

BACKWARD DOUBLY STOCHASTIC EQUATIONS WITH JUMPS AND COMPARISON THEOREMS*

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ABSTRACT

In this work we mainly prove the existence and pathwise uniqueness of solutions to general backward doubly stochastic differential equations with jumps appearing in both forward and backward integral parts. Several comparison theorems under some weak conditions are also given. Finally we apply comparison theorems in proving the existence of solution to some special backward doubly stochastic differential equations with drift coefficient increasing linearly.

1 Introduction

Backward stochastic differential equations (BSDEs) in the linear case were introduced in Kushner (1972), Bismut (1976), Bensoussan (1982) and Haussmann (1986) as adjoint processes in the maximum principle for stochastic control problems and the pricing of options. Since the important work of Pardoux and Peng (1990), the interest in BSDEs has increased considerably in recent years. The significance of BSDEs is not only proved by the considerably important role they are playing in the study of partial differential equations (PDEs); see Peng (1991), Pardoux and Peng (1992) and Darling and Pardoux (1997), but also can be found in many other fields such as mathematical economics, financial mathematics, insurance and stochastic control. Here we just list several important works in every field. Duffie and Epstein (1992a,b) used BSDEs as a powerful tool to study stochastic differential utility. Moreover, in the insurance market BSDEs are used in pricing and hedging insurance equity-linked claims and asset-liability management problems, see El Karoui et al. (1997) and Delong (2013). Peng (1993) studied stochastic optimal control systems, where the state variables are described by a system of ordinary-SDE and BSDEs, and derived a local form of the maximum principle.

As further extensions of BSDEs, backward doubly stochastic differential equations (BDSDEs) contain both forward and backward stochastic integrals. Those equations were first introduced by Pardoux and Peng (1994) in the study of quasi-linear parabolic stochastic partial differential equations (SPDEs). Compared to BSDEs, much less results about BDSDEs can be found in the literature and most of the results established are about BDSDEs driven by Brownian motions. For the details about applications of BDSDEs to SPDEs driven by Brownian motion, one can refer

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to Zhang and Zhao (2007) which studied the existence and uniqueness of solution to BDSDEs on infinite horizons, and the stationary solutions to SPDEs by virtue of the solutions to BDSDEs on infinite horizons. Moreover, some work about BDSDEs with jumps appearing in the system of ordinary-SDE have been published recently. For instance, Zhu and Shi (2012) studied BDSDEs driven by Brownian motions and Poisson process with non-Lipschitz coefficients on random time interval. Aman (2012), Aman and Owo (2012) and Ren et al. (2009) study a special reflected generalized BDSDEs (driven by Teugel's martingales associated with Lévy process) with means of the penalization method and the fixed-point theorem. Existence and uniqueness of the solution to the BDSDE with jumps in the forward integral are studies in Sow (2011) for the case of non-Lipschitz coefficients. Recently, some results about stochastic control problems of BDSDEs have been obtained by Han et al. (2010) and Bahlali and Gherbal (2010).

This work is motivated by Xiong (2013) and He et al. (2014) which mainly studied the distribution function valued process of super-Brownian motions and super-Lévy processes characterized as the pathwise unique solution to a SPDE. For any super-Lévy process with transition semigroup $(Q_t)_{t \geq 0}$ defined by (1.4) in He et al. (2014), they proved that its distribution function valued process solves the following stochastic integral equation: for any $x \in \mathbb{R}$,

$$\begin{aligned} X_t(x) = & X_0(x) + \int_0^t A^* X_s(x) ds + \sqrt{c} \int_0^t \int_0^{X_{s-}(x)} W(ds, du) \\ & + \int_0^t \int_0^\infty \int_0^{X_{s-}(x)} z \tilde{N}(ds, dz, du) - b \int_0^t X_s(x) ds, \end{aligned} \quad (1.1)$$

where $\{W(ds, du); t \geq 0, u > 0\}$ is a Gaussian white noise with intensity $ds\pi(du)$, $\{N(dt, dz, du) : t \geq 0, u > 0\}$ is a Poisson random measure with intensity $dt\mu(dz)du$ and A^* is the dual operator of A defined by: for any $f(x) \in C_0^2(\mathbb{R})$,

$$Af(x) = \beta f'(x) + \frac{1}{2}\sigma^2 f''(x) + \int_{\mathbb{R}} [f(x+z) - f(x) - f'(x)z\mathbf{1}_{\{|z| \leq 1\}}] \nu(dz).$$

Furthermore, for any fixed $T > 0$, define the Gaussian white noise $\{W^T(dt, du) : 0 \leq t \leq T, u \in E\}$ by

$$W^T([T-t, T] \times A) = W([0, t] \times A), \quad 0 \leq t \leq T, A \in \mathcal{B}(E);$$

and the Poisson random measures $\{N^T(dt, du) : 0 \leq t \leq T, u \in U_0\}$ by:

$$N^T([T-t, T] \times B) = N([0, t] \times B), \quad 0 \leq t \leq T, B \in \mathcal{B}(U_i).$$

In the proof of the pathwise uniqueness of solutions to (1.1), they established its connection to the following BDSDE:

$$\begin{aligned} X_{T-t}(L_t^r + x) = & X_0(L_t^r + x) - b \int_t^T X_{T-s}(L_s^r + x) ds + \sqrt{c} \int_{t-}^{T-} \int_0^{X_{T-s}(L_s^r + x)} W(ds, du) \\ & + \int_{t-}^{T-} \int_0^\infty \int_0^{X_{T-s}(L_s^r + x)} z \tilde{N}_0^T(\overleftarrow{ds}, dz, du) - \sigma \int_t^T \nabla X_{T-s}(L_s^r + x) dB_s \\ & - \int_t^T \nabla [X_{T-s}(L_s^r + x - z) - X_{T-s}(L_s^r + x)] \tilde{M}(ds, dz), \end{aligned}$$

where $L_t^r = L_t - L_r$ and $\{L_t : t \geq 0\}$ is a Lévy process with generator A^* .

The purpose of this work is extending the above equations into more general BDSDEs with jumps appearing not only in the forward stochastic integral part but also in the backward stochastic integral part; see (2.2) in Section 2. Pathwise uniqueness and existence of their solutions are proved in Section 2 and 3 respectively under Lipschitz conditions. In addition, several comparison

theorems for BDSDEs will also be given in Section 4, since they play an important role in both theory and applications; see Shi et al. (2005). Effected by random terms in the backward integrals, classical methods are not applicable, we use another method to get comparison theorems with some reasonable and weak conditions. The main difficulty is to deal with the influence of forward integrals to the drift coefficient and backward integrals. As an applications of comparison theorems, in Section 5 we prove that solutions to a special kind of BDSDEs with drift coefficient increasing linearly exist.

Notation: For any n -dimensional vector $X = (x_1, \dots, x_n)$, $Y = (y_1, \dots, y_n)$ and $n \times n$ -matrix $A = (a_{i,j})$, let $\|X\|^2 = \sum_{i=1}^n x_i^2$, $\langle X, Y \rangle = \sum_{i=1}^n x_i y_i$, $\mathbf{T}(A) = (a_{11}, \dots, a_{nn})$ and $\|A\|^2 = \text{Tr}(A^T A) = \sum_{i,j=1}^n a_{ij}^2$, where $\text{Tr}(A)$ is the trace of A . For any $f \in C^2(\mathbb{R}^n)$, let

$$Df(x) = \left(\frac{\partial f(x)}{\partial x_1}, \dots, \frac{\partial f(x)}{\partial x_n} \right) \quad \text{and} \quad D^2 f(x) = \left(\frac{\partial^2 f(x)}{\partial x_1^2}, \dots, \frac{\partial^2 f(x)}{\partial x_n^2} \right).$$

Throughout this paper, we make the conventions

$$\int_a^b = \int_{(a,b]} \quad , \quad \int_a^\infty = \int_{(a,\infty)} \quad \text{and} \quad \int_{a-}^{b-} = \int_{[a,b)}$$

for any $b \geq a \geq 0$.

2 Pathwise Uniqueness

In this section, we mainly study the pathwise uniqueness of solutions to general backward doubly-stochastic equations. Suppose that $T > 0$ is a fixed constant and $(\Omega, \mathcal{F}, \mathbf{P})$ is a complete probability space endowed with filtration $\{\mathcal{G}_t^{(1)}\}_{0 \leq t \leq T}$ satisfying the usual hypotheses. Let $B(s)$ is a n -dimensional $(\mathcal{G}_t^{(1)})$ -Brownian motion and $\{M(dt, du) : 0 \leq t \leq T, u \in F\}$ a $(\mathcal{G}_t^{(1)})$ -Poisson random measure with intensity $dt\nu(du)$, where $\nu(du)$ is a σ -finite Borel measure on the Polish spaces F . Let $\{\mathcal{G}_t^{(2)}\}_{0 \leq t \leq T}$ be another filtration on $(\Omega, \mathcal{F}, \mathbf{P})$ satisfying the usual hypotheses and independent from $\{\mathcal{G}_t^{(1)}\}_{0 \leq t \leq T}$. Let $\{W(ds, du) = (W_1(ds, du), \dots, W_n(ds, du))^T; 0 \leq t \leq T, u \in E\}$ be a n -dimensional $(\mathcal{G}_t^{(2)})$ -Gaussian white noise constructed with n orthogonal white noises $W_i(ds, du)$ on $\mathbb{R}^+ \times E$ with intensity $ds\pi_i(du)$ respectively. Here we denote $\pi(du) = (\pi_1(du), \dots, \pi_n(du))^T$. Suppose $\mu_0(du)$ is a σ -finite Borel measure on the Polish space U_0 and $\mu_1(du)$ is a finite Borel measure on the Polish space U_1 . Moreover, For each $i = 0, 1$, let $\{N_i(dt, du) : 0 \leq t \leq T, u \in U_i\}$ be a $(\mathcal{G}_t^{(2)})$ -Poisson random measure with intensity $dt\mu_i(du)$. Obviously, all the random elements introduced above are independent of each other.

Denote $\mathcal{G}_t^r = \sigma(\mathcal{G}_t^{(1)} \cup \mathcal{G}_{T-r}^{(2)})$ for $0 \leq r \leq t \leq T$. Specially, $\mathcal{G}_t^0 = \sigma(\mathcal{G}_t^{(1)} \cup \mathcal{G}_T^{(2)})$ and $\mathcal{G}_T^{T-t} = \sigma(\mathcal{G}_T^{(1)} \cup \mathcal{G}_t^{(2)})$ are two filtrations satisfying the usual hypotheses. It is easily seen that $\{B_t\}$ is a (\mathcal{G}_t^0) -Brownian motion and $M(dt, du)$ is a (\mathcal{G}_t^0) -Poisson random measure. Define the Gaussian white noise $\{W^T(dt, du) : 0 \leq t \leq T, u \in E\}$ by

$$W^T([T-t, T] \times A) = W([0, t] \times A), \quad 0 \leq t \leq T, A \in \mathcal{B}(E).$$

For $i = 0, 1$, define the Poisson random measures $\{N_i^T(dt, du) : 0 \leq t \leq T, u \in U_i\}$ by:

$$N_i^T([T-t, T] \times B) = N_i([0, t] \times B), \quad 0 \leq t \leq T, B \in \mathcal{B}(U_i).$$

Roughly speaking, we can consider $W^T(dt, du)$ and $N_i^T(dt, du)$ as the time reversal of $W(dt, du)$ and $N_i(dt, du)$, respectively.

A real process $\{\xi_s\}_{0 \leq s \leq T}$ is said to be (\mathcal{G}_t^r) -progressive if for any $0 \leq r \leq t \leq T$, the mapping $(s, \omega) \mapsto \xi_s(\omega)$ restricted to $[r, t] \times \Omega$ is $\mathcal{B}([r, t]) \times \mathcal{G}_t^r$ -measurable. A two-parameter real process $\{\zeta_s(u)\}_{0 \leq s \leq T, u \in E}$ is said to be (\mathcal{G}_t^r) -progressive if for any $0 \leq r \leq t \leq T$, the restriction of $(s, u, \omega) \mapsto \zeta_s(u, \omega)$ to $[r, t] \times E \times \Omega$ is $\mathcal{B}([r, t]) \times \mathcal{B}(E) \times \mathcal{G}_t^r$ -measurable.

