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Multiple and nodal solutions for parametric Neumann problems with nonhomogeneous differential operator and critical growth ☆

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A B S T R A C T

We consider a parametric Neumann problem with nonhomogeneous differential operator and critical growth. Combining variational methods based on critical point theory, with suitable truncation techniques and flow invariance arguments, we show that for all large λ, the problem has at least three nontrivial smooth solutions, two of constant sign (one positive, the other negative) and the third nodal. We also study the asymptotic behavior of all solutions obtained when λ converges to infinity. The interesting point is that we do not impose any restrictions to the behavior of the nonlinear term f at infinity. Our work unifies and sharply improves several recent papers.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$, $1 < p < N$, and $p^\star = \frac{Np}{N-p}$. In this paper we study the following nonlinear parametric Neumann problem:

$$\begin{cases} -\mathrm{div} A(x, \nabla u) + \beta(x)\,|u|^{p-2}\,u = \lambda f(x,u) + g(x)\,|u|^{p^\star-2}\,u & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where $A : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous, strictly monotone map that satisfies certain regularity conditions. The precise hypotheses on the map are listed in the hypotheses H(a) below. These conditions incorporate in our framework many differential operators of interest such as the p-Laplacian. We stress that the differential

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operator need not be homogeneous and this is a source of difficulties, especially when we look for nodal (that is, sign changing) solutions. In problem (P_λ), $\lambda > 0$ is a parameter and in the reaction $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \overline{\Omega} \to \mathbb{R}$ are assumed to be continuous functions with $g(x) \geq 0$ in $\overline{\Omega}$. Finally, we mention that the $\beta \in L^\infty(\Omega), \beta(x) \geq 0$ a.e. in $\Omega, \beta \neq 0$, and in the boundary condition n denotes the outward unit normal vector on $\partial\Omega$. Our aim is to prove a multiplicity theorem for problem (P_λ) providing sign information for all solutions provided $\lambda > 0$ is sufficiently large.

Equations driven by nonhomogeneous differential operators have been widely investigated in the subcritical case by variational methods under Dirichlet [1,17,31–33], Neumann [2,13,16,19,23,25,29], or Robin [28] boundary condition. We mention that the works [2,13,17,19,23,29,32,33] produce nodal solutions. On the other hand, it is generally hard to handle nonlinear nonhomogeneous equations without the subcritical growth condition, and thus, the results in the direction are very rare (see [18,26,30]). In [18,30], the right hand side nonlinearity is assumed to be odd near zero, and the authors produced a whole sequence of distinct nodal solutions. Based on variational methods combining invariant sets of descending flow, Motreanu and Tanaka [26] obtained the existence of a positive solution, a negative solution and a sign-changing solution for equation (P_λ) with $\beta = g = 0$ if $\lambda > 0$ is sufficiently large. They assumed that problem (P_λ) admits an ordered pair of super and lower solution. In the present paper we prove a similar three-solutions-theorem for problem (P_λ) providing sign information for all solutions obtained. Moreover, we obtain the asymptotic behavior of the three solutions when λ converges to infinity. The interesting feature of our work here, is that in problem (P_λ) the nonlinearity f satisfies a superlinear growth condition just in a neighborhood of zero. By using variational methods together with suitable truncation techniques and flow invariance arguments, we are able to avoid restrictions on the behavior of the nonlinearity f at infinity. Then we can handle nonlinearities $f(x,u)$ containing terms like $|u|^{r-2}u$ and $|u|^{r-2}ue^u$ with $p < r < \infty$.

Throughout this paper, we assume that the map A and the function f satisfy the following hypotheses H(a) and H(f), respectively:

H(a). $A(x,y) = h(x,|y|)y$, where $h(x,t) > 0$ for all $(x,t) \in \overline{\Omega} \times (0,+\infty)$, and

(i) $A \in C^{0,\epsilon}_{\text{loc}}(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^N) \cap C^1(\overline{\Omega} \times \mathbb{R}^N \setminus \{0\}, \mathbb{R}^N)$ with some $0 < \epsilon \leq 1$;

(ii) there exist constants $C_1 > 0$ and $1 < p < +\infty$ such that

$$\left|\nabla_y A(x,y)\right| \leq C_1 |y|^{p-2} \quad \text{for every } x \in \overline{\Omega}, \text{ and } y \in \mathbb{R}^N \setminus \{0\};$$

(iii) there exists $C_0 > 0$ such that

$$(\nabla_y A(x,y)\xi, \xi)_{\mathbb{R}^N} \geq C_0 |y|^{p-2} |\xi|^2 \quad \text{for every } x \in \overline{\Omega},\ y \in \mathbb{R}^N \setminus \{0\} \text{ and } \xi \in \mathbb{R}^N;$$

(iv) for all $(x,y) \in \overline{\Omega} \times \mathbb{R}^N$, we have

$$pG(x,y) \geq (A(x,y),y)_{\mathbb{R}^N},$$

where $G(x,y)$ is the primitive of $A(x,y)$, i.e., $\nabla_y G(x,y) = A(x,y)$ for every $x \in \overline{\Omega}, y \in \mathbb{R}^N$, and $G(x,0) = 0$.

In the above hypotheses by $|\cdot|$ we denote the Euclidean norm in $\mathbb{R}^N$. And the notation $\nabla_y A$ means the differential of the mapping $A(x,y)$ with respect to the variable $y \in \mathbb{R}^N$. Similar conditions are used widely in the literature (see, e.g., [1,9,24,25,31]).

H(f). $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function with primitive $F(x,t) = \int_0^t f(x,s)\mathrm{d}s$ satisfying $f(x,0) = 0$ for a.a. $x \in \Omega$ and

(i) there exists $\tau \in (p, p^\star)$ such that $\lim_{t\to 0} \frac{f(x,t)}{|t|^{\tau-2}t} = 0$ uniformly for a.a. $x \in \Omega$;

(ii) there exists $r \in (p, p^\star)$ with $r > \tau$ such that $\lim_{t\to 0} \frac{F(x,t)}{|t|^r} = \infty$ uniformly for a.a. $x \in \Omega$;

(iii) there exist $\mu \in (p, p^\star)$ and $\delta > 0$ such that

$$0 < \mu F(x,t) \le t f(x,t) \quad \text{for a.a. } x \in \Omega \text{ and all } 0 < t \le \delta;$$

(iv) there exists $m > 0$ such that for a.a. $x \in \Omega$, the function $t \mapsto f(x,t) + m|t|^{p-2}t$ is nondecreasing on $[-\delta, \delta]$, where δ is as in (iii).

The main result of this paper is the following (for the precise meaning of the notation refer to Section 2).

Theorem 1. *Assume that hypotheses* H(a) *and* H(f) *hold. Then, there exists $\lambda_\star > 0$ such that for any $\lambda \ge \lambda_\star$, problem (P_λ) admits at least three nontrivial smooth solutions*

$$u_{\lambda,1} \in \mathrm{int}C_+, \quad u_{\lambda,2} \in -\mathrm{int}C_+ \quad \textit{and} \quad u_{\lambda,3} \in C^1(\overline{\Omega}) \textit{ nodal},$$

satisfying

$$\lim_{\lambda\to+\infty} \|u_{\lambda,i}\| = 0, \quad i = 1, 2, 3.$$

Moreover, problem (P_λ) has extremal nontrivial constant sign solutions.

It is worth pointing out that similar results to ours with critical growth can be seen in [3,5,12,14]. In all these works, the nonlinearity f satisfies

$$\lim_{t\to+\infty} \frac{f(x,t)}{t^{r-1}} = 0 \tag{1}$$

uniformly in $x \in \Omega$, where $r \in (p, p^\star)$. By using the well-known concentration-compactness principle of Lions, they derived the existence of a positive solution for all $\lambda \ge \lambda_\star$ with $\lambda_\star > 0$. In our results, we drop the hypothesis (1) and add two solutions (one negative, the other nodal). So our results can be considered as a significant extension of the above mention papers in the sense that we are considering only superlinear conditions in a neighborhood of the origin.

In the next section we recall various notions and results which will be used later. In Section 3, we prove the existence of two constant-sign solutions for problem (P_λ). Finally, in Section 4, we prove the existence of a nodal solution for problem (P_λ).

2. Mathematical background

In the analysis of problem (P_λ) in addition to the Sobolev space $W^{1,p}(\Omega)$, we will also use the Banach space $C^1(\overline{\Omega})$. This is an ordered Banach space with positive cone $C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \ge 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior given by

$$\mathrm{int}C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

The inner product in $\mathbb{R}^N$ and the usual norm in $L^s(\Omega)$ $(1 \le s \le +\infty)$ will be denoted by $(\cdot,\cdot)_{\mathbb{R}^N}$ and $\|\cdot\|_s$, respectively. In what follows the Sobolev space $W^{1,p}(\Omega)$ is endowed with the norm

$$\|u\| = \big(\|\nabla u\|_p^p + \int_\Omega |u(x)|^p \,\mathrm{d}x\big)^{\frac{1}{p}} \quad \text{for all } u \in W^{1,p}(\Omega).$$

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Fixing $s \in [1, p^\star]$, by the Sobolev Embedding Theorem, there exists a positive constant κ_s such that

$$\|u\|_s \le \kappa_s \|u\|, \quad u \in W^{1,p}(\Omega). \tag{2}$$

In the sequel, by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Given $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then for $u \in W^{1,p}(\Omega)$, we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W^{1,p}(\Omega), \qquad |u| = u^+ + u^- \qquad \text{and} \qquad u = u^+ - u^-.$$

Also, if $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function (for example, a Carathéodory function), then we set

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \quad \text{for all } u \in W^{1,p}(\Omega).$$

Using the hypotheses H(a), we can easily prove the following lemma that summarizes some significant facts regarding the operator A.

