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On a general class of optimal order multipoint methods for solving nonlinear equations

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A R T I C L E I N F O

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A B S T R A C T

We develop a class of n-point iterative methods with optimal 2^n order of convergence for solving nonlinear equations. Newton's second order and Ostrowski's fourth order methods are special cases corresponding to $n = 1$ and $n = 2$. Eighth and sixteenth order methods that correspond to $n = 3$ and $n = 4$ of the class are special cases of the eighth and sixteenth order methods proposed by Sharma et al. [25]. The methodology is based on employing the previously obtained $(n - 1)$-step scheme and modifying the n-th step by using rational Hermite interpolation. Unlike that of existing higher order techniques the proposed technique is attractive since it leads to a simple implementation. Local convergence analysis is provided to show that the iterations are locally well defined and convergent. Theoretical results are verified through numerical experimentations. The performance is also compared with already established methods in literature. It is observed that new algorithms are more accurate than existing counterparts and very effective in high precision computations.

1. Introduction

Multipoint iterative methods for solving nonlinear equation $f(x) = 0$, were initially studied in Ostrowski's book [19] and then they appeared extensively in Traub's book [29] and in recently published book by Petković et al. [21]. These methods are of great practical importance since they overcome the theoretical limits of one-point iterative methods regarding the computational order and efficiency. The multipoint methods were mainly introduced with the objective to achieve as high as possible order of convergence using a fixed number of function evaluations, which is closely connected to the optimal order of convergence in the sense of the Kung–Traub hypothesis. Kung and Traub [15] conjectured that multipoint methods without memory

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based on $n+1$ function evaluations have the order of convergence at most 2^n. Multipoint methods with this property are called optimal methods.

The construction of multipoint iterative methods is mainly done by following two techniques, one is by using the weight functions and the second is by interpolation. The application of rational Hermite interpolation has been investigated by a number of authors including Ostrowski [19], Traub [29], Jarratt and Nudds [11] and Tornheim [28]. In particular, Ostrowski proposed a two-point method of optimal fourth order in which a rational function of order [1/1], i.e. a linear fraction

$$y(x) = \frac{(x - x_i) + a}{b(x - x_i) + c}, \tag{1}$$

is fitted at three points, two of which are coincident. Thus, a step in the iteration consists of matching f and y at two points x_i and w_i, where $w_i = x_i - f(x_i)/f'(x_i)$ is the Newton's point, and f' and y' at x_i only. The next approximation being given by zero of iteration function (1). In this way the following iterative method was obtained

$$\begin{cases} w_i = \ x_i - \frac{f(x_i)}{f'(x_i)}, \\ x_{i+1} = \ x_i - \frac{f(w_i)-f(x_i)}{f(w_i)f'(x_i)-f(x_i)f[x_i,w_i]} f(x_i), \quad i = 0,\, 1,\, 2, \ldots \end{cases} \tag{2}$$

where x_0 is an initial approximation closer to a root (say, x^*) and $f[\cdot, \cdot]$ is first order divided difference. The error equation of this method is given as

$$e_{i+1} = A_2(A_2^2 - A_3)e_i^4 + O(e_i^5), \tag{3}$$

wherein $e_i = x_i - x^*$ and $A_k = (1/k!)f^{(k)}(x^*)/f'(x^*),\ k = 2, 3$.

In recent years, based on Ostrowski or Ostrowski-like optimal two-point fourth order methods many researchers have developed multipoint methods of optimal higher order of convergence using various techniques (see [3–7,9,10,13,14,16–18,20,23–27,30,31]). A more extensive list of references as well as a survey on progress made on the class of multipoint methods may be found in the recent book by Petković et al. [21].

Motivated by optimization considerations, here we derive a simple yet efficient class of n-point methods possessing optimal convergence order 2^n. The procedure is based on the simple application of rational approximants of different order at each step. Well-known classical Newton's and Ostrowski's methods are special cases of the class corresponding to $n = 1$ and $n = 2$. Eighth order and sixteenth order methods which correspond to $n = 3$ and $n = 4$ of the class are special cases of the eighth and sixteenth order methods proposed by Sharma et al. [25]. Analysis of the three-point eighth order and four-point sixteenth order methods finally pave the way for introducing the general n-point family. Numerical examples are considered to check the performance of new algorithms and to verify the theoretical results. Computational results including the elapsed CPU-time, confirm the efficient and robust character of the algorithms.

The rest of the paper is organized as follows. In Section 2, the three-point eighth order and four-point sixteenth order methods are presented and their convergence is discussed. The general n-point family is introduced in Section 3. Local convergence analysis of the general family is presented in Section 4. In Section 5, some numerical examples are considered to verify the theoretical results and to compare the performance of proposed schemes with some existing optimal methods. Concluding remarks are given in Section 6.

2. Optimal eighth and sixteenth order methods

In what follows, we will present the methods of optimal eighth and sixteenth order of convergence. The methodologies are based on Ostrowski's method (2) and further developed by using rational approximants of order $[1/(n-1)]$, $n = 3, 4$.

2.1. The eighth order method

We derive a three-point optimal eighth order scheme based on the two-point Ostrowski's method (2). Let us write the Ostrowski's method as

$$\begin{cases} w_i = x_i - \frac{D_1}{\Delta_1} f(x_i), \\ z_i = x_i - \frac{D_2}{\Delta_2} f(x_i), \end{cases} \tag{4}$$

where D_n and Δ_n $(n = 1, 2)$ are defined as follows

$$D_1 = 1, \quad \Delta_1 = f'(x_i), \quad D_2 = \begin{vmatrix} 1 & f(x_i) \\ 1 & f(w_i) \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} f'(x_i) & f(x_i) \\ f[x_i, w_i] & f(w_i) \end{vmatrix}.$$

In order to obtain an approximation x_{i+1} to a root we proceed as follows. Consider the rational approximant of order [1/2]

$$y(x) = \frac{(x - x_i) + a}{b(x - x_i)^2 + c(x - x_i) + d}, \tag{5}$$

such that

$$y(x_i) = f(x_i), \quad y'(x_i) = f'(x_i), \quad y(w_i) = f(w_i), \quad y(z_i) = f(z_i). \tag{6}$$

From (5) and first condition of (6), it follows that

$$a = d\, f(x_i). \tag{7}$$

Assuming that the next approximation x_{i+1} is obtained from the zero of (5), then $y(x_{i+1}) = 0$. Thus, from (5) and (7) we obtain

$$x_{i+1} = x_i - d\, f(x_i). \tag{8}$$

Using (7) in (5) and applying the last three conditions of (6), we have the system

$$\begin{cases} d\, f'(x_i) + c\, f(x_i) = 1, \\ d\, f[x_i, w_i] + c\, f(w_i) + b\, f(w_i)(w_i - x_i) = 1, \\ d\, f[x_i, z_i] + c\, f(z_i) + b\, f(z_i)(z_i - x_i) = 1. \end{cases} \tag{9}$$

In order to find the unknown d in (8), we solve the system (9) by Cramer's rule. Thus, we have that

$$d = \frac{D_3}{\Delta_3}, \tag{10}$$

where D_3 and Δ_3 are the determinants of order 3 given as

$$D_3 = \begin{vmatrix} 1 & f(x_i) & 0 \\ 1 & f(w_i) & f(w_i)(w_i - x_i) \\ 1 & f(z_i) & f(z_i)(z_i - x_i) \end{vmatrix} [\text{Operate}\, C_3 \to C_3 + x_i\, C_2] = \begin{vmatrix} 1 & f(x_i) & x_i f(x_i) \\ 1 & f(w_i) & w_i f(w_i) \\ 1 & f(z_i) & z_i f(z_i) \end{vmatrix},$$

$$\Delta_3 = \begin{vmatrix} f'(x_i) & f(x_i) & 0 \\ f[x_i, w_i] & f(w_i) & f(w_i)(w_i - x_i) \\ f[x_i, z_i] & f(z_i) & f(z_i)(z_i - x_i) \end{vmatrix} [\text{Operate}\, C_3 \to C_3 + x_i\, C_2] = \begin{vmatrix} f'(x_i) & f(x_i) & x_i f(x_i) \\ f[x_i, w_i] & f(w_i) & w_i f(w_i) \\ f[x_i, z_i] & f(z_i) & z_i f(z_i) \end{vmatrix}.$$

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Then, combining (8) and (10), we get

$$x_{i+1} = x_i - \frac{D_3}{\Delta_3} f(x_i). \tag{11}$$

Thus, we have presented a three-point method based on Ostrowski's method (4) and then followed by (11) obtained by using rational Hermite interpolation. We state the following theorem to show the eighth order convergence of scheme (11).

Theorem 1. *Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function. If f has a simple zero $x^* \in D$ and x_0 is sufficiently close to x^*, then the method defined by (11) is of order eight.*

Proof. For the proof of theorem readers are referred to the paper [25]. In [25], Sharma et al. obtained a class of eighth order methods based on King's one-parameter (a) family of two-point methods. The method (11) is a special case for $a = 0$ of the improved King's family of eighth order methods. Following the proof as given in [25], the error equation of (11) can be written as

$$e_{i+1} = A_2^2(A_2^2 - A_3)(A_2^3 - 2A_2A_3 + A_4)e_i^8 + O(e_i^9), \tag{12}$$

where $A_k = (1/k!)f^{(k)}(x^*)/f'(x^*),\ k = 2, 3, 4.$

This shows the eighth order convergence. □

Remark 1. The scheme (11) defines a three-point eighth order method which utilizes four function evaluations, namely $f(x_i)$, $f(w_i)$, $f(z_i)$ and $f'(x_i)$. This scheme is, therefore, optimal in the sense of Kung–Traub hypothesis [15].