Let \mathcal{P} denote the σ -algebra on $\Omega \times [0, T]$ generated by all real-valued left continuous processes which are (\mathcal{G}_t^r) -progressive. A process $(\xi_s)_{0 \leq s \leq T}$ is said to be *predictable* if the mapping $(\omega, s) \mapsto \xi_s(\omega)$ is \mathcal{P} -measurable. Also a two-parameter process $\{\zeta_s(u)\}_{0 \leq s \leq T, u \in E}$ is said to be *predictable* if the mapping $(\omega, s, x) \mapsto \zeta_s(\omega, x)$ is $(\mathcal{P} \times \mathcal{B}(E))$ -measurable. For the theory of time-space stochastic integrals of predictable two parameter processes with respect to point processes or random measures, readers can refer to Section II.3 in Ikeda and Watanabe (1989). The stochastic integrals with respect to martingale measures were discussed in Section 7.3 of Li (2011). We make the convention that the stochastic integral of a progressive process refers to a predictable version of the integrand. The existence of such a version was briefly discussed in Section 2 of He et al. (2014). To simplify the following statements, we introduce several Banach spaces:

- (1) $\mathbb{S}_{\mathcal{G}, T}^2 := \{(\xi_s)_{0 \leq s \leq T} : \xi_s \text{ is } (\mathcal{G}_t^r)\text{-progressive and } \|\xi\|_{\mathbb{S}_T^2} < \infty\}$, where

$$\|\xi\|_{\mathbb{S}_T^2}^2 = \mathbf{E} \left[\sup_{s \in [0, T]} \|\xi_s\|^2 \right].$$

- (2) $\mathcal{L}_{\mathcal{G}, T}^2 := \{(\beta_s)_{0 \leq s \leq T} : \beta_s \text{ is } (\mathcal{G}_t^r)\text{-progressive and } \|\beta\|_{\mathcal{L}_T^2} < \infty\}$, where

$$\|\beta\|_{\mathcal{L}_T^2}^2 = \mathbf{E} \left\{ \int_0^T \|\beta_s\|^2 ds \right\}.$$

- (3) $\mathcal{L}_{\mathcal{G}, T}^2(E) := \{(\sigma(s, u))_{0 \leq s \leq T, u \in E} : \sigma(s, u) \text{ is } (\mathcal{G}_t^r)\text{-progressive and } \|\sigma\|_{\mathcal{L}_T^2(E)} < \infty\}$, where

$$\|\sigma\|_{\mathcal{L}_T^2(E)}^2 = \mathbf{E} \left\{ \int_0^T \|\sigma(s, \cdot)\|_{\mathcal{L}^2(E)}^2 ds \right\} = \mathbf{E} \left\{ \int_0^T ds \int_E \mathbf{T}(\sigma^T(s, u)) \pi(du) \right\}.$$

- (4) $\mathcal{L}_{\mathcal{G}, T}^2(U_0) := \{g(s, u)_{0 \leq s \leq T, u \in U_0} : g(s, u) \text{ is } (\mathcal{G}_t^r)\text{-progressive and } \|g\|_{\mathcal{L}_T^2(U_0)} < \infty\}$, where

$$\|g\|_{\mathcal{L}_T^2(U_0)}^2 = \mathbf{E} \left\{ \int_0^T \|g(s, \cdot)\|_{\mathcal{L}^2(U_0)}^2 ds \right\} = \mathbf{E} \left\{ \int_0^T ds \int_{U_0} \|g(s, u)\|^2 \mu_0(du) \right\}.$$

- (5) $\mathcal{L}_{\mathcal{G}, T}^2(U_1) := \{g(s, u)_{0 \leq s \leq T, u \in U_1} : g(s, u) \text{ is } (\mathcal{G}_t^r)\text{-progressive and } \|g\|_{\mathcal{L}_T^2(U_1)} < \infty\}$, where

$$\|g\|_{\mathcal{L}_T^2(U_1)}^2 = \mathbf{E} \left\{ \int_0^T \|g(s, \cdot)\|_{\mathcal{L}^2(U_1)}^2 ds \right\} = \mathbf{E} \left\{ \int_0^T ds \int_{U_1} \|g(s, u)\|^2 \mu_1(du) \right\}.$$

- (6) $\mathcal{L}_{\mathcal{G}, T}^2(F) := \{\zeta_s(u) : \zeta_s(u) \text{ is } (\mathcal{G}_t^r)\text{-progressive and } \|\zeta\|_{\mathcal{L}_T^2(F)} < \infty\}$, where

$$\|\zeta\|_{\mathcal{L}_T^2(F)}^2 = \mathbf{E} \left\{ \int_0^T \|\zeta_s\|_{\mathcal{L}^2(F)}^2 ds \right\} = \mathbf{E} \left\{ \int_0^T ds \int_F \|\zeta_s(u)\|^2 \nu(du) \right\}.$$

Before giving the main results, we extend Itô formula to the general case. Let X_t be a m -dimensional stochastic process defined by:

$$X_t = X_T + \int_t^T b(s) ds + \int_t^T \int_E a(s, u) W^T(\overleftarrow{ds}, du) + \int_{t-}^{T-} \int_{U_0} \gamma_0(s, u) \tilde{N}_0^T(\overleftarrow{ds}, du)$$

$$+ \int_{t-}^{T-} \int_{U_1} \gamma_1(s, u) N_1^T(\overleftarrow{ds}, du) - \int_t^T Z_s dB(s) - \int_t^T \int_F \zeta_s(u) \tilde{M}(ds, du), \quad (2.1)$$

where $b(s)$, $a(s, u)$, $\gamma_0(s, u)$, $\gamma_1(s, u)$ and $\zeta_s(u)$ are m -dimensional (\mathcal{G}_t^r) -progressive processes, $a(s, u)$ and Z_s are (\mathcal{G}_t^r) -progressive $(m \times n)$ -matrix-valued processes.

Proposition 2.1 *For any $f \in C^2(\mathbb{R}^m, \mathbb{R})$, we have*

$$\begin{aligned} f(X_t) &= f(X_T) + \int_t^T Df(X_s) b_i(s) ds + \int_t^T \int_E Df(X_s) a(s, u) W^T(\overleftarrow{ds}, du) \\ &\quad + \int_t^T ds \int_E \mathbf{T}[a^T(s, u) D^2 f(X_s) a(s, u)] \pi(du) \\ &\quad + \int_t^T \int_{U_0} [f(X_s + \gamma_0(s, u)) - f(X_s)] \tilde{N}_0^T(\overleftarrow{ds}, du) \\ &\quad + \int_t^T ds \int_{U_0} [f(X_s + \gamma_0(s, u)) - f(X_s) - Df(X_s) \gamma_0(s, u)] \mu_0(du) \\ &\quad + \int_t^T \int_{U_1} [f(X_s + \gamma_1(s, u)) - f(X_s)] N_1^T(\overleftarrow{ds}, du) \\ &\quad - \int_t^T Df(X_s) Z_s dB(s) - \frac{1}{2} \int_t^T \text{Tr}(Z^T(s) D^2 f(X_s) Z_s) ds \\ &\quad - \int_t^T \int_F [f(X_s + \zeta_s(u)) - f(X_s)] \tilde{M}(ds, du) \\ &\quad - \int_t^T ds \int_F [f(X_s + \zeta_s(u)) - f(X_s) - Df(X_s) \zeta_s(u)] \nu(du). \end{aligned}$$

Remark 2.2 *As in He et al. (2014), we make the convention that the stochastic integral of a progressive process always refers to that of a predictable version of the integrand. Here we emphasize that the integrals in (2.2) denote by $W^T(\overleftarrow{ds}, du)$, $\tilde{N}_0^T(\overleftarrow{ds}, du)$ and $N_1^T(\overleftarrow{ds}, du)$ are backward ones, which can be defined as the time-reversal of the corresponding forward stochastic integrals; see He et al. (2014) for more precise explanations. Of course, the integrals with respect to $dB(s)$ and $\tilde{M}(ds, du)$ in (2.2) are forward ones.*

Now let us introduce the backward doubly stochastic integral equation to work with. Suppose that we have the following measurable mappings:

$$\begin{aligned} \beta &: [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \times \mathcal{L}_{\mathcal{G}, T}^2(F) \mapsto \mathbb{R}^m; \\ \sigma &: [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \times \mathcal{L}_{\mathcal{G}, T}^2(F) \times E \mapsto \mathbb{R}^{m \times n}; \\ g_0 &: [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \times \mathcal{L}_{\mathcal{G}, T}^2(F) \times U_0 \mapsto \mathbb{R}^m; \\ g_1 &: [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \times \mathcal{L}_{\mathcal{G}, T}^2(F) \times U_1 \mapsto \mathbb{R}^m. \end{aligned}$$

Given $Y_T \in \mathcal{G}_T^0$, we consider the equation:

$$\begin{aligned} Y_t &= Y_T + \int_t^T \beta(s, Y_s, Z_s, \zeta_s) ds + \int_t^T \int_E \sigma(s, Y_s, Z_s, \zeta_s, u) W^T(\overleftarrow{ds}, du) \\ &\quad + \int_{t-}^{T-} \int_{U_0} g_0(s, Y_s, Z_s, \zeta_s, u) \tilde{N}_0^T(\overleftarrow{ds}, du) + \int_{t-}^{T-} \int_{U_1} g_1(s, Y_s, Z_s, \zeta_s, u) N_1^T(\overleftarrow{ds}, du) \\ &\quad - \int_t^T Z_s dB_s - \int_t^T \int_F \zeta_s(u) \tilde{M}(ds, du). \end{aligned} \quad (2.2)$$

Definition 2.3 *We call the process $(Y_t, Z_t, \zeta_t(u))_{0 \leq t \leq T}$ a solution to (2.2) if it is (\mathcal{G}_t^r) -progressive and for any $0 \leq r \leq t \leq T$ the equation (2.2) is satisfied almost surely.*

Condition 2.4 *There exist constants $C > 0$ and $0 < \alpha < 1$ such that for any $s \in [0, T]$ and $(x_i, y_i, z_i, \zeta_i) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \times \mathcal{L}_{\mathcal{G}, T}^2(F)$ with $i = 1, 2$,*

$$\|\beta(s, y_1, z_1, \zeta_1) - \beta(s, y_2, z_2, \zeta_2)\|^2 \leq C(\|y_1 - y_2\|^2 + \|z_1 - z_2\|^2 + \|\zeta_1 - \zeta_2\|_{\mathcal{L}^2(F)}^2) \quad (2.3)$$

and

$$\begin{aligned} & \|\sigma(s, y_1, z_1, \zeta_1, \cdot) - \sigma(s, y_2, z_2, \zeta_2, \cdot)\|_{\mathcal{L}^2(E)}^2 \\ & + \|\bar{g}_0(s, y_1, z_1, \zeta_1, \cdot) - \bar{g}_0(s, y_2, z_2, \zeta_2, \cdot)\|_{\mathcal{L}^2(U_0)}^2 \\ & + \|\bar{g}_1(s, y_1, z_1, \zeta_1, \cdot) - \bar{g}_1(s, y_2, z_2, \zeta_2, \cdot)\|_{\mathcal{L}^2(U_1)}^2 \\ & \leq C\|y_1 - y_2\|^2 + \alpha\|z_1 - z_2\|^2 + \alpha\|\zeta_1 - \zeta_2\|_{\mathcal{L}^2(F)}^2. \end{aligned} \quad (2.4)$$

Theorem 2.5 *Suppose Condition 2.4 holds. If $(Y_t^{(1)}, Z_t^{(1)}, \zeta_t^{(1)}(u))$ and $(Y_t^{(2)}, Z_t^{(2)}, \zeta_t^{(2)}(u))$ are solutions to (2.2) with $Y_T^{(1)} = Y_T^{(2)}$ a.s., then*

$$\mathbf{P}\left(Y_t^{(1)} = Y_t^{(2)} \text{ for all } t \in [0, T]\right) = 1 \quad (2.5)$$

and

$$\|Z^{(1)} - Z^{(2)}\|_{\mathcal{L}_T^2} + \|\zeta^{(1)} - \zeta^{(2)}\|_{\mathcal{L}_T^2(F)} = 0. \quad (2.6)$$

Proof. Let $(\bar{Y}_t, \bar{Z}_t, \bar{\zeta}_t(u)) = (Y_t^{(1)} - Y_t^{(2)}, Z_t^{(1)} - Z_t^{(2)}, \zeta_t^{(1)}(u) - \zeta_t^{(2)}(u))$. From (2.2) we get

$$\begin{aligned} \bar{Y}_t &= \int_t^T \bar{\beta}(s) ds + \int_t^T \int_E \bar{\sigma}(s, u) W^T(\bar{ds}, du) + \int_{t-}^{T-} \int_{U_0} \bar{g}_0(s, u) \tilde{N}_0^T(\bar{ds}, du) \\ &+ \int_{t-}^{T-} \int_{U_1} \bar{g}_1(s, u) N_1^T(\bar{ds}, du) - \int_t^T \bar{Z}_s dB_s - \int_t^T \int_F \bar{\zeta}_s(u) \tilde{M}(ds, du), \end{aligned} \quad (2.7)$$

where

$$\begin{aligned} \bar{\beta}(s) &= \beta(s, Y_t^{(1)}, Z_t^{(1)}, \zeta_t^{(1)}) - \beta(s, Y_t^{(2)}, Z_t^{(2)}, \zeta_t^{(2)}), \\ \bar{\sigma}(s, u) &= \sigma(s, Y_t^{(1)}, Z_t^{(1)}, \zeta_t^{(1)}, u) - \sigma(s, Y_t^{(2)}, Z_t^{(2)}, \zeta_t^{(2)}, u), \\ \bar{g}_0(s, u) &= g_0(s, Y_t^{(1)}, Z_t^{(1)}, \zeta_t^{(1)}, u) - g_0(s, Y_t^{(2)}, Z_t^{(2)}, \zeta_t^{(2)}, u), \\ \bar{g}_1(s, u) &= g_1(s, Y_t^{(1)}, Z_t^{(1)}, \zeta_t^{(1)}, u) - g_1(s, Y_t^{(2)}, Z_t^{(2)}, \zeta_t^{(2)}, u). \end{aligned}$$

By Proposition 2.1, we have

$$\begin{aligned} \|\bar{Y}_t\|^2 &= 2 \int_t^T \langle \bar{Y}_s, \bar{\beta}(s) \rangle ds + 2 \int_t^T \int_E \langle \bar{Y}_s, \bar{\sigma}(s, u) \rangle W^T(\bar{ds}, du) + \int_t^T \|\bar{\sigma}(s, \cdot)\|_{\mathcal{L}^2(E)}^2 ds \\ &+ \int_t^T \int_{U_0} [2 \langle \bar{Y}_s, \bar{g}_0(s, u) \rangle + \|\bar{g}_0(s, u)\|^2] \tilde{N}_0^T(\bar{ds}, du) + \int_t^T \|\bar{g}_0(s, \cdot)\|_{\mathcal{L}^2(U_0)}^2 ds \\ &+ 2 \int_t^T \int_{U_1} \langle \bar{Y}_s, \bar{g}_1(s, u) \rangle N_1^T(\bar{ds}, du) + \int_t^T \int_{U_1} \|\bar{g}_1(s, u)\|^2 N_1^T(\bar{ds}, du) \\ &- 2 \int_t^T \langle \bar{Y}_s, \bar{Z}_s \rangle dB_s - \int_t^T \|\bar{Z}_s\|^2 ds - \int_t^T \|\bar{\zeta}_s\|_{\mathcal{L}^2(F)}^2 ds \\ &- \int_t^T \int_F [2 \langle \bar{Y}_s, \bar{\zeta}_s(u) \rangle + \|\bar{\zeta}_s(u)\|^2] \tilde{M}(ds, du). \end{aligned}$$