Lemma 2. *If hypotheses* H(a) *hold, then*

(i) *for all $x \in \overline{\Omega}, y \to A(x, y)$ is maximal monotone and strictly monotone;*
(ii) *for all $(x, y) \in \overline{\Omega} \times \mathbb{R}^N, |A(x, y)| \le \frac{C_1}{p-1} |y|^{p-1}$;*
(iii) *for all $x \in \overline{\Omega} \times \mathbb{R}^N, (A(x, y), y)_{\mathbb{R}^N} \ge \frac{C_0}{p-1} |y|^p$.*

A straightforward consequence of the above Lemma, is the following result:

Lemma 3. *If hypotheses* H(a) *hold, then for all $(x, y) \in \overline{\Omega} \times \mathbb{R}^N$ we have*

$$\frac{C_0}{p(p-1)} |y|^p \le G(x, y) \le \frac{C_1}{p(p-1)} |y|^p .$$

Now, let $V : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^\star$ be the nonlinear map defined by

$$\langle V(u), v\rangle = \int_\Omega (A(x, \nabla u), \nabla v)_{\mathbb{R}^N} \mathrm{d}x, \quad \text{for all } u, v \in W^{1,p}(\Omega). \tag{3}$$

Here, $\langle \cdot, \cdot\rangle$ denotes the duality pairing between $W^{1,p}(\Omega)$ and its dual $(W^{1,p}(\Omega))^\star$. The following result mentioning an essential property of the differential operator $\mathrm{div}A(x, \nabla(\cdot))$ corresponding to the map A is important for the proof of the Palais–Smale condition for the Euler functional associated to problem (P_λ).

Proposition 4. *(See [15].) If hypotheses* H(a) *hold, then the nonlinear map $V : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^\star$ defined by (3) is maximal monotone, strictly monotone and of type $(S)_+$ (i.e., if $w_n \to w$ weakly in $W^{1,p}(\Omega)$ and*

$$\limsup_{n\to\infty} \langle V(w_n), w_n - w\rangle \le 0,$$

then $w_n \to w$ in $(W^{1,p}(\Omega))^\star$).

The following lemma can be found in [27, Lemma 4.11] and will be useful in the estimations that follow.

Lemma 5. *If $\beta \in L^\infty(\Omega)$, $\beta \geq 0$ a.e. in Ω, and $\beta \neq 0$, then there exists $\xi_0 > 0$ such that*

$$\|\nabla u\|_p^p + \int_\Omega \beta\, |u|^p \,\mathrm{d}x \geq \xi_0 \,\|u\|^p \quad \text{for all } u \in W^{1,p}(\Omega).$$

This lemma leads to the introduction of

$$\lambda_1 = \inf\Big\{ \frac{\|\nabla u\|_p^p + \int_\Omega \beta\, |u|^p \,\mathrm{d}x}{\|u\|_p^p} : u \in W^{1,p}(\Omega), u \neq 0 \Big\},$$

which is the first eigenvalue of

$$\begin{cases} -\Delta_p u(x) + \beta(x)\,|u(x)|^{p-2}\,u(x) = \lambda\,|u(x)|^{p-2}\,u(x), & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases}$$

We know that λ_1 is simple and is the only eigenvalue with eigenfunctions of constant sign. All the higher eigenvalues have nodal eigenfunctions. We denote by $\varphi_1 \in C_+ \setminus \{0\}$ the positive eigenfunction corresponding to λ_1. Note that $\beta \in L^\infty(\Omega)$. It is also known that $\varphi_1 \in \mathrm{int} C_+ \cap C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ (see [36]).

Since hypotheses H(f)(i)–(iii) give the behavior of f just in a neighborhood of zero, the functional

$$\int_\Omega (\lambda F(x,u)\mathrm{d}x + \frac{1}{p^\star} g(x)\,|u|^{p^\star})\mathrm{d}x$$

is not well defined in $W^{1,p}(\Omega)$. To overcome this difficulty, we use here a penalization technique in the spirit of the argument developed in [8] to obtain a new functional well defined in $W^{1,p}(\Omega)$. For this purpose, we first observe that hypotheses H(f)(i)–(ii) imply that for $x \in \Omega$ and $|t|$ small,

$$\big|f(x,t)\big| \leq \big|t\big|^{\tau-1}, \quad F(x,t) \leq \frac{1}{\tau}\,|t|^\tau \tag{4}$$

and

$$F(x,t) \geq |t|^r\,. \tag{5}$$

It is clear that for $|t|$ small,

$$\frac{|t|^{p^\star}}{p^\star} \leq \frac{1}{\tau}\,|t|^\tau\,. \tag{6}$$

Now, let $\rho(t) \in C^1(\mathbb{R},[0,1])$ be an even cut-off function verifying $t\rho'(t) \leq 0, |t\rho'(t)| \leq \frac{2}{\sigma}$ and

$$\rho(t) = \begin{cases} 1 & \text{if } |t| \leq \sigma, \\ 0 & \text{if } |t| \geq 2\sigma, \end{cases}$$

where $\sigma \in (0, \frac{\delta}{2})$ is chosen such that (4)–(6) and H(f)(iii) hold for $|t| \leq 2\sigma$. Using ρ, we define

$$\tilde{F}(x,t) = \rho(t)F(x,t) + (1-\rho(t))\frac{|t|^\tau}{\tau}, \qquad \tilde{H}(x,t) = \rho(t)g(x)\frac{|t|^{p^\star}}{p^\star} + (1-\rho(t))\frac{|t|^\tau}{\tau}$$

and $\tilde{f}(x,t) = \frac{\partial}{\partial t}\tilde{F}(x,t), \tilde{h}(x,t) = \frac{\partial}{\partial t}\tilde{H}(x,t)$. We now introduce the following auxiliary problem

$$\begin{cases} -\mathrm{div} A(x,\nabla u) + \beta(x)\,|u|^{p-2}\,u = \lambda \tilde{f}(x,u) + \tilde{h}(x,u) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Let $J_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional associated to problem (7), defined by

$$J_\lambda(u) = \int_\Omega G(x,\nabla u)\mathrm{d}x + \frac{1}{p}\int_\Omega \beta\,|u|^p\,\mathrm{d}x - \lambda\int_\Omega \tilde{F}(x,u)\mathrm{d}x - \int_\Omega \tilde{H}(x,u)\mathrm{d}x, \quad u \in W^{1,p}(\Omega). \tag{8}$$

Then $J_\lambda \in C^1(W^{1,p}(\Omega),\mathbb{R})$, and the derivative of J is given by

$$\langle J_\lambda'(u), v\rangle = \int_\Omega (A(x,\nabla u),\nabla v)_{\mathbb{R}^N}\,\mathrm{d}x + \int_\Omega \beta\,|u|^{p-2}\,uv\mathrm{d}x - \lambda\int_\Omega \tilde{f}(x,u)v\mathrm{d}x - \int_\Omega \tilde{h}(x,u)v\mathrm{d}x \tag{9}$$

for $u, v \in W^{1,p}(\Omega)$. Thus weak solutions of problem (7) are critical points of the functional J_λ on $W^{1,p}(\Omega)$. We note that critical points of J_λ with L^∞ norm less than or equal to σ are also solutions of the original problem (P_λ).

In order to study the critical points of J_λ, we now recall an abstract critical point theorem. Let E be a real Banach space and $J \in C^1(E,\mathbb{R})$. We say that J satisfies the Palais–Smale condition ((PS) condition for short) if for every sequence $\{u_n\} \subset E$ such that $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$, there exists a subsequence of $\{u_n\}$ which is convergent in E.

Lemma 6. *(Mountain pass lemma; see [34].) Let E be a real Banach space and $J \in C^1(E,\mathbb{R})$ with $J(0) = 0$ satisfy the (PS) condition. Suppose that*

(i) *there exist constants $\rho > 0, a > 0$ such that $J(u) \geq a$ for all $u \in \partial B_\rho$, where $B_\rho = \{u \in E : \|u\| < \rho\}$, ∂B_ρ denotes the boundary of B_ρ;*
(ii) *there exists $e \in E$ such that $\|u\| > \rho$ and $J(e) \leq 0$.*

Then J possesses a critical value $c \geq a$ and

$$c = \inf_{h\in\Gamma}\max_{s\in[0,1]} J(h(s)),$$

where $\Gamma = \{h \in C([0,1],E) : h(0) = 0, h(1) = e\}$.

3. Solutions of constant sign

In this section, first we establish the existence of at least two nontrivial constant sign smooth solutions (one positive and the other negative). Then we show that in fact we have "extremal" constant sign solutions, i.e., there exist a smallest nontrivial positive solution and a biggest nontrivial negative solution.

Lemma 7. *Assume that* H(f) *be satisfied. Then*

(i) *there exists a constant $C > 0$ such that*

$$\left|\tilde{f}(x,t)\right| \leq C\,|t|^{\tau-1}, \quad \left|\tilde{h}(x,t)\right| \leq C\,|t|^{\tau-1} \quad \text{for any } x \in \Omega \text{ and } t \in \mathbb{R}.$$

(ii) *it holds that*

$$0 < \theta\tilde{F}(x,t) \leq t\tilde{f}(x,t), \quad 0 < \theta\tilde{H}(x,t) \leq t\tilde{h}(x,t) \quad \text{for any } x \in \Omega \text{ and } t \in \mathbb{R}\backslash\{0\},$$

where $\theta = \min\{\mu, \tau\}$.

Proof. It follows from the definition of $\tilde{f}$ that

$$t\tilde{f}(x,t) = t\rho(t)f(x,t) + t\rho'(t)\big(F(x,t) - \frac{|t|^\tau}{\tau}\big) + (1-\rho(t))\,|t|^\tau \tag{10}$$

Since $|t\rho'(t)| \leq \frac{2}{\sigma}$, by (4) we get $\left|\tilde{f}(x,t)\right| \leq C\,|t|^{\tau-1}$. Similarly, we see that $\left|g(x)\tilde{h}(x,t)\right| \leq C\,|t|^{\tau-1}$.

