal{A}}^*)$  holds for several  $C^*$ -algebras:

- If  $\mathcal{A} = C^*(G)$  for an exponential Lie group  $G$ , then  $\text{RR}(\mathcal{A}) = \dim(\Gamma_{\mathcal{A}}) = \dim(\Gamma_{\mathcal{A}}^*)$  by [Theorem 3.5](#) and the short exact sequence in the proof of [Theorem 4.4](#), which shows that  $\Gamma_{\mathcal{A}} = [\mathfrak{g}, \mathfrak{g}]^\perp$ .
- If  $\mathcal{A}$  is any AF-algebra, then  $\text{RR}(\mathcal{A}) = 0 = \dim(\Gamma_{\mathcal{A}}^*)$ . In fact, this follows since it is well known that the quotients of AF-algebras are AF-algebras, and a commutative  $C^*$ -algebra is an AF-algebra if and only if its spectrum is totally disconnected, hence its covering dimension is equal to zero.
- If  $\mathcal{A}$  is the  $C^*$ -algebra generated by the Toeplitz operators with continuous symbols on the unit circle  $\mathbb{T}$  (equivalently,  $\mathcal{A}$  is the  $C^*$ -algebra generated by the unilateral shift operator), then one has  $\mathcal{J}(\mathcal{A}) = \mathcal{K}(L^2(\mathbb{T}))$  and  $\Gamma_{\mathcal{A}} = \mathbb{T}$ , and it is known from [\[11, Cor. 1.13\(i\)\]](#) that  $\text{RR}(\mathcal{A}) = 1 = \dim(\mathbb{T})$ .

It is clear that the equality  $\text{RR}(\mathcal{A}) = \dim(\Gamma_{\mathcal{A}}^*)$  fails to be true in general. For instance, if  $\mathcal{A}$  is a simple  $C^*$ -algebra, then  $\mathcal{J}(\mathcal{A}) = \mathcal{A}$ , hence  $\Gamma_{\mathcal{A}} = \emptyset$ , and then  $\dim(\Gamma_{\mathcal{A}}^*) = 0$ ; on the other hand, examples of simple  $C^*$ -algebras are known, having positive real rank (see [\[18, Thm. 10\]](#)). Nevertheless, if  $\mathcal{A}$  is a  $C^*$ -algebra of real rank zero, we have that  $\dim(\Gamma_{\mathcal{A}}^*) = 0$ .

#### 4. On projections in the $C^*$ -algebras of exponential Lie groups

The main result of this section provides a kind of estimates on the size of the closed two-sided ideal generated by the projections in the  $C^*$ -algebra of an exponential Lie group  $G$ . That ideal is strictly smaller than  $C^*(G)$ , as already noted in [Remark 2.2](#).

**Notation 4.1.** For any  $C^*$ -algebra  $\mathcal{A}$  we denote  $\text{Gr}(\mathcal{A}) := \{p \in \mathcal{A} \mid p = p^2 = p^*\}$ .

**Lemma 4.2.** *Let  $\mathcal{A}$  be any  $C^*$ -algebra whose spectrum  $\widehat{\mathcal{A}}$  is a Hausdorff space. If no connected component of  $\widehat{\mathcal{A}}$  is compact, then  $\text{Gr}(\mathcal{A}) = \{0\}$ .*

**Proof.** Let  $p \in \text{Gr}(\mathcal{A})$  and  $\Gamma$  be any connected component of  $\widehat{\mathcal{A}}$ . Since  $\Gamma$  is not compact, it follows by [\[10, 3.3.7–9\]](#) that the function  $\Gamma \rightarrow [0, \infty)$ ,  $[\pi] \mapsto \|\pi(p)\|$ , is continuous and  $\lim_{\Gamma \ni [\pi] \rightarrow \infty} \|\pi(p)\| = 0$ . But  $\pi(p) \in \mathcal{B}(\mathcal{H}_\pi)$  is an orthogonal projection, hence  $\|\pi(p)\| \in \{0, 1\}$  for all  $[\pi] \in \Gamma$ . This implies that  $\pi(p) = 0$  for all  $[\pi] \in \widehat{\mathcal{A}}$ , and then by [\[10, 2.7.3\]](#) we obtain  $p = 0$ , which completes the proof.  $\square$