2.2. The sixteenth order method

Based on the proposed three-point optimal eighth order scheme, we now derive a four-point optimal sixteenth order method. Thus, consider the eighth order scheme (11), which is now expressed as

$$\begin{cases} w_i = x_i - \frac{D_1}{\Delta_1} f(x_i), \\ z_i = x_i - \frac{D_2}{\Delta_2} f(x_i), \\ u_i = x_i - \frac{D_3}{\Delta_3} f(x_i). \end{cases} \tag{13}$$

In order to find the approximation x_{i+1} to a root we consider the rational approximant of order [1/3] given as

$$y(x) = \frac{(x - x_i) + \lambda}{\mu(x - x_i)^3 + \nu(x - x_i)^2 + \xi(x - x_i) + \eta}, \tag{14}$$

where the parameters λ, μ, ν, ξ and η are to be determined by imposing the conditions

$$y(x_i) = f(x_i), \quad y'(x_i) = f'(x_i), \quad y(w_i) = f(w_i), \quad y(z_i) = f(z_i), \quad y(u_i) = f(u_i). \tag{15}$$

Employing first condition of (15) in (14), we find that

$$\lambda = \eta f(x_i). \tag{16}$$

Assuming that the next approximation x_{i+1} is obtained from the zero of (14), which implies $y(x_{i+1}) = 0$. Thus, from (14) and (16) we obtain

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$$x_{i+1} = x_i - \eta f(x_i). \tag{17}$$

Using (16) in (14) and then imposing last four conditions of (15), we obtain the system of corresponding equations as

$$\begin{cases} \eta\, f'(x_i) + \xi\, f(x_i) = 1, \\ \eta\, f[x_i, w_i] + \xi\, f(w_i) + \nu\, f(w_i)(w_i - x_i) + \mu\, f(w_i)(w_i - x_i)^2 = 1, \\ \eta\, f[x_i, z_i] + \xi\, f(z_i) + \nu\, f(z_i)(z_i - x_i) + \mu\, f(z_i)(z_i - x_i)^2 = 1, \\ \eta\, f[x_i, u_i] + \xi\, f(u_i) + \nu\, f(u_i)(u_i - x_i) + \mu\, f(u_i)(u_i - x_i)^2 = 1. \end{cases} \tag{18}$$

Solving the above system for η by Cramer's rule, we have that

$$\eta = \frac{D_4}{\Delta_4}, \tag{19}$$

where

$$\begin{aligned} D_4 &= \begin{vmatrix} 1 & f(x_i) & 0 & 0 \\ 1 & f(w_i) & f(w_i)(w_i - x_i) & f(w_i)(w_i - x_i)^2 \\ 1 & f(z_i) & f(z_i)(z_i - x_i) & f(z_i)(z_i - x_i)^2 \\ 1 & f(u_i) & f(u_i)(u_i - x_i) & f(u_i)(u_i - x_i)^2 \end{vmatrix} [\text{Operate}\, C_3 \to C_3 + x_i\, C_2] \\ &= \begin{vmatrix} 1 & f(x_i) & x_i f(x_i) & 0 \\ 1 & f(w_i) & w_i f(w_i) & f(w_i)(w_i - x_i)^2 \\ 1 & f(z_i) & z_i f(z_i) & f(z_i)(z_i - x_i)^2 \\ 1 & f(u_i) & u_i f(u_i) & f(u_i)(u_i - x_i)^2 \end{vmatrix} [\text{Operate}\, C_4 \to C_4 - x_i^2\, C_2 + 2x_i\, C_3] \\ &= \begin{vmatrix} 1 & f(x_i) & x_i f(x_i) & x_i^2 f(x_i) \\ 1 & f(w_i) & w_i f(w_i) & w_i^2 f(w_i) \\ 1 & f(z_i) & z_i f(z_i) & z_i^2 f(z_i) \\ 1 & f(u_i) & u_i f(z_i) & u_i^2 f(u_i) \end{vmatrix}, \\ \Delta_4 &= \begin{vmatrix} f'(x_i) & f(x_i) & 0 & 0 \\ f[x_i, w_i] & f(w_i) & f(w_i)(w_i - x_i) & f(w_i)(w_i - x_i)^2 \\ f[x_i, z_i] & f(z_i) & f(z_i)(z_i - x_i) & f(z_i)(z_i - x_i)^2 \\ f[x_i, u_i] & f(u_i) & f(u_i)(u_i - x_i) & f(u_i)(u_i - x_i)^2 \end{vmatrix} [\text{Operate}\, C_3 \to C_3 + x_i\, C_2] \\ &= \begin{vmatrix} f'(x_i) & f(x_i) & x_i f(x_i) & 0 \\ f[x_i, w_i] & f(w_i) & w_i f(w_i) & f(w_i)(w_i - x_i)^2 \\ f[x_i, z_i] & f(z_i) & z_i f(z_i) & f(z_i)(z_i - x_i)^2 \\ f[x_i, u_i] & f(u_i) & u_i f(u_i) & f(u_i)(u_i - x_i)^2 \end{vmatrix} [\text{Operate}\, C_4 \to C_4 - x_i^2\, C_2 + 2x_i\, C_3] \\ &= \begin{vmatrix} f'(x_i) & f(x_i) & x_i f(x_i) & x_i^2 f(x_i) \\ f[x_i, w_i] & f(w_i) & w_i f(w_i) & w_i^2 f(w_i) \\ f[x_i, z_i] & f(z_i) & z_i f(z_i) & z_i^2 f(z_i) \\ f[x_i, u_i] & f(u_i) & u_i f(z_i) & u_i^2 f(u_i) \end{vmatrix}. \end{aligned}$$

Combining (17) and (19), we obtain the iterative formula

$$x_{i+1} = x_i - \frac{D_4}{\Delta_4} f(x_i). \tag{20}$$

The scheme (20) defines a four-point iterative method with the base as three-point scheme (13). In the following theorem we prove the sixteenth order of convergence of this scheme.

Theorem 2. *Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function. If f has a simple zero $x^* \in D$ and x_0 is sufficiently close to x^*, then the method defined by (20) is of order sixteen.*

Proof. The method (20) is a special case for $a = 0$ of improved King's family of sixteenth order methods developed in [25]. Therefore, for the proof readers are referred to the paper [25]. However, the error equation showing sixteenth order convergence of (20) is given as

$$e_{i+1} = A_2^4(A_2^2 - A_3)^2(A_2^3 - 2A_2A_3 + A_4)(A_2^4 - 3A_2^2A_3 + A_3^2 + 2A_2A_4 - A_5)e_i^{16} + O(e_i^{17}), \tag{21}$$

where $A_k = (1/k!)f^{(k)}(x^*)/f'(x^*),\ k = 2, 3, 4, 5.$ $\square$

Remark 2. It is clear that the method (20) requires five function evaluations viz. $f(x_i)$, $f(w_i)$, $f(z_i)$, $f(u_i)$ and $f'(x_i)$ per iteration and possesses sixteenth order of convergence. Thus, the method is optimal according to Kung–Traub hypothesis.

3. The general optimal order family

The above approach of employing previously obtained scheme and then generating the new step by using rational Hermite interpolation can be applied to obtain a generalized n-point scheme of optimal order 2^n. The rational approximant is such that its numerator is a linear function whereas denominator is a polynomial of degree $n - 1$. Thus, to obtain the general step of n-point scheme consider the rational approximant of order $[1/(n - 1)]$,

$$y(x) = \frac{(x - x_{i,1}) + a_0}{a_1(x - x_{i,1})^{n-1} + a_2(x - x_{i,1})^{n-2} + \cdots + a_{n-1}(x - x_{i,1}) + a_n}, \tag{22}$$

where $a_0, a_1, \ldots, a_n$ are $n + 1$ parameters to be determined. In order to find these parameters consider the sequence of iterates $\{x_{i,k}\}_{k=1}^{n}$ such that

$$y(x_{i,k}) = f(x_{i,k}),\ k = 1, 2, 3, \ldots, n \tag{23}$$

and

$$y'(x_{i,1}) = f'(x_{i,1}). \tag{24}$$

Employing the condition $y(x_{i,1}) = f(x_{i,1})$ in (22), we find that

$$a_0 = a_n f(x_{i,1}). \tag{25}$$

Assume that the approximation $x_{i,n+1}$ is obtained from the zero of (22), which implies that $y(x_{i,n+1}) = 0$. Then, from (22) and (25) we obtain

$$x_{i,n+1} = x_{i,1} - a_n f(x_{i,1}). \tag{26}$$

Applying the remaining conditions of (23) and (24) in (22), we obtain the system of equations as

$$\begin{cases} a_n f'(x_{i,1}) + a_{n-1} f(x_{i,1}) = 1, \\ a_n f[x_{i,1}, x_{i,2}] + \big(a_{n-1} + a_{n-2}(x_{i,2} - x_{i,1}) + \cdots + a_1(x_{i,2} - x_{i,1})^{n-2}\big) f(x_{i,2}) = 1, \\ a_n f[x_{i,1}, x_{i,3}] + \big(a_{n-1} + a_{n-2}(x_{i,3} - x_{i,1}) + \cdots + a_1(x_{i,3} - x_{i,1})^{n-2}\big) f(x_{i,3}) = 1, \\ \vdots \\ a_n f[x_{i,1}, x_{i,n}] + \big(a_{n-1} + a_{n-2}(x_{i,n} - x_{i,1}) + \cdots + a_1(x_{i,n} - x_{i,1})^{n-2}\big) f(x_{i,n}) = 1. \end{cases} \tag{27}$$