From Cauchy's inequality, for any $a, b > 0$ we have

$$\mathbf{E}[\|\bar{Y}_t\|^2] + \mathbf{E}\left[\int_t^T \|\bar{Z}_s\|^2 ds\right] + \mathbf{E}\left[\int_t^T \|\bar{\zeta}_s\|_{\mathcal{L}^2(F)}^2 ds\right]$$

$$\begin{aligned}
 &= \mathbf{E} \left\{ 2 \int_t^T \langle \bar{Y}_s, \bar{\beta}(s) \rangle ds \right\} + \mathbf{E} \left\{ \int_t^T \|\bar{\sigma}(s, \cdot)\|_{\mathcal{L}^2(E)}^2 ds \right\} + \mathbf{E} \left\{ \int_t^T \|\bar{g}_0(s, \cdot)\|_{\mathcal{L}^2(U_0)}^2 ds \right\} \\
 &\quad + \mathbf{E} \left\{ 2 \int_t^T ds \int_{U_1} \langle \bar{Y}_s, \bar{g}_1(s, u) \rangle \mu_1(du) \right\} + \mathbf{E} \left\{ \int_t^T \|\bar{g}_1(s, \cdot)\|_{\mathcal{L}^2(U_1)}^2 ds \right\} \\
 &\leq \left(\frac{1}{a} + \frac{\mu_1(U_1)}{b} \right) \mathbf{E} \left\{ \int_t^T \|\bar{Y}_s\|^2 ds \right\} + a \mathbf{E} \left\{ \int_t^T \|\bar{\beta}(s)\|^2 ds \right\} + \mathbf{E} \left\{ \int_t^T \|\bar{\sigma}(s, \cdot)\|_{\mathcal{L}^2(E)}^2 ds \right\} \\
 &\quad + \mathbf{E} \left\{ \int_t^T \|\bar{g}_0(s, \cdot)\|_{\mathcal{L}^2(U_0)}^2 ds \right\} + (1+b) \mathbf{E} \left\{ \int_t^T \|\bar{g}_1(s, \cdot)\|_{\mathcal{L}^2(U_1)}^2 ds \right\}.
 \end{aligned}$$

Since μ_1 is a finite measure, by Hölder's inequality and Condition 2.4,

$$\begin{aligned}
 &\mathbf{E}[\|\bar{Y}_t\|^2] + \mathbf{E} \left[\int_t^T \|\bar{Z}_s\|^2 ds \right] + \mathbf{E} \left[\int_t^T \|\bar{\zeta}_s\|_{\mathcal{L}^2(F)}^2 ds \right] \\
 &\leq \left(\frac{1}{a} + \frac{\mu_1(U_1)}{b} \right) \mathbf{E} \left\{ \int_t^T \|\bar{Y}_s\|^2 ds \right\} + Ca \mathbf{E} \left\{ \int_t^T [\|\bar{Y}_s\|^2 + \|\bar{Z}_s\|^2 + \|\bar{\zeta}_s\|_{\mathcal{L}^2(F)}^2] ds \right\} \\
 &\quad + (1+b) \mathbf{E} \left\{ \int_t^T [C\|\bar{Y}_s\|^2 + \alpha\|\bar{Z}_s\|^2 + \alpha\|\bar{\zeta}_s\|_{\mathcal{L}^2(F)}^2] ds \right\}.
 \end{aligned}$$

Here we can choose a, b small enough such that $\hat{\alpha} := Ca + \alpha + b\alpha < 1$. Then

$$\begin{aligned}
 &\mathbf{E}[\|\bar{Y}_t\|^2] + (1 - \hat{\alpha}) \mathbf{E} \left\{ \int_t^T \|\bar{Z}_s\|^2 ds \right\} + (1 - \hat{\alpha}) \mathbf{E} \left\{ \int_t^T \|\bar{\zeta}_s\|_{\mathcal{L}^2(F)}^2 ds \right\} \\
 &\leq [1/a + 1/b + C(1 + a + b)] \mathbf{E} \left\{ \int_t^T \|\bar{Y}_s\|^2 ds \right\}.
 \end{aligned}$$

By Gronwall's lemma, we have

$$\mathbf{E}[\|\bar{Y}_t\|^2] + \mathbf{E} \left\{ \int_t^T \|\bar{Z}_s\|^2 ds \right\} + \mathbf{E} \left\{ \int_t^T \|\bar{\zeta}_s\|_{\mathcal{L}^2(F)}^2 ds \right\} = 0.$$

This implies (2.6). Then for any fixed $t \in [0, T]$, the six terms on the right-hand side of (2.7) vanish almost surely. Since each of the six terms is right-continuous or left-continuous, they almost surely vanish for all $t \in [0, T]$. We have finished the proof. \square

3 Existence

In this section, we study the existence of solutions to (2.2). For any $0 \leq r \leq t \leq T$ we define the natural σ -algebras:

$$\begin{aligned}
 \mathcal{F}_{r,t}^{BM} &= \sigma(\{B(s) - B(r), M((r, s] \times A) : r \leq s \leq t, A \in \mathbf{B}(F)\}) \vee \mathcal{N}, \\
 \mathcal{F}_{r,t}^{WN} &= \sigma(\{W((r, s] \times A), N_0((r, s] \times B), N_1((r, s] \times C) : \\
 &\quad r \leq s \leq t, A \in \mathcal{B}(E), B \in \mathcal{B}(U_0), C \in \mathcal{B}(U_1)\}) \vee \mathcal{N},
 \end{aligned}$$

where \mathcal{N} denotes the totality of \mathbf{P} -null sets. For simplicity, we write $\mathcal{F}_t^{BM} = \mathcal{F}_{0,t}^{BM}$ and $\mathcal{F}_t^{WN} = \mathcal{F}_{0,t}^{WN}$. Let $\mathcal{F}_t^r = \mathcal{F}_t^{BM} \vee \mathcal{F}_{T-r}^{WN}$ for $0 \leq r \leq t \leq T$. Similarly, we can define $\mathbb{S}_{\mathcal{F},T}^2, \mathcal{L}_{\mathcal{F},T}^2, \mathcal{L}_{\mathcal{F},T}^2(E), \mathcal{L}_{\mathcal{F},T}^2(U_0), \mathcal{L}_{\mathcal{F},T}^2(U_1), \mathcal{L}_{\mathcal{F},T}^2(F)$ like those in the last section but with $\{\mathcal{G}_t^r : 0 \leq r \leq t \leq T\}$ replaced by $\{\mathcal{F}_t^r : 0 \leq r \leq t \leq T\}$.

Theorem 3.1 *Suppose Condition 2.4 holds. Then there exists a solution $(Y_t, Z_t, \zeta_t(u))$ to (2.2) in $\mathbb{S}_{\mathcal{F},T}^2 \times \mathcal{L}_{\mathcal{F},T}^2 \times \mathcal{L}_{\mathcal{F},T}^2(F)$.*

Obviously, combining this theorem with Theorem 2.5, we have solution to (2.2) exists uniquely in $\mathbb{S}_{\mathcal{G},T}^2 \times \mathcal{L}_{\mathcal{G},T}^2 \times \mathcal{L}_{\mathcal{G},T}^2(F)$. Before giving the proof of Theorem 3.1, we introduce a lemma about the solution to some simple backward doubly stochastic equation, which is very important in the proof of this theorem.

Lemma 3.2 *Let $\beta \in \mathcal{L}_{\mathcal{F},T}^1$, $\sigma \in \mathcal{L}_{\mathcal{F},T}^2(E)$, $g_0 \in \mathcal{L}_{\mathcal{F},T}^2(U_0)$ and $g_1 \in \mathcal{L}_{\mathcal{F},T}^2(U_1)$. Then for any $Y_T \in \mathcal{F}_T^0$ with finite second moment, there exists a unique solution $(Y_t, Z_t, \zeta_t(u)) \in \mathbb{S}_{\mathcal{F},T}^2 \times \mathcal{L}_{\mathcal{F},T}^2 \times \mathcal{L}_{\mathcal{F},T}^2(F)$ to the following equation:*

$$\begin{aligned} Y_t = Y_T &+ \int_t^T \beta(s)ds + \int_t^T \int_E \sigma(s,u)W^T(\overleftarrow{ds}, du) + \int_{t-}^{T-} \int_{U_0} g_0(s,u)\tilde{N}_0^T(\overleftarrow{ds}, du) \\ &+ \int_{t-}^{T-} \int_{U_1} g_1(s,u)N_1^T(\overleftarrow{ds}, du) - \int_t^T Z_s dB_s - \int_t^T \int_F \zeta_s(u)\tilde{M}(ds, du). \end{aligned} \quad (3.1)$$

Proof. The uniqueness of the solution follows from Theorem 2.5. Recall $\mathcal{F}_t^0 = \mathcal{F}_t^{BM} \vee \mathcal{F}_t^{WN}$ for $0 \leq t \leq T$. Observe that

$$\begin{aligned} \Psi_T := Y_T &+ \int_0^T \int_E \sigma(s,u)W^T(\overleftarrow{ds}, du) + \int_{0-}^{T-} \int_{U_0} g_0(s,u)\tilde{N}_0^T(\overleftarrow{ds}, du) \\ &+ \int_0^T \beta(s)ds + \int_{0-}^{T-} \int_{U_1} g_1(s,u)N_1^T(\overleftarrow{ds}, du) \end{aligned} \quad (3.2)$$

is \mathcal{F}_T^0 -measurable. Then we can define a Doob's martingale:

$$M_t = \mathbf{E}[\Psi_T | \mathcal{F}_t^0], \quad 0 \leq t \leq T.$$

Since $\mathcal{F}_t^t \subset \mathcal{F}_t^0$, from (3.2) we have

$$\begin{aligned} M_t = Y_t &+ \int_0^t \beta(s)ds + \int_0^t \int_E \sigma(s,u)W^T(\overleftarrow{ds}, du) \\ &+ \int_{0-}^{t-} \int_{U_0} g_0(s,u)\tilde{N}_0^T(\overleftarrow{ds}, du) + \int_{0-}^{t-} \int_{U_1} g_1(s,u)N_1^T(\overleftarrow{ds}, du), \end{aligned} \quad (3.3)$$

where $Y_t = \mathbf{E}[\Xi(t) | \mathcal{F}_t^0]$ and

$$\begin{aligned} \Xi(t) = Y_T &+ \int_t^T \beta(s)ds + \int_t^T \int_E \sigma(s,u)W^T(\overleftarrow{ds}, du) \\ &+ \int_{t-}^{T-} \int_{U_0} g_0(s,u)\tilde{N}_0^T(\overleftarrow{ds}, du) + \int_{t-}^{T-} \int_{U_1} g_1(s,u)N_1^T(\overleftarrow{ds}, du). \end{aligned} \quad (3.4)$$

By the martingale representation theorem, see Lemma 2.3 in Tang and Li (1994), there exist (\mathcal{F}_t^0) -progressive processes $\{Z_s\}$ and $\{\zeta_s(u)\}$ such that

$$M_t = M_0 + \int_0^t Z_s dB_s + \int_0^t \int_F \zeta_s(u)\tilde{M}(ds, du)$$

and hence

$$M_T = M_t + \int_t^T Z_s dB_s + \int_t^T \int_F \zeta_s(u)\tilde{M}(ds, du). \quad (3.5)$$

Since $M_T = \Psi_T$, we can substitute (3.2) and (3.3) into (3.5) to obtain (3.1). Finally, we need to prove for any $0 \leq r \leq T$ the process $(Y_t, Z_t, \zeta_t(u))_{r \leq t \leq T, u \in F}$ is (\mathcal{F}_t^r) -progressive. Observe that

$$Y_r = \mathbf{E}[\Xi(r) | \mathcal{F}_r^0] = \mathbf{E}[\Xi(r) | \mathcal{F}_r^{BM} \vee \mathcal{F}_r^{WN}] = \mathbf{E}[\Xi(r) | \mathcal{F}_r^r \vee \mathcal{F}_{T-r,T}^{WN}],$$

where \mathcal{F}_{T-r}^{WN} and $\mathcal{F}_{T-r,T}^{WN}$ are independent. By (3.4) it is easy to see that $\Xi(r)$ is independent of $\mathcal{F}_{T-r,T}^{WN}$. Then we have $Y_r = \mathbf{E}[\Xi(r)|\mathcal{F}_r^r]$, which is \mathcal{F}_r^r -measurable. By (3.1) we have

$$\begin{aligned} \int_r^T Z_s dB_s + \int_r^T \int_F \zeta_s(u) \tilde{M}(ds, du) &= \int_r^T \int_E \sigma(s, u) W^T(\overleftarrow{ds}, du) + \int_{r-}^{T-} \int_{U_0} g_0(s, u) \tilde{N}_0^T(\overleftarrow{ds}, du) \\ &\quad + Y_T - Y_r + \int_r^T \beta(s) ds + \int_{r-}^{T-} \int_{U_1} g_1(s, u) N_1^T(\overleftarrow{ds}, du). \end{aligned}$$

Then by the uniqueness of the martingale representation, the process $(Z_t, \zeta_t(u))$ has an (\mathcal{F}_t^r) -progressive version. Since each term in (3.1) is right or left continuous, the process (Y_t) is (\mathcal{F}_t^r) -progressive. \square

Proof of Theorem 3.1. We shall use a Picard iteration argument to construct a solution to (2.2). Let $Y_t^{(0)} = Z_t^{(0)} = \zeta_t^{(0)}(u) \equiv 0$. By Lemma 3.2, for any $n \geq 0$ there exists a unique solution $(Y_t^{(n+1)}, Z_t^{(n+1)}, \zeta_t^{(n+1)}(u))$ to the following equation:

$$\begin{aligned} Y_t^{(n+1)} &= Y_T + \int_t^T \beta(s, Y_s^{(n)}, Z_s^{(n)}, \zeta_s^{(n)}) ds + \int_t^T \int_E \sigma(s, Y_s^{(n)}, Z_s^{(n)}, \zeta_s^{(n)}, u) W^T(\overleftarrow{ds}, du) \\ &\quad + \int_{t-}^{T-} \int_{U_0} g_0(s, Y_s^{(n)}, Z_s^{(n)}, \zeta_s^{(n)}, u) \tilde{N}_0^T(\overleftarrow{ds}, du) - \int_t^T Z_s^{(n+1)} dB_s \\ &\quad + \int_{t-}^{T-} \int_{U_1} g_1(s, Y_s^{(n)}, Z_s^{(n)}, \zeta_s^{(n)}, u) N_1^T(\overleftarrow{ds}, du) - \int_t^T \int_F \zeta_s^{(n+1)}(u) \tilde{M}(ds, du). \end{aligned}$$