In order to (ii) we observe that

$$\theta\tilde{F}(x,t) \leq \frac{\theta}{\mu}\rho(t)tf(x,t) + \theta(1-\rho(t))\frac{|t|^\tau}{\tau} \leq \rho(t)tf(x,t) + (1-\rho(t))\,|t|^\tau$$

(see (H(f)(iii)). This together with (10) implies that

$$\theta\tilde{F}(x,t) - t\tilde{f}(x,t) \leq -t\rho'(t)\big(F(x,t) - \frac{|t|^\tau}{\tau}\big) \leq 0$$

(see (4) and recall that $t\rho'(t) \leq 0$). Similarly, we see that $0 < \theta\tilde{H}(x,t) \leq t\tilde{h}(x,t)$. Thus, the proof the lemma is over. □

In order to show that solutions of penalized problem (7) are solutions of the original problem (P_λ), we will use the following L^∞ estimate. From now on, it will be assumed that $\lambda \geq 1$ (which is not very restrictive since we are looking for nontrivial solutions when λ is large). And we denote by $C, C_0, C_1, , \cdots$ positive (possibly different) constants which do not depend on λ.

Lemma 8. *If* $u \in W^{1,p}(\Omega)$ *is a solution of problem (7), then* $u \in L^\infty(\Omega)$ *and there exists a constant* $C = C(\tau, N, \Omega) > 0$ *such that*

$$\|u\|_\infty \leq C\lambda^{\frac{1}{p^\star-\tau}}\,\|u\|^{\frac{p^\star-p}{p^\star-\tau}}.$$

Proof. The proof relies on the Moser iteration technique (cf. [7,20]). Let $u \in W^{1,p}(\Omega)$ be a solution of problem (7). We can assume, without lost of generality, that u is nonnegative. Otherwise, we argue with the positive and negative parts of u separately. We define $u_T(x) := \min\{u(x), T\}$ for $T > 0$. It is clear that $0 \leq u_T \leq u, \nabla u\nabla u_T \geq 0$ and $\left|\nabla u_T\right| \leq \left|\nabla u\right|$. For $\alpha > 1$, we note $\varphi = u_T^{p(\alpha-1)}u$ and $\psi = u_T^{\alpha-1}u$. Then, by taking as test function φ and using Lemma 7, we have

$$\begin{aligned}\int_\Omega (A(x,\nabla u), \nabla\varphi)_{\mathbb{R}^N}\mathrm{d}x + \int_\Omega \beta|u|^{p-2}u\varphi\mathrm{d}x &= \lambda\int_\Omega \tilde{f}(x,u)\varphi\mathrm{d}x + \int_\Omega \tilde{h}(x,u)\varphi\mathrm{d}x \\ &\leq 2\lambda C\int_\Omega u^{\tau-p}\psi^p\mathrm{d}s.\end{aligned} \tag{11}$$

By Lemma 2 and the definition of u_T, we have

$$\begin{aligned}\int_\Omega (A(x,\nabla u), \nabla\varphi)_{\mathbb{R}^N}\mathrm{d}x &\geq \frac{C_0}{p-1}\int_\Omega u_T^{p(\alpha-1)}|\nabla u|^p\mathrm{d}x + \frac{C_0 p(\alpha-1)}{p-1}\int_{\{u\leq T\}} u^{p(\alpha-1)}|\nabla u|^p\mathrm{d}x \\ &\geq \frac{C_0}{p-1}\int_\Omega u_T^{p(\alpha-1)}|\nabla u|^p\mathrm{d}x.\end{aligned} \tag{12}$$

Since $\int_\Omega \beta|u|^{p-2}u\varphi \mathrm{d}x \geq 0$, from (11) and (12) we see that

$$\int_\Omega u_T^{p(\alpha-1)}|\nabla u|^p \mathrm{d}x \leq \lambda C_1 \int_\Omega u^{\tau-p}\psi^p \mathrm{d}s. \tag{13}$$

By (11) and (12) again, we have

$$\int_\Omega \beta|u|^{p-2}u\varphi \mathrm{d}x \leq 2\lambda C \int_\Omega u^{\tau-p}\psi^p \mathrm{d}s. \tag{14}$$

From (2), (13), (14) and Lemma 5, it follows the inequality

$$\begin{aligned}
\Big(\int_\Omega \psi^{p^\star}\mathrm{d}x\Big)^{\frac{p}{p^\star}} &\leq C_2\|\psi\|^p \leq C_3\int_\Omega |\nabla\psi|^p \mathrm{d}x + C_3\int_\Omega \beta|\psi|^p \mathrm{d}x \\
&\leq C_3\int_\Omega \big(u_T^{p(\alpha-1)}|\nabla u|^p + (\alpha-1)^p u^p u_T^{p(\alpha-2)}|\nabla u_T|^p\big)\mathrm{d}x + 2\lambda C C_3\int_\Omega u^{\tau-p}\psi^p \mathrm{d}s \\
&\leq C_4\alpha^p \int_\Omega u_T^{p(\alpha-1)}|\nabla u|^p \mathrm{d}x + 2\lambda C C_3 \int_\Omega u^{\tau-p}\psi^p \mathrm{d}s \\
&\leq \lambda C_5 \alpha^p \int_\Omega u^{\tau-p}\psi^p \mathrm{d}x
\end{aligned}$$

(since $\alpha^p \geq 1$ and $1 + (\alpha-1)^p \leq \alpha^p$ for $\alpha \geq 1$). We now use Hölder's inequality, with exponents $\frac{p^\star}{\tau-p}$ and $\frac{p^\star}{p^\star-(\tau-p)}$, to obtain

$$\Big(\int_\Omega \psi^{p^\star}\mathrm{d}x\Big)^{\frac{p}{p^\star}} \leq \lambda C_5\alpha^p \Big(\int_\Omega u^{p^\star}\mathrm{d}x\Big)^{\frac{\tau-p}{p^\star}} \Big(\int_\Omega \psi^{\frac{pp^\star}{p^\star-(\tau-p)}}\mathrm{d}x\Big)^{\frac{p^\star-(\tau-p)}{p^\star}}.$$

By (2), we obtain

$$\|\psi\|_{p^\star}^p \leq \lambda C_6 \alpha^p \|u\|^{\tau-p}\|\psi\|_{\gamma^\star}^p,$$

where $\gamma^\star = \frac{pp^\star}{p^\star-(\tau-p)}$. Considering $\alpha = 1 + \frac{p^\star-\tau}{p}$, we obtain

$$\Big(\int_\Omega |u_T^{\alpha-1}u|^{p^\star}\mathrm{d}x\Big)^{\frac{p}{p^\star}} \leq \lambda C_6\alpha^p\|u\|^{\tau-p}\Big(\int_\Omega u^{\alpha\gamma^\star}\mathrm{d}x\Big)^{\frac{p}{\gamma^\star}}.$$

We now apply the Fatou's lemma to the variable T to obtain

$$\|u\|_{\alpha p^\star}^{\alpha p} \leq \lambda C_6\alpha^p\|u\|^{\tau-p}\,\|u\|_{\alpha\gamma^\star}^{\alpha p}$$

and so

$$\|u\|_{\alpha p^\star} \leq \big(\lambda C_6\alpha^p\|u\|^{\tau-p}\big)^{\frac{1}{\alpha p}}\,\|u\|_{\alpha\gamma^\star}. \tag{15}$$

Taking $\alpha_0 = \alpha$ and inductively $\alpha_{n+1} = \frac{p^\star\alpha_n}{\gamma^\star}$ for $n = 0, 1, 2, \cdots$, and applying the previous processes for α_1, by (15) we have

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$$\begin{aligned}\|u\|_{\alpha_1 p^\star} &\le \left(\lambda C_6 \alpha_1^p \|u\|^{\tau-p}\right)^{\frac{1}{\alpha_1 p}} \|u\|_{\alpha_1 \gamma^\star}\\ &\le \left(\lambda C_6 \alpha_1^p \|u\|^{\tau-p}\right)^{\frac{1}{\alpha_1 p}} \left(\lambda C_7 \alpha^p \|u\|^{\tau-p}\right)^{\frac{1}{\alpha p}} \|u\|_{\alpha\gamma^\star}\\ &\le \left(\lambda C_6 \|u\|^{\tau-p}\right)^{\frac{1}{\alpha p}+\frac{1}{\alpha_1 p}} \alpha^{\frac{1}{\alpha}} \alpha_1^{\frac{1}{\alpha_1}} \|u\|_{p^\star}.\end{aligned}$$

Note that $\alpha_{n+1} = \chi\alpha_n$, where $\chi = \frac{p^\star}{\gamma^\star}$. An iterative process leads to

$$\|u\|_{\alpha_n p^\star} \le \left(\lambda C_6 \|u\|^{\tau-p}\right)^{\frac{\sigma_n}{\alpha p}} \alpha^{\frac{\sigma_n}{\alpha}} \chi^{\frac{\varsigma_n}{\alpha}} \|u\|_{p^\star},$$

where $\sigma_n = \sum_{k=0}^n \chi^{-k}$ and $\varsigma_n = \sum_{k=0}^n k\chi^{-k}$. Since $\lim_{n\to\infty}\sigma_n = \frac{p^\star}{p^\star-\gamma^\star}$ and $\lim_{n\to\infty}\varsigma_n = \frac{\gamma^\star}{p^\star-\gamma^\star}$, we let $n\to\infty$ to conclude that

$$\|u\|_\infty \le \left(\lambda C_6 \|u\|^{\tau-p}\right)^{\frac{p^\star}{\alpha p(p^\star-\gamma^\star)}} \alpha^{\frac{p^\star}{\alpha(p^\star-\gamma^\star)}} \chi^{\frac{\gamma^\star}{\alpha(p^\star-\gamma^\star)}} \|u\| = C\lambda^{\frac{1}{p^\star-\tau}} \|u\|^{\frac{p^\star-p}{p^\star-\tau}}$$

and the proof the lemma is over. □

In the two following lemmas, we prove some well-known results from critical point theory for the energy functional J_λ defined by (8).

