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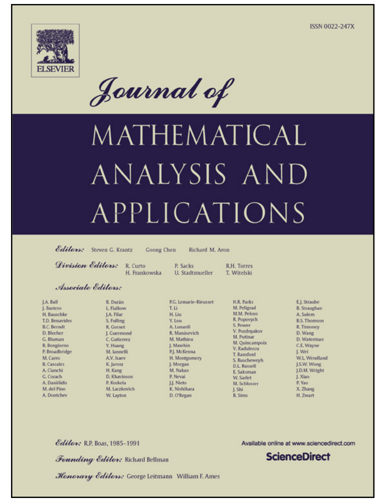

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An approach for the Convex Feasibility Problem via Monotropic Programming

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Abstract

In this note, we use recent zero duality results arising from Monotropic Programming problem for analyzing consistency of the convex feasibility problem in Hilbert spaces. We characterize consistency in terms of the lower semicontinuity of the infimal convolution of the associated support functions.

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1 Introduction

The Convex Feasibility Problem (CFP), consisting of finding a point in the intersection of a finite family of closed and convex sets $C_1, \ldots, C_m$ in a Hilbert space H, recasts numerous problems in mathematics and physical sciences. Many iterative methods have been studied and proved to converge to a point in the intersection provided it is nonempty. However, in applications it may not be a priori clear whether or not the intersection is nonempty. Hence, our aim is to analyse the feasibility problem for sets that may have an empty intersection.

It is well-known that the m-set CFP in a Hilbert space H is equivalent to a simpler problem involving two convex and closed sets in the cartesian product H^m consisting of m copies of H, with the additional advantage that one of these sets is a linear subspace (see, e.g., [1]). Namely, it can be shown that

$$\cap_{i=1}^m C_i \neq \emptyset \quad \Longleftrightarrow \quad \bar{C} \cap D \neq \emptyset,$$

where $\bar{C} := \{(x_1, \cdots, x_m) \in H^m : x_i \in C_i, i = 1, \ldots, m\} = C_1 \times C_2 \times \ldots \times C_m$, the cartesian product of the sets, and $D := \{(x_1, \cdots, x_m) \in H^m : x_1 = x_2 = \cdots = x_m\}$, the closed diagonal subspace of H^m. The product space H^m is a Hilbert space in which the scalar product, the norm, and the distance, can be defined in a standard form using the scalar product, the norm, and the distance provided by H, respectively.

Therefore, in our analysis we can assume that we are dealing with the convex feasibility problem involving only two (possibly disjoint) closed convex sets.

In our analysis, we will pose the convex feasibility problem as a *monotropic programming problem*. Monotropic programming consists of a class of convex optimization problems wherein duality results are as powerful as those known for linear programs. The early signs of research on monotropic programs appear on the seminal works of Minty [7] and Rockafellar [8, 9, 10]. In its more general setting, this problem is formulated as follows.

$$\inf \sum_{i=i}^{m} f_i(x_i) \quad \text{subject to} \quad (x_1, \cdots, x_m) \in S, \tag{P}$$

where $f_i : X_i \to \overline{\mathbb{R}} = [-\infty, +\infty]$ are proper and convex functions defined on separated locally convex spaces X_i, $i = 1, \cdots, m$, and $S \subseteq \prod_{i=1}^{m} X_i$ is a closed linear subspace such that $S \cap \prod_{i=1}^{m} \mathrm{dom} f_i \neq \emptyset$. Problem ($P$) is called the *extended monotropic programming* problem. We then use Fenchel duality to pose the dual problem. Under some constraint qualifications, we will be able to study both the *consistent* (i.e, when the intersection of the sets is not empty), and the inconsistent case. The dual problem of (P) is:

$$\inf \sum_{i=i}^{m} -f_i^*(x_i^*) \quad \text{subject to} \quad (x_1^*, \cdots, x_m^*) \in S^{\perp}, \tag{D}$$

where $f^*(x^*) := \sup_{x \in X}\{\langle x^*, x\rangle - f(x)\}$ is the *Fenchel conjugate* of f at $x^* \in X^*$ and $S^{\perp} = \{x^* \in X^* : \langle x^*, x\rangle = 0, \forall x \in S\}$ is the orthogonal subspace of S, being X^* the topological dual space of a separated locally convex space X, endowed with the weak* topology.

Denoting the optimal objective values of (P) and (D) by $v(P)$ and $v(D)$, respectively, the situation $v(P) = v(D)$ is called *zero duality gap*. Not all convex problems have zero duality gap. Rockafellar was the first to prove a zero duality gap result for the original class of monotropic programs when each space X_i is $\mathbb{R}$ (see [9, 10]), by using a variant of the ε-descent method. More recently, Bertsekas [3] generalized this result to extended monotropic programs in which the X_i's are finite-dimensional spaces. In [3], the f_i's are assumed to be lower semicontinuous in $\mathrm{dom} f_i$, for all $i = 1, \cdots, m$, and the set

$$S^{\perp} + \prod_{i=1}^{m} \partial_{\varepsilon} f_i(x_i), \tag{1.1}$$

is assumed to be closed, $\forall \varepsilon > 0$ and $\forall (x_1, \cdots, x_m) \in \prod_{i=1}^m \operatorname{dom} f_i \bigcap S$, where $\partial_\varepsilon f_i$ denotes the ε-subdifferential of f_i. Afterwards, Boţ and Csetnek [4] showed that each f_i has to be lower semicontinuous in the whole space (otherwise the statement is false) and extended Bertsekas' result to the general case, under weaker topological assumptions.

