



On strongly separately continuous functions on sequence spaces



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ABSTRACT

We study strongly separately continuous real-valued function defined on the Banach spaces ℓ_p . Determining sets for the class of strongly separately continuous functions on ℓ_p are characterized. We prove that for every $2 \leq \alpha < \omega_1$ there exists a strongly separately continuous function which belongs to the α 'th Baire class and does not belong to the β 'th Baire class on ℓ_p for $\beta < \alpha$. We show that any open set in ℓ_p is the set of discontinuities of a strongly separately continuous real-valued function.

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1. Introduction

Let $(X_t : t \in T)$ be a family of sets X_t with $|X_t| > 1$ for all $t \in T$. For $S \subseteq T$ we put $X_S = \prod_{t \in S} X_t$. If $S \subseteq S_1 \subseteq T$, $x = (x_t)_{t \in T} \in X_T$, $a = (a_t)_{t \in S_1} \in X_{S_1}$, then we denote by x_S^a the point $(y_t)_{t \in T} \in X_T$ defined by

$$y_t = \begin{cases} a_t, & t \in S, \\ x_t, & t \in T \setminus S. \end{cases}$$

In the case of $S = \{s\}$ we write x_s^a instead of $x_{\{s\}}^a$.

If $S \subseteq T$, then we put $\pi_S(x) = (x_t)_{t \in S}$.

For $n \in \mathbb{N}$ let $\sigma_n(x) = \{y = (y_t)_{t \in T} \in X_T : |\{t \in T : y_t \neq x_t\}| \leq n\}$ and $\sigma(x) = \bigcup_{n=1}^{\infty} \sigma_n(x)$. Each of the sets of the form $\sigma(x)$ we call a σ -product of X_T .

A set $X \subseteq X_T$ is called $\mathcal{S}$ -open [6] if $\sigma_1(x) \subseteq X$ for all $x \in X$. Notice that the definition of an $\mathcal{S}$ -open set develops the definition of a set of the type (P_1) introduced in [1]. Observe that σ -products of two distinct

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points of X_T either coincide, or do not intersect. Thus, the family of all σ -products of an arbitrary $\mathcal{S}$ -open set $X \subseteq X_T$ generates a partition of X on mutually disjoint $\mathcal{S}$ -open sets, which we will call $\mathcal{S}$ -components of X .

Let $X \subseteq X_T$ be an $\mathcal{S}$ -open set, τ be a topology on X and let (Y, d) be a metric space. A mapping $f : (X, \tau) \rightarrow Y$ is called *strongly separately continuous at a point $a \in X$ with respect to the t 'th variable* if

$$\lim_{x \rightarrow a} d(f(x), f(x_t^a)) = 0.$$

A mapping $f : X \rightarrow Y$ is *strongly separately continuous at a point $a \in X$* if f is strongly separately continuous at a with respect to every variable $t \in T$; and f is *strongly separately continuous on the set X* if f is strongly separately continuous at every point $a \in X$ with respect to every variable $t \in T$. Strongly separately continuous functions we will also call *ssc-functions* for short.

The notion of real-valued ssc-function defined on $\mathbb{R}^n$ was introduced by Omar Dzagnidze [2], who proved that a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is strongly separately continuous at x^0 if and only if f is continuous at x^0 .

In [1] the authors extended the notion of the strong separate continuity to functions defined on the Hilbert space ℓ_2 equipped with the norm topology. They proved that there exists a real-valued ssc-function on ℓ_2 which is everywhere discontinuous. Determining sets (see Definition 2.1) for the class of all ssc-functions were also studied in [1].

The second named author continued to study properties of strongly separately continuous functions on ℓ_2 in [9] and constructed a strongly separately continuous function $f : \ell_2 \rightarrow \mathbb{R}$ which belongs to the third Baire class and is not quasi-continuous at every point. Moreover, he gave a sufficient condition for a strongly separately continuous function to be continuous on ℓ_2 and a sufficient condition for a subset of ℓ_2 to be determining in the class of real-valued ssc-functions.

The first named author extended the concept of an ssc-function on any $\mathcal{S}$ -open subset of a product of topological spaces [6]. A characterization of the set of all points of discontinuity of strongly separately continuous functions defined on a σ -product of a sequence of finite-dimensional normed spaces was given in [6]. Further, the Baire classification of ssc-functions defined on the space $\mathbb{R}^\omega$ equipped with the topology of pointwise convergence was investigated in [4]. Moreover, it was shown in [4] that if X is a product of normed spaces and $a \in X$ then for any open set $G \subseteq \sigma(a)$ there is a strongly separately continuous function $f : \sigma(a) \rightarrow \mathbb{R}$ such that the discontinuity point set of f is equal to G . Strongly separately continuous functions defined on a box-product of topological spaces were considered in [5].