**Lemma 4.3.** *For any short exact sequence of  $C^*$ -algebras*

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{A} \xrightarrow{Q} \mathcal{A}/\mathcal{J} \rightarrow 0$$

*if  $\text{Gr}(\mathcal{A}/\mathcal{J}) = \{0\}$ , then  $\text{Gr}(\mathcal{J}) = \text{Gr}(\mathcal{A})$ .*

**Proof.** Since  $\mathcal{J} \subseteq \mathcal{A}$ , we have  $\text{Gr}(\mathcal{J}) \subseteq \text{Gr}(\mathcal{A})$ . For the opposite inclusion, if  $p \in \text{Gr}(\mathcal{A})$  then  $Q(p) \in \text{Gr}(\mathcal{A}/\mathcal{J}) = \{0\}$ , hence  $p \in \mathcal{J}$ , and then  $p \in \text{Gr}(\mathcal{J})$ .  $\square$

**Theorem 4.4.** *Let  $G$  be any exponential Lie group with its  $C^*$ -algebra  $\mathcal{A} := C^*(G)$ . We denote by  $\mathcal{J}_0 \subset \mathcal{A}$  the intersection of kernels of all characters of  $G$  extended to 1-dimensional  $*$ -representations of  $\mathcal{A}$ . Then  $\text{Gr}(\mathcal{A}) = \text{Gr}(\mathcal{J}_0)$ , and moreover  $\mathcal{J}_0 \subsetneq \mathcal{A}$  if  $\dim G > 0$ .*

**Proof.** It follows by the method of coadjoint orbits that we have a short exact sequence

$$0 \rightarrow \mathcal{J}_0 \rightarrow \mathcal{A} \rightarrow \mathcal{C}_0([\mathfrak{g}, \mathfrak{g}]^\perp) \rightarrow 0$$

where  $[\mathfrak{g}, \mathfrak{g}]^\perp = \{\chi \in \mathfrak{g}^* \mid [\mathfrak{g}, \mathfrak{g}] \subseteq \text{Ker } \chi\}$  is the space of characters of the Lie algebra  $\mathfrak{g}$ . The vector space  $[\mathfrak{g}, \mathfrak{g}]^\perp$  has no compact connected components, hence we may use [Lemma 4.3](#) to obtain  $\text{Gr}(\mathcal{A}) = \text{Gr}(\mathcal{J}_0)$ .

Finally, since  $\mathfrak{g}$  is a solvable Lie algebra, it follows that  $[\mathfrak{g}, \mathfrak{g}] \subsetneq \mathfrak{g}$  if  $\dim \mathfrak{g} > 0$ , hence in this case  $[\mathfrak{g}, \mathfrak{g}]^\perp \neq \{0\}$ , and the above short exact sequence implies  $\mathcal{J}_0 \subsetneq \mathcal{A}$ , which concludes the proof.  $\square$

The above [Theorem 4.4](#) shows that nontrivial projections appear as soon as the group has open coadjoint orbits. The next proposition shows that there could be only a finite number of such orbits, and give a description of them.

