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Closed convex sets of Minkowski type

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ABSTRACT

In this paper we provide several characterizations of Minkowski sets, i.e. closed, possibly unbounded, convex sets which are representable as the convex hulls of their sets of extreme points. The equality between the relative boundary of a closed convex set containing no lines and its Pareto-like associated set ensures the Minkowski property of the set. In two dimensions this equality characterizes the Minkowski sets containing no lines.

1. Introduction

It is well-known that every closed convex subset of $\mathbb{R}^n$ which is not an affine variety or a closed half of an affine variety (*flat* or a *closed halfflat* in the terminology of [13, pp. 83, 84]) is the convex hull of its relative boundary [12, Theorem 18.4, p. 166]. In fact the relative boundary of the closed convex subsets of $\mathbb{R}^n$ can be decreased, sometimes properly, to some of its subsets which still elongate the given set. Indeed, every closed convex set is the convex hull of its primitive faces, i.e. the convex hull of its faces which are affine varieties or closed half of affine varieties (flats or closed halfflats) [13, Theorem 2.6.13, Corollary 2.6.14, p. 85]. If we restrict ourselves to the closed convex sets that contain no lines, such sets are representable as the convex hulls of their extreme points and their extreme half-lines [12, Theorem 18.7, p. 168], [13, Corollary 2.6.15, p. 86]. Going one step further towards compactness, every compact convex subset C of $\mathbb{R}^n$ can be represented

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as $C = \text{conv ext } C$, where ext C stands for the set of extreme points of C, as the compact sets do not contain half-lines. In fact this is Minkowski's theorem which was first proved by Minkowski in [10] (see also [2, Theorem 2.7.2]). In the infinite dimensional context a result of a similar flavour, due to Krein–Milman, works. Indeed, a compact subset K of a locally convex topological vector space can be represented as $K = \text{cl conv ext } K$ (see e.g. [1, Theorem III.4.1]). An interesting application of the Krein–Milman theorem shows that $L^1(0,1)$ is not isometric to the dual of a normed space as its unit ball has no extreme points at all (see e.g. [3, pp. 168, 169]).

In this paper we are interested about closed, possibly unbounded, sets which can be represented as the convex hull of their extreme points.

Definition 1.1. A closed convex subset C of $\mathbb{R}^n$ is said to be a *Minkowski set* if $C = \text{conv ext } C$.

We will use the following characterization of convexity in $\mathbb{R}^n$:

Theorem 1.2. *[7, p. 258] (see also [9, Theorem 1.1]). A subset C of $\mathbb{R}^n$ is convex if and only if for every $x \in \mathbb{R}^n \setminus C$ there exists an $n \times n$ matrix A_x such that $A_x y <_L A_x x$ for all $y \in C$.*

Theorem 1.3. *A closed convex subset C of $\mathbb{R}^n$ is a Minkowski subset of $\mathbb{R}^n$ if and only if $C = \text{cl conv ext } C$.*

Proof. The only if statement is obvious. To prove the opposite implication, assume that $C \neq \text{conv ext } C$, i.e. $\text{conv ext } C \subsetneq C$, as the inclusion $\text{conv ext } C \subseteq C$ is obvious, and consider $\bar{x} \in C \setminus \text{conv ext } C$ and the set $F := \{x \in C \,|\, Ax = A\bar{x}\}$, where A is the matrix $A_{\bar{x}}$ given by Theorem 1.2. Recall that F is a (nonempty) closed face of C (see [8, Theorem 2]) and $\text{ext } F \neq \emptyset$ since F contains no lines (otherwise ext C would be empty, and hence the set $C = \text{conv ext } C$ would be empty too). On the other hand $F \cap \text{conv ext } C = \emptyset$ as $Ay <_L A\bar{x} = Ax$ for all $x \in F$ and all $y \in \text{conv ext } C = \emptyset$. Consequently $\text{ext } F = \text{ext } F \cap \text{ext } C \subseteq \text{ext } F \cap \text{conv ext } C \subseteq F \cap \text{conv ext } C = \emptyset$, which shows that $\text{ext } F = \emptyset$, which is absurd. $\square$

We present here several characterizations of Minkowski sets. One of the characterizations provides a proof for the Minkowski theorem. The equality between the relative boundary and the Pareto-like set associated with a closed convex set ensures the Minkowski property of the set. Recall that the Pareto-like set associated with an M-decomposable set with no lines is the smallest compact component in the Motzkin decompositions of the given set [5,11].