Solving the above system for a_n by Cramer's rule, we have that

$$a_n = \frac{D_n}{\Delta_n}, \tag{28}$$

wherein D_n and Δ_n are n-th order determinants expressed as

$$D_n = \begin{vmatrix} 1 & f(x_{i,1}) & 0 & 0 & \dots & 0 \\ 1 & f(x_{i,2}) & f(x_{i,2})(x_{i,2}-x_{i,1}) & f(x_{i,2})(x_{i,2}-x_{i,1})^2 & \dots & f(x_{i,2})(x_{i,2}-x_{i,1})^{n-2} \\ 1 & f(x_{i,3}) & f(x_{i,3})(x_{i,3}-x_{i,1}) & f(x_{i,3})(x_{i,3}-x_{i,1})^2 & \dots & f(x_{i,3})(x_{i,3}-x_{i,1})^{n-2} \\ 1 & f(x_{i,4}) & f(x_{i,4})(x_{i,4}-x_{i,1}) & f(x_{i,4})(x_{i,4}-x_{i,1})^2 & \dots & f(x_{i,4})(x_{i,4}-x_{i,1})^{n-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & f(x_{i,n}) & f(x_{i,n})(x_{i,n}-x_{i,1}) & f(x_{i,n})(x_{i,n}-x_{i,1})^2 & \dots & f(x_{i,n})(x_{i,n}-x_{i,1})^{n-2} \end{vmatrix}$$

operating $C_3 \to C_3 + (-1)^2 \binom{1}{0} x_{i,1} C_2$

$$= \begin{vmatrix} 1 & f(x_{i,1}) & x_{i,1}f(x_{i,1}) & 0 & \dots & 0 \\ 1 & f(x_{i,2}) & x_{i,2}f(x_{i,2}) & f(x_{i,2})(x_{i,2}-x_{i,1})^2 & \dots & f(x_{i,2})(x_{i,2}-x_{i,1})^{n-2} \\ 1 & f(x_{i,3}) & x_{i,3}f(x_{i,3}) & f(x_{i,3})(x_{i,3}-x_{i,1})^2 & \dots & f(x_{i,3})(x_{i,3}-x_{i,1})^{n-2} \\ 1 & f(x_{i,4}) & f(x_{i,4})x_{i,4} & f(x_{i,4})(x_{i,4}-x_{i,1})^2 & \dots & f(x_{i,4})(x_{i,4}-x_{i,1})^{n-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & f(x_{i,n}) & x_{i,n}f(x_{i,n}) & f(x_{i,n})(x_{i,n}-x_{i,1})^2 & \dots & f(x_{i,n})(x_{i,n}-x_{i,1})^{n-2} \end{vmatrix}$$

operating $C_4 \to C_4 + (-1)^3 \binom{2}{0} x_{i,1}^2 C_2 + (-1)^2 \binom{2}{1} x_{i,1} C_3$

$$= \begin{vmatrix} 1 & f(x_{i,1}) & x_{i,1}f(x_{i,1}) & x_{i,1}^2 f(x_{i,1}) & \dots & 0 \\ 1 & f(x_{i,2}) & x_{i,2}f(x_{i,2}) & x_{i,2}^2 f(x_{i,2}) & \dots & f(x_{i,2})(x_{i,2}-x_{i,1})^{n-2} \\ 1 & f(x_{i,3}) & x_{i,3}f(x_{i,3}) & x_{i,3}^2 f(x_{i,3}) & \dots & f(x_{i,3})(x_{i,3}-x_{i,1})^{n-2} \\ 1 & f(x_{i,4}) & x_{i,4}f(x_{i,4}) & x_{i,4}^2 f(x_{i,4}) & \dots & f(x_{i,4})(x_{i,4}-x_{i,1})^{n-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & f(x_{i,n}) & x_{i,n}f(x_{i,n}) & x_{i,n}^2 f(x_{i,n}) & \dots & f(x_{i,n})(x_{i,n}-x_{i,1})^{n-2} \end{vmatrix}.$$

$\vdots$

operating $C_n \to C_n + (-1)^{n-1} \binom{n-2}{0} x_{i,1}^{n-2} C_2 + (-1)^{n-2} \binom{n-2}{1} x_{i,1}^{n-3} C_3$

$$+ \dots + (-1)^{n-(n-2)} \binom{n-2}{n-3} x_{i,1} C_{n-1}$$

$$= \begin{vmatrix} 1 & f(x_{i,1}) & x_{i,1}f(x_{i,1}) & \dots & x_{i,1}^{n-2} f(x_{i,1}) \\ 1 & f(x_{i,2}) & x_{i,2}f(x_{i,2}) & \dots & x_{i,2}^{n-2} f(x_{i,2}) \\ 1 & f(x_{i,3}) & x_{i,3}f(x_{i,3}) & \dots & x_{i,3}^{n-2} f(x_{i,3}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & f(x_{i,n}) & x_{i,n}f(x_{i,n}) & \dots & x_{i,n}^{n-2} f(x_{i,n}) \end{vmatrix}, \ n \geq 2,$$

$$\Delta_n = \begin{vmatrix} f'(x_{i,1}) & f(x_{i,1}) & 0 & \dots & 0 \\ f[x_{i,1}, x_{i,2}] & f(x_{i,2}) & f(x_{i,2})(x_{i,2}-x_{i,1}) & \dots & f(x_{i,2})(x_{i,2}-x_{i,1})^{n-2} \\ f[x_{i,1}, x_{i,3}] & f(x_{i,3}) & f(x_{i,3})(x_{i,3}-x_{i,1}) & \dots & f(x_{i,3})(x_{i,3}-x_{i,1})^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f[x_{i,1}, x_{i,n}] & f(x_{i,n}) & f(x_{i,n})(x_{i,n}-x_{i,1}) & \dots & f(x_{i,n})(x_{i,n}-x_{i,1})^{n-2} \end{vmatrix}$$

operating as in the case of D_n

$$= \begin{vmatrix} f'(x_{i,1}) & f(x_{i,1}) & x_{i,1}f(x_{i,1}) & \dots & x_{i,1}^{n-2} f(x_{i,1}) \\ f[x_{i,1}, x_{i,2}] & f(x_{i,2}) & x_{i,2}f(x_{i,2}) & \dots & x_{i,2}^{n-2} f(x_{i,2}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f[x_{i,1}, x_{i,n}] & f(x_{i,n}) & x_{i,n}f(x_{i,n}) & \dots & x_{i,n}^{n-2} f(x_{i,n}) \end{vmatrix}, \ n \geq 2.$$

Thus, we can write the general n-point scheme as

$$\begin{cases} x_{i,1} = & x_i, \ i \geq 0, \\ x_{i,2} = & x_{i,1} - \frac{D_1}{\Delta_1} f(x_{i,1}), \\ x_{i,3} = & x_{i,1} - \frac{D_2}{\Delta_2} f(x_{i,1}), \\ x_{i,4} = & x_{i,1} - \frac{D_3}{\Delta_3} f(x_{i,1}), \\ \vdots & \\ x_{i,n+1} = x_{i+1} = & x_{i,1} - \frac{D_n}{\Delta_n} f(x_{i,1}), \end{cases} \tag{29}$$

where $x_{0,1} = x_0$ is given starting point and $n \in N$.