Let $\bar{Y}_t^{(n+1)} = Y_t^{(n+1)} - Y_t^{(n)}$, $\bar{Z}_t^{(n+1)} = Z_t^{(n+1)} - Z_t^{(n)}$ and $\bar{\zeta}_t^{(n+1)}(u) = \zeta_t^{(n+1)}(u) - \zeta_t^{(n)}(u)$. From (2.2) we have

$$\begin{aligned} \bar{Y}_t^{(n+1)} &= \int_t^T \bar{\beta}^{(n)}(s) ds + \int_t^T \int_E \bar{\sigma}^{(n)}(s, u) W^T(\overleftarrow{ds}, du) + \int_{t-}^{T-} \int_{U_0} \bar{g}_0^{(n)}(s, u) \tilde{N}_0^T(\overleftarrow{ds}, du) \\ &\quad + \int_{t-}^{T-} \int_{U_1} \bar{g}_1^{(n)}(s, u) N_1^T(\overleftarrow{ds}, du) - \int_t^T \bar{Z}_s^{(n+1)} dB_s - \int_t^T \int_F \bar{\zeta}_s^{(n+1)}(u) \tilde{M}(ds, du), \end{aligned}$$

where

$$\begin{aligned} \bar{\beta}^{(n)}(s) &= \beta(s, Y_s^{(n)}, Z_s^{(n)}, \zeta_s^{(n)}) - \beta(s, Y_s^{(n-1)}, Z_s^{(n-1)}, \zeta_s^{(n-1)}), \\ \bar{\sigma}^{(n)}(s, u) &= \sigma(s, Y_s^{(n)}, Z_s^{(n)}, \zeta_s^{(n)}, u) - \sigma(s, Y_s^{(n-1)}, Z_s^{(n-1)}, \zeta_s^{(n-1)}, u), \\ \bar{g}_0^{(n)}(s, u) &= g_0(s, Y_s^{(n)}, Z_s^{(n)}, \zeta_s^{(n)}, u) - g_0(s, Y_s^{(n-1)}, Z_s^{(n-1)}, \zeta_s^{(n-1)}, u), \\ \bar{g}_1^{(n)}(s, u) &= g_1(s, Y_s^{(n)}, Z_s^{(n)}, \zeta_s^{(n)}, u) - g_1(s, Y_s^{(n-1)}, Z_s^{(n-1)}, \zeta_s^{(n-1)}, u). \end{aligned}$$

According to Proposition 2.1., we have

$$\begin{aligned} \|\bar{Y}_t^{(n+1)}\|^2 &= 2 \int_t^T \langle \bar{Y}_s^{(n+1)}, \bar{\beta}^{(n)}(s) \rangle ds + 2 \int_t^T \int_E \langle \bar{Y}_s^{(n+1)}, \bar{\sigma}^{(n)}(s, u) \rangle W^T(\overleftarrow{ds}, du) \\ &\quad + \int_t^T \|\bar{\sigma}^{(n)}(s, \cdot)\|_{\mathcal{L}^2(E)}^2 ds + \int_t^T \|\bar{g}_0^{(n)}(s, \cdot)\|_{\mathcal{L}^2(U_0)}^2 ds \\ &\quad + \int_{t-}^{T-} \int_{U_0} [2 \langle \bar{Y}_s^{(n+1)}, \bar{g}_0^{(n)}(s, u) \rangle + \|\bar{g}_0^{(n)}(s, u)\|^2] \tilde{N}_0^T(\overleftarrow{ds}, du) \\ &\quad + \int_{t-}^{T-} \int_{U_1} [2 \langle \bar{Y}_s^{(n+1)}, \bar{g}_1^{(n)}(s, u) \rangle + \|\bar{g}_1^{(n)}(s, u)\|^2] N_1^T(\overleftarrow{ds}, du) \\ &\quad - 2 \int_t^T \langle \bar{Y}_s^{(n+1)}, \bar{Z}_s^{(n+1)} \rangle dB_s - \int_t^T \|\bar{Z}_s^{(n+1)}\|^2 ds - \int_t^T \|\bar{\zeta}_s^{(n+1)}\|_{\mathcal{L}^2(F)}^2 ds \\ &\quad - \int_t^T \int_F [2 \langle \bar{Y}_s^{(n+1)}, \bar{\zeta}_s^{(n+1)}(u) \rangle + \|\bar{\zeta}_s^{(n+1)}(u)\|^2] \tilde{M}(ds, du). \end{aligned}$$

It follows that

$$\begin{aligned} & \mathbf{E}[|\bar{Y}_t^{(n+1)}|^2] + \mathbf{E}\left\{\int_t^T \|\bar{Z}_s^{(n+1)}\|^2 ds\right\} + \mathbf{E}\left\{\int_t^T \|\bar{\zeta}_s^{(n+1)}\|_{\mathcal{L}^2(F)}^2 ds\right\} \\ &= \mathbf{E}\left\{2\int_t^T \langle \bar{Y}_s^{(n+1)}, \bar{\beta}^{(n)}(s) \rangle ds\right\} + \mathbf{E}\left\{\int_t^T \|\bar{\sigma}^{(n)}(s, \cdot)\|_{\mathcal{L}^2(E)}^2 ds\right\} \\ &+ \mathbf{E}\left\{\int_t^T \|\bar{g}_0^{(n)}(s, \cdot)\|_{\mathcal{L}^2(U_0)}^2 ds\right\} + \mathbf{E}\left\{\int_t^T \|\bar{g}_1^{(n)}(s, \cdot)\|_{\mathcal{L}^2(U_1)}^2 ds\right\} \\ &+ \mathbf{E}\left\{2\int_t^T ds \int_{U_1} \langle \bar{Y}_s^{(n+1)}, \bar{g}_1^{(n)}(s, u) \rangle \mu_1(du)\right\}. \end{aligned}$$

By integration by parts, one can see, for any $\lambda > 0$,

$$\begin{aligned} \lambda \int_t^T e^{\lambda s} \mathbf{E}[\|\bar{Y}_s^{(n+1)}\|^2] ds &= e^{\lambda s} \mathbf{E}[\|\bar{Y}_s^{(n+1)}\|^2] \Big|_t^T - \int_t^T e^{\lambda s} d\mathbf{E}[\|\bar{Y}_s^{(n+1)}\|^2] \\ &= -e^{\lambda t} \mathbf{E}[\|\bar{Y}_t^{(n+1)}\|^2] + \mathbf{E}\left\{2\int_t^T e^{\lambda s} \langle \bar{Y}_s^{(n+1)}, \bar{\beta}^{(n)}(s) \rangle ds\right\} \\ &+ \mathbf{E}\left\{\int_t^T e^{\lambda s} \|\bar{\sigma}^{(n)}(s, \cdot)\|_{\mathcal{L}^2(E)}^2 ds\right\} \\ &+ \mathbf{E}\left\{\int_t^T e^{\lambda s} \|\bar{g}_0^{(n)}(s, \cdot)\|_{\mathcal{L}^2(U_0)}^2 ds\right\} \\ &+ \mathbf{E}\left\{\int_t^T e^{\lambda s} \|\bar{g}_1^{(n)}(s, \cdot)\|_{\mathcal{L}^2(U_1)}^2 ds\right\} \\ &+ \mathbf{E}\left\{2\int_t^T e^{\lambda s} ds \int_{U_1} \langle \bar{Y}_s^{(n+1)}, \bar{g}_1^{(n)}(s, u) \rangle \mu_1(du)\right\} \\ &- \mathbf{E}\left\{\int_t^T \|\bar{Z}_s^{(n+1)}\|^2 e^{\lambda s} ds\right\} - \mathbf{E}\left\{\int_t^T e^{\lambda s} \|\bar{\zeta}_s^{(n+1)}\|_{\mathcal{L}^2(F)}^2 ds\right\}. \end{aligned}$$

By Hölder's inequality, for any $a, b > 0$ we have

$$\begin{aligned} & \lambda \int_t^T e^{\lambda s} \mathbf{E}[\|\bar{Y}_s^{(n+1)}\|^2] ds + \mathbf{E}\left\{\int_t^T e^{\lambda s} \|\bar{Z}_s^{(n+1)}\|^2 ds\right\} + \mathbf{E}\left\{\int_t^T e^{\lambda s} \|\bar{\zeta}_s^{(n+1)}\|_{\mathcal{L}^2(F)}^2 ds\right\} \\ &\leq (1/a + 1/b) \int_t^T e^{\lambda s} \mathbf{E}[\|\bar{Y}_s^{(n+1)}\|^2] ds + \mathbf{E}\left\{a \int_t^T e^{\lambda s} \|\bar{\beta}^{(n)}(s)\|^2 ds\right\} \\ &+ \mathbf{E}\left\{\int_t^T e^{\lambda s} [\|\bar{\sigma}^{(n)}(s, u)\|_{\mathcal{L}^2(E)}^2 + \|\bar{g}_0^{(n)}(s, u)\|_{\mathcal{L}^2(U_0)}^2] ds\right\} \\ &+ (1+b) \mathbf{E}\left\{\int_t^T e^{\lambda s} \|\bar{g}_1^{(n)}(s, u)\|_{\mathcal{L}^2(U_1)}^2 ds\right\}. \end{aligned}$$

Using Condition 2.4, we have

$$\begin{aligned} & \lambda \int_t^T e^{\lambda s} \mathbf{E}[\|\bar{Y}_s^{(n+1)}\|^2] ds + \mathbf{E}\left\{\int_t^T e^{\lambda s} \|\bar{Z}_s^{(n+1)}\|^2 ds\right\} + \mathbf{E}\left\{\int_t^T e^{\lambda s} \|\bar{\zeta}_s^{(n+1)}\|_{\mathcal{L}^2(F)}^2 ds\right\} \\ &\leq (1/a + 1/b) \int_t^T e^{\lambda s} \mathbf{E}[\|\bar{Y}_s^{(n+1)}\|^2] ds + aC \mathbf{E}\left\{\int_t^T e^{\lambda s} [\|\bar{Y}_s^{(n)}\|^2 + \|\bar{Z}_s^{(n)}\|^2 + \|\bar{\zeta}_s^{(n)}\|_{\mathcal{L}^2(F)}^2] ds\right\} \\ &+ (1+b) \mathbf{E}\left\{\int_t^T e^{\lambda s} [C\|\bar{Y}_s^{(n)}\|^2 + \alpha\|\bar{Z}_s^{(n)}\|^2 + \alpha\|\bar{\zeta}_s^{(n)}\|_{\mathcal{L}^2(F)}^2] ds\right\}. \end{aligned}$$

Then

$$\mathbf{E}\left\{\int_t^T e^{\lambda s} [(\lambda - 1/a - 1/b)\|\bar{Y}_s^{(n+1)}\|^2 + \|\bar{Z}_s^{(n+1)}\|^2 + \|\bar{\zeta}_s^{(n+1)}\|_{\mathcal{L}^2(F)}^2] ds\right\}$$

$$\leq \mathbf{E} \left\{ \int_t^T e^{\lambda s} [(a+b+1)C \|\bar{Y}_s^{(n)}\|^2 + (aC + b\alpha + \alpha)(\|\bar{Z}_s^{(n)}\|^2 + \|\bar{\zeta}_s^{(n)}\|_{\mathcal{L}^2(F)}^2)] ds \right\}.$$

Let a, b be small enough such that $aC + b\alpha + \alpha < 1$. Then choose $\lambda > 0$ large enough such that

$$\lambda - \frac{1}{a} - \frac{1}{b} > \frac{(a+b+1)C}{aC + b\alpha + \alpha} > 0.$$

It follows that

$$\begin{aligned} & \mathbf{E} \left\{ \int_t^T e^{\lambda s} \left[\left(\lambda - \frac{1}{a} - \frac{1}{b} \right) \|\bar{Y}_s^{(n+1)}\|^2 + \|\bar{Z}_s^{(n+1)}\|^2 + \|\bar{\zeta}_s^{(n+1)}\|_{\mathcal{L}^2(F)}^2 \right] ds \right\} \\ & \leq (aC + b\alpha + \alpha) \mathbf{E} \left\{ \int_t^T e^{\lambda s} \left[\left(\lambda - \frac{1}{a} - \frac{1}{b} \right) \|\bar{Y}_s^{(n)}\|^2 + \|\bar{Z}_s^{(n)}\|^2 + \|\bar{\zeta}_s^{(n)}\|_{\mathcal{L}^2(F)}^2 \right] ds \right\} \\ & \leq \dots \\ & \leq (aC + b\alpha + \alpha)^n \mathbf{E} \left\{ \int_t^T e^{\lambda s} \left[\left(\lambda - \frac{1}{a} - \frac{1}{b} \right) \|\bar{Y}_s^{(1)}\|^2 + \|\bar{Z}_s^{(1)}\|^2 + \|\bar{\zeta}_s^{(1)}\|_{\mathcal{L}^2(F)}^2 \right] ds \right\}. \end{aligned}$$

Since the right-hand side of the inequality is summable, we see that $\{(Y_s^{(n)}, Z_s^{(n)}, \zeta_s^{(n)}(u))\}$ is a Cauchy sequence. By Burkholder-Davis-Gundy Inequality, it is easy to see $Y_s^{(n)}$ is also a Cauchy sequence in $\mathbb{S}_{\mathcal{F}, T}^2$. Then it converges in $\mathbb{S}_{\mathcal{F}, T}^2 \times \mathcal{L}_{\mathcal{F}, T}^2 \times \mathcal{L}_{\mathcal{F}, T}^2(F)$ to some process $(Y_s, Z_s, \zeta_s(u))$, which is clearly a solution to (2.2). Then we have finished the proof. \square

4 Comparison theorems

Comparison theorems are very important in both theory and applications. For instance, if you want to earn more money from a complete capital market in the future time T , you should either invest more money in the market at time 0 or improve your investment policy. This section will mainly introduce several comparison theorems under Condition 2.4. There are two classical ways to prove comparison theorems in the theory of BSDEs; see Situ (2005, p.243-250). One is transforming the BSDE into a summation of a non-negative processes and a martingale under a new probability measure. Then the desired results can be gotten by taking conditional expectation under the new probability measure. Another one is called "a duality method" which mainly by constructing a relative forward SDE (FSDE). Applying Itô formula to the multiplication of the solutions of these two stochastic equations (FBSDE), we will get a new process which is a summation of a non-negative processes and a martingale. Similarly, we get the comparison theorem by taking conditional expectation. Actually, both of these two methods come from the same ideas.