Here we study ssc-functions defined on the spaces ℓ_p with $1 \leq p < +\infty$ of all sequences $(x_n)_{n=1}^\infty$ of real numbers for which the series $\sum_{n=1}^\infty |x_n|^p$ is convergent. In Section 2 we find a necessary and sufficient condition on a subset of ℓ_p to be determining in the class of all real-valued ssc-functions on ℓ_p . In the third section we show that for any ordinal $\alpha \in [2, \omega_1)$ there exists an $\mathcal{S}$ -open set $E \subseteq \ell_p$ which is of the α 'th additive Borel class and does not belong to the α 'th multiplicative Borel class. Using this fact we construct a real-valued ssc-function on ℓ_p which belongs to the α 'th Baire class and does not belong to the β 'th Baire class for $\beta < \alpha$. In Section 4 we prove that for any open nonempty set $G \subseteq \ell_p$ and $1 \leq p < \infty$ there exists a strongly separately continuous function $f : \ell_p \rightarrow \mathbb{R}$ which is discontinuous exactly on G .

2. Determining sets for strongly separately continuous functions

Let (X, Y) be a pair of topological spaces and $\mathcal{F}(X, Y)$ be a class of mappings between X and Y .

Definition 2.1. A set $E \subseteq X$ is called *determining for the class $\mathcal{F}(X, Y)$* if for any mappings $f, g \in \mathcal{F}(X, Y)$ the equality $f|_E = g|_E$ implies that $f = g$ on X .

It is well-known that any everywhere dense subset E of a topological space X is determining in the class $C(X, \mathbb{R})$ of all continuous real-valued functions on X . The theorem of Sierpiński [8] tells us that any everywhere dense subset $E \subseteq \mathbb{R}^2$ is determining in the class $CC(\mathbb{R}^2, \mathbb{R})$ of all separately continuous real-valued functions on $\mathbb{R}^2$.

In this section we give necessary and sufficient conditions on a subset of an $\mathcal{S}$ -open set $X \subseteq \prod_{n=1}^{\infty} X_n$ to be determining for the class $\text{SSC}(X)$ of all strongly separately continuous real-valued functions on X .

The following two notions were introduced in [6].

Definition 2.2. A set $A \subseteq X_T$ is called *projectively symmetric with respect to a point* $a \in A$ if $x_t^a \in A$ for all $t \in T$ and $x \in A$.

Definition 2.3. Let $X \subseteq X_T$ and τ be a topology on X . Then (X, τ) is said to be *locally projectively symmetric* if every $x \in X$ has a base of projectively symmetric neighborhoods with respect to x .

Definition 2.4. Let $(X_t : t \in T)$ be a family of topological spaces and $X \subseteq X_T$ be an $\mathcal{S}$ -open set. We say that a topology τ on X is *finitely generated* if for every $a \in X$ and every finite set $S \subseteq T$ the space $(X_S \times \prod_{t \in T \setminus S} \{a_t\}, \tau)$ is homeomorphic to the space X_S with the topology of pointwise convergence.

Notice that an arbitrary $\mathcal{S}$ -open subset of a product X_T of topological spaces X_t equipped with the topology of pointwise convergence is a locally projectively symmetric space with a finitely generated topology. All classical spaces of sequences as the space c of all convergence sequences or the spaces ℓ_p with $0 < p \leq \infty$ are locally projectively symmetric with a finitely generated topology.

Throughout the paper we consider only finitely generated topologies.

We need the following result [6, Theorem 3.4].

Theorem 2.1. Let $X \subseteq X_T$ be an $\mathcal{S}$ -open set equipped with a locally projectively symmetric topology τ and $x_0 \in X$.

- (i) If $f : X \rightarrow \mathbb{R}$ and $g : X \rightarrow \mathbb{R}$ are ssc-functions at x_0 , then $f(x) \pm g(x)$, $f(x) \cdot g(x)$, $|f(x)|$, $\min\{f(x), g(x)\}$ and $\max\{f(x), g(x)\}$ are ssc-functions at x_0 .
- (ii) If $(f_n)_{n=1}^{\infty}$ is a sequence of ssc-functions at the point x_0 and the series $\sum_{n=1}^{\infty} f_n(x)$ is convergent uniformly on X , then the sum $f(x)$ is an ssc-function at x_0 .
- (iii) If $(f_i)_{i \in I}$ is a locally finite family of ssc-functions $f_i : X \rightarrow \mathbb{R}$ at x_0 , then $f(x) = \sum_{i \in I} f_i(x)$ is an ssc-function at x_0 .

It is worth noting that the quotient of two real-valued ssc-functions need not be an ssc-function [3].

Definition 2.5. Let $(X_t : t \in T)$ be a family of topological spaces X_t and $a \in X_T$. A set $W \subseteq \sigma(a)$ is called *nearly open in $\sigma(a)$* if for any finite set $T_0 \subseteq T$ the set

$$W_{T_0} = \{x \in X_{T_0} : a_{T_0}^x \in W\}$$

is open in the space X_{T_0} equipped with the topology of pointwise convergence.

Definition 2.6. Let $(X_t : t \in T)$ be a family of topological spaces X_t , $X \subseteq X_T$ be an $\mathcal{S}$ -open set and $(\sigma_i : i \in I)$ be a partition of X on $\mathcal{S}$ -components. A set $W \subseteq X$ is called *nearly open in X* if $W \cap \sigma_i$ is nearly open in σ_i for every $i \in I$.