The determinants D_n and Δ_n can easily be computed through first column. The expansion of D_n through first column is given by

$$\begin{aligned} D_n = & \begin{vmatrix} f(x_{i,2}) & x_{i,2}f(x_{i,2}) & \dots & x_{i,2}^{n-2}f(x_{i,2}) \\ f(x_{i,3}) & x_{i,3}f(x_{i,3}) & \dots & x_{i,3}^{n-2}f(x_{i,3}) \\ \vdots & \vdots & \ddots & \vdots \\ f(x_{i,n}) & x_{i,n}f(x_{i,n}) & \dots & x_{i,n}^{n-2}f(x_{i,n}) \end{vmatrix} - \begin{vmatrix} f(x_{i,1}) & x_{i,1}f(x_{i,1}) & \dots & x_{i,1}^{n-2}f(x_{i,1}) \\ f(x_{i,3}) & x_{i,3}f(x_{i,3}) & \dots & x_{i,3}^{n-2}f(x_{i,3}) \\ \vdots & \vdots & \ddots & \vdots \\ f(x_{i,n}) & x_{i,n}f(x_{i,n}) & \dots & x_{i,n}^{n-2}f(x_{i,n}) \end{vmatrix} \\ & + \cdots\cdots + (-1)^{n-1} \begin{vmatrix} f(x_{i,1}) & x_{i,1}f(x_{i,1}) & \dots & x_{i,1}^{n-2}f(x_{i,1}) \\ f(x_{i,2}) & x_{i,2}f(x_{i,2}) & \dots & x_{i,2}^{n-2}f(x_{i,2}) \\ \vdots & \vdots & \ddots & \vdots \\ f(x_{i,n-1}) & x_{i,n-1}f(x_{i,n-1}) & \dots & x_{i,n-1}^{n-2}f(x_{i,n-1}) \end{vmatrix} \\ = & f(x_{i,2})f(x_{i,3})\cdots f(x_{i,n})\, V(x_{i,2}, x_{i,3}, \dots, x_{i,n}) - f(x_{i,1})f(x_{i,3})\cdots f(x_{i,n})\, V(x_{i,1}, x_{i,3}, \dots, x_{i,n}) \\ & + \cdots\cdots + (-1)^{n-1} f(x_{i,1})f(x_{i,2})\cdots f(x_{i,n-1})\, V(x_{i,1}, x_{i,2}, \dots, x_{i,n-1}), \end{aligned}$$

where V denotes Vandermonde's determinant. For example, the determinant $V(x_{i,2}, x_{i,3}, \dots, x_{i,n})$ is given by

$$V(x_{i,2}, x_{i,3}, \dots, x_{i,n}) = \begin{vmatrix} 1 & x_{i,2} & x_{i,2}^2 & \dots & x_{i,2}^{n-2} \\ 1 & x_{i,3} & x_{i,3}^2 & \dots & x_{i,3}^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{i,n} & x_{i,n}^2 & \dots & x_{i,n}^{n-2} \end{vmatrix} = \prod_{\substack{k,\, l=2 \\ l<k}}^{n} (x_{i,k} - x_{i,l}).$$

Thus, we have

$$D_n = \sum_{j=1}^{n} (-1)^{j-1} \prod_{\substack{k=1 \\ k\neq j}}^{n} f(x_{i,k}) \prod_{\substack{k,\, l=1 \\ k,\, l\neq j \\ l<k}}^{n} (x_{i,k} - x_{i,l}).$$

Similarly, the expansion of Δ_n is given as

$$\begin{aligned} \Delta_n = & f[x_{i,1}, x_{i,1}]\, f(x_{i,2})f(x_{i,3})\cdots f(x_{i,n})\, V(x_{i,2}, x_{i,3}, \dots, x_{i,n}) - f[x_{i,1}, x_{i,2}]\, f(x_{i,1})f(x_{i,3})\cdots f(x_{i,n}) \\ & \times V(x_{i,1}, x_{i,3}, \dots, x_{i,n}) + \cdots + (-1)^{n-1} f[x_{i,1}, x_{i,n}]\, f(x_{i,1})f(x_{i,2})\cdots f(x_{i,n-1})\, V(x_{i,1}, x_{i,2}, \dots, x_{i,n-1}) \\ = & \sum_{j=1}^{n} (-1)^{j-1} f[x_{i,1}, x_{i,j}] \prod_{\substack{k=1 \\ k\neq j}}^{n} f(x_{i,k}) \prod_{\substack{k,\, l=1 \\ k,\, l\neq j \\ l<k}}^{n} (x_{i,k} - x_{i,l}), \end{aligned}$$

wherein $f[x_{i,1}, x_{i,1}] = f'(x_{i,1})$.

From now on, we use the following short form of (29)

$$\begin{cases} x_{i,1} = \ x_i, \ i \geq 0, \\ x_{i,k+1} = x_{i+1} = \ x_{i,1} - \frac{D_k}{\Delta_k} f(x_{i,1}), \quad k = 1, 2, \dots, n, \end{cases} \tag{30}$$

where $D_1 = 1$, $\Delta_1 = f'(x_{i,1})$, and D_j and Δ_j $(j \geq 2)$ are the leading principal minors of D_n and Δ_n, respectively.

Remark 3. Note that the first step of general scheme (30) is the well-known Newton's scheme. The first two steps are the steps of Ostrowski's scheme. Similarly, the first three and four steps are that of eighth and sixteenth order schemes derived in previous section.

4. Local convergence analysis

The local convergence analysis that follows is based on some scalar functions and parameters. Let $L_0 > 0$, $L > 0$, $b > 0$, $b_0 > 0$ and $M > 0$ be given parameters. Define functions g_2, p_2 and h_{p_2} on the interval $[0, \frac{1}{bL_0})$ by

$$g_2(t) = \frac{bLt}{2(1 - bL_0t)},$$
$$p_2(t) = b\left(\frac{L_0}{2}\big(1 + g_2(t)\big) + \frac{bM^2 g_2(t)}{1 - \frac{bL_0t}{2}}\right)t,$$
$$h_{p_2}(t) = p(t) - 1$$

and parameter r_A by

$$r_A = \frac{2}{b(2L_0 + L)}. \tag{31}$$

Then, we have $0 < r_A$, $g_2(r_A) = 1$ and $0 \leq g_2(t) < 1$ for each $t \in [0, r_A)$, $h_{p_2}(0) = -1$ and $h_p(t) \to \infty$ as $t \to \frac{1}{bL_0}^-$. It then follows from the intermediate value theorem that function h_p has zeros in the interval $(0, \frac{1}{bL_0})$. Denote by r_{p_2} the smallest such zero.

Moreover, define functions g_3 and h_3 on the interval $[0, r_{p_2})$ by

$$g_3(t) = \frac{1}{2(1 - bL_0t)}\Big(L + M\delta_2(t)\gamma_1(t)t\Big)t$$

and

$$h_3(t) = g_3(t) - 1, \tag{32}$$

where $\delta_2(t) = \frac{b^2}{t(1 - \frac{bL_0t}{2})(1 - p_2(t))}$, $\gamma_1(t) = \frac{L_0M}{2}\Big(5g_2(t) + 3\Big)$. Then, we get that $h_3(0) = -1$ and $h_3(t) \to +\infty$ as $t \to r_{p_2}^-$. Denote by r_3 the smallest zero of function h_3 in the interval $(0, r_{p_2})$.

Furthermore, for each $m = 4, 5, \dots, n+1$, define functions g_m and h_m on the interval $[0, r_{\gamma_2}^m)$ by

$$g_m(t) = \frac{1}{2(1 - bL_0t)}\left(Lt + \frac{bM\gamma_3(t)}{1 - \gamma_2(t)}\right)$$

and

$$h_m(t) = g_m(t) - 1, \tag{33}$$

where

$$\gamma_2(t) = b(L_0 t + b_0) \sum_{j=1}^{m} 2^j M^j t^j, \quad \gamma_3(t) = L_0 \sum_{j=1}^{m} 2^{j+1} M^j t^{j+1}.$$

Finally, we have that $h_m(0) = -1 < 0$ and $h_m(t) \to \infty$ as $t \to (r_{\gamma_2}^m)^-$, where $r_{\gamma_2}^m$ is the smallest positive zero of function h_{γ_2}. Denote by r_m the smallest zero of function h_m on the interval $(0, \bar{r})$.

Define the radius of convergence r by

$$r = \min\{r_A, r_i\}, \ i = 3, 4, \ldots, n+1. \tag{34}$$

Then, we have that

$$0 < r \le r_A < \frac{1}{bL_0}, \tag{35}$$

$$0 \le g_i(t) < 1, \ i = 2, 3, \ldots, n+1, \tag{36}$$

$$0 \le p_2(t) < 1 \tag{37}$$

and

$$0 \le \gamma_2(t) < 1 \text{ for each } t \in [0, r). \tag{38}$$

Let $U(v, \delta)$ and $\bar{U}(v, \delta)$ stand, respectively for the open and closed balls in $\mathbb{R}$ with center $v \in \mathbb{R}$ and of radius $\delta > 0$. Let also $\mathcal{L}(\mathbb{R}, \mathbb{R})$ stand for the space of linear functions from $\mathbb{R}$ into $\mathbb{R}$. A mapping $f[x, y] : D^2 \to \mathcal{L}(\mathbb{R}, \mathbb{R})$ is called a first order divided difference at the point $(x, y) \in D^2$, if $f[x, y](x-y) = f(x) - f(y)$ for each $(x, y) \in D^2$ with $x \neq y$. Moreover, if function f is differentiable at $x \in D$, then $f[x, x] = f'(x)$. Furthermore, $f[\cdot, \cdot]$ is called a first order divided difference on D^2 if $f[\cdot, \cdot]$ is a first order divided difference for each $(x, y) \in D^2$.

Next, we present the local convergence analysis of method (30) using the preceding notation.