The aim of the present paper is to present a reformulation of the CFP as an instance of problem (P), and use the resulting dual problem (D) to identify cases on empty or non-empty intersection.

Our paper is organized as follows. In Section 2 we present the basic definitions and results, including the dual reformulation of the CFP. In the last section, Section 3, we show how the dual problem provides us information on the intersection of the sets, under a constraint qualification. More precisely, in Section 3.2, the original two-set CFP is posed as a monotropic programming problem. Section 3.3 is devoted to prove the zero duality gap, strong duality and the optimality conditions for such a monotropic problem. In section 3.4 disjointness of the sets is analyzed in terms of the optimal value of the dual problem, getting a characterization by means of the lower semicontinuity of the infimal convolution of the support function of the sets. Focusing on the most critical case, when the distance of the sets is 0, in section 3.5 we provide information about the consistency regarding the dual solutions. In the last Section 3.6, the set of solutions is characterized in both the consistent and inconsistent case.

2 Preliminaries

This section collects the mathematical setting, as well as some definitions and results to be used in the sequel. For more details about these and other well-known notions see, for instance, [2]. Let us establish the notation used throughout the paper. From now on, we are focusing on the framework of a Hilbert space H with norm denoted by $\|\cdot\|$ and scalar product denoted by $\langle \cdot, \cdot \rangle$. The unit ball of H is denoted by $B := \{x \in H : \|x\| \leq 1\}$.

For a subset C of H, we denote by $\operatorname{cl} C$, $\operatorname{int} C$, $\operatorname{Bd} C$, the *closure*, the *interior* and the *boundary* of the set C, respectively. The *indicator function* of C is denoted as $\iota_C(x) := 0$ if $x \in C$ and $\iota_C(x) := +\infty$ otherwise. The *support function* of C is $\sigma_C(v) := \sup\limits_{y \in C} \langle v, y \rangle$, for any $v \in H$. The *normal cone* operator of C at x is $N_C(x) := \left\{x^* \in X^* : \sup\limits_{c \in C} \langle c - x, x^* \rangle \leq 0\right\}$ if $x \in C$ and $N_C(x) := \emptyset$ otherwise. If C is nonempty and convex, the *recession cone* of C, denoted as $\operatorname{rec} C$ is the set

$$\operatorname{rec} C := \{y \in H \; : \; \forall\, x \in C \text{ and } \forall\, t \geq 0, \;\text{ we have } x + t\, y \in C\}.$$

Given a subspace S, the *orthogonal subspace* is $S^\perp := \{v \in H : \langle v, x \rangle = 0, \forall x \in S\}$. Denote by $\mathbb{R}_{--} := (-\infty, 0)$, $\mathbb{R}_+ := [0, +\infty)$ and $\overline{\mathbb{R}} := [-\infty, +\infty]$.

Let $f : H \to \overline{\mathbb{R}}$ be a function with *domain* $\operatorname{dom} f := \{x \in H : f(x) < +\infty\}$ and *epigraph* $\operatorname{epi} f := \{(x, r) \in H \times \mathbb{R} : f(x) \le r\}$. The *Fenchel conjugate* of f is the function $f^* : H \to \overline{\mathbb{R}}$ defined by

$$f^*(v) := \sup_{x \in H}\{\langle v, x\rangle - f(x)\}.$$

The *closure* of f is the function $\operatorname{cl} f : X \to \overline{\mathbb{R}}$ defined as the *lower semicontinuous hull* of the function f, which has as epigraph $\operatorname{cl}\operatorname{epi} f$. Namely, $\operatorname{cl} f$ is the largest lower semicontinuous function majorized by f, and $\operatorname{cl} f = \liminf_{y \to x} f(y)$, $\forall x \in H$. Hence, f is lower semicontinuous at $x \in H$ if and only if $\operatorname{cl} f = f$.

Given f proper, for $\varepsilon \ge 0$, the *ε-subdifferential* of f is the set-valued operator $\partial_\varepsilon f : H \rightrightarrows H$ defined by

$$\partial_\varepsilon f(x) := \begin{cases} \{v \in H \mid \langle v, y - x\rangle - \varepsilon \le f(y) - f(x), \text{ for all } y \in H\} & \text{if } f(x) \in \mathbb{R} \\ \emptyset & \text{otherwise} \end{cases}$$

When $\varepsilon = 0$, $\partial_0 f(x) = \partial f(x)$ is the *subdifferential of f* at x.

For functions ψ_1, ψ_2 such that $\psi_i : H \to \overline{\mathbb{R}}$, their *infimal convolution* is the function $(\psi_1 \Box \psi_2) : H \to \overline{\mathbb{R}}$ defined by

$$\big(\psi_1 \Box \psi_2\big) z := \inf_{z_1 + z_2 = z} \big\{\psi(z_1) + \psi_2(z_2)\big\}. \tag{2.1}$$

2.1 Separation of two convex sets

We recall here diverse notions of separation between two sets with the aim of analyzing the different possible cases of CFP, according to the optimal value in the monotropic programming dual program. Note that, in general, these notions do not require disjointness of the sets.