**Proposition 4.5.** *Let  $G$  be any connected Lie group with its Lie algebra  $\mathfrak{g}$  and the duality pairing  $\langle \cdot, \cdot \rangle: \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$ . For any basis  $\{X_1, \dots, X_m\}$  in  $\mathfrak{g}$  define the polynomial function*

$$P: \mathfrak{g}^* \rightarrow \mathbb{R}, \quad P(\xi) := \det(\langle \xi, [X_j, X_k] \rangle)_{1 \leq j, k \leq m}.$$

Then the following assertions hold:

- (i) *If  $\xi \in \mathfrak{g}^*$ , then the coadjoint orbit  $\mathcal{O}_\xi := \text{Ad}_G^*(\xi)$  is an open subset of  $\mathfrak{g}^*$  if and only if  $P(\xi) \neq 0$ .*
- (ii) *The set of open coadjoint orbits of  $G$  is finite and their union is a Zariski open subset of  $\mathfrak{g}^*$  which may be empty.*

**Proof.** For Assertion (i) denote  $\mathfrak{g}(\xi) := \{X \in \mathfrak{g} \mid \langle \xi, [X, \cdot] \rangle = \{0\}\}$ , the coadjoint isotropy subalgebra at  $\xi$ . Then the tangent space at  $\xi \in \mathcal{O}_\xi$  can be computed as  $T_\xi(\mathcal{O}_\xi) = \mathfrak{g}/\mathfrak{g}(\xi)$ , hence  $\mathcal{O}_\xi$  is an open subset of  $\mathfrak{g}^*$  if and only if  $\mathfrak{g}(\xi) = \{0\}$ , and this is equivalent to the condition that the bilinear map

$$\mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}, \quad (X, Y) \mapsto B_\xi := \langle \xi, [X, Y] \rangle$$

be nondegenerate. As  $\{X_1, \dots, X_m\}$  is a basis in  $\mathfrak{g}$ , this condition is further equivalent to  $P(\xi) \neq 0$ .

For Assertion (ii), use Assertion (i) to see that the union of all open coadjoint orbits of  $G$  is  $P^{-1}(\mathbb{R} \setminus \{0\}) = \{\xi \in \mathfrak{g}^* \mid P(\xi) \neq 0\}$ , and this is a (maybe empty) Zariski open subset of  $\mathfrak{g}^*$  since  $P: \mathfrak{g}^* \rightarrow \mathbb{R}$  is a polynomial function.

To see that the set of open coadjoint orbits of  $G$  is finite, first note that every coadjoint orbit of  $G$  is path connected since  $G$  is path connected. Hence the open coadjoint orbits of  $G$  can be equivalently described as the path connected components of the algebraic set  $P^{-1}(\mathbb{R} \setminus \{0\})$ . Then we may use for instance [\[9, Thm. 4.1\]](#), which says in particular that the set of all path components of any algebraic variety over  $\mathbb{R}$  is finite, and this concludes the proof.  $\square$

**Remark 4.6.** It follows by [\[13, Thm. 2.7 and Rem. 2.8\]](#) that the solvable Lie groups of type  $HN$  that arise from Iwasawa decompositions of complex semisimple Lie groups (that is, Borel subgroups) may have at most one open coadjoint orbit, and such an open orbit exists if and only if  $-1$  belongs to the corresponding Weyl group.

**Remark 4.7** (*Theorem 4.4 is sharp*). Let us resume the notation of [Theorem 4.4](#) and assume that the set  $\Sigma_0$  of open coadjoint orbits of  $G$  is nonempty. Then  $\Sigma_0$  is a finite set and  $\mathcal{J}_0$  contains a closed ideal  $\mathcal{J}_{00}$  of  $\mathcal{A}$  which is  $*$ -isomorphic to a direct sum of  $|\Sigma_0|$  copies of the  $C^*$ -algebra  $\mathcal{K}$  of compact operators on a separable infinite-dimensional complex Hilbert space, hence  $\text{Gr}(\mathcal{J}_{00}) \neq \{0\}$ .

Moreover, the specific example of the  $(ax + b)$ -group shows that we may have  $\mathcal{J}_{00} = \mathcal{J}_0$ , hence in this case  $\text{Gr}(\mathcal{A}) = \text{Gr}(\mathcal{J}_0) = \text{Gr}(\mathcal{J}_{00}) \neq \{0\}$ .