Definition 2.7. Let $X \subseteq X_T$ be an $\mathcal{S}$ -open set. A set $H \subseteq X$ is called *nearly closed in X* if the complement $X \setminus H$ is nearly open in X .

For a set $A \subseteq \sigma(a)$ we put

$$\bar{A}^\bullet = \{x \in \sigma(a) : W \cap A \neq \emptyset \quad \forall W - \text{nearly open}, x \in W\}.$$

Definition 2.8. A set $A \subseteq \sigma(a)$ is said to be *super dense* (*p-dense*) in $\sigma(a)$ if $\bar{A}^\bullet = \sigma(a)$ (respectively, $\bar{A}^p = \sigma(a)$), where $\bar{A}^p$ stands for the closure of A in the topology of pointwise convergence on $\sigma(a)$.

Definition 2.9. A subset A of an $\mathcal{S}$ -open set $X = \bigcup_{i \in I} \sigma_i \subseteq X_T$, where $(\sigma_i : i \in I)$ is a family of all $\mathcal{S}$ -components of X , is called *super dense in X* if for every $i \in I$ the set $A_i = A \cap \sigma_i$ is super dense in σ_i whenever $A_i \neq \emptyset$.

Observe that A is nearly closed if and only if $\bar{A}^\bullet = A$. Notice also that every super dense set is p-dense.

Theorem 2.2. (See [5, Theorem 1].) Let $(X_t)_{t \in T}$ be a family of topological spaces, $a \in X_T$, τ be a topology on $\sigma(a)$ and $f : \sigma(a) \rightarrow \mathbb{R}$ be an ssc-function. Then $f^{-1}(V)$ is nearly open in $\sigma(a)$ for any open set $V \subseteq \mathbb{R}$.

Proposition 2.3. Let $(X_t : t \in T)$ be a family of topological spaces and E be a super dense set in an $\mathcal{S}$ -open space $X \subseteq X_T$ equipped with a locally projectively symmetric topology τ . Then E is determining set for the class $\text{SSC}(X)$.

Proof. Consider ssc-functions $f, g : X \rightarrow \mathbb{R}$ such that $f|_E = g|_E$ and let $(\sigma_i : i \in I)$ be a partition of X on mutually disjoint $\mathcal{S}$ -components. Denote

$$H = \{x \in X : f(x) - g(x) = 0\}$$

and for every $i \in I$ we put $E_i = E \cap \sigma_i$ and $H_i = H \cap \sigma_i$. Since the function $h(x) = f(x) - g(x)$ is strongly separately continuous on X by Theorem 2.1, every set $H_i = (h^{-1}(0)) \cap \sigma_i$ is nearly closed in σ_i according to Theorem 2.2. Then the inclusion $E_i \subseteq H_i$ implies that $\sigma_i = \bar{E}_i^\bullet \subseteq \bar{H}_i^\bullet = H_i$. Consequently, $H_i = \sigma_i$ for every $i \in I$. Therefore, $H = \bigcup_{i \in I} H_i = \bigcup_{i \in I} \sigma_i = X$. $\square$

Lemma 2.4. Let $(X_n)_{n=1}^\infty$ be a sequence of locally compact Hausdorff spaces, $a = (a_n)_{n=1}^\infty \in \prod_{n=1}^\infty X_n$ and $W \subseteq \sigma(a)$ be a nearly open set. Then for every $w \in W$ there exists a sequence $(U_n)_{n=1}^\infty$ of functionally open sets $U_n \subseteq X_n$ such that $w \in (\prod_{n=1}^\infty U_n) \cap \sigma(a) \subseteq W$.

Proof. Without loss of generality we may assume that $w = a$, since $\sigma(a) = \sigma(w)$.

For every $n \in \mathbb{N}$ we denote $Y_n = \prod_{k=1}^n X_k$ and $W_n = W_{\{1, \dots, n\}} = \{y \in Y_n : a_{\{1, \dots, n\}}^y \in W\}$. Notice that every W_n is an open subset of the locally compact Hausdorff (and, consequently, completely regular) space Y_n . For $n = 1$ we take a functionally open neighborhood U_1 of a_1 with compact closure in Y_1 such that $K_1 = \bar{U}_1 \subseteq W_1$. For every $x \in K_1$ we take a functionally open neighborhood $V_x \times G_x$ of the point (a_1, a_2) with compact closure in Y_2 such that $(a_1, a_2) \in V_x \times G_x \subseteq W_2$. Since the set $K_1 \times \{a_2\}$ is compact, there exists a finite set $I \subseteq K_1$ such that $K_1 \times \{a_2\} \subseteq \bigcup_{x \in I} (V_x \times G_x)$. Put $U_2 = \bigcap_{x \in I} G_x$. Then U_2 is a functionally open neighborhood of a_2 in X_2 and $K_2 = K_1 \times \bar{U}_2 \subseteq W_2$. Proceeding inductively in this way we obtain a sequence $(U_n)_{n=1}^\infty$ of functionally open sets $U_n \subseteq X_n$ with $a \in \prod_{n=1}^\infty U_n \subseteq W$. $\square$

The following result follows from [5, Lemma 2].