2. Examples of unbounded Minkowski sets

For unbounded Minkowski sets we rely on epigraphs of some lower semi-continuous convex functions. If $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{\infty\}$ is a given function, we define its *effective domain*, its *graph* and its *epigraph* in the following way:

$$\text{dom}(f) = \{z \in \mathbb{R}^n : f(z) < \infty\}$$
$$\text{Graph}(f) = \{(x, f(x)) \ : \ x \in \text{dom}(f)\}$$
$$\text{epi}(f) = \{(x,r) \in \mathbb{R}^n \times \mathbb{R} : r \geq f(x)\}.$$

Note that $\text{Graph}(f) \subset \text{epi}(f)$ and $\pi(\text{epi}(f)) = \pi(\text{Graph}(f)) = \text{dom}(f)$, where $\pi : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ stands for the projection. Recall also that f is lower semi-continuous (on $\mathbb{R}^n$) if and only if its epigraph is closed in $\mathbb{R}^n \times \mathbb{R}$ (see e.g. [6, p. 78]).

Observe that not all epigraphs of lower semi-continuous functions are Minkowski sets, as the example $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $f(x) = \|x\|$ shows. However we shall prove the following:

Proposition 2.1. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}\cup\{\infty\}$ be a lower semi-continuous function whose effective domain* $\mathrm{dom}(f)$ *is open.*

1. *If f is convex and* $\mathrm{dom}(f)$ *is bounded, then* $\mathrm{epi}(f)$ *is a Minkowski set.*
2. *If f is strictly convex on* $\mathrm{dom}(f)$*, then* $\mathrm{epi}(f)$ *is a Minkowski set.*

In order to prove Proposition 2.1, we need the following:

Lemma 2.2. *If $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{\infty\}$ is a lower semi-continuous function whose effective domain* $\mathrm{dom}(f)$ *is open in $\mathbb{R}^n$, then* $\mathrm{bd}\,(\mathrm{epi}(f)) = \mathrm{Graph}(f)$*.*

Proof. We first observe that $\mathrm{Graph}(f) \subseteq \mathrm{epi}(f) = \mathrm{cl}\,(\mathrm{epi}(f)) = \mathrm{int}\,(\mathrm{epi}(f)) \cup \mathrm{bd}\,(\mathrm{epi}(f))$. The inclusion $\mathrm{Graph}(f) \subseteq \mathrm{bd}\,(\mathrm{epi}(f))$ follows due to the relation $\mathrm{Graph}(f) \cap \mathrm{int}\,(\mathrm{epi}(f)) = \emptyset$, i.e.

$$\mathrm{int}\,(\mathrm{epi}(f)) \subseteq \left(\mathbb{R}^{n-1} \times \mathbb{R}\right) \setminus \mathrm{Graph}(f). \tag{1}$$

In order to justify the inclusion (1), we consider $(p, q) \in \mathrm{int}\,(\mathrm{epi}(f))$ and some open set $U \subseteq \mathrm{dom}(f)$ and open interval $I \subseteq \mathbb{R}$ with the properties $(p, q) \in U \times I \subseteq \mathrm{int}\,(\mathrm{epi}(f))$. Consider now $q' \in I$ such that $q' < q$ and observe that $q > q' \geq f(p)$ as $(p, q') \in U \times I \subseteq \mathrm{int}\,(\mathrm{epi}(f))$. Thus $(p, q) \notin \mathrm{Graph}(f)$.