Theorem 3. *Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function and $f[x, y] : D^2 \to \mathcal{L}(\mathbb{R}, \mathbb{R})$ be a first order divided difference on D^2. Suppose that there exist $x^* \in D$, $L_0 > 0$, $b > 0$, $b_0 > 0$ such that for each $x \in D$:*

$$f(x^*) = 0, \ f'(x^*) \neq 0, \ |f'(x^*)| \le b_0, \ |f'(x^*)^{-1}| \le b \tag{39}$$

and

$$|f'(x) - f'(x^*)| \le L_0 |x - x^*|. \tag{40}$$

Moreover, suppose there exist $L > 0$, $M > 0$ such that for each $x, y \in D_0 = D \cap U(x^*, \frac{1}{bL_0})$

$$|f'(x) - f'(y)| \le L|x - y|, \tag{41}$$

$$|f'(x)| \le M, \tag{42}$$

$$|\Delta - f'(x^*)| \le |\Delta| \tag{43}$$

and

$$\bar{U}(x^*, r) \subseteq D, \tag{44}$$

where Δ is the continuous form of Δ_n and r is the radius of convergence defined by (34). Then, sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover the following estimates hold

$$|x_{i,2} - x^*| \le g_2(|x_{i,1} - x^*|)|x_{i,1} - x^*| \le |x_{i,1} - x^*| < r \tag{45}$$

$$|x_{i,3} - x^*| \le g_3(|x_{i,2} - x^*|)|x_{i,2} - x^*| \le |x_{i,2} - x^*| \tag{46}$$

and

$$|x_{i,m} - x^*| \le g_m(|x_{i,m-1} - x^*|)|x_{i,m-1} - x^*| \le |x_{i,m-1} - x^*|, \tag{47}$$

where the "g" functions are defined previously. Furthermore, for $T \in [r, \frac{2}{bL_0})$ the limit point x^* is the only solution of equation $f(x) = 0$ in $D_1 = D \cap U(x^*, T)$.

Proof. We shall show using mathematical induction that the sequence $\{x_n\}$ is well defined and convergent to x^* so the estimates (45)–(47) hold.

Let $i = 0$. By hypotheses (35), (39), (40) and $x_0 \in U(x^*, r) - x^*$, we have that

$$b|f'(x_{0,1}) - f'(x^*)| = b|f'(x_0) - f'(x^*)| \le bL_0|x_0 - x^*| \le bL_0 r < 1. \tag{48}$$

It follows from (48) and the Banach lemma on invertible functions [1] that $f'(x_{0,1}) \neq 0$,

$$|f'(x_{0,1})^{-1}| \le \frac{b}{1 - bL_0|x_{0,1} - x^*|} \tag{49}$$

and $x_{0,2}$ is well defined. Then, we have by the first substep of method (30) that

$$x_{0,2} - x^* = x_{0,1} - x^* - f'(x_{0,1})^{-1} f(x_{0,1}). \tag{50}$$

Using (34), (36), (39), (41), (49) and (50), we get in turn that

$$\begin{aligned} |x_{0,2} - x^*| &\le |f'(x_0)^{-1}| \left| \int_0^1 \Big(f'(x^* + \theta(x_{0,1} - x^*)) - f'(x_{0,1})\Big)(x_{0,1} - x^*) d\theta \right| \\ &\le \frac{bL|x_{0,1} - x^*|^2}{2(1 - bL_0|x_{0,1} - x^*|)} = g_2(|x_{0,1} - x^*|)|x_{0,1} - x^*| \le |x_{0,1} - x^*| < r, \end{aligned} \tag{51}$$

which shows (45) and $x_{0,2} \in U(x^*, r)$.

We can write by (39)

$$f(x_{0,1}) - f(x^*) = \int_0^1 f'(x^* + \theta(x_{0,1} - x^*))(x_{0,1} - x^*) d\theta. \tag{52}$$

Notice that $x^* + \theta(x_{0,1} - x^*) \in U(x^*, r)$, since $|x^* + \theta(x_{0,1} - x^*) - x^*| = \theta|x_{0,1} - x^*| < r$.

Then by (42) and (52), we get that

$$|f(x_{0,1}| \le M|x_{0,1} - x^*|. \tag{53}$$

We must show the existence of $x_{0,3}$. To achieve this, notice that we can rewrite $\frac{D_2}{\Delta_2}$ as

$$\frac{D_2}{\Delta_2} = \frac{1 - \frac{f(x_{0,2})}{f(x_{0,1})}}{A_2}, \tag{54}$$

where $A_2 = f[x_{0,1}, x_{0,2}] - \frac{f(x_{0,2})}{f(x_{0,1})} f'(x_{0,1})$ for $f(x_{0,1}) \neq 0$ (if $f(x_{0,1}) = 0$, then $x_{0,1} = x^*$ and the iteration is terminated). By (40) we have that

$$\left|(f'(x^*)(x_{0,1} - x^*))^{-1}(f(x_{0,1}) - f(x^*) - f'(x^*)(x_{0,1} - x^*))\right| \leq \frac{bL_0}{2}|x_{0,1} - x^*| < \frac{bL_0}{2} r < 1, \tag{55}$$

by the choice of r. Hence, $f(x_{0,1}) \neq 0$ and

$$|f(x_{0,1})^{-1}| \leq \frac{b}{|x_0 - x^*|\left(1 - \frac{bL_0}{2}|x_0 - x^*|\right)}. \tag{56}$$

Next, we show that $A_2 \neq 0$. We have in turn by (34), (35), (37), (40) (51), (53) and (55) that

$$\begin{aligned} b\left|f[x_{0,1}, x_{0,2}] - f'(x^*) - \frac{f(x_{0,2})}{f(x_{0,1})} f'(x_{0,1})\right| &\leq \frac{bL_0}{2}(|x_{0,1} - x^*| + |x_{0,2} - x^*|) + \frac{bM^2|x_{0,2} - x^*|}{|x_{0,1} - x^*|\left(1 - \frac{bL_0}{2}|x_{0,1} - x^*|\right)} \\ &\leq b\left(\frac{L_0}{2}(1 + g_2(|x_{0,1} - x^*|)) + \frac{M^2 g_2(|x_{0,1} - x^*|)}{1 - \frac{bL_0}{2}|x_{0,1} - x^*|}\right) \\ &= p_2(|x_0 - x^*|) < p_2(r) < 1. \end{aligned} \tag{57}$$

Hence, $A_2 \neq 0$ and

$$|A_2^{-1}| \leq \frac{b}{1 - p_2(|x_0 - x^*|)}, \tag{58}$$

so $x_{0,3}$ is well defined. Then, we have that

$$\begin{aligned} \left|\frac{D_2}{\Delta_2}\right| \leq \frac{b\left(1 + \left|\frac{f(x_{0,2}}{f(x_{0,1})}\right|\right)}{1 - p_2(|x_0 - x^*|)} \leq \frac{b\left(1 + \frac{M|x_{0,2} - x^*|}{|x_{0,1} - x^*|\left(1 - \frac{bL_0}{2}|x_{0,1} - x^*|\right)}\right)}{1 - p_2(|x_{0,1} - x^*|)} &\leq \frac{b\left(1 + \frac{Mg_2(|x_{0,1} - x^*|)}{1 - \frac{bL_0}{2}|x_{0,1} - x^*|}\right)}{1 - p_2(|x_{0,1} - x^*|)} \\ &\leq \frac{b\left(1 + \frac{Mg_2(r)}{1 - \frac{bL_0}{2} r}\right)}{1 - p_2(r)}. \end{aligned} \tag{59}$$

It follows from (59) that $\Delta_2 \neq 0$ and

$$|\Delta_2^{-1}| \leq \delta_2(|x_{0,1} - x^*|). \tag{60}$$

We can write by the second substep of method (30) for $i = 0$:

$$x_{0,3} - x^* = x_{0,1} - x^* - f'(x_{0,1})^{-1} f(x_{0,1}) + f'(x_{0,1})^{-1}(\Delta_2 - f'(x_{0,1}) D_2)\Delta_2^{-1} f(x_{0,1}). \tag{61}$$

Using (40), (51) and (53) we get in turn that

$$\begin{aligned}|\Delta_2 - f'(x_{0,1})D_2| &= |(f'(x_{0,1}) - f'(x^*))f(x_{0,2}) + (f'(x^*) - f[x_{0,1}, x_{0,2}])f(x_{0,1})\\ &\quad + (f'(x^*) - f'(x_{0,1}))f(x_{0,2}) + (f'(x_{0,1}) - f'(x^*))f(x_{0,1})|\\ &\le ML_0|x_{0,1} - x^*||x_{0,2} - x^*| + \frac{L_0M}{2}\Big(|x_{0,1} - x^*| + |x_{0,2} - x^*|\Big)|x_{0,1} - x^*|\\ &\quad + L_0M|x_{0,1} - x^*||x_{0,2} - x^*| + L_0M|x_{0,1} - x^*||x_{0,1} - x^*| \le \gamma_1(r)|x_{0,1} - x^*|^2. \end{aligned} \tag{62}$$