Proposition 2.5. *Let $(X_n)_{n=1}^\infty$ be a sequence of topological spaces, $a \in \prod_{n=1}^\infty X_n$, $(U_n)_{n=1}^\infty$ be a sequence of functionally open sets $U_n \subseteq X_n$, $W = (\prod_{n=1}^\infty U_n) \cap \sigma(a)$ and let τ be the topology of pointwise convergence on $\sigma(a)$. Then there exists an ssc-function $f : (\sigma(a), \tau) \rightarrow [0, 1]$ such that $W = f^{-1}((0, 1])$.*

Proposition 2.6. *Let $X \subseteq \prod_{n=1}^\infty X_n$ be an $\mathcal{S}$ -open subset of the product of a sequence of locally compact Hausdorff spaces X_n , $\mathcal{T}$ be a topology on X which is finer than the topology τ of pointwise convergence and $E \subseteq X$ be a determining set for the class $\text{SSC}(X)$. Then E is super dense in X .*

Proof. Consider a partition $(\sigma_i : i \in I)$ of X on mutually disjoint $\mathcal{S}$ -open components. Assume that E is not super dense in X . Then there exists $i \in I$ such that $\emptyset \neq \overline{E \cap \sigma_i}^\bullet \neq \sigma_i$. Since $W = \sigma_i \setminus \overline{E \cap \sigma_i}^\bullet$ is a nonempty nearly open set in σ_i , by Lemma 2.4 there exists a sequence $(U_n)_{n=1}^\infty$ of functionally open sets $U_n \subseteq X_n$ such that $G = (\prod_{n=1}^\infty U_n) \cap \sigma_i \subseteq W$. According to Proposition 2.5 there exists an ssc-function $f : (\sigma_i, \tau) \rightarrow [0, 1]$ such that $G = f^{-1}((0, 1])$. Since $\tau \subseteq \mathcal{T}$, f is strongly separately continuous on $(\sigma_i, \mathcal{T})$. Notice that $f|_{E \cap \sigma_i} = 0$ and $f(x) > 0$ for every $x \in G$, which implies a contradiction, since $E \cap \sigma_i$ is determining in σ_i . $\square$

Since every space ℓ_p is an $\mathcal{S}$ -open subset of a countable product $\mathbb{R}^\omega$ and the standard topology on ℓ_p is finer than the topology of pointwise convergence, Propositions 2.3 and 2.6 immediately imply the following result.

Theorem 2.7. *For $p \in [1, +\infty)$ a set $E \subseteq \ell_p$ is determining for the class $\text{SSC}(\ell_p)$ if and only if E is super dense in ℓ_p .*

3. Baire classification of ssc-functions on ℓ_p

For $p \in [1, +\infty)$ and $x = (x_n)_{n=1}^\infty, y \in \ell_p$ we denote

$$\|x\|_p = \left(\sum_{n=1}^\infty |x_n|^p \right)^{\frac{1}{p}} \quad \text{and} \quad d_p(x, y) = \|x - y\|_p.$$

Let $\pi_n(x) = x_n$ for every $x = (x_n)_{n=1}^\infty \in \ell_p$ and $n \in \mathbb{N}$.

For $\alpha \in [1, \omega_1)$ we will use the symbols Σ_α^0 and Π_α^0 for the collections of sets of the α 'th additive or multiplicative class in X , respectively (see [7, p. 68]).

Lemma 3.1. *For any $\alpha \in [2, \omega_1)$ and $p \in [1, +\infty)$ there exists an $\mathcal{S}$ -open set E in ℓ_p such that $E \in \Sigma_\alpha^0 \setminus \Pi_\alpha^0$.*

Proof. Fix $p \in [1, +\infty)$.

We define inductively sequences $(\tilde{A}_\alpha)_{2 \leq \alpha < \omega_1}$ and $(\tilde{B}_\alpha)_{2 \leq \alpha < \omega_1}$ of subsets of $\mathbb{R}^\omega$ in the following way. Put

$$\tilde{A}_2 = \{x = (x_n)_{n=1}^\infty \in \mathbb{R}^\omega : \exists m \forall n \geq m \ x_n = 0\} \quad \text{and} \quad \tilde{B}_2 = \mathbb{R}^\omega \setminus \tilde{A}_2.$$

Let $\mathbb{N} = \bigcup_{n=1}^\infty T_n$ be a union of a sequence of mutually disjoint infinite sets $T_n = \{t_{n1}, t_{n2}, \dots\}$, where $(t_{nm})_{m=1}^\infty$ is a strictly increasing sequence of numbers $t_{nm} \in \mathbb{N}$. For every $n \in \mathbb{N}$ we denote by $\tilde{A}_2^n / \tilde{B}_2^n$ the copy of the set $\tilde{A}_2 / \tilde{B}_2$, which is contained in the space $\mathbb{R}^{T_n}$. Assume that for some $\alpha > 2$ we have