Definition 2.1. Given C_1 and C_2 nonempty sets in a Hilbert space, a hyperplane $\mathcal{H}$ is said to *separate* C_1 and C_2 if C_1 is contained in one of the closed half-spaces defined by $\mathcal{H}$, while C_2 lies in the opposite closed half-space; in other words, C_1 and C_2 *are separated* if there exist $v \in H$, $\|v\| = 1$, and $\delta \in \mathbb{R}$ such that

$$C_1 \subseteq \mathcal{H}^{\le}_{v,\delta} := \{x \in H : \langle v, x\rangle \le \delta\},$$

$$C_2 \subseteq \mathcal{H}^{\ge}_{v,\delta} := \{x \in H : \langle v, x\rangle \ge \delta\},$$

where the sets on the right-hand side denote the half-spaces corresponding to the separating hyperplane, denoted by

$$\mathcal{H}_{v,\delta} := \{x \in H : \langle v, x\rangle = \delta\}.$$

This separation is said to be:

- *proper* if C_1 and C_2 are not contained in $\mathcal{H}$;
- *nice* if the hyperplane $\mathcal{H}$ is disjoint from C_1 or C_2;
- *strict* if the hyperplane $\mathcal{H}$ is disjoint from both C_1 and C_2;
- *strong* if there exist $\varepsilon > 0$ such that $C_1 + \varepsilon B$ is contained in one of the open half-spaces bounded by $\mathcal{H}$ and $C_2 + \varepsilon B$ is contained in the other open half-space, where B is the unit ball; that is,

$$C_1 + \varepsilon B \subseteq \mathcal{H}^{<}_{v,\delta} := \{x \in H : \langle v, x \rangle < \delta\},$$

$$C_2 + \varepsilon B \subseteq \mathcal{H}^{>}_{v,\delta} := \{x \in H : \langle v, x \rangle > \delta\}.$$

The standard (or basic) separation theorem asserts that C_1 and C_2, two nonempty convex sets such that the interior of C_1 is nonempty, $\operatorname{int} C_1 \neq \emptyset$, are properly separated if and only if

$$(\operatorname{int} C_1) \cap C_2 = \emptyset.$$

This result is true in any topological vector space. See for instance [11].

We recall now a lemma from [5], originally written for a Banach space.

Lemma 2.2. *Let C and D be closed convex subsets of H. Then $C \cap D \neq \emptyset$ if and only if* $(0, -1) \notin \operatorname{cl}(\operatorname{epi} \sigma_C + \operatorname{epi} \sigma_D)$.

3 Duality Properties for problems (P) and (D)

In this section we recall the duality properties for problems (P) and (D). The first focus of this section is on *zero duality gap*, which entails equality of the optimal values of the primal and dual problem, respectively.

3.1 Zero duality gap

In the following theorem, Boţ and Csetnek establish *zero duality gap* for the primal-dual problems (P)–(D) (i.e., $v(P) = v(D)$), under the assumption that the set given in (1.1) is closed. As mentioned in the introduction, Bertsekas provided a similar result in the setting of Euclidean spaces (see [3, Proposition 4.1]). We quote next [4, Theorem 3.2].

Theorem 3.1 (Boţ and Csetnek [4])**.** *Let X_i be separately locally convex spaces, $f_i : X_i \to \overline{\mathbb{R}} := [-\infty, +\infty]$ proper and convex functions, $i = 1, \cdots, m$, $S \subseteq \prod_{i=1}^m X_i$ a linear closed subspace such that $\prod_{i=1}^m \operatorname{dom} f_i \cap S \neq \emptyset$ and $g : \prod_{i=1}^m X_i \to \overline{\mathbb{R}}$ defined by $g(x_1, \cdots, x_m) = \sum_{i=1}^m f_i(x_i)$. Suppose further that $\operatorname{cl} f_i$, $i = 1, \cdots, m$, are proper* *functions and* $g(x) = \operatorname{cl} g(x)$ *for all* $x \in \operatorname{dom} \operatorname{cl} g \cap S$. *If for all* $(x_1, \cdots, x_m) \in \prod_{i=1}^m \operatorname{dom} f_i \cap S$ *and all* $\varepsilon > 0$ *the set* (1.1) *is closed, then* $v(P) = v(D)$.

If the functions f_i are lower semicontinuous on X_i, $i = 1, \cdots, m$, then the topological assumptions regarding the functions $\operatorname{cl} f_i$ in Theorem 3.1 are fulfilled (see [4]).

3.2 Duality for CFP

Let C_1 and C_2 be two closed convex sets in a Hilbert space H whose intersection may be empty. In general, the CFP, written as

$$\text{find } x \in C_1 \cap C_2, \tag{3.1}$$

may not have a solution. Thus we pose an optimization problem that allows to find the best substitute for a point in the intersection, the so-called *best approximation solution*. A natural way to find such best approximation solution is to solve the following optimization problem:

$$\inf_{x \in H} d_{C_1}(x) + d_{C_2}(x). \tag{O}$$

Remark 3.2. Unlike the CFP, problem (O) always has a finite optimal value, as long as each of the sets is nonempty. Indeed, it is easy to check that the optimal value of (O) is exactly the distance between C_1 and C_2, $d(C_1, C_2)$ (which is finite when $C_1 \neq \emptyset \neq C_2$). Indeed, for any $x_1 \in C_1$ and $x_2 \in C_2$, set $z = \alpha x_1 + (1-\alpha)x_2 \in H$, for some $\alpha \in (0,1)$. Hence $\|x_1 - z\| + \|x_2 - z\| = \|x_1 - x_2\|$, and

$$\begin{aligned} \inf_{x \in H} d_{C_1}(x) + d_{C_2}(x) &\leq d_{C_1}(z) + d_{C_2}(z) \\ &\leq \|x_1 - z\| + \|x_2 - z\| = \|x_1 - x_2\|. \end{aligned}$$

Then

$$\inf_{x \in H} d_{C_1}(x) + d_{C_2}(x) \leq d(C_1, C_2).$$

The opposite inequality follows from the triangular inequality.