For the opposite inclusion we observe the following relations:

$$\begin{aligned} \mathrm{bd}\,(\mathrm{epi}(f)) &\subseteq \mathrm{cl}\,(\mathrm{epi}(f)) = \mathrm{epi}(f) \\ &= \mathrm{Graph}(f) \cup \{(p, q) \in \mathrm{dom}(f) \times \mathbb{R} \ : \ q > f(p)\} \\ &\subseteq \mathrm{Graph}(f) \cup \mathrm{int}\,(\mathrm{epi}(f))\,; \end{aligned} \tag{2}$$

the latter inclusion follows from

$$\{(p, q) \in \mathrm{dom}(f) \times \mathbb{R} \ : \ q > f(p)\} \subseteq \mathrm{int}\,(\mathrm{epi}(f))\,. \tag{3}$$

The inclusion $\mathrm{bd}\,(\mathrm{epi}(f)) \subseteq \mathrm{Graph}(f) \cup \mathrm{int}\,(\mathrm{epi}(f))$, proved by the relations (2), shows that $\mathrm{bd}\,(\mathrm{epi}(f)) \subseteq \mathrm{Graph}(f)$, as $\mathrm{bd}\,(\mathrm{epi}(f)) \cap \mathrm{int}\,(\mathrm{epi}(f)) = \emptyset$. In order to justify the relation (3), we consider $(p, q) \in \mathrm{dom}(f) \times \mathbb{R}$ such that $q > f(p)$, i.e. $q - \varepsilon > f(p)$ for $\varepsilon > 0$ sufficiently small. The convexity of f ensures its continuity on $\mathrm{dom}(f)$ (see e.g. [4, Corollary 2.3, p. 12]) and therefore the existence of some open neighbourhood $U \subseteq \mathrm{dom}(f)$ of p such that $f(x) > q - \varepsilon$, for all $x \in U$. Thus, the neighbourhood $U \times (q - \varepsilon, +\infty)$ of (p, q) is contained in $\mathrm{epi}(f)$, namely $(p, q) \in \mathrm{int}\,(\mathrm{epi}(f))$. □

Proof of Proposition 2.1. In both cases we only need to show, according to [13, Corollary 2.6.15], that the boundary of $\mathrm{epi}(f)$, i.e. $\mathrm{Graph}(f)$, has no half-lines.

(1) If $h = (a, f(a)) + \{(tp, tq) \ : \ t \geq 0\}$ would be a half-line in $\mathrm{Graph}(f)$, then $(p, q) \neq (0, 0)$ and $f(a) + tq = f(a + tp)$, for all $t \geq 0$. Thus $a + tp \in \mathrm{dom}(f)$ for every $t \geq 0$. This shows that $p = 0$ due to the boundedness of $\mathrm{dom}(f)$ and $f(a) + tq = f(a)$, for all $t \geq 0$, i.e. $q = 0$ as well, a contradiction with the condition $(p, q) \neq (0, 0)$.

(2) The strict convexity of f implies that its graph $\mathrm{Graph}(f) = \mathrm{bd}((\mathrm{epi}(f))$ contains no straight line segments and therefore no half-lines. In fact $\mathrm{Graph}(f) \subseteq \mathrm{ext}\,(\mathrm{epi}(f))$ [6, Fact 5.3.3, p. 239]. On the other hand $\mathrm{ext}\,(\mathrm{epi}(f)) \subseteq \mathrm{bd}((\mathrm{epi}(f)) = \mathrm{Graph}(f)$, namely $\mathrm{Graph}(f) = \mathrm{bd}((\mathrm{epi}(f)) = \mathrm{ext}\,(\mathrm{epi}(f))$ in this case. □

Example 2.3. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{\infty\}$ be a lower semi-continuous function whose effective domain $\operatorname{dom}(f)$ is open. The epigraph $\operatorname{epi}(f)$ of f is a Minkowski set if f is C^1-smooth and the gradient of f is strictly monotone on $\operatorname{dom}(f)$; in particular, if f is C^2-smooth and the Hessian matrices of f at all points of $\operatorname{dom}(f)$ are positive definite. Indeed, in such a case f is strictly convex (see e.g. [6, Theorem 4.1.4]).

3. Characterizations of Minkowski sets

In this section we provide several characterizations of Minkowski sets, most of which involve the facial structure of the set.