Unfortunately, effected by backward integral parts in (2.2), neither of these two methods works. Here we use another method to get comparison theorems under some conditions which are not really stronger than those in BSDEs. The main difficulty is to deal with the influence of ζ_s to the drift coefficient and backward integrals. We divide the influence into several parts and deal with them one by one. Here we only consider the one-dimensional case, comparison theorem for multi-dimensional case is still an open problem; see Peng (1999). Firstly, we give a simple comparison theorem about the non-positivity of solution to the following one-dimensional BDSDEs, which can be used to derive other results.

$$\begin{aligned} Y_t = & Y_T + \int_t^T \beta(s, Y_s, Z_s, \zeta_s) ds + \int_t^T \int_E \sigma(s, Y_s, Z_s, u) W^T(\overleftarrow{ds}, du) \\ & + \int_{t-}^{T-} \int_{U_0} g_0(s, Y_s, Z_s, u) \tilde{N}_0^T(\overleftarrow{ds}, du) + \int_{t-}^{T-} \int_{U_1} g_1(s, Y_s, Z_s, u) \tilde{N}_1^T(\overleftarrow{ds}, du) \\ & - \int_t^T Z_s dB_s - \int_t^T \int_F \zeta_s(u) \tilde{M}(ds, du). \end{aligned} \quad (4.1)$$

Lemma 4.1 Suppose Condition 2.4 holds, (Y_t, Z_t, ζ_t) is a solution to BDSDE (4.1) and

- (1) both $y + g_0(s, y, z, u)$ and $y + g_1(s, y, z, u)$ are non-positive for any $y \in (-\infty, 0]$;
- (2) there exist some constants $C > 0$ and $0 < \alpha < 1$ such that for any $s \in [0, T]$,

$$\|\sigma(s, y, z, \cdot)\|_{\mathcal{L}^2(E)}^2 + \|g_0(s, y, z, \cdot)\|_{\mathcal{L}^2(U_0)}^2 + \|g_1(s, y, z, \cdot)\|_{\mathcal{L}^2(U_1)}^2 \leq C|y|^2 + \alpha|z|^2;$$

- (3) for some constant $K > 0$ we have $\beta(s, y, z, \zeta) = h(s, y, z) + \int_F C(s, u)\zeta_s(u)\nu(du)$ with

$$|h(s, y, z)| \leq K(|y| + |z|),$$

where $C(s, u) \geq -1$ and $\int_F |C(s, u)|^2 \nu(du) \leq K$ for any $s \in [0, T]$.

If $Y_T \leq 0$ a.s., we have $\mathbf{P}(Y_t \leq 0 : t \in [0, T]) = 1$.

Proof. Here we just prove $\mathbf{P}(Y_t \leq 0 : t \in [0, T]) = 1$ under the corresponding conditions. It suffices to prove this theorem with $\nu(du)$ to be a finite Borel measure. Actually, for the general case we can always find a sequence $F_n \nearrow F$ such that $\nu_n(F) = \nu(F_n) < \infty$ and $\nu_n(\cdot) = \mathbf{1}_{\{\cdot \in F_n\}} \nu(\cdot) \rightarrow \nu(\cdot)$. For any $n \geq 1$, from Theorem 2.5 and 3.1 there exists a unique solution $(Y_s^{(n)}, Z_s^{(n)}, \zeta_s^{(n)}(u))$ to (4.1) with $M(ds, du)$ replaced by $M_n(ds, du)$, which has intensity $d\nu_n(du)$. Like the proof of Theorem 3.1, we also have $Y_t^{(n)} \rightarrow Y_t$ in $\mathbb{S}_{\mathcal{F}, T}^2$. For any integer $n \geq 0$, let

$$\alpha_n = \exp \left\{ -\frac{n(n+1)}{2} \right\}.$$

Then $\alpha_n \rightarrow 0$ decreasingly as $n \rightarrow \infty$ and

$$\int_{\alpha_n}^{\alpha_{n-1}} z^{-1} dz = n.$$

Let $x \mapsto g_n(x)$ be a positive continuous function supported by (α_n, α_{n-1}) such that

$$\int_{\alpha_n}^{\alpha_{n-1}} g_n(x) dx = 1 \quad \text{and} \quad x g_n(x) \leq \frac{2}{n}.$$

Moreover, for any $n > 0$, define

$$f_n(z) = \left| \int_0^z dy \int_0^y g_n(x) dx \right|^2, \quad z \in \mathbb{R}.$$

It is easy to see that

(a) $f_n(z) \rightarrow |z^+|^2$ increasingly.

(b) $|f'_n(z)| = \begin{cases} 2 \int_0^z g_n(x) dx \int_0^z dy \int_0^y g_n(x) dx \leq 2z, & z > 0; \\ 0, & z \leq 0 \end{cases}$ and $\lim_{n \rightarrow \infty} f'_n(z) = 2z^+$.

(c) $f''_n(z) = \begin{cases} 2 \left| \int_0^z g_n(x) dx \right|^2 + 2g_n(z) \int_0^z dy \int_0^y g_n(x) dx \leq 2 + \frac{4}{n}, & z > 0; \\ 0, & z \leq 0 \end{cases}$

and $\lim_{n \rightarrow \infty} f''_n(z) = 2\mathbf{1}_{\{z > 0\}}$.

Applying Proposition 2.1 to $f_n(Y_t)$, Since $Y_T \leq 0$ a.s. we have

$$\begin{aligned}
 f_n(Y_t) = & \int_t^T f'_n(Y_s) \beta(s, Y_s, Z_s, \zeta_s) ds + \int_t^T \int_E f'_n(Y_s) \sigma(s, Y_s, Z_s, u) W^T(\overleftarrow{ds}, du) \\
 & + \frac{1}{2} \int_t^T ds \int_E f''_n(Y_s) |\sigma(s, Y_s, Z_s, u)|^2 \pi(du) \\
 & + \int_{t-}^{T-} \int_{U_0} [f_n(Y_s + g_0(s, Y_s, Z_s, u)) - f_n(Y_s)] \tilde{N}_0^T(\overleftarrow{ds}, du) \\
 & + \int_t^T ds \int_{U_0} [f_n(Y_s + g_0(s, Y_s, Z_s, u)) - f_n(Y_s) - f'_n(Y_s) g_0(s, Y_s, Z_s, u)] \mu_0(du) \\
 & + \int_{t-}^{T-} \int_{U_1} [f_n(Y_s + g_1(s, Y_s, Z_s, u)) - f_n(Y_s)] N_1^T(\overleftarrow{ds}, du) \\
 & - \int_t^T f'_n(Y_s) Z_s dB_s - \frac{1}{2} \int_t^T f''_n(Y_s) |Z_s|^2 ds \\
 & - \int_t^T \int_F [f_n(Y_s + \zeta_s(u)) - f_n(Y_s)] \tilde{M}(ds, du) \\
 & - \int_t^T ds \int_F [f_n(Y_s + \zeta_s(u)) - f_n(Y_s) - f'_n(Y_s) \zeta_s(u)] \nu(du).
 \end{aligned}$$

Taking expectation to the above inequality, we have

$$\begin{aligned}
 \mathbf{E}[f_n(Y_t)] = & \int_t^T \mathbf{E}[f'_n(Y_s) \beta(s, Y_s, Z_s, \zeta_s)] ds + \frac{1}{2} \int_t^T ds \int_E \mathbf{E}[f''_n(Y_s) |\sigma(s, Y_s, Z_s, u)|^2] \pi(du) \\
 & + \int_t^T ds \int_{U_0} \mathbf{E}[f_n(Y_s + g_0(s, Y_s, Z_s, u)) - f_n(Y_s) - f'_n(Y_s) g_0(s, Y_s, Z_s, u)] \mu_0(du) \\
 & + \int_t^T ds \int_{U_1} \mathbf{E}[f_n(Y_s + g_1(s, Y_s, Z_s, u)) - f_n(Y_s)] \mu_1(du) - \frac{1}{2} \int_t^T \mathbf{E}[f''_n(Y_s) |Z_s|^2] ds \\
 & - \int_t^T ds \int_F \mathbf{E}[f_n(Y_s + \zeta_s(u)) - f_n(Y_s) - f'_n(Y_s) \zeta_s(u)] \nu(du).
 \end{aligned}$$

Since $Y_t \in \mathbb{S}_{\mathcal{F}, T}^2$, from (a)-(c) and dominated convergence theorem, we have as $n \rightarrow \infty$

$$\begin{aligned}
 \mathbf{E}[|Y_t^+|^2] \leq & \int_t^T \mathbf{E}[2Y_s^+ \beta(s, Y_s, Z_s, \zeta_s)] ds + \int_t^T ds \int_E \mathbf{E}[|\sigma(s, Y_s, Z_s, u)|^2 \mathbf{1}_{\{Y_s > 0\}}] \pi(du) \\
 & + \int_t^T ds \int_{U_0} \mathbf{E}[|(Y_s + g_0(s, Y_s, Z_s, u))^+|^2 - |Y_s^+|^2 - 2Y_s^+ g_0(s, Y_s, Z_s, u)] \mu_0(du) \\
 & + \int_t^T ds \int_{U_1} \mathbf{E}[|(Y_s + g_1(s, Y_s, Z_s, u))^+|^2 - |Y_s^+|^2] \mu_1(du) - \int_t^T \mathbf{E}[|Z_s|^2 \mathbf{1}_{\{Y_s > 0\}}] ds \\
 & - \int_t^T ds \int_F \mathbf{E}[|(Y_s + \zeta_s(u))^+|^2 - |Y_s^+|^2 - 2Y_s^+ \zeta_s(u)] \nu(du).
 \end{aligned}$$

From condition (1) we have

$$\begin{aligned}
 \mathbf{E}[|Y_t^+|^2] \leq & \int_t^T \mathbf{E}[2Y_s^+ \beta(s, Y_s, Z_s, \zeta_s)] ds + \int_t^T ds \int_E \mathbf{E}[|\sigma(s, Y_s, Z_s, u)|^2 \mathbf{1}_{\{Y_s > 0\}}] \pi(du) \\
 & + \int_t^T ds \int_{U_1} \mathbf{E}[2Y_s^+ g_1(s, Y_s, Z_s, u) + |g_1(s, Y_s, Z_s, u)|^2 \mathbf{1}_{\{Y_s > 0\}}] \mu_1(du) \\
 & + \int_t^T ds \int_{U_0} \mathbf{E}[|g_0(s, Y_s, Z_s, u)|^2 \mathbf{1}_{\{Y_s > 0\}}] \mu_0(du) - \int_t^T \mathbf{E}[|Z_s|^2 \mathbf{1}_{\{Y_s > 0\}}] ds \\
 & - \mathbf{E}\left[\int_t^T ds \int_F [|(Y_s + \zeta_s(u))^+|^2 - |Y_s^+|^2 - 2Y_s^+ \zeta_s(u)] \nu(du)\right]. \tag{4.2}
 \end{aligned}$$

Let η denote the integrand in the last term of (4.2). Then

$$\eta = \begin{cases} |(Y_s + \zeta_s(u))^+|^2 \geq 0, & \text{if } Y_s \leq 0; \\ |\zeta_s(u)|^2 & \text{if } Y_s > 0 \text{ and } \zeta_s(u) \geq -Y_s; \\ -|Y_s|^2 - 2Y_s\zeta_s(u) & \text{if } Y_s > 0 \text{ and } \zeta_s(u) < -Y_s. \end{cases} \quad (4.3)$$

Otherwise, by Cauchy's inequality and Hölder's inequality, for any $b > 0$,

$$\begin{aligned} 2Y_s^+ \int_F C(s, u)\zeta_s(u)\nu(du) &= 2Y_s^+ \int_F C(s, u)\zeta_s(u)\mathbf{1}_{\{\zeta_s(u) \geq -Y_s\}}\nu(du) \\ &\quad + 2Y_s^+ \int_F C(s, u)\zeta_s(u)\mathbf{1}_{\{\zeta_s(u) < -Y_s\}}\nu(du) \\ &\leq b \left| \int_F C(s, u)\zeta_s(u)\mathbf{1}_{\{Y_s > 0, \zeta_s(u) \geq -Y_s\}}\nu(du) \right|^2 \\ &\quad + \frac{1}{b}|Y_s^+|^2 + 2Y_s^+ \int_F C(s, u)\zeta_s(u)\mathbf{1}_{\{\zeta_s(u) < -Y_s\}}\nu(du) \\ &\leq b \int_F |C(s, u)|^2\nu(du) \int_F |\zeta_s(u)|^2\mathbf{1}_{\{Y_s > 0, \zeta_s(u) \geq -Y_s\}}\nu(du) \\ &\quad + \frac{1}{b}|Y_s^+|^2 + 2Y_s^+ \int_F C(s, u)\zeta_s(u)\mathbf{1}_{\{\zeta_s(u) < -Y_s\}}\nu(du). \end{aligned} \quad (4.4)$$

From (4.3), (4.4) and conditions in this theorem, for $a, c > 0$ we have

$$\begin{aligned} \mathbf{E}[|Y_t^+|^2] &\leq \left(\frac{1}{a} + \frac{1}{b} + \frac{\mu_1(U_1)}{c} + \nu(F)\right) \int_t^T \mathbf{E}[|Y_s^+|^2] ds + a \int_t^T \mathbf{E}[|h(s, Y_s, Z_s)|^2 \mathbf{1}_{\{Y_s > 0\}}] ds \\ &\quad + \mathbf{E}\left[\int_t^T \left(b \int_F |C(s, u)|^2\nu(du) - 1\right) ds \int_F |\zeta_s(u)|^2 \mathbf{1}_{\{Y_s > 0, \zeta_s(u) \geq -Y_s\}}\nu(du)\right] \\ &\quad + \mathbf{E}\left[\int_t^T 2Y_s^+ ds \int_F [C(s, u) + 1]\zeta_s(u)\mathbf{1}_{\{\zeta_s(u) < -Y_s\}}\nu(du)\right] \\ &\quad + \int_t^T ds \int_E \mathbf{E}[|\sigma(s, Y_s, Z_s, u)|^2 \mathbf{1}_{\{Y_s > 0\}}] \pi(du) \\ &\quad + \int_t^T ds \int_{U_0} \mathbf{E}[|g_0(s, Y_s, Z_s, u)|^2 \mathbf{1}_{\{Y_s > 0\}}] \mu_0(du) - \int_t^T \mathbf{E}[|Z_s|^2 \mathbf{1}_{\{Y_s > 0\}}] ds \\ &\quad + \int_t^T ds \int_{U_1} \mathbf{E}[(1+c)|g_1(s, Y_s, Z_s, u)|^2 \mathbf{1}_{\{Y_s > 0\}}] \mu_1(du). \end{aligned}$$