Lemma 9. *Under the hypotheses of Theorem 1, the functional J_λ exhibits the mountain-pass geometry:*

(i) *there exist constants $\rho_\lambda > 0, a_\lambda > 0$ such that $J_\lambda(u) \ge a_\lambda$ for all $\|u\| = \rho_\lambda$.*
(ii) *let $e = \frac{\sigma}{2} \in W^{1,p}(\Omega)$, then there exists $\lambda_0 \ge 1$ such that $\|e\| > \rho_\lambda$ and $J_\lambda(e) < 0$ for all $\lambda \ge \lambda_0$.*

Proof. It follows from Lemmas 3, 5 and 7 and (2) that

$$\begin{aligned}J_\lambda(u) &\ge \frac{C_0}{p(p-1)} \|\nabla u\|_p^p + \frac{1}{p}\int_\Omega \beta |u|^p \,\mathrm{d}x - (\lambda C_3 + C_4)\|u\|_\tau^\tau\\ &\ge \|u\|^p \left(C_5 - \lambda C_6 \|u\|^{\tau-p}\right)\end{aligned}$$

(recall that $\lambda \ge 1$ and $\tau > p$). Taking $\rho_\lambda = \left(\frac{C_5}{2\lambda C_6}\right)^{\frac{1}{\tau-p}}$, we have

$$J_\lambda(u) \ge a_\lambda := \frac{C_5}{2}\rho_\lambda^p > 0 \quad \text{for } u \in W^{1,p}(\Omega),\ \|u\| = \rho_\lambda.$$

Next we prove (ii). Since $\rho_\lambda \to 0$ as $\lambda \to \infty$, we get $\|e\| > \rho_\lambda$ for λ large. By virtue of (5) and the definition of $\tilde{F}$, we know that $\tilde{F}(x,e) \ge |e|^r$ for any $x \in \Omega$. Then

$$J_\lambda(e) \le \frac{1}{p}\int_\Omega \beta |e|^p \,\mathrm{d}x - \lambda \int_\Omega \tilde{F}(x,e)\mathrm{d}x \le C_7 |e|^p - \lambda C_8 |e|^r,$$

which implies that there exists $\lambda_0 \ge 1$ such that $J_\lambda(e) < 0$ for $\lambda \ge \lambda_0$. □

Lemma 10. *If hypotheses* H(a) *and* H(f) *hold, then, for every $\lambda > 0$, the functional J_λ satisfies (PS) condition.*

Proof. Let $\{u_n\} \subset W^{1,p}(\Omega)$ be such that $d := \sup_{n\in\mathbb{N}} J_\lambda(u_n) < \infty$, and $J'_\lambda(u_n) \to 0$ in $W^{1,p}(\Omega)^\star$ as $n \to \infty$. For n sufficiently large, and by H(a)(iv) and Lemma 7, one has

$$
\begin{aligned}
d+1+\|u_n\| &\geq J_\lambda(u_n) - \frac{1}{\theta}\langle J'_\lambda(u_n), u_n\rangle \\
&= \int_\Omega G(x,\nabla u_n)\mathrm{d}x + \frac{1}{p}\int_\Omega \beta\,|u_n|^p\,\mathrm{d}x - \lambda\int_\Omega \tilde{F}(x,u_n)\mathrm{d}x \\
&\quad - \int_\Omega \tilde{H}(x,u_n)\mathrm{d}x - \frac{1}{\theta}\int_\Omega (A(x,\nabla u_n),\nabla u_n)_{\mathbb{R}^N}\,\mathrm{d}x \\
&\quad - \frac{1}{\theta}\int_\Omega \beta\,|u_n|^p\,\mathrm{d}x + \frac{\lambda}{\theta}\int_\Omega \tilde{f}(x,u_n)u_n\mathrm{d}x + \frac{1}{\theta}\int_\Omega \tilde{h}(x,u_n)u_n\mathrm{d}x \\
&\geq (1-\frac{p}{\theta})\int_\Omega G(x,\nabla u_n)\mathrm{d}x + (\frac{1}{p}-\frac{1}{\theta})\int_\Omega \beta\,|u_n|^p\,\mathrm{d}x \\
&\geq C_1\,\|u_n\|^p
\end{aligned}
\tag{16}
$$

(see Lemmas 3 and 5). Therefore $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$. Passing if necessary to a suitable subsequence, we may assume that

$$
u_n \to u \text{ weakly in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^s(\Omega), \tag{17}
$$

where $s \in [p, p^\star)$. Since $|\langle J'_\lambda(u_n), v\rangle| \leq \varepsilon_n \|v\|$ for all $v \in W^{1,p}(\Omega)$ with $\varepsilon_n \downarrow 0^+$, we have

$$
\Big|\langle V(u_n), u_n-u\rangle + \int_\Omega \big(\beta\,|u_n|^{p-2}\,u_n - \lambda\tilde{f}(x,u_n) - \tilde{h}(x,u_n)\big)(u_n-u)\mathrm{d}x\Big| \leq \varepsilon_n\,\|u_n-u\|\,. \tag{18}
$$

From (17) and Lemma 7, it follows that

$$
\int_\Omega \big(\beta\,|u_n|^{p-2}\,u_n - \lambda\tilde{f}(x,u_n) - \tilde{h}(x,u_n)\big)(u_n-u)\mathrm{d}x \to 0.
$$

So, if in (18) we pass to the limit as $n\to\infty$, then we obtain $\langle V(u_n), u_n-u\rangle \to 0$. This and Proposition 4 mean that $\{u_n\}$ strongly converges to u in $W^{1,p}(\Omega)$. Therefore, J_λ satisfies (PS) condition for all $\lambda > 0$. □

Lemma 11. *Let hypotheses* H(a) *and* H(f) *be satisfied. Let $u \in W^{1,p}(\Omega)$ be a critical point of J_λ. Then there exists $C > 0$, independent of λ, such that $\|u\|^p \leq CJ_\lambda(u)$.*

Proof. Let $u \in W^{1,p}(\Omega)$ be a critical point of J_λ. Similarly to (16), we have $C_1\,\|u\|^p \leq J_\lambda(u)$, where $C_1 > 0$ is independent on λ, and the proof the lemma is over. □

Lemma 12. *Assume that hypotheses* H(a) *and* H(f) *hold. Then, there exists $\lambda_\star > 0$ such that for all $\lambda \geq \lambda_\star$ problem (P_λ) has at least one nontrivial solution $u_\lambda \in C^1(\overline{\Omega}) \cap [-\sigma,\sigma]$ and $\lim_{\lambda\to+\infty}\|u_\lambda\| = 0$, where $[-\sigma,\sigma] = \{u \in W^{1,p}(\Omega) : -\sigma \leq u(x) \leq \sigma$ a.e. in $\Omega\}$.*

Proof. Lemmas 9 and 10 guarantee that for any $\lambda \geq \lambda_0$ the energy functional J_λ satisfies all the assumptions of the Mountain Pass Lemma (see Lemma 6). Hence, for any $\lambda \geq \lambda_0$ there exists a nontrivial critical point $u_\lambda \in W^{1,p}(\Omega)$ of J_λ with critical value c_λ and

$$
J_\lambda(0) = 0 < a_\lambda \leq c_\lambda = \inf_{h\in\Gamma}\max_{s\in[0,1]} J_\lambda(h(s)),
$$

where $\Gamma = \{h \in C([0,1], E) : h(0) = 0, h(1) = e\}$. In order to get the estimate of the critical level c_λ, we introduce the following energy functional

$$I_\lambda(u) = \int_\Omega G(x, \nabla u)\mathrm{d}x + \frac{1}{p}\int_\Omega \beta |u|^p \,\mathrm{d}x - \lambda \int_\Omega |u|^r \,\mathrm{d}x, \quad u \in W^{1,p}(\Omega).$$

Let $\varphi_\lambda(t) = \frac{t^p}{p}\|\beta\|_\infty \|e\|_p^p - \lambda t^r \|e\|_r^r$. We can obtain through straightforward calculations that

$$\max_{t\geq 0} \varphi_\lambda(t) = \frac{r-p}{pr} r^{-\frac{p}{r-p}} \|\beta\|_\infty^{\frac{r}{r-p}} \|e\|_p^{\frac{rp}{r-p}} \|e\|_r^{-\frac{rp}{r-p}} \lambda^{-\frac{p}{r-p}}.$$

Then, using (8) and the fact $e = \frac{\sigma}{2}$, we have

$$c_\lambda \leq \max_{t\in[0,1]} J_\lambda(te) \leq \max_{t\in[0,1]} I_\lambda(te) \leq \max_{t\geq 0} \varphi_\lambda(t) \leq C\lambda^{-\frac{p}{r-p}}, \tag{19}$$

where $C > 0$ is independent of $\lambda \geq 1$. By virtue of (19) and Lemmas 8 and 11, we obtain

$$\|u_\lambda\|_\infty \leq C\lambda^{-\frac{p^\star - r}{(p^\star - \tau)(r-p)}}, \tag{20}$$

where the exponent of λ is negative, so that there exists $\lambda_\star \geq \lambda_0$ such that $C\lambda^{-\frac{p^\star - r}{(p^\star - \tau)(r-p)}} \leq \sigma$ for any $\lambda \geq \lambda_\star$. Hence, $u_\lambda \in [-\sigma, \sigma]$ is a nontrivial solution of the original problem (P_λ), and the nonlinear regularity theory (see [21, Theorem 2]) implies that $u_\lambda \in C^1(\overline{\Omega})$. Using again (19) and Lemma 11, we conclude that $\lim_{\lambda\to+\infty} \|u_\lambda\| = 0$. The proof is complete. □