Proposition 3.1. *For a closed convex set $C \subseteq \mathbb{R}^n$, the following statements are equivalent:*

1. *C is Minkowski.*
2. *There exists a smallest set $S \subseteq \mathbb{R}^n$ such that* $\operatorname{conv} S = C$.
3. *There exists a minimal set $S \subseteq \mathbb{R}^n$ such that* $\operatorname{conv} S = C$.

In (2) *and* (3), *one has* $S = \operatorname{ext} C$.

Proof. (1) $\Longrightarrow$ (2). Let $S \subseteq \mathbb{R}^n$ be such that $\operatorname{conv} S = C$. By [12, Corollary 18.3.1], we have $\operatorname{ext} C \subseteq \operatorname{ext} \operatorname{conv} S \subseteq S$, which, as C is Minkowski, shows that *ext* C is the smallest set S such that $\operatorname{conv} S = C$.

Implication (2) $\Longrightarrow$ (3) is obvious.

(3) $\Longrightarrow$ (1). Let S be as in (3). By [12, Corollary 18.3.1], we have $\operatorname{ext} C = \operatorname{ext} \operatorname{conv} S \subseteq S$. It will thus suffice to prove the opposite inclusion. Let $x \in S$, and suppose $x \notin \operatorname{ext} C$. Since

$$C = \operatorname{conv} S = \{(1-\lambda)\, y + \lambda x : y \in \operatorname{conv}(S \setminus \{x\}), \lambda \in [0,1]\},$$

there exist $y, z \in \operatorname{conv}(S \setminus \{x\})$, $\lambda, \mu \in [0,1[$ and $\alpha \in]0,1[$ such that $x = (1-\alpha)((1-\lambda)\, y + \lambda x) + \alpha((1-\mu)\, z + \mu x)$. An easy computation shows that

$$x = \frac{(1-\alpha)(1-\lambda)}{(1-\alpha)(1-\lambda) + \alpha(1-\mu)} y + \frac{\alpha(1-\mu)}{(1-\alpha)(1-\lambda) + \alpha(1-\mu)} z \in \operatorname{conv}(S \setminus \{x\}),$$

from which we deduce that

$$C = \operatorname{conv} S = \operatorname{conv}(\operatorname{conv}(S \setminus \{x\}) \cup \{x\}) = \operatorname{conv} \operatorname{conv}(S \setminus \{x\}) = \operatorname{conv}(S \setminus \{x\}),$$

thus contradicting with the minimality of S. This proves the required inclusion $S \subseteq \operatorname{ext} C$. $\square$

The following lemma is an immediate consequence of [12, Theorem 18.2 and Corollary 18.1.3].

Lemma 3.2. *The relative boundary of a closed convex set $\emptyset \neq C \subseteq \mathbb{R}^n$ is the union of the proper faces of C.*

Proposition 3.3. *A closed convex set $C \subseteq \mathbb{R}^n$ of dimension at least two is Minkowski if and only if all of its proper faces are Minkowski.*

Proof. The direct statement is an immediate corollary of [12, Theorem 18.3]. For the opposite statement, denote by $\mathcal{F}(C)$ the collection of all faces of a given closed convex set C. Since $F = \operatorname{conv} \operatorname{ext} F$ for every

$F \in \mathcal{F}^*(C) := \mathcal{F}(C) \setminus \{C\}$, it follows that F is neither an affine variety nor a closed half of such an affine variety and therefore, by [12, Theorem 18.4] and Lemma 3.2, one obtains:

$$\begin{aligned} C &= \text{conv rbd } C = \text{conv}\left(\bigcup\nolimits_{F\in\mathcal{F}^*(C)} F\right) \\ &= \text{conv}\left(\bigcup\nolimits_{F\in\mathcal{F}^*(C)} \text{conv ext } F\right) \\ &= \text{conv}\left(\bigcup\nolimits_{F\in\mathcal{F}^*(C)} \text{ext } F\right). \end{aligned}$$

Since a face of a face of a convex set is a face of the convex set itself [12, p. 163], it follows that the extreme points of all faces $F \in \mathcal{F}^*(C)$ are extreme points of C itself. In fact the union $\bigcup_{F\in\mathcal{F}^*(C)} \text{ext } F$ covers the whole set of extreme points of C, as such extreme points are among the faces in $\mathcal{F}^*(C)$. Thus $C = \text{conv ext } C$. □