To introduce duality, we write Problem (O) as an instance of the monotropic programming problem. Namely, consider the "diagonal" subspace $S := \{(x,y) \in H^2 : x = y\}$, and write Problem (O) as

$$\inf_{(x,y) \in S} d_{C_1}(x) + d_{C_2}(y). \tag{P}$$

The objective function in this problem consists of the sum of the terms $f_1 := d_{C_1}(\cdot)$ and $f_2 := d_{C_2}(\cdot)$. Both $f_1, f_2 : H \to \mathbb{R}$ are convex and continuous everywhere, and are functions of the separate variables x and y, respectively.

Following [3] (see also problem (D) in the introduction), the primal problem (P) has for dual:

$$\sup_{(v,w)\in S^{\perp}} - d^*_{C_1}(v) - d^*_{C_2}(w), \tag{D}$$

where $S^{\perp}$ denotes the orthogonal subspace of S and $d^*_{C_i}$ is the Fenchel conjugate of d_{C_i}, $i = 1, 2$. It is straightforward to see that

$$S^{\perp} = \{(v, w) \in H^2 : v + w = 0\}. \tag{3.2}$$

It is well-known that, for a convex set C, the conjugate function of the distance function is

$$d^*_C(v) = \sigma_C(v) + \imath_B(v), \tag{3.3}$$

where $\sigma_C(\cdot)$ is the support function of C, and $\imath_B(\cdot)$ is the indicator function of the unit ball B of H.

3.3 Strong duality and Optimality Conditions

Let $H^2 := H \times H$ be the product of two Hilbert spaces. Consider the following pair of primal-dual problems arising in the context of Fenchel duality.

$$\inf_{(x,y)\in H^2} f(x, y) + g(x, y), \tag{P_0}$$

and

$$\sup_{(v,w)\in H^2} -f^*(v, w) - g^*(-v, -w). \tag{D_0}$$

To derive strong duality (i.e., existence of a dual solution) and first order optimality conditions for our primal-dual problems (P) and (D), we will use [2, Proposition 15.22 and Theorem 19.1], respectively. For convenience of the reader, we quote these results next, after adapting them to our particular context. To state one of these, we need the definition of core of a set.

Definition 3.3. Let C be a convex subset of H, the *core* of C is

$$\text{core}(C) := \{x \in C \ : \ cone(C - x) = H\},$$

where $cone(C - x)$ denotes the smallest cone containing the set $C - x$.

The following result is [2, Proposition 15.22], and establishes strong duality for (P_0) and (D_0).

Proposition 3.4. *Let $f, g : H^2 \to \overline{\mathbb{R}}$ be proper, convex and lsc functions such that*

$$0 \in \operatorname{core}(\operatorname{dom} g - \operatorname{dom} f).$$

Then

$$\inf_{x \in H^2} f(x) + g(x) = -\min_{v \in H^2} f^*(v) + g^*(-v),$$

i.e., the optimal values of (P_0) and (D_0) coincide, and (D_0) has a solution.

The theorem we quote next is [2, Theorem 19.1], and establishes first order optimality conditions for (P_0) and (D_0).

Theorem 3.5. *Let $f, g : H^2 \to \overline{\mathbb{R}}$ be proper, convex and lsc functions such that $\operatorname{dom} g \cap \operatorname{dom} f \neq \emptyset$. Let $(x_1, x_2) \in H^2$, and let $(v_1, v_2) \in H^2$. Then the following are equivalent:*

(i) *(x_1, x_2) solves (P_0), and (v_1, v_2) solves (D_0), with the same optimal values.*

(ii) *$(v_1, v_2) \in \partial f(x_1, x_2)$ and $-(v_1, v_2) \in \partial g(x_1, x_2)$.*

The following technical lemma connects the optimal values and solutions of $(P_0)-(D_0)$ with those of $(P)-(D)$ in section 3.2.

Lemma 3.6. *With the notation used in problems (P) and (D), define $f : H^2 \to \overline{\mathbb{R}}$ as $f(x, y) := d_{C_1}(x) + \iota_S(x, y)$, and $g : H^2 \to \overline{\mathbb{R}}$ as $g(x, y) := d_{C_2}(y)$. The following hold.*

(a)

$$\inf_{(x,y) \in S} d_{C_1}(x) + d_{C_2}(y) = \inf_{(x,y) \in H^2} f(x, y) + g(x, y),$$

(b) *$(z_1, z_2) \in S$ solves (P) if and only if (z_1, z_2) solves (P_0).*

(c)

$$\sup_{(v,w) \in S^\perp} -d^*_{C_1}(v) - d^*_{C_2}(w) = \sup_{(v,w) \in H^2} -f^*(v, w) - g^*(-v, -w), \tag{3.4}$$

(d) *$(u, -u) \in H^2$ solves (D) if and only if $(0, u)$ solves (D_0).*

Proof. The proof of (a) follows directly from the definitions. The proof of (b) follows directly from (a). For parts (c) and (d), we note that a direct calculation yields $f^*(v_1, v_2) = d^*_{C_1}(v_1 + v_2)$ and

$$g^*(v_1, v_2) = \begin{cases} d^*_{C_2}(v_2) & \text{if } v_1 = 0, \\ +\infty & \text{if } v_1 \neq 0. \end{cases}$$

Hence, if we have

$$-f^*(u,w) - g^*(-u,-w) > -\infty \quad \Longrightarrow \quad u = 0. \tag{3.5}$$

This implies that

$$\sup_{u\in H} -f^*(0,u) - g^*(0,-u) \;=\; \sup_{(w,u)\in H^2} -f^*(w,u) - g^*(-w,-u) =: v(D_0),$$

where the equality between the supremums follows from (3.5). On the other hand, by definition, we have

$$v(D_0) = \sup_{u\in H} -f^*(0,u) - g^*(0,-u) = \sup_{u\in H} -d^*_{C_1}(u) - d^*_{C_2}(-u) = v(D),$$

and thus (c) is proved. Assume now that u is such that $-d^*_{C_1}(u) - d^*_{C_2}(-u) = v(D)$. Then $(u,-u)$ solves (D) and the above expression implies that

$$(0,u) \in \operatorname{Argmax}_{(v,w)\in H^2} - f^*(v,w) - g^*(-v,-w).$$

The proof is complete. □

We can now use these results to derive the following first order optimality conditions for our primal dual problems. We point out that, even though the result stated in the next proposition is well-known, we include the details here for convenience of the reader.