As we already mentioned in the Introduction, our method of proof involves also truncation techniques. So, we introduce the following truncations:

$$Q_\lambda^+(x,t) = \begin{cases} Q_\lambda(x,t) & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases} \quad \text{and} \quad Q_\lambda^-(x,t) = \begin{cases} Q_\lambda(x,t) & \text{if } t < 0 \\ 0 & \text{if } t \geq 0, \end{cases}$$

where $Q_\lambda(x,t) := \lambda \tilde{F}(x,t) + \tilde{H}(x,t)$ (see Section 2), and consider the C^1-functionals $J_\lambda^\pm : W^{1,p}(\Omega) \to \mathbb{R}$ defined for all $u \in W^{1,p}(\Omega)$ by

$$J_\lambda^\pm(u) = \int_\Omega G(x, \nabla u)\mathrm{d}x + \frac{1}{p}\int_\Omega \beta |u|^p \,\mathrm{d}x - \int_\Omega Q_\lambda^\pm(x,u)\mathrm{d}x.$$

Lemma 13. *Assume that hypotheses* H(a) *and* H(f) *hold and* $\lambda \geq \lambda_\star$ *(*$\lambda_\star > 0$ *as in Lemma 12). Then problem* (P_λ) *has at least two nontrivial constant sign smooth solutions:* $u_{\lambda,1} \in \mathrm{int}C_+ \cap [0,\sigma]$ *and* $u_{\lambda,2} \in -\mathrm{int}C_+ \cap [-\sigma, 0]$*, with* $\lim_{\lambda\to+\infty} \|u_{\lambda,i}\| = 0$ *for* $i = 1, 2$.

Proof. First we produce the positive smooth solution. Arguing as in the proofs of Lemmas 9–12, we obtain the corresponding results for the functional J_λ^+. So, using the Mountain Pass Lemma, we can find $u_{\lambda,1} \in C^1(\overline{\Omega}) \cap [-\sigma, \sigma]$, $u_{\lambda,1} \neq 0$ such that $J_\lambda'(u_{\lambda,1}) = 0$, thus

$$\langle V(u_{\lambda,1}), v\rangle + \int_\Omega \beta |u_{\lambda,1}|^{p-2} u_{\lambda,1} v \mathrm{d}x = \int_\Omega q_\lambda^+(x, u_{\lambda,1}) v \mathrm{d}x, \qquad v \in W^{1,p}(\Omega), \tag{21}$$

where $q_\lambda^+(x,t) = \frac{\partial}{\partial t} Q_\lambda^+(x,t)$. In (21), we choose $v = -u_{\lambda,1}^- \in W^{1,p}(\Omega)$. Using Lemmas 2 and 5, we have

$$\xi_1 \|u^-_{\lambda,1}\|^p \le \frac{C_0}{p-1}\|\nabla u^-_{\lambda,1}\|^p_p + \int_\Omega \beta |u^-_{\lambda,1}|^p \mathrm{d}x \le \langle V(u_{\lambda,1}), -u^-_{\lambda,1}\rangle + \int_\Omega \beta |u^-_{\lambda,1}|^p \mathrm{d}x = 0,$$

where $\xi_1 > 0$, so $u_{\lambda,1} \ge 0$, $u_{\lambda,1} \ne 0$. Note that

$$-\mathrm{div}A(x, \nabla u_{\lambda,1}) + \beta\, |u_{\lambda,1}|^{p-2}\, u_{\lambda,1} = \lambda f(x, u_{\lambda,1}) + g(x)\, |u_{\lambda,1}|^{p^\star-2}\, u_{\lambda,1} \ge 0 \quad \text{a.e.} \quad \text{in} \quad \Omega, \tag{22}$$

by hypothesis H(f)(iii). From the maximum principle of [24] (see also [9]) and (22), we get $u_{\lambda,1} \in \mathrm{int}C_+ \cap [0, \sigma]$. By Lemma 12, we have that $\lim_{\lambda\to+\infty} \|u_{\lambda,1}\| = 0$.

Similarly, working with the functional J^-_λ and using the Mountain Pass Lemma, we show that for every $\lambda \ge \lambda_\star$, problem (P_λ) has another solution $u_{\lambda,2} \in -\mathrm{int}C_+ \cap [-\sigma, 0]$ and $\lim_{\lambda\to+\infty} \|u_{\lambda,2}\| = 0$. The proof is complete. □

In fact, we can show that there exist extremal constant sign solutions for problem (P_λ) with $\lambda \ge \lambda_\star$, i.e., there is a smallest nontrivial positive solution $u_\star \in \mathrm{int}C_+$ and a biggest nontrivial negative solution $v_\star \in -\mathrm{int}C_+$.

Lemma 14. *If hypotheses* H(a) *and* H(f) *hold and* $\lambda \ge \lambda_\star$*, then problem* (P_λ) *admits a smallest nontrivial positive solution* $u_\star \in \mathrm{int}C_+ \cap [0, \sigma]$ *and a biggest nontrivial negative solution* $v_\star \in -\mathrm{int}C_+ \cap [-\sigma, 0]$.

Proof. We do the proof for $u_\star$, the proof for $v_\star$ being similar. Let

$$S^+_\lambda = \{u \in W^{1,p}(\Omega) : u \text{ is a positive solution of problem } (P_\lambda) \text{ with } u \in [0, \sigma] \cap \mathrm{int}C_+\}.$$

Since $\lambda \ge \lambda_\star$, using Lemma 13 we know that $S^+_\lambda \ne \varnothing$. Let $C \subseteq S^+_\lambda$ be a chain (i.e., a totally ordered subset of S^+_λ). From Dunford–Schwartz [11, p. 336], we know that we can find $\{u_n\} \subseteq C$ such that

$$\inf C = \inf_{n\ge1} u_n.$$

Because of $u_n \in S^+_\lambda$ for every $n \in \mathbb{N}$, we have $0 \le u_n \le \sigma$ and

$$V(u_n) + \beta u_n^{p-1} = \lambda N_{\tilde{f}}(u_n) + N_{\tilde{h}}(u_n). \tag{23}$$

Hence $\{u_n\}_{n\ge1} \subseteq W^{1,p}(\Omega)$ is bounded. So, we may assume without loss of generality that

$$u_n \to u \text{ weakly in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^p(\Omega). \tag{24}$$

On (23) we act with $u_n - u \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (24). We immediately obtain

$$\lim_{n\to\infty} \langle V(u_n), u_n - u\rangle = 0.$$

This and Proposition 4 mean that $\{u_n\}$ strongly converges to u in $W^{1,p}(\Omega)$. Using this fact we can pass again to the limit as $n \to \infty$ in (23) which gives

$$V(u) + \beta u^{p-1} = \lambda N_{\tilde{f}}(u) + N_{\tilde{h}}(u).$$

Then by (24), $u \in [0, \sigma]$ is a solution of problem (P_λ). We now show that $u \ne 0$. Arguing indirectly, suppose that $u = 0$. We have $u_n \to 0$ in $W^{1,p}(\Omega)$. Then hypothesis H(f)(i) implies that

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$$\int_\Omega \frac{1}{\|u_n\|^p}\tilde{f}(x,u_n)u_n\mathrm{d}x \to 0 \quad \text{and} \quad \int_\Omega \frac{1}{\|u_n\|^p}\tilde{h}(x,u_n)u_n\mathrm{d}x \to 0$$

(recall that $u_n \in [0,\sigma], n \geq 1$). Acting on (23) with $\frac{u_n}{\|u_n\|^p} \in W^{1,p}(\Omega)$ and passing to the limit as $n \to \infty$, one gets, thanks to Lemmas 5 and 2,

$$\begin{aligned} 0 < C &\leq \frac{1}{\|u_n\|^p}\Big(\frac{C_0}{p-1}\int_\Omega |\nabla u_n|^p\,\mathrm{d}x + \int_\Omega \beta\,|u_n|^p\,\mathrm{d}x\Big) \\ &\leq \frac{1}{\|u_n\|^p}\Big(\int_\Omega (A(x,\nabla u_n),\nabla u_n)_{\mathbb{R}^N}\mathrm{d}x + \int_\Omega \beta\,|u_n|^p\,\mathrm{d}x\Big) \\ &= \frac{1}{\|u_n\|^p}\Big(\lambda\int_\Omega \tilde{f}(x,u_n)u_n\mathrm{d}x + \int_\Omega \tilde{h}(x,u_n)u_n\mathrm{d}x\Big) \\ &\to 0, \end{aligned}$$

a contradiction. This proves that $u \neq 0$. As before, via the nonlinear maximum principle (see [9,24]), we have that $u \in \mathrm{int}C_+$, and so $u \in S_\lambda^+$. Since C is an arbitrary chain, from the Kuratowski–Zorn lemma we infer that S_λ^+ has a minimal element $u_\star \in S_\lambda^+$. Note that S_λ^+ is downward directed, we obtain that $u_\star$ is the smallest nontrivial positive solution of problem (P_λ). Similarly, we produce $v_\star \in -\mathrm{int}C_+$, the biggest nontrivial negative solution of problem (P_λ). $\square$

4. Sign-changing solution

In this section by virtue of flow invariance arguments, we use the two extremal solutions $u_\star \in \mathrm{int}C_+$ and $v_\star \in -\mathrm{int}C_+$ obtained in Lemma 14 to produce sign-changing (nodal) solution for problem (P_λ). Throughout this section, we denote the set of critical points of J_λ by K_{J_λ}, that is, $K_{J_\lambda} := \big\{u \in W^{1,p}(\Omega) : J_\lambda'(u) = 0\big\}$. For convenience, we denote $\{u \in W^{1,p}(\Omega) : v_\star(x) \leq u(x) \leq u_\star(x) \text{ a.e. in } \Omega\}$ by $[v_\star, u_\star]$. The sets $\mathrm{int}_{C^1(\overline{\Omega})}[v_\star, u_\star]$ and $\partial_{C^1(\overline{\Omega})}[v_\star, u_\star]$ are the interior and boundary of the order interval $[v_\star, u_\star]$ in $C^1(\overline{\Omega})$, respectively.