Theorem 3.4. *For a closed convex set $\emptyset \neq C \subseteq \mathbb{R}^n$, the following statements are equivalent:*

1. *C is Minkowski.*
2. *C contains no lines, and every one-dimensional face of C is a segment.*
3. *For every $x \in C \setminus \text{ext } C$ there exists a straight line $L \subseteq \mathbb{R}^n$ such that $L \cap C$ is a line segment and $x \in \text{rint}\,(L \cap C)$.*

Proof. (1) $\Longrightarrow$ (2). If $C = \emptyset$, then (ii) clearly holds. If $C \neq \emptyset$, then we have ext $C \neq \emptyset$, and hence C contains no lines (indeed, if C would contain a line L then for every $x \in C$ the line through x parallel to C would be contained in C, which is impossible if x is an extreme point of C). Moreover, by Proposition 3.3, every one-dimensional face of C is a segment, since segments are obviously the only one-dimensional Minkowski sets.

(2) $\Longrightarrow$ (3). Let $x \in C \setminus \text{ext } C$. By [12, Theorem 18.2], we have $x \in \text{rint } G$ for some face G of C. Since $x \notin \text{ext } C$, the dimension of G is at least 1. By [12, Theorem 18.4], the point x lies on some open line segment $]a, b[$ joining two relative boundary points of G. Let L be the straight line that contains this segment. We obviously have $[a, b] \subseteq L \cap C$. We will now prove that this inclusion actually holds as an equality. Assume, to the contrary, that there exists some $y \in L \cap C \setminus [a, b]$. Without loss of generality, we can assume that $a \in]y, x[$. Since G is a face of C, we have $y \in G$. But then, by [12, Theorem 6.1], it turns out that $a \in \text{rint } G$, which is a contradiction. We thus have $L \cap C = [a, b]$ and $x \in]a, b[= \text{rint}\,(L \cap C)$.

(3) $\Longrightarrow$ (1). Statement (i) clearly holds if C is either empty or a singleton, so we will assume that p, the dimension of C, is at least 1, and we will proceed by induction on p. If $p = 1$, statement (3) clearly implies that C is a line segment, and hence it is Minkowski. Assume that $p > 1$, and let $x \in C$. If $x \in \text{ext } C$, then obviously $x \in \text{conv ext } C$. If $x \notin C$, then, by (3), there exists a straight line $L \subseteq \mathbb{R}^n$ such that $L \cap C$ is a line segment $[a, b]$ and $x \in \text{rint}\,(L \cap C) =]a, b[$. Clearly, $a, b \in \text{rbd } C$, since otherwise we could easily prove the nonemptiness of $L \cap C \setminus [a, b]$, thus obtaining a contradiction. Hence, by [12, Theorem 11.6], there exist non-trivial supporting hyperplanes H_1 and H_2 to C containing a and b, respectively. By [12, Corollary 18.1.3], the dimension of the exposed faces $C \cap H_1$ and $C \cap H_2$ are strictly smaller than p; hence, by the induction hypothesis, $C \cap H_1$ and $C \cap H_2$ are Minkowski, so that

$$\begin{aligned} x \in [a, b] = \text{conv}(\{a\} \cup \{b\}) &\subseteq \text{conv}\,(\text{conv ext}\,(C \cap H_1) \cup \text{conv ext}\,(C \cap H_2)) \\ &\subseteq \text{conv conv}\,(\text{ext}\,(C \cap H_1) \cup \text{ext}\,(C \cap H_2)) \\ &= \text{conv}\,(\text{ext}\,(C \cap H_1) \cup \text{ext}\,(C \cap H_2)) \subseteq \text{conv ext } C. \end{aligned}$$

We have thus proved that $C \subseteq \text{conv ext } C$, that is, C is Minkowski. □

Remark 3.5.

1. Implication (2) $\Longrightarrow$ (1) is also a direct consequence of [12, Theorem 18.5].
2. A closed convex set which contains no lines and has no one dimensional faces is Minkowski.