Proposition 3.7. *Problems (P) and (D) satisfy the zero duality gap property, and the dual problem always has a solution. In this situation, (x,y) is a solution of Problem (P) and (u,v) is a solution of Problem (D), if and only if (x,y) and (u,v) verify the following properties:*

$$(x,y) \in S,\ u \in \partial d_{C_1}(x), v \in \partial d_{C_2}(y),\ \text{ and } (u,v) \in S^{\perp}.$$

Proof. The fact that zero duality gap holds for the above primal-dual pairs follows from Theorem 3.1 (or Proposition 3.4). Indeed, each C_i is closed, and the functions $f_i = d_{C_i} : H \to \mathbb{R}_+$, $i = 1,2$, are continuous. Thus the functions f_i satisfy the assumptions of Theorem 3.1. Moreover, since every function f_i is real-valued, the ε-subdifferential $\partial_\varepsilon f_i(x_i)$, $i = 1,2$, $x_i \in H$, is a nonempty and weakly compact set. Thus the set (1.1) is weakly closed. Since every weakly closed convex set is closed for the strong topology, we have that all hypotheses in Theorem 3.1 hold. Therefore, (P) and (D) have the same optimal value. To verify strong duality, it is enough to check that the assumptions of Proposition 3.4 hold for a suitable choice of f and g. Indeed, take $f : H^2 \to \overline{\mathbb{R}}$ defined as $f(x_1,x_2) := d_{C_1}(x_1) + \iota_S(x_1,x_2)$, where $S := \{(x,x) : x \in H\}$ nonempty. Take $g : H^2 \to \overline{\mathbb{R}}$ defined as $g(x_1,x_2) := d_{C_2}(x_2)$. We can write

$$\operatorname{dom} g - \operatorname{dom} f = H^2 - S = H^2,$$

Hence we have $\operatorname{core}(\operatorname{dom} g - \operatorname{dom} f) = H^2$ and we are in conditions of the Proposition 3.4, and therefore there exists $(v_1, v_2) \in H^2$ such that

$$-(f^*(v_1, v_2) + g^*(-v_1, -v_2)) = \inf_{(x_1,x_2)\in H^2} f(x_1, x_2) + g(x_1, x_2).$$

The above equality entails strong duality, where $(v_1, v_2) \in H^2$ is a dual solution and $(x_1, x_2) \in H^2$ is a primal one.

To complete the proof, assume that $(x, y) \in S$ is a primal solution and $(u, v) \in S^\perp$ is a dual one, therefore, $y = x$ and $v = -u$. By Lemma 3.6(b)(d), we have that (x, x) solves (P_0) and $(0, u)$ solves (D_0). Now using the latter fact in Theorem 3.5 we deduce that $(0, u) \in \partial f(x, x)$ and $(0, -u) \in \partial g(x, x)$. Since f is the sum of two functions, one of which has full domain, the subdifferential sum formula holds, and hence we have

$$\begin{aligned} (0, u) \in \partial f(x, x) &= N_S(x, x) + \partial d_{C_1}(x) \times \{0\} \\ &= S^\perp + \partial d_{C_1}(x) \times \{0\} \\ &= \{(z + w, -z) \,:\, w \in \partial d_{C_1}(x), z \in H\}, \end{aligned}$$

where we also used (3.2). In a similar way, we have

$$(0, -u) \in \partial g(x, x) \;=\; \{0\} \times \partial d_{C_2}(x).$$

The last equality gives $-u \in \partial d_{C_2}(x)$. This fact, used in the expression for the subgradient of f gives $u = -z$ and

$$0 = z + w = -u + w, \text{ with } w \in \partial d_{C_1}(x),$$

and hence $w = u \in \partial d_{C_1}(x)$. Altogether, we obtained that if (x, y) is a solution of Problem (P) and (u, v) is a solution of Problem (D) then, clearly $(x, y) \in S$ and $(u, v) \in S^\perp$, and $u \in \partial d_{C_1}(x)$ and $v \in \partial d_{C_2}(y)$.

To prove the converse implication, if there exist $(x, y) \in S$ and $(u, v) \in S^\perp$ such that $u \in \partial d_{C_1}(x)$ and $v \in \partial d_{C_2}(y)$, therefore $-u \in \partial d_{C_2}(x)$, and let us prove that (x, x) is a solution of Problem (P) and $(u, -u)$ is a solution of Problem (D). First, we have that

$$\begin{aligned} (0, u) &= (-u + u, u) \\ &= (-u, u) + (u, 0) \\ &\in S^\perp + \partial d_{C_1}(x) \times \{0\} \\ &= \partial f(x, x). \end{aligned}$$

Since $(0, -u) \in \{0\} \times \partial d_{C_2}(x)$, by the equivalence in Theorem 3.5 we have that (x, x) solves (P_0) and $(0, u)$ solves (D_0). Then by Lemma 3.6(b)(d) we prove that (x, x) is a solution of (P) and $(u, -u)$ is a solution of (D). This completes the proof. □