already defined sequences $(\tilde{A}_\beta)_{2 \leq \beta < \alpha}$ and $(\tilde{B}_\beta)_{2 \leq \beta < \alpha}$ (and their copies $(\tilde{A}_\beta^n)_{2 \leq \beta < \alpha}$ and $(\tilde{B}_\beta^n)_{2 \leq \beta < \alpha}$ in $\mathbb{R}^{T_n}$) of subsets of $\mathbb{R}^\omega$. Now we put

$$\tilde{A}_\alpha = \begin{cases} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \pi_{T_n}^{-1}(\tilde{B}_\beta^n), & \alpha = \beta + 1, \\ \bigcup_{n=1}^{\infty} \pi_{T_n}^{-1}(\tilde{A}_{\beta_n}^n), & \alpha = \sup \beta_n, \end{cases}$$

$$\tilde{B}_\alpha = \mathbb{R}^\omega \setminus \tilde{A}_\alpha.$$

Let for every $\alpha \in [2, \omega_1)$

$$A_\alpha = \tilde{A}_\alpha \cap \ell_p, \quad B_\alpha = \tilde{B}_\alpha \cap \ell_p,$$

$$A_\alpha^n = \tilde{A}_\alpha^n \cap \ell_p(\mathbb{R}^{T_n}), \quad B_\alpha^n = \tilde{B}_\alpha^n \cap \ell_p(\mathbb{R}^{T_n}).$$

Claim 1. For every $\alpha \in [2, \omega_1)$ the following statements are true:

- (1) the sets A_α and B_α are $\mathcal{S}$ -open in ℓ_p ;
- (2) for any $y = (y_n)_{n=1}^\infty \in \ell_p$ with $y_n \neq 0$ for all $n \in \mathbb{N}$ we have

$$x = (x_n)_{n=1}^\infty \in A_\alpha \iff z = (x_n \cdot y_n)_{n=1}^\infty \in A_\alpha.$$

Proof. (1). Evidently, A_2 and B_2 are $\mathcal{S}$ -open. Assume that for some $\alpha < \omega_1$ the claim is valid for all $\beta < \alpha$. Let $\alpha = \beta + 1$ be an isolated ordinal. Take any $x \in A_\alpha$ and $y \in \sigma_1(x)$. Then there exists $m \in \mathbb{N}$ such that $\pi_{T_n}(x) \in B_\beta^n$ for all $n \geq m$. Since $\pi_{T_n}(y) \in \sigma_1(\pi_{T_n}(x))$ and B_β^n is $\mathcal{S}$ -open, $\pi_{T_n}(y) \in B_\beta^n$. Therefore, $y \in A_\alpha$. We argue similarly in the case where α is a limit ordinal.

(2). Fix $y = (y_n)_{n=1}^\infty \in \ell_p$ such that $y_n \neq 0$ for all $n \in \mathbb{N}$. For $\alpha = 2$ the statement is true, since $A_2 = \sigma(0)$. Assume that for some $\alpha < \omega_1$ the property is valid for all $\beta < \alpha$. Let $\alpha = \beta + 1$ be an isolated ordinal (for a limit α we argue analogously). The inductive assumption implies that

$$x \in A_\alpha \iff \exists m \in \mathbb{N} \forall n \geq m \pi_{T_n}(x) \in B_\beta^n \iff \exists m \in \mathbb{N} \forall n \geq m \pi_{T_n}(z) \in B_\beta^n \iff z \in A_\alpha. \quad \square$$

Consider the equivalent metric

$$d(x, y) = \min\{d_p(x, y), 1\}$$

on the space ℓ_p .

Claim 2. For every $\alpha \in [2, \omega_1)$ the following condition holds:

- (*) for every set $C \subseteq (\ell_p, d)$ of the additive /multiplicative/ class α there exists a contracting mapping $f : (\ell_p, d) \rightarrow (\ell_p, d)$ with the Lipschitz constant $q = \frac{1}{2}$ such that

$$C = f^{-1}(A_\alpha) \quad /C = f^{-1}(B_\alpha)/, \quad (3.1)$$

$$|\pi_n(f(x))| \leq 1 \quad \forall x \in \ell_p \quad \forall n \in \mathbb{N}. \quad (3.2)$$

Proof. We will argue by the induction on α . Let $\alpha = 2$ and C be an arbitrary F_σ -subset of (ℓ_p, d) . Then $C = \bigcup_{n=1}^\infty C_n$ is a union of an increasing sequence of closed sets $C_n \subseteq (\ell_p, d)$. For every $x \in \ell_p$ we put

$$f(x) = \left(\frac{1}{3}d(x, C_1), \dots, \frac{1}{3^n}d(x, C_n), \dots\right).$$

Since every $d(x, C_n) \leq 1$, $f(x) \in \ell_p$ for every $x \in \ell_p$. Show that $C = f^{-1}(A_2)$. Take $x \in C$ and choose $m \in \mathbb{N}$ such that $x \in C_n$ for all $n \geq m$. Then $d(x, C_n) = 0$ and $\pi_n(f(x)) = 0$ for all $n \geq m$. Hence, $x \in f^{-1}(A_2)$. The inverse inclusion follows from the closedness of C_n . Since

$$d(f(x), f(y)) \leq d_p(f(x), f(y)) = \left(\sum_{n=1}^{\infty} \frac{1}{3^{np}} |d(x, C_n) - d(y, C_n)|^p\right)^{\frac{1}{p}} \leq d(x, y) \left(\sum_{n=1}^{\infty} \frac{1}{3^{np}}\right)^{\frac{1}{p}} \leq \frac{1}{2}d(x, y)$$

for all $x, y \in \ell_p$, the mapping $f : (\ell_p, d) \rightarrow (\ell_p, d)$ is contracting with the Lipschitz constant $q = \frac{1}{2}$. Moreover, $|\pi_n(f(x))| = \frac{1}{3^n}d(x, C_n) \leq 1$ for every $n \in \mathbb{N}$.