Corollary 3.6 *(Minkowski). Every compact convex subset of $\mathbb{R}^n$ is Minkowski.*

Proof. This well known result follows immediately from Theorem 3.4 in two different ways, since a compact convex set C obviously satisfies Theorem 3.4(2) and Theorem 3.4(3). $\square$

4. The role of the Pareto like sets in the setting of Minkowski sets

The *Pareto-like set* of a closed convex set $\emptyset \neq C \subset \mathbb{R}^n$ is

$$M(C) := \{x \in C : (x - 0^+C) \cap C \subseteq x + 0^+C\} = \{x \in C : (x - 0^+C) \cap C = x + \text{lin } C\},$$

where $0^+C = \{y \in \mathbb{R}^n : y + C \subseteq C\}$ stands for the *recession cone* of C. Recall that 0^+C is a convex cone [12, Theorem 18.1] and $\text{lin } C := 0^+C \cap (-0^+C)$ is a subspace of $\mathbb{R}^n$ called *the lineality* of C.

In this section we basically show that the relations $\emptyset \neq M(C) = \text{rbd } C$ on a closed convex set $C \subseteq \mathbb{R}^n$ ensure the Minkowski property of the convex set. In dimension two, these relations characterize the Minkowski property.

Remark 4.1. Let $\emptyset \neq C \subseteq \mathbb{R}^n$ be a closed convex set.

1. The equality $v + C = C$ holds for every $v \in \text{lin } C$ [13, Theorem 2.5.7].
2. C contains no lines if and only if $\text{lin } C$ is trivial. Indeed, according to [13, Theorem 2.5.8], if $\text{lin } C$ is trivial, then C contains no lines. Conversely, if C contains no lines, then, by 1, the subspace $\text{lin } C$ is trivial.

Remark 4.2. Let $\emptyset \neq C \subseteq \mathbb{R}^n$ be a closed unbounded convex set.

1. If $x \in \text{rbd } C$ and $y \in 0^+C$, then either $x + \mathbb{R}^*_+ y \subseteq \text{rint } C$ or $x + \mathbb{R}^*_+ y \subseteq \text{rbd } C$, where $\mathbb{R}^*_+$ stands for $(0, +\infty)$. Indeed, if $x_0 = x + t_0 y \in \text{rint } C$ for some $t_0 > 0$, then, according to [12, Theorem 8.3, p. 52], one has $x_0 + (0, +\infty)\, y \subseteq \text{rint } C$, and, according to [12, Theorem 6.1, p. 36], the half-open segment $]x, x_0] := \{(1 - \lambda)x + \lambda x_0 : 0 < \lambda \leq 1\}$ is also contained in $\text{rint } C$. Consequently, one obtains $x + (0, +\infty)\, y =]x, x_0] \cup (x_0 + (0, +\infty)\, y) \subseteq \text{rint } C \cup \text{rint } C = \text{rint } C$.
2. For $x \in \text{rbd } C$, the following relations are equivalent:
   (a) $(x + 0^+C) \cap \text{rbd } C = \{x\}$.
   (b) $(x + 0^+C) \setminus \{x\} \subseteq \text{rint } C$.

Remark 4.3. The inclusion $M(C) \subseteq \text{rbd } C$ holds whenever 0^+C is not a subspace or, equivalently, the set $C \cap (\text{lin } C)^\perp$ is unbounded. Note that the equivalence between the boundedness of $C \cap (\text{lin } C)^\perp$ and the quality of 0^+C to be a subspace follows from [5, Lemma 5]. On the other hand, if $M(C) \cap \text{rint } C \neq \emptyset$, then $-0^+C \subseteq 0^+C$, which is equivalent to 0^+C being a subspace, as $\text{lin } C = 0^+C$ in such a case. Indeed, if $x \in M(C) \cap \text{rint } C$ and $u \in 0^+C \subseteq -x + aff(C)$, then $x - tu \in \text{rint } C \subseteq C$ for some $t > 0$ small enough. Therefore $x - tu \in (x - 0^+C) \cap C \subseteq x + 0^+C$, which shows that $-tu \in 0^+C$, i.e. $-u \in 0^+C$ or, equivalently, $u \in -0^+C$.