3.4 Consistency of CFP and the optimal dual values

In this section, we study all the possible cases for the set $C_1 \cap C_2$ in terms of the optimal value of the dual problem. From (3.3) and the fact that the support function of a set is homogenous of degree 1, we write the dual problem (D) as:

$$\begin{aligned}
\sup_{v\in H} - d^*_{C_1}(v) - d^*_{C_2}(-v) &= \sup_{v\in H} - [\sigma_{C_1}(v) + \imath_B(v)] - [\sigma_{C_2}(-v) + \imath_B(-v)] \\
&= \sup_{\|v\|\le 1} - \sigma_{C_1}(v) - \sigma_{C_2}(-v) \\
&= \max_{t\in[0,1]} t \left(\sup_{\|v\|=1} - \sigma_{C_1}(v) - \sigma_{C_2}(-v) \right) \\
&= \max_{t\in[0,1]} t \left(- \inf_{\|v\|=1} \sigma_{C_1}(v) + \sigma_{C_2}(-v) \right) \\
&= - \min_{t\in[0,1]} t \left(\inf_{\|v\|\le 1} \sigma_{C_1}(v) + \sigma_{C_2}(-v) \right)
\end{aligned} \tag{3.6}$$

For $t > 0$, denote by $\Phi(t) := \inf_{\|v\|\le t} \sigma_{C_1}(v) + \sigma_{C_2}(-v)$. Since all functions involved are positively homogeneous and 0 belongs to the constraint set, it is direct to check that $0 \ge \Phi(t) = t\Phi(1)$ for all $t > 0$. Altogether, the above expression becomes

$$\sup_{v\in H} - d^*_{C_1}(v) - d^*_{C_2}(-v) = - \min_{t\in[0,1]} \{t\Phi(1)\} = \begin{cases} -\Phi(1)(>0) & \text{if } \Phi(1) < 0, \\ 0 & \text{if } \Phi(1) = 0. \end{cases} \tag{D'}$$

Indeed, the optimal value in (3.6) will be always greater than, or equal to zero.

The expression in (D') indicates that, by studying the different values of $\Phi(1)$, we can obtain information about the set $C_1 \cap C_2$. We make this fact precise in the following result.

Proposition 3.8. *Let $\Phi(1) = \inf_{\|v\|\le 1} \{\sigma_{C_1}(v) + \sigma_{C_2}(-v)\}$ be defined as in (D').*

1. *If $\Phi(1) < 0$, then $C_1 \cap C_2 = \emptyset$. In this situation, C_1 and C_2 are strongly separated. This situation is equivalent to $0 \notin \mathrm{cl}\,(C_2 - C_1)$.*

2. *If $\Phi(1) = 0$, then this situation is equivalent to $0 \in \mathrm{cl}\,(C_2 - C_1)$. We can have two possible cases:*

   2.1 *If $(\sigma_{C_1} \Box \sigma_{C_2})(\cdot)$ is lower-semicontinuous at 0 we have $C_1 \cap C_2 \neq \emptyset$. Equivalently, $0 \in (C_2 - C_1)$.*

   2.2 *If $(\sigma_{C_1} \Box \sigma_{C_2})(\cdot)$ is not lower-semicontinuous at 0 we have $C_1 \cap C_2 = \emptyset$. In this situation, there exists a closed hyperplane separating the sets, and this separation may not be proper. This situation is equivalent to $0 \in \mathrm{Bd}\,(C_2 - C_1) \setminus (C_2 - C_1)$.*

Proof. We start by relating Φ with the infimal convolution of the support functions. We can write

$$\begin{aligned}(\sigma_{C_1}\Box\sigma_{C_2})(0) &= \inf_{v\in H}\{\sigma_{C_1}(v)+\sigma_{C_2}(-v)\} \\ &= \inf_{t>0}\inf_{\|v\|\le t}\{\sigma_{C_1}(v)+\sigma_{C_2}(-v)\} \\ &= \inf_{t>0} t\Phi(1).\end{aligned} \tag{3.7}$$

1. $\Phi(1)<0$. The above expression yields $(\sigma_{C_1}\Box\sigma_{C_2})(0)=-\infty$. This implies that

$$\mathrm{cl}\,(\sigma_{C_1}\Box\sigma_{C_2})(0)\le(\sigma_{C_1}\Box\sigma_{C_2})(0)=-\infty. \tag{3.8}$$

Recall from [12, Theorem 2.2(e)] that

$$\mathrm{epi}\,\mathrm{cl}\,(\sigma_{C_1}\Box\sigma_{C_2})=\mathrm{cl}\,(\mathrm{epi}\,\sigma_{C_1}+\mathrm{epi}\,\sigma_{C_2}).$$

Combining this fact with (3.8) we deduce that $(0,-1)\in\mathrm{cl}\,(\mathrm{epi}\,\sigma_{C_1}+\mathrm{epi}\,\sigma_{C_2})$ and Lemma 2.2 yields $C_1\cap C_2=\emptyset$. To show the strong separation, fix $a>0$ such that $\Phi(1)<-a<0$. Hence there exists a nonzero u such that $\|u\|\le 1$ for which

$$\sigma_{C_1}(u)<-a-\sigma_{C_2}(-u).$$

Because the functions are positively homogeneous and u is not zero, we can assume $\|u\|=1$. The above inequality translates directly into strong separation of the sets. By the zero duality gap property, this case entails $d(C_1,C_2)>0$, which is equivalent to $0\notin\mathrm{cl}\,(C_2-C_1)$.