Assume that for some $\alpha < \omega_1$ the condition $(*)$ is valid for all $\beta < \alpha$. Let $C \subseteq (\ell_p, d)$ be any set of the α 'th additive class. Take an increasing sequence of sets C_n such that $C = \bigcup_{n=1}^{\infty} C_n$, where every C_n belongs to the multiplicative class β if $\alpha = \beta + 1$, and in the case $\alpha = \sup \beta_n$ we can assume that C_n belongs to the additive class β_n for every $n \in \mathbb{N}$. By the inductive assumption for every $n \in \mathbb{N}$ there exists a contracting mapping $f_n : (\ell_p, d) \rightarrow (\ell_p, d)$ with the Lipschitz constant $q = \frac{1}{2}$ such that

$$C_n = \begin{cases} f_n^{-1}(B_\beta), & \alpha = \beta + 1, \\ f_n^{-1}(A_{\beta_n}), & \alpha = \sup \beta_n, \end{cases} \quad (3.3)$$

$$|\pi_m(f_n(x))| \leq 1 \quad \forall x \in \ell_p \quad \forall n, m \in \mathbb{N}. \quad (3.4)$$

For every $k \in \mathbb{N}$ we choose a unique pair $(n(k), m(k)) \in \mathbb{N}^2$ such that

$$k = t_{n(k)m(k)} \in T_{n(k)}.$$

For all $x \in \ell_p$ and $n, m \in \mathbb{N}$ we put $f_{nm}(x) = \pi_m(f_n(x))$. Let

$$f(x) = \left(\frac{1}{3^2}f_{n(1)m(1)}(x), \dots, \frac{1}{3^{k+1}}f_{n(k)m(k)}(x), \dots\right)$$

for all $x \in \ell_p$. It is easy to see that $f(x) \in \ell_p$ for every $x \in \ell_p$.

Since

$$\frac{1}{3}|f_{nm}(x) - f_{nm}(y)| \leq d_p(f_n(x), f_n(y)) \quad \text{and} \quad \frac{1}{3}|f_{nm}(x) - f_{nm}(y)| \leq \frac{2}{3} \leq 1,$$

we have

$$\frac{1}{3}|f_{nm}(x) - f_{nm}(y)| \leq d(f_n(x), f_n(y)) \leq \frac{1}{2}d(x, y)$$

for all $x, y \in \ell_p$ and $n, m \in \mathbb{N}$. Consequently,

$$\begin{aligned} d(f(x), f(y)) &\leq d_p(f(x), f(y)) = \left(\sum_{k=1}^{\infty} \frac{1}{3^{kp}} \left(\frac{1}{3}|f_{n(k)m(k)}(x) - f_{n(k)m(k)}(y)|\right)^p\right)^{\frac{1}{p}} \leq \\ &\leq \frac{1}{2}d(x, y) \left(\sum_{k=1}^{\infty} \frac{1}{3^{kp}}\right)^{\frac{1}{p}} \leq \frac{1}{2}d(x, y) \end{aligned}$$

for all $x, y \in \ell_p$. Therefore, f has the Lipschitz constant $q = \frac{1}{2}$.

Finally, we show that $C = f^{-1}(A_\alpha)$. Consider the case $\alpha = \beta + 1$. Then

$$x \in C \iff \exists m \in \mathbb{N} \forall n \geq m \ x \in C_n \iff \exists m \in \mathbb{N} \forall n \geq m \ f_n(x) \in B_\beta.$$

Since

$$\pi_{T_n}(f(x)) = \left(\frac{1}{3^{k+1}} \cdot \pi_k(f_n(x)) \right)_{k \in T_n},$$

we have

$$f_n(x) \in B_\beta \iff \pi_{T_n}(f(x)) \in B_\beta^n,$$

by statement (2) of [Claim 1](#). Therefore, $C = f^{-1}(A_\alpha)$. We argue similarly when α is limit. $\square$

Claim 3. For every $\alpha \in [2, \omega_1)$ the set A_α belongs to the additive class α and does not belong to the multiplicative class α in ℓ_p .

Proof. We first prove that the set $\tilde{A}_\alpha / \tilde{B}_\alpha /$ is of the α 'th additive /multiplicative/ class in $\mathbb{R}^\omega$.