Consider the map $T : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^\star$ defined for all $u, v \in W^{1,p}(\Omega)$ by

$$\langle T(u), v\rangle = \int_\Omega (A(x,\nabla u),\nabla v)_{\mathbb{R}^N}\mathrm{d}x + \int_\Omega (\beta + m)\,|u|^{p-2}\,uv\mathrm{d}x$$

with $m > 0$ as in hypothesis H(f) (iv). Then, the inverse $T^{-1} : (W^{1,p}(\Omega))^\star \to W^{1,p}(\Omega)$ of T exists and it is continuous (see [26, Proposition 9]). Let

$$B_\lambda(u) = T^{-1}(q_\lambda(\cdot,u) + m\,|u|^{p-2}\,u) \quad \text{for } u \in W^{1,p}(\Omega), \tag{25}$$

where $q_\lambda(\cdot,t) = \frac{\partial}{\partial t}(\lambda\tilde{F}(\cdot,t) + \tilde{H}(\cdot,t))$ (see Section 2). Due to Lemma 7 and the Sobolev Embedding Theorem, B_λ is a compact operator (continuous and maps bounded sets into relatively compact sets) from $W^{1,p}(\Omega) \to W^{1,p}(\Omega)$. Moreover, critical points of the energy functional J_λ correspond to fixed points of B_λ. From the regularity result in [21] we have $B_\lambda(C^1(\overline{\Omega})) \subseteq C^1(\overline{\Omega})$.

Lemma 15. *If hypotheses* H(a) *and* H(f) *hold,* $\lambda \geq \lambda_\star$*, then* $B_\lambda(\pm C_+ \setminus \{0\}) \subseteq (\pm\mathrm{int}C_+)$ *and* $B_\lambda([v_\star, u_\star]) \subseteq [v_\star, u_\star]$.

Proof. We first do the proof for $u \in C_+ \setminus \{0\}$. The proof is similar for $u \in -C_+ \setminus \{0\}$. Let $v = B_\lambda(u)$. We have $v \in C^1(\overline{\Omega})$ and

$$-\mathrm{div}A(x,\nabla v)+(\beta(x)+m)\,|v|^{p-2}\,v=q_\lambda(x,u)+m\,|u|^{p-2}\,u,\quad \text{in }\Omega.$$

As before, taking $-v^-$as a test function, we obtain

$$\begin{aligned}C\left\|v^-\right\|^p&\le\int_\Omega(A(x,\nabla v),-\nabla v^-)_{\mathbb{R}^N}\mathrm{d}x+\int_\Omega(\beta(x)+m)\left|v^-\right|^p\mathrm{d}x\\&=-\int_\Omega(q_\lambda(x,u)+m\,|u|^{p-2}\,u)v^-\mathrm{d}x\le 0\end{aligned}$$

(see Lemmas 2 and 5). Therefore $v^-=0$ a.e. in Ω. Evidently $q_\lambda(\cdot,u)+m\,|u|^{p-2}\,u\ne 0$ in $(W^{1,p}(\Omega))^\star$. Then $v\ne 0$ due to $v=T^{-1}(q_\lambda(\cdot,u)+m\,|u|^{p-2}\,u)$. Consequently, $v\in C_+\setminus\{0\}$ (see [21]). Note that

$$-\mathrm{div}A(x,\nabla v)+(\beta(x)+m)\,|v|^{p-2}\,v=q_\lambda(x,u)+m\,|u|^{p-2}\,u\ge 0,\quad \text{in }\Omega. \tag{26}$$

From the maximum principle of [24] (see also [9]) and (26), we get $v=B_\lambda(u)\in\mathrm{int}C_+$.

Next, we claim that $B_\lambda(u_0)\in[v_\star,u_\star]$ for every $u_0\in[v_\star,u_\star]$. Indeed, we have $v_0:=B_\lambda(u_0)\in C^1(\overline{\Omega})$ and

$$V(v_0)+(\beta(x)+m)\,|v_0|^{\,p-2}v_0=N_{q_\lambda}(u_0)+m\,|u_0|^{p-2}\,u_0. \tag{27}$$

As before, on (27) we act with $(v_0-u_\star)^+\in W^{1,p}(\Omega)$. Then, using hypothesis H(f)(iv) and recalling that $u_\star\in\mathrm{int}C_+$ solves problem (P_λ), we have

$$\begin{aligned}&\left\langle V(v_0),(v_0-u_\star)^+\right\rangle+\int_\Omega(\beta(x)+m)\,|v_0|^{p-2}\,v_0(v_0-u_\star)^+\mathrm{d}x\\&=\int_\Omega(q_\lambda(x,u_0)+m\,|u_0|^{p-2}\,u_0)(v_0-u_\star)^+\mathrm{d}x\\&\le\int_\Omega(q_\lambda(x,u_\star)+m\,|u_\star|^{p-2}\,u_\star)(v_0-u_\star)^+\mathrm{d}x\\&=\left\langle V(u_\star),(u_0-u_\star)^+\right\rangle+\int_\Omega(\beta(x)+m)\,|u_\star|^{\,p-2}u_\star(v_0-u_\star)^+\mathrm{d}x,\end{aligned}$$

so

$$\left\langle V(v_0)-V(u_\star),(v_0-u_\star)^+\right\rangle+\int_\Omega(\beta(x)+m)(|v_0|^{\,p-2}v_0-|u_\star|^{\,p-2}u_\star)(v_0-u_\star)^+\mathrm{d}x\le 0$$

and thus $|\{v_0>u_\star\}|_N=0$, i.e. $v_0\le u_\star$. Similarly, acting on (27) with $(v_\star-v_0)^+\in W^{1,p}(\Omega)$, we obtain $v_\star\le v_0$. Therefore, $v_0\in[v_\star,u_\star]$ and the claim holds. □

The proof of the following lemma can be shown by the argument in [4, Lemmas 3.7 and 3.8]. Thus, we omit the proof.

Lemma 16. *Let $\lambda\ge\lambda_\star$. Then, there exist $a_i=a_i(\lambda)>0$ $(i=1,2)$ such that for all $u\in W^{1,p}(\Omega)$,*

(i) *if $1<p\le 2$, then*

$$\langle J_\lambda'(u), u - B_\lambda(u)\rangle \geq a_1 \|u - B_\lambda(u)\|^2 (\|u\| + \|B_\lambda(u)\|)^{p-2},$$
$$\|J_\lambda'(u)\| \leq a_2 \|u - B_\lambda(u)\|^{p-1};$$

(ii) *if* $p \geq 2$, *then*

$$\langle J_\lambda'(u), u - B_\lambda(u)\rangle \geq a_1 \|u - B_\lambda(u)\|^p,$$
$$\|J_\lambda'(u)\| \leq a_2 \|u - B_\lambda(u)\| (\|u\| + \|B_\lambda(u)\|)^{p-2}.$$

We need to construct a special descending flow of J_λ. Since it is not assumed that B_λ is locally Lipschitz continuous, we first construct a locally Lipschitz continuous operator A_λ on $X = W^{1,p}(\Omega) \setminus K_{J_\lambda}$, which inherits the properties of B_λ. The next result follows from a similar argument as in [4, Lemma 4.1]; [26, Lemma 17] using the properties of B_λ described in Lemmas 15 and 16.

Lemma 17. *Let* $\lambda \geq \lambda_\star$. *Then, there exists a locally Lipschitz continuous operator* $A_\lambda : X \to W^{1,p}(\Omega)$ *with the following properties:*

(i) $A_\lambda(\pm C_+ \setminus \{0\}) \subseteq (\pm \mathrm{int} C_+)$ *and* $A_\lambda([v_\star, u_\star]) \subseteq [v_\star, u_\star]$;
(ii) *for all* $u \in X$,

$$\frac{1}{2}\|u - B_\lambda(u)\| \leq \|u - A_\lambda(u)\| \leq 2\|u - B_\lambda(u)\|;$$

(iii) *for all* $u \in X$ *and* a_1 *as in Lemma 16,*

$$\langle J_\lambda'(u), u - A_\lambda(u)\rangle \geq \frac{a_1}{2} \|u - B_\lambda(u)\|^2 (\|u\| + \|B_\lambda(u)\|)^{p-2} \quad \text{if } 1 < p \leq 2,$$
$$\langle J_\lambda'(u), u - A_\lambda(u)\rangle \geq \frac{a_1}{2} \|u - B_\lambda(u)\|^p \quad \text{if } p \geq 2.$$

For $u \in X$, we consider the following initial value problem in X:

$$\begin{cases} \frac{\mathrm{d}\phi_\lambda(t,u)}{\mathrm{d}t} = -\phi_\lambda(t,u) + A_\lambda(\phi_\lambda(t,u)) \\ \phi_\lambda(0,u) = u, \end{cases} \tag{28}$$

where $\lambda \geq \lambda_\star$. By the theory of ordinary differential equations in Banach spaces, (28) has a unique solution in X, still denoted by $\phi_\lambda(t,u)$, with right maximal interval of existence $[0, \tau(u))$. Note that $J_\lambda(\phi_\lambda(t,u))$ is strictly decreasing in $t \in [0, \tau(u))$ and therefore $\phi_\lambda(t,u) (0 \leq t < \tau(u))$ is called a descending flow curve. The flow is given by

$$\phi_\lambda(t,u) = e^{-t}u + \int_0^t e^{-(t-s)} A_\lambda(\phi_\lambda(s,u))\mathrm{d}s \quad \text{for } 0 \leq t < \tau(u). \tag{29}$$

Definition 18. (See [22].) A nonempty subset M of $W^{1,p}(\Omega)$ is said to be positive invariant for the descending flow ϕ_λ if

$$\{\phi_\lambda(t,u) : 0 \leq t < \tau(u)\} \subseteq M$$

for all $u \in M \setminus K_{J_\lambda}$.