2. $\Phi(1)=0$. Using again the zero duality gap property, this case entails $d(C_1,C_2)=0$. Equivalently, $0\in\mathrm{cl}\,(C_2-C_1)$

By (3.7) we have that $(\sigma_{C_1}\Box\sigma_{C_2})(0)=0$. Now we consider two cases.
Case 2.1: $(\sigma_{C_1}\Box\sigma_{C_2})(\cdot)$ is lower-semicontinuous at 0. In this case, we have that the infimal convolution coincides with its closure at 0, i.e.,

$$\mathrm{cl}\,(\sigma_{C_1}\Box\sigma_{C_2})(0)=(\sigma_{C_1}\Box\sigma_{C_2})(0)=0.$$

So $(0,-1)\notin\mathrm{epi}\,(\mathrm{cl}\,(\sigma_{C_1}\Box\sigma_{C_2}))$ and the lemma yields $C_1\cap C_2\neq\emptyset$. In this case, the primal problem has a solution $z\in C_1\cap C_2$. Since the problem is consistent, we do not analyze the separation of these sets and we clearly have $0\in(C_2-C_1)$. Conversely, if $0\in(C_2-C_1)$ we can write,

$$\mathrm{cl}\,(\sigma_{C_1}\Box\sigma_{C_2})=(\sigma^*_{C_1}+\sigma^*_{C_2})^*=(\iota_{C_1\cap C_2})^*=\sigma_{C_1\cap C_2},$$

where we used in the first equality a classical property (see, e.g., [12, Theorem 3.2(c)]) and the fact that all involved functions are proper. Hence,

$$\mathrm{cl}\,(\sigma_{C_1}\Box\sigma_{C_2})(0)=\sigma_{C_1\cap C_2}(0)=0\le(\sigma_{C_1}\Box\sigma_{C_2})(0)=0.$$

Which implies that $(\sigma_{C_1}\Box\sigma_{C_2})$ is lower-semicontinuous at 0. Therefore, this case is characterized by the fact that $0 \in (C_2 - C_1)$.

CASE 2.2: $(\sigma_{C_1}\Box\sigma_{C_2})(\cdot)$ is not lower-semicontinuous at 0. In this situation, the infimal convolution must be strictly greater than its closure at 0, i.e.,

$$\mathrm{cl}\,(\sigma_{C_1}\Box\sigma_{C_2})(0) < (\sigma_{C_1}\Box\sigma_{C_2})(0) = 0.$$

This means that there exists $a < 0$ such that $(0, a) \in \mathrm{epi}\,\mathrm{cl}\,(\sigma_{C_1}\Box\sigma_{C_2})$. The set $\mathrm{epi}\,\mathrm{cl}\,(\sigma_{C_1}\Box\sigma_{C_2})$ is a closed cone because the function $(\sigma_{C_1}\Box\sigma_{C_2})$ is sublinear. Indeed, the epigraph of $\mathrm{cl}\,(\sigma_{C_1}\Box\sigma_{C_2})$, being the closure of $\mathrm{epi}\,(\sigma_{C_1}\Box\sigma_{C_2})$, is a closed cone too. Consequently, we must have

$$(0, -1) \in \mathrm{epi}\,(\mathrm{cl}\,(\sigma_{C_1}\Box\sigma_{C_2})) = \mathrm{cl}\,(\mathrm{epi}\,\sigma_{C_1} + \mathrm{epi}\,\sigma_{C_2}), \tag{3.9}$$

and hence $C_1 \cap C_2 = \emptyset$ with $d(C_1, C_2) = 0$. Let us study the separation properties for this case. Since for $v = 0$ we have $0 = \sigma_{C_1}(v) + \sigma_{C_2}(-v)$, the infimal convolution is exact at 0 and $\{0\} \subset \mathrm{Argmin}_{\|v\|\le 1}\{\sigma_{C_1}(v) + \sigma_{C_2}(-v)\}$. We claim that we cannot have $\mathrm{Argmin}_{\|v\|\le 1}\{\sigma_{C_1}(v) + \sigma_{C_2}(-v)\} = \{0\}$. Indeed, this implies that for every v such that $\|v\| \neq 0$ we must have

$$s(v) := \sigma_{C_1}(v) + \sigma_{C_2}(-v) > 0.$$

Hence, for every v such that $\|v\| \neq 0$, $\sigma_{C_1}(v) \in \mathbb{R}$ and $\sigma_{C_2}(-v) \in \mathbb{R}$, we can write

$$(v, \sigma_{C_1}(v)) + (-v, \sigma_{C_2}(-v)) = (0, \sigma_{C_1}(v) + \sigma_{C_2}(-v)) \subset \mathrm{epi}\,(\sigma_{C_1}) + \mathrm{epi}\,(\sigma_{C_2}).$$

This implies that

$$(\{0\} \times \mathbb{R}) \cap (\mathrm{epi}\,(\sigma_{C_1}) + \mathrm{epi}\,(\sigma_{C_2})) \subset \{0\} \times [0, +\infty). \tag{3.10}$$

By (3.9) we have that $(0, -1) \in \mathrm{cl}\,(\mathrm{epi}\,(\sigma_{C_1}) + \mathrm{epi}\,(\sigma_{C_2}))$. Altogether, for every $t \in (0, 1)$ we have

$$(1 - t)(0, -1) + t(0, s(v)) \in \mathrm{epi}\,(\sigma_{C_1}) + \mathrm{epi}\,(\sigma_{C_2}). \tag{3.11}$$