If $\alpha = 2$, then $\tilde{A}_2 = \bigcup_{n=1}^{\infty} \sigma_n(0)$ is an F_σ -subset of $\mathbb{R}^\omega$, since every $\sigma_n(0)$ is closed in $\mathbb{R}^\omega$. Consequently, $\tilde{B}_2$ is G_δ in $\mathbb{R}^\omega$. Suppose that for some $\alpha \geq 2$ the set $\tilde{A}_\beta / \tilde{B}_\beta /$ belongs to the additive /multiplicative/ class β in $\mathbb{R}^\omega$ for every $\beta < \alpha$. Since every projection $\pi_{T_n} : \mathbb{R}^\omega \rightarrow \mathbb{R}^{T_n}$ is continuous, the set $\tilde{A}_\alpha$ belongs to the additive class α in $\mathbb{R}^\omega$ and the set $\tilde{B}_\beta$ belongs to the multiplicative class α in $\mathbb{R}^\omega$.

Since the topology of pointwise convergence on ℓ_p is weaker than the topology generated by the norm $\|\cdot\|_p$, for every α the set $A_\alpha / B_\alpha /$ is of the α 'th additive /multiplicative/ class in ℓ_p .

Fix $\alpha \in [2, \omega_1)$. In order to show that A_α does not belong to the α 'th multiplicative class we assume the contrary. Then there exists a contracting mapping $f : (\ell_p, d) \rightarrow (\ell_p, d)$ such that $A_\alpha = f^{-1}(B_\alpha)$. By the Contraction Map Principle, there exists a fixed point for the mapping f , which implies a contradiction.

It remains to put $E = A_\alpha$. $\square$

Recall that a function $f : X \rightarrow \mathbb{R}$ on a metric space X belongs to the α 'th Baire class, if the preimage $f^{-1}(V)$ belongs to $\Sigma_{\alpha+1}^0$ for any open set $V \subseteq \mathbb{R}$.

Theorem 3.2. Let $\alpha \in [2, \omega_1)$ and $p \in [1, +\infty)$. Then there exists an ssc-function $f : \ell_p \rightarrow \mathbb{R}$ which belongs to the α 'th Baire class and does not belong to the β 'th Baire class for $\beta < \alpha$.

Proof. By [Lemma 3.1](#) there exists an $\mathcal{S}$ -open set E in ℓ_p such that $E \in \Sigma_{\alpha+1}^0 \setminus \Pi_{\alpha+1}^0$. Then the function $f : \ell_p \rightarrow \mathbb{R}$,

$$f(x) = \chi_E(x) = \begin{cases} 1, & x \in E, \\ 0, & x \notin E, \end{cases}$$

satisfies the required properties. $\square$

The existence of an ssc-function $f : \mathbb{R}^\omega \rightarrow \mathbb{R}$ which is not Baire measurable was proved in [\[4, Proposition 3.2\]](#).

4. Discontinuities of ssc-functions on ℓ_p

We denote by ℓ_p^ω the set $\ell_p \subseteq \mathbb{R}^\omega$ endowed with the topology of pointwise convergence induced from $\mathbb{R}^\omega$. Evidently, $\text{SSC}(\ell_p^\omega) \subseteq \text{SSC}(\ell_p)$. The converse is not true as the following example shows.

Example 1. There exists a continuous function $f : \ell_2 \rightarrow \mathbb{R}$ which is not strongly separately continuous on ℓ_2^ω .

Proof. For every $n \in \mathbb{N}$ we set

$$x_n = (\underbrace{\frac{1}{n}, 0, \dots, 0}_{n-1}, n, 0, \dots) \text{ and } y_n = (\underbrace{0, \dots, 0}_{n-1}, n, 0, \dots).$$

Since the sets $F_1 = \{x_n : n \in \mathbb{N}\}$ and $F_2 = \{y_n : n \in \mathbb{N}\}$ are disjoint and closed in ℓ_2 , the function $f : \ell_2 \rightarrow \mathbb{R}$ defined by the formula

$$f(x) = \frac{d_2(x, F_1)}{d_2(x, F_1) + d_2(x, F_2)}$$

is continuous and $F_1 = f^{-1}(0)$, $F_2 = f^{-1}(1)$. Notice that $x_n \rightarrow 0$ in ℓ_2^ω and $y_n = (x_n)_1^0$. But $f(x_n) - f(y_n) = 1$ for every n , which implies that f is not strongly separately continuous on ℓ_2^ω at the point $x = 0$ with respect to the first variable. $\square$

By $C(f)$ ($D(f)$) we denote the set of all points of continuity (discontinuity) of a mapping f .

Theorem 4.1. For any open nonempty set $G \subseteq \ell_p$ with $1 \leq p < \infty$ there exists a strongly separately continuous function $f : \ell_p \rightarrow \mathbb{R}$ such that $D(f) = G$.

Proof. Fix $p \in [1, +\infty)$ and let $\sigma = \sigma(0) = \{(x_n)_{n=1}^\infty \in \ell_p : (\exists k \in \mathbb{N}) (\forall n \geq k) (x_n = 0)\}$. Denote $F = \ell_p \setminus G$. For every $x = (x_n)_{n \in \mathbb{N}} \in \ell_p$ we put

$$\varphi(x) = \begin{cases} \min\{d_p(x, F), 1\}, & F \neq \emptyset, \\ 1, & F = \emptyset, \end{cases}$$

$$g(x) = \begin{cases} \exp(-\sum_{n=1}^\infty |x_n|), & x = (x_n)_{n=1}^\infty \in \sigma, \\ 1, & x \in \ell_p \setminus \sigma, \end{cases}$$

and let

$$f(x) = \varphi(x) \cdot g(x).$$

Claim 1. $F \subseteq C(f)$.