Then, by (34), (35), (36), (51), (60) and (62), we obtain in turn that

$$\begin{aligned}|x_{0,3} - x^*| &\le \frac{bL|x_{0,1} - x^*|^2}{2(1 - bL_0|x_{0,1} - x^*|)} + \frac{M\delta_2(r)\gamma_1(r)|x_{0,1} - x^*|^3}{1 - bL_0|x_{0,1} - x^*|}\\ &= g_3(|x_{0,1} - x^*|)|x_0 - x^*| \le |x_0 - x^*| < r, \end{aligned} \tag{63}$$

which shows (46) and $x_{0,3} \in U(x^*, r)$. It follows from (43) that $\Delta_m \neq 0$, $m = 3, 4, \ldots, n+1$, since otherwise $f'(x^*) = 0$ contradicting (33). If the iterates are equal to each other, then $\Delta_m = 0$ but then iterate $x_{0,m} = x^*$, so the iteration has been terminated. In view of (38), (40), (42) and (43), we also get in turn that

$$\begin{aligned}b|\Delta_m - f'(x^*)| &\le b\Big|\sum_{j=1}^{m}(-1)^{j-1}(f[x_{0,1}, x_{0,j}] - f'(x^*)) \prod_{\substack{k=1\\k\neq j}}^{m} f(x_{0,k}) \prod_{\substack{k,l=1\\k,l\neq j\\l<k}}^{m} ((x_{0,k} - x^*) + (x^* - x_{0,l}))\\ &\quad + \sum_{j=1}^{m}(-1)^{j-1}f'(x^*) \prod_{\substack{k=1\\k\neq j}}^{m} f(x_{0,k}) \prod_{\substack{k,l=1\\k,l\neq j\\l<k}}^{m} ((x_{0,k} - x^*) + (x^* - x_{0,l}))\Big|\\ &\le b\Big(\frac{L_0}{2}\sum_{j=1}^{m}(-1)^{j-1}(|x_{0,1} - x^*| + |x_{0,j} - x^*|)M^m \prod_{\substack{k,l=1\\k,l\neq j\\l<k}}^{m} (|x_{0,k} - x^*| + |x_{0,l} - x^*|)\\ &\quad + \sum_{j=1}^{m}(-1)^{j-1}b_0M^m \prod_{\substack{k,l=1\\k,l\neq j\\l<k}}^{m} (|x_{0,k} - x^*| + |x_{0,l} - x^*|)\Big)\\ &\le \gamma_2(|x_{0,1} - x^*|) < \gamma_2(r) < 1. \end{aligned} \tag{64}$$

Hence, we get

$$|\Delta_m^{-1}| \le \frac{1}{1 - \gamma_2(|x_{0,1} - x^*|)}. \tag{65}$$

Let

$$E_m = \Delta_m - f'(x_{0,1})D_m. \tag{66}$$

Then using (66), we have as in (64) that

$$\begin{aligned}|E_m| &= \Big|\sum_{j=1}^{m}(-1)^{j-1}\Big[(f[x_{0,1}, x_{0,j}] - f'(x^*)) + (f'(x^*) - f'(x_{0,1}))\Big] \prod_{\substack{k=1\\k\neq j}}^{m} f(x_{0,k}) \prod_{\substack{k,l=1\\k,l\neq j\\l<k}}^{m} (x_{0,k} - x_{0,l})\Big|\\ &\le \gamma_3(|x_{0,1} - x^*|). \end{aligned} \tag{67}$$

Then, we have as in (61) that

$$\begin{aligned}|x_{0,m}-x^*| &= |x_{0,1}-x^*-f'(x_{0,1})^{-1}f(x_{0,1})+f'(x_{0,1})^{-1}(\Delta_m-f'(x_{0,m})D_m)\Delta_m^{-1}f(x_{0,m})| \\ &\le \frac{bL|x_{0,1}-x^*|^2}{2(1-bL_0|x_{0,1}-x^*|)}+\frac{bM\gamma_3(|x_{0,1}-x^*|)|x_{0,1}-x^*|}{(1-bL_0|x_{0,1}-x^*|)(1-\gamma_2(|x_{0,1}-x^*|)} \\ &= g_m(|x_{0,1}-x^*|)|x_{0,1}-x^*| < |x_{0,1}-x^*| < r, \end{aligned} \tag{68}$$

which shows (47) and $x_{0,m} \in U(x^*, r)$. In particular, we have that

$$|x_1 - x^*| < |x_0 - x^*|. \tag{69}$$

By simply replacing $x_{0,1}, x_{0,2}, \ldots, x_{0,n+1}$ by $x_{\mu,1}$, $x_{\mu,2}, \ldots, x_{\mu,n+1}$, $\mu = 1, 2, \ldots$ we complete the induction.

In view of the estimate

$$|x_{m+1}-x^*| \le C\,|x_m - x^*| < r, \quad C = g_m(|x_{0,1}-x^*|) \in [0,\,1), \tag{70}$$

we deduce that $\lim_{m\to\infty} x_m = x^*$ and $x_{m+1} \in U(x^*, r)$.

Finally, to show the uniqueness part, let $Q = \int_0^1 f'(y^* + \theta(x^* - y^*))d\theta$ for some $y^* \in D_1 = D \cap U(x^*, \frac{2}{bL_0})$. Then, using (39) and (40), we get that

$$b|Q - f'(x^*)| \le \frac{bL_0}{2}T < 1. \tag{71}$$

It follows from (71) that $Q \neq 0$. Then, from the identity, $0 = f(x^*) - f(y^*) = Q(x^* - y^*)$, we conclude that $x^* = y^*$. □

Remark 4. (a) In view of (40) condition (42) can be dropped and replaced by $M(t) = 1 + bL_0 t$ or simply by $M = 2$, since $t \in [0, \frac{1}{bL_0})$.

(b) It follows from (35) that the radius of convergence r cannot be larger than the radius of convergence for Newton's method r_A defined by (31). The radius of convergence r_A is found by Argyros in [1,2] that improves upon the radius of convergence given independently by Traub [29] and Rheinboldt [22]

$$r_{TR} = \frac{2}{3bL_1}, \tag{72}$$

where L_1 is the Lipschitz constant on the whole domain D. Notice however that

$$L \le L_1. \tag{73}$$

Consequently, since $L_0 \le L_1$, we have that

$$r_{TR} \le r_A. \tag{74}$$

Hence, our ball is the largest if $L_0 < L_1$ or $L < L_1$. For an example, let us consider the equation $f(x) = e^x - 1 = 0$ on $D = \bar{U}(0,1)$. Since $x^* = 0$, we have $L_0 = e - 1$, $L = e^{\frac{1}{L_0}}$, $b_0 = b = 1$ and $L_1 = e$, so $L_0 < L < L_1$. From (31) and (72), it follows that

$$r_{TR} = 0.2452\ldots < 0.3827\ldots = r_A.$$

(c) The convergence order is found next by using Taylor expansions and hypotheses reaching up to the $n+1$-th derivative. These hypotheses limit the applicability of method (30). As a motivational example, let us define function f on $D = [-1/2, 5/2]$ by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, \ x \neq 0, \\ 0, \ x = 0. \end{cases}$$

Then, we have that

$$f'''(x) = 6 \ln x^2 + 60x^2 - 24x + 22.$$

Then function $f'''(x)$ is not bounded on D. However, the results of Theorem 3 can apply for $L_0 = L = 441$ and $M = 6$. Notice that the order of convergence can be found without using the Taylor expansion using the COC as well as the ACOC (see [12,32]).

(d) Hypothesis (43) can be replaced by

$$b\,|\Delta - f'(x^*)| \leq q < 1 \text{ for some } q \in [0, 1). \tag{75}$$

Then $\Delta \neq 0$, since otherwise $1 \leq q < 1$, which is a contradiction. Define $\bar{\gamma}_2(t) = q$. Then, the conclusions of Theorem 3 hold with (75) and the new $\bar{\gamma}_2$ replacing (43) and the old γ_2, respectively.

Below we obtain the optimal convergence order of general scheme.

Theorem 4. *Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function. Suppose that the hypotheses of Theorem 3 hold. Then the n-point family (30) converges to x^* with at least 2^n-th order.*

Proof. According to Theorem 3 the method (30) is well defined and converges to x^*. Let $e_{i,1} = x_{i,1} - x^*$, $e_{i,2} = x_{i,2} - x^*, \ldots, e_{i,n} = x_{i,n} - x^*$ be the errors in the i-th iteration. In order to find the error of the family (30), first we write error equations of the schemes of convergence order 2, 4, 8 and 16 in the more systemic forms.

The error equation of one-point scheme, which is well-known Newton's iteration, is given by

$$e_{i,2} = C_1 e_{i,1}^2 + O(e_{i,1}^3), \tag{76}$$

where $C_1 = A_2$.