Proposition 4.4. *Let $\emptyset \neq C \subseteq \mathbb{R}^n$ be a closed convex set. Then $M(C) + \text{lin } C = M(C)$.*

Proof. Since $0 \in \text{lin } C$, we have $M(C) \subseteq M(C) + \text{lin } C$. To prove the opposite inclusion, let $y \in M(C)$ and $l \in \text{lin } C$. We have $(y + l - 0^+C) \cap C = (y - 0^+C) \cap C \subseteq y + 0^+C = y + l + 0^+C$, which shows that $y + l \in M(C)$. $\square$

Proposition 4.5. *If $\emptyset \neq C \subseteq \mathbb{R}^n$ is a closed convex set, then $M(C) = M(C \cap (\text{lin } C)^\perp) + \text{lin } C$.*

Proof. Recall that C can be represented as $C = C \cap (\text{lin } C)^\perp + \text{lin } C$ [12, p. 65], which shows that $x \in M(C)$ can be represented (in a unique way) as $x = y + l$, where $y \in C \cap (\text{lin } C)^\perp$ and $l \in \text{lin } C$. We need to show that $x \in M(C)$ if and only if $y \in M(C \cap (\text{lin } C)^\perp)$, i.e.

$$(x - 0^+C) \cap C = x + \text{lin } C \Longleftrightarrow \left(y - 0^+C \cap (\text{lin } C)^\perp\right) \cap C = \{y\}.$$

By considering the translation φ_{-l} of vector $-l$, we obtain

$$\begin{aligned}
(y + l - 0^+C) \cap C = y + l + \text{lin } C &\Leftrightarrow \varphi_{-l}\left((y + l - 0^+C) \cap C\right) = \varphi_{-l}(y + l + \text{lin } C) \\
&\Leftrightarrow \varphi_{-l}\left(y + l - 0^+C\right) \cap \varphi_{-l}(C) = \varphi_{-l}(y + \text{lin } C) \\
&\Leftrightarrow (y - 0^+C) \cap C = y + \text{lin } C \\
&\Leftrightarrow (y - 0^+C) \cap C \cap (\text{lin } C)^\perp = (y + \text{lin } C) \cap (\text{lin } C)^\perp \\
&\Leftrightarrow (y - 0^+C) \cap (y - (\text{lin } C)^\perp) \cap C = \{y\} \\
&\Leftrightarrow \left(y - 0^+C \cap (\text{lin } C)^\perp\right) \cap C = \{y\}.
\end{aligned}$$

Thus the equality $M(C) = M(C \cap (\text{lin } C)^\perp) + \text{lin } C$ is now completely proved. $\square$

Proposition 4.6. *If $\emptyset \neq C \subseteq \mathbb{R}^n$ is a closed convex set, then* $\text{rbd } C + \text{lin } C = \text{rbd } C$.

Proof. Since $0 \in \text{lin } C$, we only have to prove the inclusion $\text{rbd } C + \text{lin } C = \text{rbd } C$. Let $x \in \text{rbd } C$ and $v \in \text{lin } C$. We have $x + v \in C$, and if we had $x + v \in \text{rint } C$ then, since $x - v \in C$, by [12, Theorem 6.1], we would have $x \in \text{rint } C$, which is a contradiction. Hence $x + v \in \text{rbd } C$. $\square$

Proposition 4.7. *Let $\emptyset \neq C \subseteq \mathbb{R}^n$ be a closed convex set which contains no lines. Then $\emptyset \neq M(C) = \text{rbd } C$ if and only if C is unbounded and $(x + 0^+C) \cap \text{rbd } C = \{x\}$, for every $x \in \text{rbd } C$.*

Proof. Assume that $\emptyset \neq M(C) = \text{rbd } C$. The unboundedness of C is obvious, as for closed convex and bounded sets one has $M(C) = C$. If $(x + 0^+C) \cap \text{rbd } C \neq \{x\}$ for some $x \in \text{rbd } C$, consider $x \neq y \in (x + 0^+C) \cap \text{rbd } C$, namely $y = x + u$ for some $0 \neq u \in 0^+C$. This shows that $x = y - u \in (y - 0^+C) \cap \text{rbd } C$. Thus $y \in \text{rbd } C \setminus M(C)$ as $(y - 0^+C) \cap \text{rbd } C \neq \{y\}$.