Lemma 19. *If hypotheses* H(a) *and* H(f) *hold and* $\lambda \geq \lambda_\star$, *then* $[v_\star, u_\star]$ *and* $\mathrm{int}_{C^1(\overline{\Omega})}[v_\star, u_\star]$ *are positive invariant for the flow* ϕ_λ.

Proof. We first show that $[v_\star, u_\star]$ is positive invariant for the flow ϕ_λ. We argue by contradiction. Thus, suppose that there exists $u \in [v_\star, u_\star] \setminus K_{J_\lambda}$ such that $\phi_\lambda(t,u) \notin [v_\star, u_\star]$ for some $t \in [0, \tau(u))$. Let $0 \leq t_1 < t_2 < \tau(u)$ be such that $\phi_\lambda(t_1, u) \in \partial_{C^1(\overline{\Omega})}[v_\star, u_\star]$ and $\phi_\lambda(t,u) \notin [v_\star, u_\star]$ for $t_1 < t < t_2$. Note that $\eta_\lambda(t) := \phi_\lambda(t + t_1, u)$ is a solution of

$$\begin{cases} \frac{\mathrm{d}\eta_\lambda(t)}{\mathrm{d}t} = -\eta_\lambda(t) + A_\lambda(\eta_\lambda(t)) \quad 0 \leq t < t_2 - t_1, \\ \eta_\lambda(0) = \phi_\lambda(t_1, u). \end{cases}$$

Moreover, the convexity of $[v_\star, u_\star]$ and Lemma 17 (i) imply that for $0 \leq s \leq 1$,

$$\eta_\lambda(0) + s(-\eta_\lambda(0) + A_\lambda(\eta_\lambda(0)) = (1-s)\eta_\lambda(0) + sA_\lambda(\eta_\lambda(0)) \in [v_\star, u_\star].$$

By Theorem 6.3 of [6], we know that there exists $0 < t_3 < t_2 - t_1$ such that $\eta_\lambda(t) = \phi_\lambda(t+t_1, u) \in [v_\star, u_\star]$ for $0 \leq t < t_3$. This is a contradiction. Therefore, for every $u \in [v_\star, u_\star] \setminus K_{J_\lambda}$, $\{\phi_\lambda(t,u) : 0 \leq t < \tau(u)\} \subseteq [v_\star, u_\star]$.

Next, we prove that $\mathrm{int}_{C^1(\overline{\Omega})}[v_\star, u_\star]$ is positive invariant for the flow ϕ_λ. Let $u_0 \in \mathrm{int}_{C^1(\overline{\Omega})}[v_\star, u_\star]$, then $\{A_\lambda(\phi_\lambda(t, u_0)) : 0 \leq t < \tau(u_0)\} \subseteq [v_\star, u_\star]$. By (29), we have

$$\phi_\lambda(t, u_0) = e^{-t}u_0 + (1 - e^{-t}) \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} A_\lambda\big(\phi_\lambda(\ln(1 + \frac{k}{n}(e^t - 1)), u_0)\big)$$

(see [22, p. 272]). Exploiting the fact that $[v_\star, u_\star]$ is closed and convex, we have that

$$w := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} A_\lambda\big(\phi_\lambda(\ln(1 + \frac{k}{n}(e^t - 1)), u_0)\big) \in [v_\star, u_\star]$$

and

$$\phi_\lambda(t, u_0) = e^{-t}u_0 + (1 - e^{-t})w \in \mathrm{int}_{C^1(\overline{\Omega})}[v_\star, u_\star].$$

Therefore, $\mathrm{int}_{C^1(\overline{\Omega})}[v_\star, u_\star]$ is positive invariant for the flow ϕ_λ. □

Lemma 20. *If hypotheses* H(a) *and* H(f) *hold and* $\lambda \geq \lambda_\star$, *then problem* (P_λ) *admits a sign-changing solution* $u_{\lambda,3} \in C^1(\overline{\Omega})$ *with* $\lim_{\lambda\to+\infty} \|u_{\lambda,3}\| = 0$.

Proof. We introduce the following set:

$$D_1 = \big\{u \in C^1(\overline{\Omega}) \setminus K_{J_\lambda} : \phi(t,u) \in \mathrm{int}_{C^1(\overline{\Omega})}[v_\star, u_\star] \text{ for some } t \in [0, \tau(u))\big\} \cup \mathrm{int}_{C^1(\overline{\Omega})}[v_\star, u_\star].$$

Evidently $0 \in D_1$, and by virtue of the continuity of $u \to \phi_\lambda(t,u)$, we see that D_1 is an open subset of $C^1(\overline{\Omega})$.

Claim 1. D_1 *and* ∂D_1 *are positive invariant for the flow* ϕ_λ.

Proof. Let us first show that D_1 is positive invariant for the descending flow ϕ_λ. Suppose by contradiction that there exist $u \in D_1 \setminus K_{J_\lambda}$ and $t_1 \in [0, \tau(u))$ such that $\phi_\lambda(t_1, u) \notin D_1$, then $\phi_\lambda(t,u) \notin \mathrm{int}_{C^1(\overline{\Omega})}[v_\star, u_\star]$ for every $t_1 \leq t < \tau(u)$. Due to $u \in D_1 \setminus K_{J_\lambda}$, there exists $0 \leq t_2 < \tau(u)$ such that $\phi_\lambda(t_2, u) \in$

$\mathrm{int}_{C^1(\overline{\Omega})}[v_\star, u_\star]$. Recall that $\mathrm{int}_{C^1(\overline{\Omega})}[v_\star, u_\star]$ are positive invariant for the flow ϕ_λ (see Lemma 19). Hence, $\phi_\lambda(t,u) \in \mathrm{int}_{C^1(\overline{\Omega})}[v_\star, u_\star]$ for every $t_2 \le t < \tau(u)$, which is a contradiction.

Next, we prove that ∂D_1 is positive invariant for the flow ϕ_λ. As in the proof of [22, Lemma 2.3], we argue by contradiction. So suppose that we can find $u_0 \in \partial D_1 \backslash K_{J_\lambda}$ and $0 < t_0 < \tau(u_0)$ such that $\phi_\lambda(t_0, u_0) \notin \partial D_1$. Then by the definition of D_1, $\phi_\lambda(t_0,u_0) \in C^1(\overline{\Omega}) \setminus \overline{D}_1$, where $\overline{D}_1$ is the closure of D_1 in $C^1(\overline{\Omega})$. Since D_1 is positive invariant for the flow ϕ_λ and $C^1(\overline{\Omega}) \setminus \overline{D}_1$ is open, we can find a neighborhood U of u_0 in $C^1(\overline{\Omega})$ such that, for any $u_1 \in U$, $\phi_\lambda(t_0, u_1) \in C^1(\overline{\Omega}) \setminus \overline{D}_1$. Taking $u_1 \in U \cap D_1$ and recalling that D_1 is positive invariant for the flow ϕ_λ, we get a contradiction. This proves the Claim 1. $\square$

Now let $E_2 = \mathrm{span}\{\varphi_1, \varphi_2\}$, where $\varphi_1 > 0, \varphi_2$ are the first and second eigenfunctions of the differential operator $u \to -\Delta_p u + \beta(x)u, u \in W^{1,p}(\Omega)$, corresponding respectively to the eigenvalues λ_1, λ_2 (see Section 2).

Claim 2. *$D_1 \cap E_2$ is a bounded set of E_2 and $\inf_{u\in\partial D_1} J_\lambda(u) > -\infty$.*

Proof. By Lemma 7 (ii), we have

$$\tilde{F}(x,t) \ge C_4 |t|^\theta - C_5 \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}.$$

Hence, using Lemma 3 and the fact that all norms of a finite dimensional normed space are equivalent, we obtain that for any $u \in S^1 := \{u \in E_2 : \|u\| = 1\}$,

$$\begin{aligned} J_\lambda(tu) &\le \frac{C_1 |t|^p}{p(p-1)} \|\nabla u\|_p^p + \frac{\|\beta\|_\infty |t|^p}{p} \|u\|_p^p - \lambda C_4 |t|^\theta \|u\|_\theta^\theta - C_6 \\ &\le \max\Big\{\frac{C_1}{p(p-1)}, \frac{\|\beta\|_\infty}{p}\Big\} |t|^p - \lambda C_7 |t|^\theta - C_6. \end{aligned}$$

Thus $J_\lambda(tu) \to -\infty$ as $|t| \to \infty$ uniformly in S^1, being $p < \theta$. Since J_λ is bounded from below on $[v_\star, u_\star]$, we have $c := \inf_{u\in[v_\star,u_\star]} J_\lambda(u) > -\infty$. Note that for $u \in D_1$, $t \to J_\lambda(\phi_\lambda(t,u))$ is decreasing. Then,

$$\inf_{u\in D_1\cap E_2} J_\lambda(u) \ge \inf_{u\in[v_\star,u_\star]\cap E_2} J_\lambda(u) \ge c.$$

It follows that $D_1 \cap E_2$ is a bounded set in E_2.