Proof. Fix $x^0 \in F$ and take any convergent sequence $(x^m)_{m=1}^\infty$ to x^0 in ℓ_p . Notice that

$$\lim_{m \rightarrow \infty} \varphi(x^m)g(x^m) = 0,$$

because $\varphi(x)$ is continuous at x^0 and $g(x)$ is bounded. Then

$$\lim_{m \rightarrow \infty} f(x^m) = 0 = f(x^0).$$

Hence, $x^0 \in C(f)$. $\square$

Claim 2. $G \subseteq D(f)$.

Proof. Fix $x^0 \in G$. Then $f(x^0) > 0$. We put $\varepsilon = \frac{1}{2}f(x^0)$ and take an arbitrary $\delta > 0$. Since the set $D = \ell_p \setminus \sigma$ is dense in ℓ_p , there exists $x = (x_n)_{n \in \mathbb{N}} \in \ell_p$ such that

$$\|x - x^0\|_p < \frac{\delta}{2} \text{ and } x \notin \sigma.$$

Take a number N such that

$$\sum_{n=1}^N |x_n| > \ln\left(\frac{1}{f(x^0) - \varepsilon}\right) \text{ and } \sum_{n=N+1}^{\infty} |x_n|^p < \left(\frac{\delta}{2}\right)^p.$$

We put

$$y = (x_1, \dots, x_N, 0, 0, \dots).$$

Then $y \in \sigma$ and

$$\|y - x^0\|_p \leq \|y - x\|_p + \|x - x^0\|_p = \left(\sum_{n=N+1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} + \|x - x^0\|_p < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

But

$$f(x^0) - f(y) = f(x^0) - \varphi(y) \cdot \exp\left(-\sum_{n=1}^N |x_n|\right) > f(x^0) - \exp\left(-\sum_{n=1}^N |x_n|\right) > f(x^0) + \varepsilon - f(x^0) = \varepsilon,$$

which implies that f is discontinuous at x^0 . $\square$

Claim 3. $f : \ell_p \rightarrow \mathbb{R}$ is strongly separately continuous.

Proof. Let $x^0 \in \ell_p$. Evidently, f is strongly separately continuous at x^0 if $x^0 \in F$. Therefore, we assume that $x^0 \in G$. Fix $k \in \mathbb{N}$ and $\varepsilon \in (0, 1)$. Take $\delta_1 > 0$ with $B(x^0, \delta_1) \subseteq G$. Since $\varphi(x)$ is continuous at x^0 , there exists $\delta_2 > 0$ such that

$$|\varphi(x) - \varphi(x^0)| < \frac{\varepsilon}{4}$$

for all $x \in B(x^0, \delta_2)$. Put

$$\delta = \min\{\delta_1, \delta_2, \ln(1 + \frac{\varepsilon}{2})\}.$$

Now let $x \in B(x^0, \delta)$ and $y = x_k^0 \in B(x^0, \delta)$. If $x \notin \sigma$, then $y \notin \sigma$. In this case

$$|f(x) - f(y)| = |\varphi(x) - \varphi(y)| \leq |\varphi(x) - \varphi(x^0)| + |\varphi(x^0) - \varphi(y)| < \varepsilon.$$

Assume that $x \in \sigma$. Then $y \in \sigma$ and

$$\begin{aligned} |f(x) - f(y)| &\leq |g(x)||\varphi(x) - \varphi(y)| + |\varphi(y)||g(x) - g(y)| < \\ &< \frac{\varepsilon}{2} + |\exp(-\sum_{n=1}^{\infty} |x_n|) - \exp(-\sum_{n=1}^{\infty} |y_n|)| < \frac{\varepsilon}{2} + |\exp(\sum_{n=1}^{\infty} |y_n| - \sum_{n=1}^{\infty} |x_n|) - 1|. \end{aligned}$$

Taking into account that

$$\exp\left(-\sum_{n=1}^{\infty} |x_n - y_n|\right) \leq \exp\left(\sum_{n=1}^{\infty} |x_n| - \sum_{n=1}^{\infty} |y_n|\right) \leq \exp\left(\sum_{n=1}^{\infty} |x_n - y_n|\right),$$

we obtain that

$$|f(x) - f(y)| < \frac{\varepsilon}{2} + \exp(|x_k - x_k^0|) - 1 < \frac{\varepsilon}{2} + \exp(\delta) - 1 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

in the case $|x_k^0| - |x_k| \geq 0$, or

$$|f(x) - f(y)| < \frac{\varepsilon}{2} + 1 - \exp(-|x_k^0 - x_k|) < \frac{\varepsilon}{2} + 1 - \exp(-\delta) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

otherwise. Hence, f is strongly separately continuous at x^0 with respect to the k 'th variable. $\square$

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