The error equation of two-point scheme (i.e. Ostrowski's iteration) using (3) can be written as

$$e_{i,3} = C_2 C_1 e_{i,1}^{2^2} + O(e_{i,1}^5), \tag{77}$$

where C_2 is the determinant of Toeplitz matrix of order 2 defined as

$$C_2 = \begin{vmatrix} A_2 & 1 \\ A_3 & A_2 \end{vmatrix}.$$

In view of (12) the error equation for three-point eighth scheme can be expressed as

$$e_{i,4} = C_3 C_2 C_1^2 e_{i,1}^{2^3} + O(e_{i,1}^9), \tag{78}$$

where C_3 is the determinant of Toeplitz matrix of lower Hessenberg form of order 3 defined as

$$C_3 = \begin{vmatrix} A_2 & 1 & 0 \\ A_3 & A_2 & 1 \\ A_4 & A_3 & A_2 \end{vmatrix}.$$

From (21) the error equation for four-point sixteenth order scheme of the family (30) can be reproduced as

$$e_{i,5} = C_4 C_3 C_2^2 C_1^{2^2} e_{i,1}^{2^4} + O(e_{i,1}^{17}), \tag{79}$$

where C_4 is the determinant of Toeplitz matrix of lower Hessenberg form of order 4 and given as

$$C_4 = \begin{vmatrix} A_2 & 1 & 0 & 0 \\ A_3 & A_2 & 1 & 0 \\ A_4 & A_3 & A_2 & 1 \\ A_5 & A_4 & A_3 & A_2 \end{vmatrix}.$$

The above process can be easily generalized to write the error equation of n-point family (30). Thus, we can write

$$e_{i,n+1} = e_{i+1} = C_n C_{n-1} C_{n-2}^2 C_{n-3}^{2^2} \cdots C_2^{2^{n-3}} C_1^{2^{n-2}} e_{i,1}^{2^n} + O(e_{i,1}^{2^{n+1}}). \tag{80}$$

Here C_n, C_{n-1} etc. are the determinants of Toeplitz matrices of lower Hessenberg form of order indicated by subscript index. In particular, the determinant C_n is given as

$$C_n = \begin{vmatrix} A_2 & 1 & 0 & \dots & 0 & 0 \\ A_3 & A_2 & 1 & \ddots & \ddots & 0 \\ A_4 & A_3 & A_2 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ A_n & A_{n-1} & A_{n-2} & \ddots & A_2 & 1 \\ A_{n+1} & A_n & A_{n-1} & \dots & \dots & A_2 \end{vmatrix}.$$

The error equation (80) shows the 2^n-th order of convergence. □

Remark 5. **(a)** Note that the family (30) requires $n+1$ function evaluations, namely $f(x_{i,1})$, $f(x_{i,2})$, $f(x_{i,3})$, $\dots$, $f(x_{i,n})$ and $f'(x_{i,1})$ per iteration and possesses convergence order 2^n. Thus, it is optimal in the sense of Kung–Traub hypothesis.

(b) General families of n-point Newton type iterative methods of optimal order of convergence 2^n have also been presented in [20,31]. First step of these schemes is the Newton or Newton-type step, whereas, in each subsequent step first derivative is approximated by using Hermite interpolation. The present family is, therefore, completely different and hence new.

5. Numerical results

In order to demonstrate the convergence behavior and to check the validity of theoretical results of the new methods, here we perform numerical tests. For demonstration let us choose the eighth order (11) and sixteenth order (20) methods, which are now denoted by N8 and N16, respectively. We also compare the methods with some existing optimal order methods. For example, the eighth order methods proposed by Bi–Wu–Ren [4], Thukral [26], Thukral–Petković [27], Cordero–Torregrosa–Vassileva [7] and Khan–Fardi–Sayevand [13], and sixteenth order method by Geum–Kim [10]. These methods are given as follows:

Bi–Wu–Ren Method (BWR8):

$$w_i = x_i - \frac{f(x_i)}{f'(x_i)},$$

$$z_i = w_i - \frac{2f(x_i) - f(w_i)}{2f(x_i) - 5f(w_i)} \frac{f(w_i)}{f'(x_i)},$$

$$x_{i+1} = z_i - \frac{f(x_i) + (\gamma+2) f(z_i)}{f(x_i) + \gamma f(z_i)} \frac{f(z_i)}{f[z_i, w_i] + f[z_i, x_i, x_i](z_i - w_i)},$$

where $\gamma \in \mathbb{R}$ and $f[z_i, x_i, x_i] = \frac{f[z_i,x_i] - f'(x_i)}{z_i - x_i}$.

Cordero–Torregrosa–Vassileva Method (CTV8):

$$\begin{aligned} w_i &= x_i - \frac{f(x_i)}{f'(x_i)}, \\ z_i &= x_i - \frac{f(x_i) - f(w_i)}{f(x_i) - 2f(w_i)} \frac{f(x_i)}{f'(x_i)}, \\ x_{i+1} &= u_i - \frac{3(\beta_2 + \beta_3)(u_i - z_i)}{\beta_1(u_i - z_i) + \beta_2(w_i - x_i) + \beta_3(z_i - x_i)} \frac{f(z_i)}{f'(x_i)}, \end{aligned}$$

where $\beta_i \in \mathbb{R}$ $(i = 1, 2, 3)$, $\beta_2 + \beta_3 \neq 0$ and $u_i = z_i - \frac{f(z_i)}{f'(x_i)} \left(\frac{f(x_i) - f(w_i)}{f(x_i) - 2f(w_i)} + \frac{1}{2} \frac{f(z_i)}{f(w_i) - 2f(z_i)} \right)^2$.

Khan–Fardi–Sayevand Method (KFS8):

$$\begin{aligned} w_i &= x_i - \frac{f(x_i)}{f'(x_i)}, \\ z_i &= w_i - \frac{f^2(x_i)}{f^2(x_i) - 2f(x_i)f(w_i) + \omega f^2(w_i)} \frac{f(w_i)}{f'(x_i)}, \\ x_{i+1} &= z_i - \frac{1}{1 + \nu q_i^2} \frac{f(z_i)}{K - C(w_i - z_i) - D(w_i - z_i)^2}, \end{aligned}$$

where $\omega, \nu \in \mathbb{R}$, $q_i = \frac{f(z_i)}{f(x_i)}$, $D = \frac{f'(x_i) - H}{(x_i - w_i)(x_i - z_i)} - \frac{H - K}{(x_i - z_i)^2}$, $C = \frac{H - K}{(x_i - z_i)} - D(x_i + w_i - 2z_i)$,

$$H = \frac{f(x_i) - f(w_i)}{x_i - w_i}, \; K = \frac{f(w_i) - f(z_i)}{w_i - z_i}.$$

Thukral Method (T8):

$$\begin{aligned} w_i &= x_i - \frac{f(x_i)}{f'(x_i)}, \\ z_i &= x_i - \frac{f(x_i)^2 + f(w_i)^2}{f'(x_i)(f(x_i) - f(w_i))}, \\ x_{i+1} &= z_i - \left(\left(\frac{1 + \mu_i^2}{1 - \mu_i} \right)^2 - 2\mu_i^2 - 6\mu_i^3 + \frac{f(z_i)}{f(w_i)} + 4\frac{f(z_i)}{f(x_i)} \right) \frac{f(z_i)}{f'(x_i)}, \end{aligned}$$

where $\mu_i = \frac{f(w_i)}{f(x_i)}$.

Thukral–Petković Method (TP8):

$$\begin{aligned} w_i &= x_i - \frac{f(x_i)}{f'(x_i)}, \\ z_i &= w_i - \frac{f(x_i) + bf(w_i)}{f(x_i) + (b-2)f(w_i)} \frac{f(w_i)}{f'(x_i)}, \\ x_{i+1} &= z_i - \left[\phi(t) + \frac{f(z_i)}{f(w_i) - af(z_i)} + \frac{4f(z_i)}{f(x_i)} \right] \frac{f(z_i)}{f'(x_i)}, \end{aligned}$$

where $a, b \in \mathbb{R}$, $\phi(t) = 1 + 2t + (5 - 2b)t^2 + (12 - 12b + 2b^2)t^3$ and $t = \frac{f(w_i)}{f(x_i)}$.

Geum–Kim Method (GK16):

$$y_i = x_i - \frac{f(x_i)}{f'(x_i)},$$

$$z_i = y_i - K_f(u_i)\frac{f(y_i)}{f'(x_i)},$$

$$s_i = z_i - H_f(u_i, v_i, w_i)\frac{f(z_i)}{f'(x_i)},$$

$$x_{i+1} = s_i - W_f(u_i, v_i, w_i, t_i)\frac{f(s_i)}{f'(x_i)},$$

where

$$K_f(u_i) = \frac{1-9u_i^2}{1-2u_i-4u_i^2}, \quad H_f(u_i, v_i, w_i) = \frac{1+2u_i}{1-v_i-2w_i},$$

$$W_f(u_i, v_i, w_i, t_i) = \frac{1+2u_i}{1-v_i-2w_i-t_i} + G(u_i, v_i, w_i),$$

$$G(u_i, v_i, w_i) = -6u_i^3 v_i + 6w_i^2 - 4u_i^4(3v_i + 17w_i) + u_i(2v_i^2 + 4v_i^3 + w_i - 2w_i^2),$$

$$u_i = \frac{f(y_i)}{f(x_i)}, \quad v_i = \frac{f(z_i)}{f(y_i)}, \quad w_i = \frac{f(z_i)}{f(x_i)}, \quad t_i = \frac{f(s_i)}{f(z_i)}.$$

For comparison let us consider Kepler's equation

$$f(x) = x - \alpha \sin(x) - K = 0,$$

where $0 \le \alpha < 1$ and $0 \le K \le \pi$.

A numerical study, for different values of α and K has been performed in [8]. Using (39)–(42), we get $L = L_0 = L_1 = \alpha$, $M = 1 + \alpha$, $b_0 = 1 - \alpha\cos(x^*)$, $b = \frac{1}{b_0}$. As a specific numerical example, let us take $\alpha = 0.9$ and $K = 0.1$. In this case the solution is $x^* = 0.63084352756315343\ldots$. Therefore, we have, $L = L_0 = L_1 = 0.9$, $M = 1.9$, $b_0 \approx 0.2732$ and $b \approx 3.6600$.