Here we choose b small enough such that $b \int_F |C(s, u)|^2\nu(du) \leq 1$. From condition (2) and (3) we have

$$\mathbf{E}[|Y_t^+|^2] \leq D \int_t^T \mathbf{E}[|Y_s^+|^2] ds + \tilde{\alpha} \int_t^T \mathbf{E}[|Z_s|^2 \mathbf{1}_{\{Y_s > 0\}}] ds,$$

where $D > 0$ is a constant and $\tilde{\alpha} = aK + (1+c)\alpha - 1$. Let a and c be small enough such that $\tilde{\alpha} < 0$, we have

$$\mathbf{E}[|Y_t^+|^2] \leq D \int_t^T \mathbf{E}[|Y_s^+|^2] ds.$$

By Gronwall's inequality, we have $\mathbf{E}[|Y_t^+|^2] = 0$, which means $\mathbf{P}(Y_t \leq 0) = 1$ for any $t \in [0, T]$. Like the proof of Theorem 2.5, we will get the desired result. \square

Proposition 4.2 *The conclusion in Lemma 4.1 remains true if (3) is replaced by*

(3') for some constant $K > 0$ have

$$|\beta(s, y, z, \zeta)| \leq K(|y| + |z|) + \int_F C(s, u) |\zeta(u)| \nu(du),$$

where $0 \leq C(s, u) \leq 1$ and $\int_F |C(s, u)|^2 \nu(du) \leq K$ a.s. for any $s \in [0, T]$.

Now let us derive the general comparison theorem from Lemma 4.1. However, since the deficiency of information about Z_s , we only consider the following case. For $i = 1, 2$, suppose $(Y_t^{(i)}, Z_t^{(i)}, \zeta_s^{(i)}(u))$ is a solution to

$$\begin{aligned} Y_t^{(i)} &= Y_T^{(i)} + \int_t^T \beta^{(i)}(s, Y_s^{(i)}, Z_s^{(i)}, \zeta_s^{(i)}(u)) ds + \int_t^T \int_E \sigma(s, Y_s^{(i)}, Z_s^{(i)}, u) W^T(\overleftarrow{ds}, du) \\ &\quad + \int_{t-}^{T-} \int_{U_0} g_0(s, Y_s^{(i)}, u) \tilde{N}_0^T(\overleftarrow{ds}, du) + \int_{t-}^{T-} \int_{U_1} g_1(s, Y_s^{(i)}, u) N_1^T(\overleftarrow{ds}, du) \\ &\quad - \int_t^T Z_s^{(i)} dB_s - \int_t^T \int_F \zeta_s^{(i)}(u) \tilde{M}(ds, du). \end{aligned} \quad (4.5)$$

Theorem 4.3 Suppose

- (1) $\beta^{(1)}(s, y, z, \zeta) \leq \beta^{(2)}(s, y, z, \zeta)$;
- (2) both $y + g_0(s, y, u)$ and $y + g_1(s, y, u)$ are nondecreasing with respect to y ;
- (3) $\sigma(s, y, z, u)$, $g_0(s, y, u)$ and $g_1(s, y, u)$ satisfy (2.4);
- (4) there exists a constant $K > 0$ such that one of following conditions satisfies:

$$(a) \beta^{(1)}(s, y, z, \zeta) = h^{(1)}(s, y, z) + \int_F C^{(1)}(s, u) \zeta_s(u) \nu(du) \text{ with}$$

$$|h^{(1)}(s, y, z) - h^{(1)}(s, y', z')| \leq K(|y - y'| + |z - z'|),$$

where $C^{(1)}(s, u) \geq -1$ and $\int_F |C^{(1)}(s, u)|^2 \nu(du) \leq K$ for any $s \in [0, T]$.

$$(b) \beta^{(2)}(s, y, z, \zeta) = h^{(2)}(s, y, z) + \int_F C^{(2)}(s, u) \zeta_s(u) \nu(du) \text{ with}$$

$$|h^{(2)}(s, y, z) - h^{(2)}(s, y', z')| \leq K(|y - y'| + |z - z'|),$$

where $C^{(2)}(s, u) \leq 1$ and $\int_F |C^{(2)}(s, u)|^2 \nu(du) \leq K$ for any $s \in [0, T]$.

If $Y_T^{(1)} \leq Y_T^{(2)}$ a.s., then $\mathbf{P}(Y_t^{(1)} \leq Y_t^{(2)} : t \in [0, T]) = 1$.

Proof. Here we assume (a) in condition (4) of this theorem holds and $\nu(du)$ to be a finite Borel measure. Let $(\bar{Y}_t, \bar{Z}_t, \bar{\zeta}_t(u)) = (Y_t^{(1)} - Y_t^{(2)}, Z_t^{(1)} - Z_t^{(2)}, \zeta_t^{(1)}(u) - \zeta_t^{(2)}(u))$. From (4.1) and condition (1) we get

$$\begin{aligned} \bar{Y}_t &\leq \bar{Y}_T + \int_t^T \bar{\beta}(s) ds + \int_t^T \int_E \bar{\sigma}(s, u) W^T(\overleftarrow{ds}, du) \\ &\quad + \int_{t-}^{T-} \int_{U_0} \bar{g}_0(s, u) \tilde{N}_0^T(\overleftarrow{ds}, du) + \int_{t-}^{T-} \int_{U_1} \bar{g}_1(s, u) N_1^T(\overleftarrow{ds}, du) \\ &\quad - \int_t^T \bar{Z}_s dB_s - \int_t^T \int_F \bar{\zeta}_s(u) \tilde{M}(ds, du), \end{aligned} \quad (4.6)$$

where $\bar{Y}_T = Y_T^{(1)} - Y_T^{(2)}$ and

$$\begin{aligned}\bar{\beta}(s) &= \beta^{(1)}(s, Y_s^{(1)}, Z_s^{(1)}, \zeta_s^{(1)}) - \beta^{(1)}(s, Y_s^{(2)}, Z_s^{(2)}, \zeta_s^{(2)}), \\ \bar{\sigma}(s, u) &= \sigma(s, Y_s^{(1)}, Z_s^{(1)}, u) - \sigma(s, Y_s^{(2)}, Z_s^{(2)}, u), \\ \bar{g}_0(s, u) &= g_0(s, Y_s^{(1)}, u) - g_0(s, Y_s^{(2)}, u), \\ \bar{g}_1(s, u) &= g_1(s, Y_s^{(1)}, u) - g_1(s, Y_s^{(2)}, u).\end{aligned}$$

It is easy to check that $\bar{\beta}(s)$, $\bar{\sigma}(s, u)$, $\bar{g}_0(s, u)$ and $\bar{g}_1(s, u)$ satisfy conditions in Lemma 4.1, so the desired result follows. \square

Remark 4.4

- (1) Condition (2) means any jumps from N_0 and N_1 will not make $Y^{(1)}$ exceed $Y^{(2)}$.
- (2) Since we do not have enough information about Z_s and ζ_s , the result does not include the case of g_0 and g_1 depending on Z and ζ . This is still an open problem.
- (3) Obviously, Condition (4) can not be weakened to (2.3) in Condition 2.4, counterexamples about BSDEs as a special case of BDSDEs can be seen in many works, such as Situ (2005, p.245).
- (4) Condition (4) can be replaced by
- (4') one of following conditions satisfies

$$|\beta^{(i)}(s, y, z, \zeta) - \beta^{(i)}(s, y', z', \zeta')| \leq K(|y - y'| + |z - z'|) + \int_F C(s, u)|\zeta(u) - \zeta'(u)|\nu(du),$$

where $0 \leq C(s, u) \leq 1$ and $\int_F |C(s, u)|^2 \nu(du) \leq K$ for any $s \in [0, T]$.

In the sequel of this section, we will show that comparison theorem still hold for the 1/2-Hölder continuous case which is studied in He et al. (2014). For $i = 1, 2$, suppose $(Y_t^{(i)}, Z_t^{(i)}, \zeta_t^{(i)})$ is a solution to the following BDSDE:

$$\begin{aligned}Y_t^{(i)} &= Y_T^{(i)} + \int_t^T \beta^{(i)}(s, Y_s^{(i)}, \zeta_s^{(i)})ds + \int_t^T \int_E \sigma(s, Y_s^{(i)}, Z_s^{(i)}, u)W^T(\overleftarrow{ds}, du) \\ &\quad + \int_{t-}^{T-} \int_{U_0} g_0(s, Y_s^{(i)}, u)\tilde{N}_0^T(\overleftarrow{ds}, du) + \int_{t-}^{T-} \int_{U_1} g_1(s, Y_s^{(i)}, u)N_1^T(\overleftarrow{ds}, du) \\ &\quad - \int_t^T Z_s^{(i)}dB_s - \int_t^T \int_F \zeta_s^{(i)}(u)\tilde{M}(ds, du).\end{aligned}\tag{4.7}$$

Theorem 4.5 Suppose

- (1) $\beta^{(1)}(s, y, \zeta) \leq \beta^{(2)}(s, y, \zeta)$ a.s..
- (2) both $y + g_0(s, y, u)$ and $y + g_1(s, y, u)$ are nondecreasing with respect to y .
- (3) For any $s \in [0, T]$ and $(y, z), (y', z') \in \mathbb{R}^2$,

$$\int_{U_0} |g_0(s, y, u) - g_0(s, y', u)|^2 \mu_0(du) + \int_{U_1} |g_1(s, y, u) - g_1(s, y', u)|\mu_1(du) \leq C|y - y'|$$

and

$$\int_E |\sigma(s, y, z, u) - \sigma(s, y', z', u)|^2 \pi(du) \leq C|y - y'| + \alpha|z - z'|^2$$

where $C > 0$ and $0 \leq \alpha \leq 1$.

(4) there exists a constant $K > 0$ such that one of the following conditions is satisfied:

(a) $\beta^{(1)}(s, y, \zeta) = h^{(1)}(s, y) + \int_F C^{(1)}(s, u) \zeta_s(u) \nu(du)$ with

$$|h^{(1)}(s, y) - h^{(1)}(s, y')| \leq K|y - y'|,$$

where $C^{(1)}(s, u) \in [-1, 0]$ and $\int_F |C^{(1)}(s, u)| \nu(du) \leq K$ for any $s \in [0, T]$.

(b) $\beta^{(2)}(s, y, \zeta) = h^{(2)}(s, y) + \int_F C^{(2)}(s, u) \zeta_s(u) \nu(du)$ with

$$|h^{(2)}(s, y) - h^{(2)}(s, y')| \leq K|y - y'|,$$

where $C^{(2)}(s, u) \in [0, 1]$ and $\int_F |C^{(2)}(s, u)| \nu(du) \leq K$ for any $s \in [0, T]$.

If $Y_T^{(1)} \leq Y_T^{(2)}$ a.s., we have $\mathbf{P}(Y_t^{(1)} \leq Y_t^{(2)} : t \in [0, T]) = 1$.

Proof. Here we just prove this theorem under condition (a) holds. Moreover like the proof of Theorem 4.3, we assume $\nu(F) < \infty$. Let $(\bar{Y}_t, \bar{Z}_t, \bar{\zeta}_t(u)) = (Y_s^{(1)} - Y_s^{(2)}, Z_s^{(1)} - Z_s^{(2)}, \zeta_s^{(1)} - \zeta_s^{(2)})$. For any $n \geq 0$, recall $g_n(z)$ defined in the proof of Theorem 4.3, let

$$f_n(z) = \int_0^z dy \int_0^y g_n(x) dx, \quad z \in \mathbb{R}.$$

It is easy to see that

(a) $f_n(z) \rightarrow z^+$ increasingly.

(b) $f'_n(z) = \begin{cases} \int_0^{z^+} g_n(x) dx \leq 1, & z > 0; \\ 0, & z \leq 0 \end{cases}$ and $\lim_{n \rightarrow \infty} f'_n(z) \rightarrow \mathbf{1}_{\{z > 0\}}$.