Conversely, assume that C is unbounded and $M(C) \neq \text{rbd } C$. Consider $x \in \text{rbd } C \setminus M(C)$, i.e. $(x - 0^+C) \cap C \neq \{x\}$. If $x \neq y \in (x - 0^+C) \cap C$, then $y = x - u$ for some $0 \neq u \in 0^+C$. Thus $x = y + u \in (y + 0^+C) \cap \text{rbd } C$. We only need to show that $y \in \text{rbd } C$, since this implies that $(y + 0^+C) \cap \text{rbd } C = \{y\}$ and hence $x = y$, which is a contradiction. Assume that $y \in \text{rint } C$. Since $0^+(\text{rint } C) = 0^+C$ [12, p. 52], it follows that $y + 0^+C = y + 0^+(\text{rint } C) \subseteq \text{rint } C$, a contradiction to $x \in (y + 0^+C) \cap \text{rbd } C$. $\square$

Proposition 4.8. *Let $C \subset \mathbb{R}^n$ be a closed convex set containing no lines such that* $\dim(C) \geq 2$. *If $\emptyset \neq M(C) = \text{rbd } C$, then C is a Minkowski set.*

Proof. Indeed, in such a case, by [12, Theorem 18.4] and [5, Lemma 8], we have $C = \text{conv rbd } C = \text{conv } M(C) = \text{conv ext } C$. $\square$

The converse statement of Proposition 4.8 is not true, as the following example shows:

Example 4.9. If $C = \{(x, y, z) \in \mathbb{R}^3 : z \geq x^2 + y^2 \text{ and } y \leq 1\}$, then one can easily see that C is a Minkowski set and the points $(u, v, w) \in C$ satisfying $w > u^2 + v^2$ and $v = 1$ are in bd $C \setminus M(C)$.

However, in the two dimensional case the converse statement of Proposition 4.8 is valid for unbounded closed convex sets.

Proposition 4.10. *Let C be a two dimensional unbounded closed convex set. If C is a Minkowski set, then $\emptyset \neq M(C) = \operatorname{rbd} C$.*

Proof. Assume that $\operatorname{rbd} C \setminus M(C) \neq \emptyset$ and consider $x \in \operatorname{rbd} C \setminus M(C)$. Since $x \notin M(C)$, it follows that $(x - 0^+C) \cap C \neq \{x\}$. Consider $x \neq y \in (x - 0^+C) \cap C$, namely $y = x - u$ for some $0 \neq u \in 0^+C$. Observe that y belongs to the relative boundary of C since otherwise we would obtain, taking into account that $0^+(\operatorname{rint} C) = 0^+C$ [12, p. 52], the relations $x = y + u \in \operatorname{rint} C + 0^+C = \operatorname{rint} C + 0^+(\operatorname{rint} C) \subseteq \operatorname{rint} C$, a contradiction to $x \in \operatorname{rbd} C$. In fact the whole half-line $[yx := y + \mathbb{R}_+(x - y)$ is, according to Remark 4.1(1), contained in the relative boundary of C, as $x = y + 1 \cdot (x - y) \in \operatorname{rbd} C$. Thus, by Lemma 3.2, the half-line $[yx$ is contained in a proper face of C, which can only be one dimensional, as the dimension of the whole set is two. This face cannot be a segment as it is unbounded, a contradiction with the characterization of Minkowski sets provided by Theorem 3.4(2). □

Corollary 4.11. *A two dimensional unbounded closed convex set C containing no lines is a Minkowski set if and only if $\emptyset \neq M(C) = \operatorname{rbd} C$.*

Corollary 4.12. *For every two dimensional unbounded face C of a Minkowski set one has $\emptyset \neq M(C) = \operatorname{rbd} C$.*

Proof. We only need to combine here Propositions 3.3 and 4.10. □

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