## 5. On the stable rank

In this section we briefly indicate how the line of reasoning that leads to [Theorem 3.5](#) could be modified in order to compute the stable rank of  $C^*(G)$  for any exponential Lie group  $G$ . This problem has been raised

in [15, Question 4.14]: If  $G$  is a Lie group, how does one compute  $\text{tsr}(C^*(G))$  in terms of the structure of  $G$ ? It was mentioned already in [15, Ex. 4.13] that if  $G$  is the  $(ax + b)$ -group, then  $\text{tsr}(C^*(G)) = 2$ .

**Remark 5.1.** For any compact space  $X$  we have

$$\text{tsr}(\mathcal{C}(X)) = 1 + [(\dim X)/2] \tag{5.1}$$

by [15, Prop. 1.7]. For any  $C^*$ -algebra  $\mathcal{A}$  and a separable infinite-dimensional complex Hilbert space  $\mathcal{H}$  we have

$$\text{tsr}(\mathcal{A} \otimes \mathcal{K}(\mathcal{H})) \leq 2 \tag{5.2}$$

by [15, Thm. 6.4]. Also, for any closed two-sided ideal  $\mathcal{J} \subseteq \mathcal{A}$  we have

$$\text{tsr}(\mathcal{A}) \geq \max\{\text{tsr}(\mathcal{J}), \text{tsr}(\mathcal{A}/\mathcal{J})\} \tag{5.3}$$

by [15, Thm. 4.3–4.4].

**Remark 5.2.** In the setting of Lemma 3.2, one has  $\text{tsr}(\mathcal{A}) \leq \max\{2, \text{tsr}(\mathcal{A}/\mathcal{J})\}$ , by [5, Thm. 3.12(i)].

**Remark 5.3.** Using Remark 5.2 above, one can prove by a similar method that, in the setting of Proposition 3.3, one has  $\text{tsr}(\mathcal{A}) \leq \max\{2, \text{tsr}(\mathcal{A}/\mathcal{J}_{n-1})\}$ .

**Theorem 5.4.** For every exponential Lie group  $G$  with its Lie algebra  $\mathfrak{g}$ , if we denote  $r := \dim(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$ , then

$$\text{tsr}(C^*(G)) = \begin{cases} 1 & \text{if and only if } G = \mathbb{R}, \\ 1 + \max\{[r/2], 1\} & \text{otherwise.} \end{cases}$$

**Proof.** We resume the notation from the proof of Theorem 3.5. By Remark 5.3 and Remark 5.1 we obtain

$$\text{tsr}(\mathcal{A}/\mathcal{J}_{n-1}) \leq \text{tsr}(\mathcal{A}) \leq \max\{2, \text{tsr}(\mathcal{A}/\mathcal{J}_{n-1})\}.$$

By [17, Lemma 3.7] we have that  $\text{tsr}(\mathcal{A}) = 1$  if and only if  $G = \mathbb{R}$ . Assume that  $\text{tsr}(\mathcal{A}) \geq 2$ . We have that  $\mathcal{A}/\mathcal{J}_{n-1} \simeq \mathcal{C}_0([\mathfrak{g}, \mathfrak{g}]^\perp)$ , hence  $\text{tsr}(\mathcal{A}/\mathcal{J}_{n-1}) = [r/2] + 1$  by (5.1). Then, using also that  $\max\{1 + a, 1 + b\} = 1 + \max\{a, b\}$  for all  $a, b \in \mathbb{R}$ , the conclusion follows directly.  $\square$

**Remark 5.5.** The formula provided by Theorem 5.4 agrees with [17, Thm. 3.9] in the case of exponential Lie groups. However, the proof of [17, Thm. 3.9] seems to be incomplete because it is based on [17, Lemma 3.2], whose proof requires the hypothesis on the finite-dimensionality of spectra of the continuous-trace algebras. A similar issue was pointed out at the top of [1, page 100].

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