(c) $zf''_n(z) = \begin{cases} zg_n(z) \leq 2/n, & z > 0; \\ 0, & z \leq 0. \end{cases}$

(d) For any $az \geq 0$,

$$|f_n(a+z) - f_n(a)| = \left| \int_{a^+}^{(a+z)^+} dy \int_0^y g_n(x) dx \right| \leq |z| \mathbf{1}_{\{a > 0\}}$$

and

$$|f_n(a+z) - f_n(a) - zf'_n(a)| \leq \frac{1}{n} z^2.$$

By Proposition 2.1, since $\bar{Y}_T \leq 0$ a.s., we have

$$\begin{aligned} f_n(\bar{Y}_t) &= \int_t^T f'_n(\bar{Y}_s) \tilde{\beta}(s) ds + \int_t^T \int_E f'_n(\bar{Y}_s) \bar{\sigma}(s, u) W^T(\overleftarrow{ds}, du) \\ &\quad + \frac{1}{2} \int_t^T ds \int_E f''_n(\bar{Y}_s) |\bar{\sigma}(s, u)|^2 \pi(du) \\ &\quad + \int_{t-}^{T-} \int_{U_0} [f_n(\bar{Y}_s + \bar{g}_0(s, u)) - f_n(\bar{Y}_s)] \tilde{N}_0^T(\overleftarrow{ds}, du) \\ &\quad + \int_t^T ds \int_{U_0} [f_n(\bar{Y}_s + \bar{g}_0(s, u)) - f_n(\bar{Y}_s) - f'_n(\bar{Y}_s) \bar{g}_0(s, u)] \mu_0(du) \end{aligned}$$

$$\begin{aligned}
 & + \int_{t-}^{T-} \int_{U_1} [f_n(\bar{Y}_s + \bar{g}_1(s, u)) - f_n(\bar{Y}_s)] N_1^T(\bar{ds}, du) \\
 & - \int_t^T f'_n(\bar{Y}_s) \bar{Z}_s dB_s - \frac{1}{2} \int_t^T f''_n(\bar{Y}_s) |\bar{Z}_s|^2 ds \\
 & - \int_t^T \int_F [f_n(\bar{Y}_s + \bar{\zeta}_s(u)) - f_n(\bar{Y}_s)] \tilde{M}(ds, du) \\
 & - \int_t^T \int_F [f_n(\bar{Y}_s + \bar{\zeta}_s(u)) - f_n(\bar{Y}_s) - f'_n(\bar{Y}_s) \bar{\zeta}_s(u)] \nu(du) ds,
 \end{aligned}$$

where $\tilde{\beta}(s) = \beta^{(1)}(s, Y_s^{(1)}, \zeta_s^{(1)}) - \beta^{(2)}(s, Y_s^{(2)}, \zeta_s^{(2)})$ and $\bar{\sigma}(s, u)$, $\bar{g}_0(s, u)$, $\bar{g}_1(s, u)$ are defined like before. Since $f'_n(z), f''_n(z) \geq 0$ and $\beta^{(1)}(s, y, z) \leq \beta^{(2)}(s, y, z)$, we have

$$\begin{aligned}
 f_n(\bar{Y}_t) & \leq \int_t^T f'_n(\bar{Y}_s) \bar{\beta}(s) ds + \int_t^T \int_E f'_n(\bar{Y}_s) \bar{\sigma}(s, u) W^T(\bar{ds}, du) \\
 & + \frac{1}{2} \int_t^T f''_n(\bar{Y}_s) (C|\bar{Y}_s| + \alpha|\bar{Z}_s|^2) ds \\
 & + \int_t^T \int_{U_0} [f_n(\bar{Y}_s + \bar{g}_0(s, u)) - f_n(\bar{Y}_s)] \tilde{N}_0^T(\bar{ds}, du) \\
 & + \int_t^T ds \int_{U_0} [f_n(\bar{Y}_s + \bar{g}_0(s, u)) - f_n(\bar{Y}_s) - f'_n(\bar{Y}_s) \bar{g}_0(s, u)] \mu_0(du) \\
 & + \int_t^T \int_{U_1} [f_n(\bar{Y}_s + \bar{g}_1(s, u)) - f_n(\bar{Y}_s)] N_1^T(\bar{ds}, du) \\
 & - \int_t^T f'_n(\bar{Y}_s) \bar{Z}_s dB_s - \frac{1}{2} \int_t^T f''_n(\bar{Y}_s) |\bar{Z}_s|^2 ds \\
 & - \int_t^T \int_F [f_n(\bar{Y}_s + \bar{\zeta}_s(u)) - f_n(\bar{Y}_s)] \tilde{M}(ds, du) \\
 & - \int_t^T ds \int_F [f_n(\bar{Y}_s + \bar{\zeta}_s(u)) - f_n(\bar{Y}_s) - f'_n(\bar{Y}_s) \bar{\zeta}_s(u)] \nu(du),
 \end{aligned}$$

where

$$\bar{\beta}(s) = \beta^{(1)}(s, Y_s^{(1)}, \zeta_s^{(1)}) - \beta^{(1)}(s, Y_s^{(2)}, \zeta_s^{(2)}).$$

Taking the expectation, since $\alpha < 1$, we have

$$\begin{aligned}
 \mathbf{E}[f_n(\bar{Y}_t)] & \leq \int_t^T \mathbf{E}[f'_n(\bar{Y}_s) \bar{\beta}(s)] ds + \frac{C}{2} \int_t^T \mathbf{E}[f''_n(\bar{Y}_s) |\bar{Y}_s|] ds \\
 & + \int_t^T ds \int_{U_0} \mathbf{E}[f_n(\bar{Y}_s + \bar{g}_0(s, u)) - f_n(\bar{Y}_s) - f'_n(\bar{Y}_s) \bar{g}_0(s, u)] \mu_0(du) \\
 & + \int_t^T ds \int_{U_1} \mathbf{E}[f_n(\bar{Y}_s + \bar{g}_1(s, u)) - f_n(\bar{Y}_s)] \mu_1(du) \\
 & - \int_t^T ds \int_F \mathbf{E}[f_n(\bar{Y}_s + \bar{\zeta}_s(u)) - f_n(\bar{Y}_s) - f'_n(\bar{Y}_s) \bar{\zeta}_s(u)] \nu(du).
 \end{aligned}$$

By the assumption of this theorem and (c), we have

$$f''_n(\bar{Y}_s) |\bar{Y}_s| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Moreover, from Lemma 3.1 in Li and Pu (2012),

$$\int_{U_0} [f_n(\bar{Y}_s + \bar{g}_0(s, u)) - f_n(\bar{Y}_s) - f'_n(\bar{Y}_s) \bar{g}_0(s, u)] \mu_0(du) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Thus by Fatou's lemma, we have as $n \rightarrow \infty$,

$$\begin{aligned} \mathbf{E}[\bar{Y}_t^+] &\leq \int_t^T \mathbf{E}[\bar{\beta}_1(s) \mathbf{1}_{\{\bar{Y}_s > 0\}}] ds + \int_t^T ds \int_{U_1} \mathbf{E}[(\bar{Y}_s + \bar{g}_1(s, u))^+ - \bar{Y}_s^+] \mu_1(du) \\ &\quad - \int_t^T ds \int_F \mathbf{E}[(\bar{Y}_s + \bar{\zeta}_s(u))^+ - \bar{Y}_s^+ - \bar{\zeta}_s(u) \mathbf{1}_{\{\bar{Y}_s > 0\}}] \nu(du). \end{aligned} \quad (4.8)$$

Now let us discuss the integrand of (4.8) denoted by η :

$$\eta = \begin{cases} (\bar{Y}_s + \bar{\zeta}(s, u))^+ \geq 0, & \text{if } \bar{Y}_s \leq 0; \\ 0, & \text{if } \bar{Y}_s > 0 \text{ and } \bar{\zeta}(s, u) \geq -\bar{Y}_s; \\ -\bar{Y}_s^+ - \bar{\zeta}_s(u) \mathbf{1}_{\{\bar{Y}_s > 0\}}, & \text{if } \bar{Y}_s > 0 \text{ and } \bar{\zeta}(s, u) < -\bar{Y}_s. \end{cases} \quad (4.9)$$

Moreover, since $-1 \leq C(s, u) \leq 0$,

$$\begin{aligned} \int_F C(s, u) \bar{\zeta}_s(u) \nu(du) &= \int_F C(s, u) \bar{\zeta}_s(u) \mathbf{1}_{\{\bar{\zeta}(s, u) \geq 0\}} \nu(du) \\ &\quad + \int_F C(s, u) \bar{\zeta}_s(u) \mathbf{1}_{\{-\bar{Y}_s \leq \bar{\zeta}(s, u) < 0\}} \nu(du) \\ &\quad + \int_F C(s, u) \bar{\zeta}_s(u) \mathbf{1}_{\{\bar{\zeta}(s, u) < -\bar{Y}_s < 0\}} \nu(du) \\ &\leq |\bar{Y}_s| \int_F |C(s, u)| \nu(du) \\ &\quad + \int_F C(s, u) \bar{\zeta}_s(u) \mathbf{1}_{\{\bar{\zeta}(s, u) < -\bar{Y}_s < 0\}} \nu(du) \\ &\leq K|\bar{Y}_s| + \int_F C(s, u) \bar{\zeta}_s(u) \mathbf{1}_{\{\bar{\zeta}(s, u) < -\bar{Y}_s < 0\}} \nu(du). \end{aligned}$$

From this, (4.8) and conditions in this theorem we have

$$\begin{aligned} \mathbf{E}[\bar{Y}_t^+] &\leq [2K + C + \nu(F)] \int_t^T \mathbf{E}[\bar{Y}_s^+] ds + \int_F [C(s, u) + 1] \bar{\zeta}_s(u) \mathbf{1}_{\{\bar{\zeta}(s, u) < -\bar{Y}_s < 0\}} \nu(du) \\ &\leq [2K + C + \nu(F)] \int_t^T \mathbf{E}[\bar{Y}_s^+] ds. \end{aligned}$$

By Gronwall's inequality, we have $\mathbf{E}[\bar{Y}_t^+] = 0$ and $\mathbf{P}(Y_t^{(1)} \leq Y_t^{(2)}) = 1$ for any $t \in [0, T]$. Like the proof of Theorem 2.5, we will get the desired result. \square

5 Maximum and Minimum Solutions

In the proof of Theorem 3.1, Picard iteration argument seriously depends on the Lipschitz condition. Actually, sometimes solutions still exist (maybe not unique), even the drift term is linear increasing which is much weaker than Lipschitz condition. As a simple application of comparison theorems, in this section we will prove the existence of solution to (4.5) under some weak conditions.

Theorem 5.1 *Suppose conditions (2), (3) in Theorem 4.3 holds and there exists a constant $K > 0$ such that one of the following conditions holds:*

- (a) $\beta(s, y, z, \zeta) = h(s, y, z) + \int_F C(s, u) \zeta_s(u) \nu(du)$ and $|h(s, y, z)| \leq K(1 + |y| + |z|)$, where $C(s, u) \in (-\infty, 1]$ (or $C(s, u) \in [-1, \infty)$) and $\int_F |C(s, u)|^2 \nu(du) \leq K$ for any $s \in [0, T]$.

(b) $|\beta(s, y, z, \zeta)| \leq K(1 + |y| + |z| + \|\zeta\|_{\mathcal{L}^2(F)})$ and for any y, z

$$|\beta(s, y, z, \zeta) - \beta(s, y, z, \zeta')| \leq \int_F C(s, u) |\zeta_s(u) - \zeta'_s(u)| \nu(du),$$

where $C(s, u) \in [0, 1]$ and $\sup_{s \in [0, T]} \int_F |C(s, u)|^2 \nu(du) \leq K$ a.s.

Then solutions to (4.5) exist. Moreover, there exist two solutions $(Y_t^I, Z_t^I, \zeta_t^I)$ and $(Y_t^S, Z_t^S, \zeta_t^S)$ such that for any solution (Y_t, Z_t, ζ_t) to (4.1) have

$$\mathbf{P}(Y_t^I \leq Y_t \leq Y_t^S : t \in [0, T]) = 1.$$

Before using comparison theorem to prove this theorem, we need to construct a suitable sequence of BDSDEs with solutions exist and satisfy the conditions of comparison theorems; see the following lemma. Since the proof is easy and similar to Lemma 1 in Lepeltier and Martin (1997), we will omit it.

Lemma 5.2 For $n \geq K$, let

$$\beta_n^I(s, y, z, \zeta) = \inf_{y', z' \in \mathbb{R}^2} \left\{ \beta(s, y', z', \zeta) + n|y - y'| + n|z - z'| \right\}$$

and

$$\beta_n^S(s, y, z, \zeta) = \min \left\{ \beta(s, y, z, \zeta) + K, \sup_{y', z' \in \mathbb{R}^2} \left\{ \beta(s, y', z', \zeta) - n|y - y'| - n|z - z'| \right\} \right\}.$$

Then $\beta_n^I(s, y, z, \zeta)$ and $\beta_n^S(s, y, z, \zeta)$ are \mathcal{F}_t^r -progressive and satisfy:

(1) For any $n \geq K$, $\beta_n^I(s, y, z, \zeta) \leq \beta_{n+1}^I(s, y, z, \zeta) \leq \beta(s, y, z, \zeta)$ and

$$\beta(s, y, z, \zeta) \leq \beta_{(n+1)}^S(s, y, z, \zeta) \leq \beta_n^S(s, y, z, \zeta) \leq \beta(s, y, z, \zeta) + K.$$

(2) If $\beta(s, y, z, \zeta)$ satisfies (a)(or (b)) in Theorem 5.1, then so do $\beta_n^I(s, y, z, \zeta)$ and $\beta_n^S(s, y, z, \zeta)$.

(3) For any $(y, z), (y', z') \in \mathbb{R}^2$ have

$$|\beta_n^I(s, y, z, \zeta) - \beta_n^I(s, y', z', \zeta)| \leq n(|y - y'| + |z - z'|),$$

$$|\beta_n^S(s, y, z, \zeta) - \beta_n^S(s, y', z', \zeta)| \leq n(|y - y'| + |z - z'|).$$

(4) if $(y_n, z_n, \zeta_n) \rightarrow (y, z, \zeta)$, then $\beta_n^I(s, y_n, z_n, \zeta_n)$ and $\beta_n^S(s, y_n, z_n, \zeta_n)$ converge to $\beta(s, y, z, \zeta)$.

From Theorem 2.5 and 3.1, there exist unique solutions to (4.5) with β replaced by β_n^I and β_n^S respectively, denoted by $(Y_n^I(t), Z_n^I(t), \zeta_n^I(t))$ and $(Y_n^S(t), Z_n^S(t), \zeta_n^S(t))$. According to this lemma and Theorem 4.3, for any $n \geq K$ we have $Y_n^I(t) \leq Y_{n+1}^I(t) \leq Y_{n+1}^S(t) \leq Y_n^S(t)$, which means both $\{Y_n^I(t)\}$ and $\{Y_n^S(t)\}$ are convergent in $\mathcal{L}_{\mathcal{F}, T}^2$. So it suffices to show $(Y_n^I(t), Z_n^I(t), \zeta_n^I(t)) \rightarrow (Y_t^I, Z_t^I, \zeta_t^I)$ and $(Y_n^S(t), Z_n^S(t), \zeta_n^S(t)) \rightarrow (Y_t^S, Z_t^S, \zeta_t^S)$ in $\mathcal{L}_{\mathcal{F}, T}^2 \times \mathcal{L}_{\mathcal{F}, T}^2 \times \mathcal{L}_{\mathcal{F}, T}^2(F)$ as $n \rightarrow \infty$. The key point is to prove for any $t \in [0, T]$,

$$\begin{aligned} \int_t^T \beta_n^I(s, Y_n^I(s), Z_n^I(s), \zeta_n^I(s)) ds &\rightarrow \int_t^T \beta(s, Y_s^I, Z_s^I, \zeta_s^I) ds, \\ \int_t^T \beta_n^S(s, Y_n^S(s), Z_n^S(s), \zeta_n^S(s)) ds &\rightarrow \int_t^T \beta(s, Y_s^S, Z_s^S, \zeta_s^S) ds. \end{aligned}$$

By condition (4) in Lemma 5.2, it suffices to prove $(Y_n^I(t), Z_n^I(t), \zeta_n^I(t))$ and $(Y_n^S(t), Z_n^S(t), \zeta_n^S(t))$ are uniformly bounded on $[0, T]$.