From the definition of D_1, we obtain that $\phi_\lambda(t,u) \in \mathrm{int}_{C^1(\overline{\Omega})}[v_\star, u_\star]$ for some $t \in [0, \tau(u))$, where $u \in D_1$. Then $J_\lambda(u) \ge J_\lambda(\phi(t,u)) \ge c$ for all $u \in D_1$. This implies that $J_\lambda(u)$ is bounded from below on D_1. Let $u_0 \in \partial D_1$. Then we can find a sequence $\{u_n\}_{n\ge1} \subseteq D_1$ such that $u_n \to u_0$ and $J_\lambda(u_n) \to J_\lambda(u_0)$. It follows that there exists $n_0 \ge 1$ such that $J_\lambda(u_n) < J_\lambda(u_0) + 1$ for all $n \ge n_0$. But recall that $J_\lambda(u_n) \ge c$ for all $n \ge 1$. Then $\inf_{u\in\partial D_1} J_\lambda(u) \ge c - 1$. This proves the Claim 2. $\square$

Now let

$$D_2 = \big\{u \in C^1(\overline{\Omega}) \setminus K_{J_\lambda} : \phi(t,u) \in \mathrm{int}\, C_+ \text{ for some } t \in [0,\tau(u))\big\} \cup \mathrm{int}\, C_+.$$

Clear, D_2 is open and $0 \in \partial D_2$. Note that $\mathrm{int}\, C_+$ is positive invariant for the flow ϕ_λ (see Lemma 17 (i) and (29)). We infer that D_2 is also positive invariant for the flow ϕ_λ. Moreover, reasoning as in the Claim 1, we can show that ∂D_2 is positive invariant for the flow ϕ_λ. Note that $D_1 \cap E_2$ is a bounded neighborhood of 0 in E_2 (see Claim 2). We may assume that $D_1 \cap E_2$ is connected. (Otherwise, we consider the connected component $D_1' \subset E_2$ of $D_1 \cap E_2$, with $(0,0) \in D_1'$, instead of $D_1 \cap E_2$). By Lemma 2 of [10], $\partial D_1 \cap E_2$ has a connected component Σ that intersects each one-sided ray in E_2 through 0 and hence contains some

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multiples of $\pm\varphi_1 \in \pm\mathrm{int}\, C_+$ and intersects ∂D_2. Since $\Sigma \subset \partial D_1$ it follows that $\partial D_1 \cap (\pm C_+) \neq \emptyset$ and $\partial D_1 \cap \partial D_2 \neq \emptyset$. Then, we can define

$$c_\star = \inf \{J_\lambda(u) : u \in \partial D_1 \cap \partial D_2\}.$$

Due to Lemma 10 and Claim 2, we have that J_λ satisfies the (PS) condition and $c_\star > -\infty$. Recall that ∂D_1 is closed positive invariant for the flow ϕ_λ (see Claim 1). By Lemma 2 in [35], we can find

$$u_{\lambda,3} \in \partial D_1 \cap \partial D_2 \cap K_{J_\lambda}.$$

Note that $u_{\lambda,3} \in C^1(\overline{\Omega})$ (see Lemma 12 and [21]). Because of $u_{\lambda,3} \in \partial D_1$, we have $u_{\lambda,3} \neq 0$. Also, since $u_{\lambda,3} \in \partial D_2$ and $\partial D_2 \cap (\pm C_+ \setminus \{0\}) = \varnothing$, we have $u_{\lambda,3} \notin \mathrm{int}\, C_+ \cup (-\mathrm{int}\, C_+)$. So by virtue of the strong maximum principle of [24,36], $u_{\lambda,3}$ cannot have constant sign and so $u_{\lambda,3}$ is the desired sign-changing solution. Using Lemma 12, we deduce that $\lim_{\lambda\to+\infty} \|u_{\lambda,3}\| = 0$. The proof is complete. □

Proof of Theorem 1. This proof is a immediate consequence of Lemmas 13, 14 and 20. □

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References

[1] S. Aizicovici, N.S. Papageorgiou, V. Staicu, On p-superlinear equations a nonhomogeneous differential operator, NoDEA Nonlinear Differential Equations Appl. 20 (2013) 151–175.
[2] S. Aizicovici, N.S. Papageorgiou, V. Staicu, Nodal solutions for nonlinear nonhomogeneous Neumann equations, Topol. Methods Nonlinear Anal. 43 (2014) 421–438.
[3] G. Autuori, A. Fiscella, P. Pucci, Stationary Kirchhoff problems involving a fractional elliptic operator and a critical nonlinearity, Nonlinear Anal. 125 (2015) 699–714.
[4] T. Bartsch, Z.L. Liu, On a superlinear elliptic p-Laplacian equation, J. Differential Equations 198 (2004) 149–175.
[5] F. Cammaroto, L. Vilasi, A critical Kirchhoff-type problem involving the $p\&q$-Laplacian, Math. Nachr. 287 (2014) 184–193.
[6] K.C. Chang, Infinite Dimensional Morse Theory and Multiple Solutions Problem, Birkhauser, Basel, 1993.
[7] F.J.S.A. Corrêa, G.M. Figueiredo, On an elliptic equation of p-Kirchhof type via variational methods, Bull. Aust. Math. Soc. 74 (2006) 263–277.
[8] D.G. Costa, Z.Q. Wang, Multiplicity results for a class of superlinear elliptic problems, Proc. Amer. Math. Soc. 133 (2005) 787–794.
[9] L. Damascelli, Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results, Ann. Inst. H. Poincaré Anal. Non Linéaire 15 (1998) 493–516.
[10] E.N. Dancer, Z.T. Zhang, Fucik spectrum, sign-changing, and multiple solutions for semilinear elliptic boundary value problems with resonance at infinity, J. Math. Anal. Appl. 250 (2000) 449–464.
[11] N. Dunford, J. Schwartz, Linear Operators I, Wiley, New York, 1958.
[12] G.M. Figueiredo, Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument, J. Math. Anal. Appl. 401 (2013) 706–713.
[13] M. Filippakis, N.S. Papageorgiou, Nodal solutions for Neumann problems with a nonhomogeneous differential operator, Funkcial. Ekvac. 56 (2013) 63–79.
[14] A. Fiscella, E. Valdinoci, A critical Kirchhoff type problem involving a nonlocal operator, Nonlinear Anal. 94 (2014) 156–170.
[15] L. Gasiński, N.S. Papageorgiou, Nonlinear Analysis, vol. 9, Chapman & Hall/CRC, Boca Raton, FL, 2006.
[16] L. Gasiński, N.S. Papageorgiou, Multiple solutions for a class of nonlinear Neumann eigenvalue problems, Commun. Pure Appl. Anal. 13 (2014) 1491–1512.
[17] L. Gasiński, D. O'Regan, N.S. Papageorgiou, A variational approach to nonlinear logistic equations, Commun. Contemp. Math. 17 (2015) 1450021.
[18] T.S. He, H.M. Yan, Z.H. Sun, M. Zhang, On nodal solutions for nonlinear elliptic equations with a nonhomogeneous differential operator, Nonlinear Anal. 118 (2015) 41–50.
[19] S.C. Hu, N.S. Papageorgiou, Nonlinear Neumann problems with indefinite potential and concave terms, Commun. Pure Appl. Anal. 14 (2015) 2561–2616.
[20] A. Lê, Eigenvalue problems for the p-Laplacian, Nonlinear Anal. 64 (2000) 1057–1099.

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[21] G.M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, Nonlinear Anal. 12 (1988) 1203–1219.
[22] Z.L. Liu, J.X. Sun, Invariant sets of descending flow in critical point theory with applications to nonlinear differential equations, J. Differential Equations 172 (2001) 257–299.
[23] S.E. Manouni, N.S. Papageorgiou, P. Winkert, Parametric nonlinear nonhomogeneous Neumann equations involving a nonhomogeneous differential operator, Monatsh. Math. 177 (2015) 203–233.
[24] M. Montenegro, Strong maximum principles for supersolutions of quasilinear elliptic equations, Nonlinear Anal. 37 (1999) 431–448.
[25] D. Motreanu, N.S. Papageorgiou, Multiple solutions for nonlinear Neumann problems driven by a nonhomogeneous differential operator, Proc. Amer. Math. Soc. 139 (2011) 3527–3535.
[26] D. Motreanu, M. Tanaka, Multiple existence results of solutions for quasilinear elliptic equations with a nonlinearity depending on a parameter, Ann. Mat. Pura Appl. 193 (2014) 1255–1282.
[27] D. Mugnai, N.S. Papageorgiou, Resonant nonlinear Neumann problems with indefinite weight, Ann. Sc. Norm. Super. Pisa Cl. Sci. 11 (2012) 729–788.
[28] N.S. Papageorgiou, V.D. Rădulescu, Bifurcation of positive solutions for nonlinear nonhomogeneous Robin and Neumann problems with competing nonlinearities, Discrete Contin. Dyn. Syst. 35 (2015) 5003–5036.
[29] N.S. Papageorgiou, V.D. Rădulescu, Coercive and noncoercive nonlinear Neumann problems with indefinite potential, Forum Math. 28 (2016) 545–571.
[30] N.S. Papageorgiou, V.D. Rădulescu, Infinitely many nodal solutions for nonlinear nonhomogeneous Robin problems, Adv. Nonlinear Stud. 16 (2016) 287–299.
[31] N.S. Papageorgiou, E.M. Rocha, On nonlinear parametric problems for p-Laplacian-like operators, Rev. R. Acad. Cienc. Ser. A Mat. 103 (2009) 177–200.
[32] N.S. Papageorgiou, P. Winkert, On a parametric nonlinear Dirichlet problem with subdiffusive and equidiffusive reaction, Adv. Nonlinear Stud. 14 (2014) 565–591.
[33] N.S. Papageorgiou, P. Winkert, Nonlinear nonhomogeneous Dirichlet equations involving a superlinear nonlinearity, Results Math. 70 (2016) 31–79.
[34] P.H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Regional Conf. Ser. in Math., vol. 65, Amer. Math. Soc., Providence, RI, 1986.
[35] J.X. Sun, J.L. Sun, Existence theorems of multiple critical points and applications, Nonlinear Anal. 61 (2005) 1303–1317.
[36] J.L. Vázquez, A strong maximum principle for some quasilinear elliptic equations, Appl. Math. Optim. 12 (1984) 191–202.

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