Next, we shall determine the convergence radius r, say for $n = 4$, so that we can choose initial points from the convergence ball $U(x^*, r)$. According to (34), we must compute r_A, r_3, r_4 and r_5. From (31), we obtain $r_A \approx 0.2024$. The parameter r_3 is the smallest zero of $h_3(t)$ expressed in (32). Thus, solving $h_3(t) = g_3(t) - 1 = 0$, we get the smallest zero $r_3 \approx 0.0230$. The parameters r_4 and r_5 are the smallest zeros of $h_m(t)$, $m = 4, 5$, expressed in (33). Thus, solving $h_m(t) = g_m(t) - 1 = 0$ taking $m = 4$, 5, we get the corresponding zeros as $r_4 \approx 0.0861$ and $r_5 \approx 0.0858$. Then, by (34)

$$r = \min\{r_A, r_3, r_4, r_5\} = \min\{0.2024, 0.0230, 0.0861, 0.0858\} = 0.0230.$$

Theorem 3 guarantees the convergence of method (30) to $x^* = 0.63084352756315343\ldots$ provided that $x_0 \in U(x^*, r)$. This condition yields very close initial approximation. We solve the Kepler's equation by selecting different initial approximations even from the outside of our convergence ball. For the parameters used in BWR8, CTV8, KFS8, TP8 we choose the same values as considered by the respective authors in their numerical work. To verify the theoretical order of convergence, we calculate the computational order of convergence COC using the formula [12]

$$\mathrm{COC} = \frac{\log|f(x_i)/f(x_{i-1})|}{\log|f(x_{i-1})/f(x_{i-2})|},$$

Table 1
Comparison of performance of methods.

Methods	$\lvert x_2 - x_1 \rvert$	$\lvert x_3 - x_2 \rvert$	$\lvert x_4 - x_3 \rvert$	i	COC	CPU-time
$x_0 = -1$						
BWR8 ($\gamma = 1$)	667.09	651.72	1.01	8	8.000	1.9937
CTV8 ($\beta_1 = 0, \beta_2 = 1, \beta_3 = 0$)	2.79	1.23(−1)	3.62(−8)	6	8.000	1.4602
KFS8 ($\omega = 1, \nu = 1$)	6.73(−1)	5.44(−10)	1.38(−74)	5	8.000	1.1722
T8	2.25(−1)	3.55(−5)	6.81(−35)	5	8.000	1.0639
TP8 ($a = 0, b = 0$)	5.30(−1)	2.43(−3)	1.77(−20)	6	8.000	1.4274
N8	1.60(−1)	4.26(−8)	1.31(−61)	5	8.000	0.5887
GK16	Failure	–	–	–	–	–
N16	1.18(−1)	1.77(−18)	1.17(−287)	4	16.000	0.4898
$x_0 = 0$						
BWR8 ($\gamma = 1$)	3.32(−2)	1.79(−12)	1.64(−94)	5	8.000	1.2168
CTV8 ($\beta_1 = 0, \beta_2 = 1, \beta_3 = 0$)	9.96(−5)	3.99(−33)	3.36(−260)	4	8.000	0.9563
KFS8 ($\omega = 1, \nu = 1$)	2.81(−2)	6.07(−13)	3.33(−98)	5	8.000	1.1575
T8	14.76	1.52(−1)	2.69(−6)	6	8.000	1.2636
TP8 ($a = 0, b = 0$)	8.01(−2)	4.71(−8)	3.54(−58)	5	8.000	1.1591
N8	9.20(−5)	6.16(−35)	2.52(−276)	4	8.000	0.4832
GK16	1.43(7)	6.62(5)	1.65(5)	12	16.000	2.8017
N16	8.66(−9)	1.72(−132)	2.79(−2110)	4	16.000	0.4281
$x_0 = 1.5$						
BWR8 ($\gamma = 1$)	3.46(−3)	3.26(−20)	1.98(−156)	5	8.000	1.2324
CTV8 ($\beta_1 = 0, \beta_2 = 1, \beta_3 = 0$)	6.35(−3)	1.36(−18)	6.20(−144)	5	8.000	1.2527
KFS8 ($\omega = 1, \nu = 1$)	7.44(−3)	1.63(−17)	8.99(−135)	5	8.000	1.1606
T8	2.74(−2)	7.27(−12)	2.11(−88)	5	8.000	1.0577
TP8 ($a = 0, b = 0$)	1.93(−2)	2.49(−13)	2.18(−100)	5	8.000	1.1653
N8	2.28(−3)	8.21(−24)	2.50(−187)	5	8.000	0.5211
GK16	2.17(−3)	1.08(−38)	1.32(−603)	4	16.000	0.8908
N16	2.22(−7)	6.05(−110)	2.31(−1751)	4	16.000	0.4886
$x_0 = 2.2$						
BWR8 ($\gamma = 1$)	1.78(−3)	1.59(−22)	6.19(−175)	5	8.000	1.2215
CTV8 ($\beta_1 = 0, \beta_2 = 1, \beta_3 = 0$)	4.91(−2)	1.55(−11)	1.75(−87)	5	8.000	1.2324
KFS8 ($\omega = 1, \nu = 1$)	4.97(−2)	5.27(−11)	1.07(−82)	5	8.000	1.1744
T8	1.30(−1)	8.88(−7)	1.05(−47)	5	8.000	1.1012
TP8 ($a = 0, b = 0$)	1.09(−1)	1.27(−7)	1.01(−54)	5	8.000	1.2223
N8	3.25(−2)	3.57(−15)	3.15(−118)	5	8.000	0.6252
GK16	1.41(−2)	1.59(−25)	6.57(−393)	4	16.000	0.9148
N16	3.32(−4)	3.74(−59)	6.64(−1877)	4	16.000	0.5578
$x_0 = 3.5$						
BWR8 ($\gamma = 1$)	1.59(−1)	1.75(−7)	1.35(−54)	5	8.000	1.9114
CTV8 ($\beta_1 = 0, \beta_2 = 1, \beta_3 = 0$)	6.67(−1)	2.39(−3)	5.64(−22)	6	8.000	2.1674
KFS8 ($\omega = 1, \nu = 1$)	1.17(−1)	3.47(−8)	3.82(−60)	5	8.000	1.8143
T8	2.66(−1)	1.01(−4)	2.83(−31)	5	8.000	1.6115
TP8 ($a = 0, b = 0$)	2.58(−1)	4.36(−5)	1.94(−34)	5	8.000	1.7995
N8	1.19(−1)	5.48(−10)	9.76(−77)	5	8.000	0.7284
GK16	2.86(−3)	7.35(−37)	2.89(−574)	4	16.000	1.3650
N16	7.02(−3)	4.80(−38)	1.37(−600)	4	16.000	0.6528

taking into consideration the last three approximations in the iterative process. In numerical results, we also include CPU time (measured in seconds) used in the execution of program which is computed by the Mathematica command "TimeUsed[]".

The absolute errors $|x_{i+1} - x_i|$ in the first three iterations are displayed in Table 1, where $a(-b)$ denotes $a \times 10^{-b}$ and $a(b)$ denotes $a \times 10^{b}$. The necessary iterations (i), the computational order of convergence COC and the mean elapsed CPU-time (CPU-time) are also presented in the table. The necessary iterations (i) and the CPU-time are calculated by selecting $|x_{i+1} - x_i| + |f(x_i)| < 10^{-200}$ as the stopping criterion. Mean CPU-time is calculated by taking the mean of 100 performances of each program.

From the numerical results displayed in Table 1, we can observe that, in general, the accuracy in numerical values of approximations to the root by the present algorithms N8 and N16 is higher than the existing algorithms. Calculated values of the computational order of convergence COC also verify the theoretical order of convergence proved in Section 2. From the values of last column we can observe that the new methods utilize less computing time in the execution of program than the existing methods of same nature.

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This verifies the highly efficient nature of the present methods. In fact, speaking about the highly efficient nature of some iterative method, we mean that the iterative method uses the smallest CPU-time. Similar numerical experimentations, carried out for a number of problems of different type, confirmed the above conclusions to a large extent.

6. Conclusions

Based on the optimal two-point fourth order Ostrowski's scheme, we have developed a three-point method of optimal order eight for solving nonlinear equations. Then, based on this three-point method a four-point method of optimal sixteen order is developed. In both the methods the new step for obtaining approximation to a root is generated by using rational Hermite interpolation. This approach of employing previously obtained scheme and generating the new step by using rational Hermite interpolation is applied to obtain a generalized n-point scheme of optimal order 2^n. The rational approximant is such that its numerator is a linear function whereas denominator is a polynomial of degree $n-1$, that means, the rational approximant of order $[1/(n-1)]$. For example, in case of three-point scheme this approximant is of order $[1/2]$. It has been seen that Newton's and Ostrowski's methods are special cases corresponding to $n=1$ and $n=2$.

The proposed methods are compared with existing optimal order methods through numerical experimentation. Superiority of presented methods over the existing methods is corroborated by numerical results including CPU-time utilized in the execution of program. Finally, we conclude that the methods presented in this paper are preferable to other recognized optimal methods because of simple design and better computational efficiency.

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