Lemma 5.3 Assume conditions in Theorem 5.1 hold. Then there exists $C > 0$ such that for any $n \geq K$ such that

$$\|Y_n^I\|_{\mathbb{S}_T^2} \vee \|Y_n^S\|_{\mathbb{S}_T^2} \vee \|Z_n^I\|_{\mathcal{L}_T^2} \vee \|Z_n^S\|_{\mathcal{L}_T^2} \vee \|\zeta_n^I\|_{\mathcal{L}_T^2(E)} \vee \|\zeta_n^S\|_{\mathcal{L}_T^2(E)} \leq C.$$

Proof. Here we just prove this lemma with (a) in Theorem 5.1 holds. First let us find a strip to cover all $Y_n^I(t)$ and $Y_n^S(t)$, which is easier to be dealt with. Define

$$\begin{aligned}\beta^*(s, y, z, \zeta) &= K(2 + |y| + |z|) + \int_F C(s, u) \zeta_s(u) \nu(du), \\ \beta_*(s, y, z, \zeta) &= -K(2 + |y| + |z|) + \int_F C(s, u) \zeta_s(u) \nu(du).\end{aligned}$$

Obviously, β^* and β_* satisfy (2.3) in Condition 2.4. From Theorem 2.5 and 3.1, solutions to (4.1) uniquely exist with β replaced by β^* and β_* respectively, denoted by $(Y_t^*, Z_t^*, \zeta_t^*)$ and $(Y_{*t}, Z_{*t}, \zeta_{*t})$. Moreover, we have $\beta_*(t, y, z, \zeta) \leq \beta_n^I(t, y, z, \zeta) \leq \beta_n^S(t, y, z, \zeta) \leq \beta^*(t, y, z, \zeta)$ for any $n \geq K$. So $Y_{*t} \leq Y_n^I(t) \leq Y_n^S(t) \leq Y_t^*$. It suffices to prove $\|Y^*\|_{\mathbb{S}_T^2} \vee \|Y^*\|_{\mathbb{S}_T^2} \leq D$. By Proposition 2.1,

$$\begin{aligned}& |Y_t^*|^2 + \int_t^T |Z_s^*|^2 ds + \int_t^T \|\zeta_s^*\|_{\mathcal{L}^2(F)}^2 ds = \\& |Y_T|^2 + 2K \int_t^T Y_s^* (2 + |Y_s^*| + |Z_s^*|) ds + 2 \int_t^T Y_s^* ds \int_F C(s, u) \zeta_s^*(u) \nu(du) \\& + 2 \int_t^T \int_E Y_s^* \sigma(s, Y_s^*, Z_s^*, u) W^T(\overleftarrow{ds}, du) + \int_t^T \|\sigma(s, Y_s^*, Z_s^*, \cdot)\|_{\mathcal{L}^2(E)}^2 ds \\& + \int_{t-}^{T-} \int_{U_0} [|Y_s^* + g_0(s, Y_s^*, u)|^2 - |Y_s^*|^2] \tilde{N}_0^T(\overleftarrow{ds}, du) + \int_t^T \|g_0(s, Y_s^*, \cdot)\|_{\mathcal{L}^2(U_0)}^2 ds \\& + \int_{t-}^{T-} \int_{U_1} [|Y_s^* + g_1(s, Y_s^*, u)|^2 - |Y_s^*|^2] \tilde{N}_1^T(\overleftarrow{ds}, du) \\& - 2 \int_t^T Y_s^* Z_s^* dB(s) - \int_t^T \int_F [|Y_s^* + \zeta_s^*(u)|^2 - |Y_s^*|^2] \tilde{M}(ds, du) \\& \leq |Y_T|^2 + 2T + (3K + K/a + 1/b + \mu_1(U_1)) \int_t^T |Y_s^*|^2 ds + a \int_t^T |Z_s^*|^2 ds \\& + bK \int_t^T \|\zeta_s^*\|_{\mathcal{L}^2(F)}^2 ds + 2 \int_t^T \int_E Y_s^* \sigma(s, Y_s^*, Z_s^*, u) W^T(\overleftarrow{ds}, du) \\& + (1 + c) \int_t^T [C|Y_s^*|^2 + \alpha|Z_s^*|^2] ds + (1 + 1/c) \int_t^T \|\sigma(s, 0, 0, \cdot)\|_{\mathcal{L}^2(E)}^2 ds \\& + 2 \int_t^T \|g_0(s, Y_s^*, \cdot) - g_0(s, 0, \cdot)\|_{\mathcal{L}^2(U_0)}^2 ds + 2 \int_t^T \|g_0(s, 0, \cdot)\|_{\mathcal{L}^2(U_0)}^2 ds \\& + 4 \int_t^T \|g_1(s, Y_s^*, \cdot) - g_1(s, 0, \cdot)\|_{\mathcal{L}^2(U_1)}^2 ds + 4 \int_t^T \|g_1(s, 0, \cdot)\|_{\mathcal{L}^2(U_1)}^2 ds \\& + \int_{t-}^{T-} \int_{U_0} [|Y_s^* + g_0(s, Y_s^*, u)|^2 - |Y_s^*|^2] \tilde{N}_0^T(\overleftarrow{ds}, du) \\& + \int_{t-}^{T-} \int_{U_1} [|Y_s^* + g_1(s, Y_s^*, u)|^2 - |Y_s^*|^2] \tilde{N}_1^T(\overleftarrow{ds}, du) \\& - 2 \int_t^T Y_s^* Z_s^* dB(s) - \int_t^T \int_F [|Y_s^* + \zeta_s^*(u)|^2 - |Y_s^*|^2] \tilde{M}(ds, du).\end{aligned}$$

Apply Burkholder-Davis-Gundy Inequality to this formula, for example, there are some constants $A, B > 0$ such that

$$\mathbf{E} \left[\sup_{t \in [0, T]} \int_t^T \int_{U_0} [|Y_s^* + g_0(s, Y_s^*, u)|^2 - |Y_s^*|^2] \tilde{N}_0^T(\overleftarrow{ds}, du) \right]$$

$$\begin{aligned}
 &\leq \mathbf{A}\mathbf{E}\left[2\left[\int_0^T |Y_s^*|^2 \|g_0(s, Y_s^*, \cdot)\|_{\mathcal{L}^2(U_0)}^2 ds\right]^{1/2}\right] + \mathbf{B}\mathbf{E}\left[\int_t^T \|g_0(s, Y_s^*, \cdot)\|_{\mathcal{L}^2(U_0)}^2 ds\right] \\
 &\leq \mathbf{A}\mathbf{E}\left[2\left[\sup_{t \in [0, T]} |Y_s^*|^2\right]^{1/2}\left[\int_0^T \|g_0(s, Y_s^*, \cdot)\|_{\mathcal{L}^2(U_0)}^2 ds\right]^{1/2}\right] + \mathbf{B}\mathbf{E}\left[\int_t^T \|g_0(s, Y_s^*, \cdot)\|_{\mathcal{L}^2(U_0)}^2 ds\right] \\
 &\leq d\|Y^*\|_{\mathbb{S}_T^2}^2 + (A/d + B)\mathbf{E}\left[\int_t^T \|g_0(s, Y_s^*, \cdot)\|_{\mathcal{L}^2(U_0)}^2 ds\right].
 \end{aligned}$$

The last inequality above comes from Cauchy's inequality. Like the proof before and choose d small enough, we have

$$C_0\|Y^*\|_{\mathbb{S}_T^2}^2 + \|Z^*\|_{\mathcal{L}_T^2}^2 + \|\zeta^*\|_{\mathcal{L}_T^2(F)}^2 \leq C_1 + C_2\|Y^*\|_{\mathcal{L}_T^2}^2 + C_3[\|Z^*\|_{\mathcal{L}_T^2}^2 + \|\zeta^*\|_{\mathcal{L}_T^2(F)}^2],$$

where $C_1, C_2, C_3 > 0$ and $C_0 \in (0, 1)$. Since $(Y_t^*, Z_t^*, \zeta_t^*) \in \mathcal{L}_{\mathcal{F}, T}^2 \times \mathcal{L}_{\mathcal{F}, T}^2 \times \mathcal{L}_{\mathcal{F}, T}^2(F)$, so there exists $C > 0$ such that $\|Y^*\|_{\mathbb{S}_T^2}^2 \leq C$. Similarly, $\|Y^*\|_{\mathbb{S}_T^2}^2 \leq C$ also can be proved. Here we have proved the first part of this lemma. For the second part, we just prove $\|Z_n^I\|_{\mathcal{L}_T^2} \vee \|\zeta_n^I\|_{\mathcal{L}_T^2} \leq C$, others are similar. Apply the Itô-Pardoux-Peng formula to $|Y_n^I(t)|^2$ and like the proof of Theorem 2.5, we have

$$\mathbf{E}[|Y_n^I(0)|^2] + a\|Y_n^I\|_{\mathcal{L}_T^2}^2 + b\|\zeta_n^I\|_{\mathcal{L}_T^2(F)}^2 \leq C_0 + C_1\|Y_n^I\|_{\mathcal{L}_T^2}^2 \leq C,$$

where $a, b \in (0, 1)$ and $C_0, C_1 > 0$ independent to n . Here we have finished the proof. \square

Since we have showed $\{Y_n^I(t)\}$ and $\{Y_n^S(t)\}$ converge, it suffices to identify (Z_n^I, ζ_n^I) and (Z_n^S, ζ_n^S) are Cauchy sequences in $\mathcal{L}_{\mathcal{F}, T}^2 \times \mathcal{L}_{\mathcal{F}, T}^2(F)$.

Lemma 5.4 Assume the conditions in Theorem 5.1 holds, then both (Z_n^I, ζ_n^I) and (Z_n^S, ζ_n^S) are convergent in $\mathcal{L}_{\mathcal{F}, T}^2 \times \mathcal{L}_{\mathcal{F}, T}^2(F)$.

Proof. Like before we just prove (Z_n^S, ζ_n^S) converges with condition (a) in Theorem 5.1 holds. For any $n > m > K$, let $(Y_{n,m}^S(t), Z_{n,m}^S(t), \zeta_{n,m}^S(t, u)) = (Y_n^S(t) - Y_m^S(t), Z_n^S(t) - Z_m^S(t), \zeta_n^S(t, u) - \zeta_m^S(t, u))$ which satisfies

$$\begin{aligned}
 Y_{n,m}^S(t) &= \int_t^T \bar{\beta}^{(n,m)}(s) ds + \int_t^T \int_E \bar{\sigma}^{(n,m)}(s, u) W^T(\bar{ds}, du) \\
 &\quad + \int_t^T \int_{U_0} \bar{g}_0^{(n,m)}(s, u) \tilde{N}_0^T(\bar{ds}, du) + \int_t^T \int_{U_1} \bar{g}_1^{(n,m)}(s, u) \tilde{N}_1^T(\bar{ds}, du) \\
 &\quad - \int_t^T Z_{n,m}^S(s) dB_s - \int_t^T \int_F \zeta_{n,m}^S(s, u) \tilde{M}(ds, du),
 \end{aligned}$$

where $\bar{\beta}^{(n,m)}(s)$, $\bar{\sigma}^{(n,m)}(s, u)$, $\bar{g}_0^{(n,m)}(s, u)$ and $\bar{g}_1^{(n,m)}(s, u)$ are defined like before.

By Proposition 2.1 and taking the expectation, we have

$$\|Z_{n,m}^S\|_{\mathcal{L}_T^2}^2 + \|\zeta_{n,m}^S\|_{\mathcal{L}_T^2}^2 \leq \mathbf{E}\left[2\int_0^T Y_{n,m}^S(s) \bar{\beta}^{(n,m)}(s) ds\right] + (3C + \mu_1(U_1))\|Y_{n,m}^S\|_{\mathcal{L}_T^2}^2 + \alpha\|Z_{n,m}^S\|_{\mathcal{L}_T^2}^2.$$

From Lemma 5.3 we have $\sup_{s \in [0, T]} \mathbf{E}[|\bar{\beta}_{n,m}^S(s)|^2] < \infty$ and

$$\mathbf{E}\left[2\int_0^T Y_{n,m}^S(s) \bar{\beta}^{(n,m)}(s) ds\right] \leq 2\|Y_{n,m}^S\|_{\mathcal{L}_T^2}^2 \left(\mathbf{E}\left[\int_0^T |\bar{\beta}^{(n,m)}(s)|^2 ds\right]\right)^{1/2} \leq 2\sqrt{CT}\|Y_{n,m}^S\|_{\mathcal{L}_{\mathcal{F}, T}^2},$$

where $C > 0$ is independent to n and m . By Hölder inequality,

$$(1 - \alpha)\|Z_{n,m}^S\|_{\mathcal{L}_{\mathcal{F}, T}^2}^2 + \|\zeta_{n,m}^S\|_{\mathcal{L}_{\mathcal{F}, T}^2(F)}^2 \leq 2\sqrt{CT}\|Y_{n,m}^S\|_{\mathcal{L}_{\mathcal{F}, T}^2} + (3C + \mu_1(U_1))\|Y_{n,m}^S\|_{\mathcal{L}_{\mathcal{F}, T}^2}^2$$

Since Y_n^S is a Cauchy sequence, so are Z_n^I and ζ_n^I . \square

Proof of Theorem 5.1. With the preparations of Lemma 5.2, 5.3 and 5.4, like the classical proof in SDE theory, we can get the desired result (omit the details). \square

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