Take $0 < t < 1/(s(v) + 1)$ we will have

$$t - 1 + t\,s(v) = t(s(v) + 1) - 1 < 1 - 1 = 0,$$

which, together with (3.11), yields

$$(0, t(s(v) + 1) - 1) \in (\mathrm{epi}\,(\sigma_{C_1}) + \mathrm{epi}\,(\sigma_{C_2})) \cap \{0\} \times (-\infty, 0),$$

contradicting (3.10). Hence our claim is true and we must have Argmin $_{\|v\|\leq 1}\{\sigma_{C_1}(v)+\sigma_{C_2}(-v)\} \supsetneq \{0\}$. Fix a nonzero $u \in$ Argmin $_{\|v\|\leq 1}\{\sigma_{C_1}(v)+\sigma_{C_2}(-v)\}$. For this u we will have

$$\sigma_{C_1}(u) = -\sigma_{C_2}(-u) =: b.$$

Because the functions are positively homogeneous and u is not zero, we can assume $\|u\| = 1$. The above equality translates directly into separation of the sets by the hyperplane $\mathcal{H}_{u,b}$. This separation may be improper if one of the sets is contained in the hyperplane.

To complete the proof, let us prove that, in this case, $0 \in \mathrm{Bd}\,(C_2 - C_1) \setminus (C_2 - C_1)$. Indeed, in this case we have $0 \in \mathrm{cl}\,(C_2 - C_1)$ with $0 \notin (C_2 - C_1)$, which yields $0 \in \mathrm{Bd}\,(C_2 - C_1) \setminus (C_2 - C_1)$. Conversely, if $0 \in \mathrm{Bd}\,(C_2 - C_1) \setminus (C_2 - C_1)$ then we have $d(C_1, C_2) = 0$ and hence by zero duality gap we are in the case $\Phi(1) = 0$. So $(\sigma_{C_1} \square \sigma_{C_2})(0) = 0$. We must show that $(\sigma_{C_1} \square \sigma_{C_2})(\cdot)$ is not lsc at 0. Indeed, if this is not the case, then by the previous case we must have $0 \in (C_2 - C_1)$. Since we assumed that $0 \in \mathrm{Bd}\,(C_2 - C_1) \setminus (C_2 - C_1)$, we cannot have $(\sigma_{C_1} \square \sigma_{C_2})(\cdot)$ lsc at 0.

□

It may not be easy to determine whether or not $(\sigma_{C_1} \square \sigma_{C_2})$ is lsc at 0. The following corollary provides a geometric condition for this to hold.

Corollary 3.9. *Assume that $(\sigma_{C_1} \square \sigma_{C_2})(0) > -\infty$. Then the function $(\sigma_{C_1} \square \sigma_{C_2})$ is proper, and the following statements are satisfied and equivalent:*

(i) $C_1 \cap C_2 \neq \emptyset$,

(ii) $(\sigma_{C_1} \square \sigma_{C_2})$ *is lsc at* 0,

(iii) $\{0\} \times \mathbb{R} \cap \mathrm{epi}\,(\sigma_{C_1} \square \sigma_{C_2}) = \{0\} \times \mathbb{R}_+$

Consequently, if $\mathrm{epi}\,\sigma_{C_1} + \mathrm{epi}\,\sigma_{C_2}$ *is closed, then* $C_1 \cap C_2 \neq \emptyset$.

Proof. From (3.7) we see that the properness assumption $(\sigma_{C_1} \square \sigma_{C_2})(0) > -\infty$ yields $\Phi(1) = 0$, and hence $(\sigma_{C_1} \square \sigma_{C_2})(0) = 0$. This implies that for every $u \in \mathrm{dom}\,(\sigma_{C_1} \square \sigma_{C_2})$ we must have $(\sigma_{C_1} \square \sigma_{C_2})(u) > -\infty$. Indeed, the fact that $(0, 0) = (0, (\sigma_{C_1} \square \sigma_{C_2})(0)) \in \mathrm{epi}\,(\sigma_{C_1} \square \sigma_{C_2})$, together with the definition of recession cone imply that the direction $y := (0, -1) \in H \times \mathbb{R}$ does not belong to the recession cone of the epigraph. Indeed, if $(0, -1) \in \mathrm{rec}\,(\mathrm{epi}\,(\sigma_{C_1} \square \sigma_{C_2}))$, we must have

$$\{0\} \times \mathbb{R} \subset \mathrm{epi}\,(\sigma_{C_1} \square \sigma_{C_2}),$$

because $(0,0)$ belongs to the epigraph. This inclusion contradicts the properness assumption. Hence $y := (0,-1) \in H \times \mathbb{R}$ is not in the recession cone of the epigraph. Equivalently, the epigraph contains no vertical lines.

Furthermore, the fact that $(\sigma_{C_1} \square \sigma_{C_2})(0) = 0$, by definition of epigraph, implies statement (iii). And (iii) implies (i) because, by Lemma 2.2, $(0,-1) \notin \operatorname{epi}(\sigma_{C_1} \square \sigma_{C_2})$ is equivalent to $C_1 \cap C_2 \neq \emptyset$. The equivalence between (i) and (ii) follows from case 2.1 in Proposition 3.8.

Finally, the closedness of the sum of the epigraphs yields the lsc of the infimal convolution in the whole space, so we can use (ii) implies (i) to obtain the last statement. $\square$

3.5 Consistency of CFP and the dual solutions

By Proposition 3.7 we know that (D) always has a solution. If the optimal value of (D) is positive, then we know by Proposition 3.8 that CFP has no solution because the sets are strongly separated. The critical case is when $v(D) = 0$, since in this case we may or may not be in the consistent case. The following result gives us information about consistency of the CFP when $v(D) = 0$.

Corollary 3.10. *Consider the (nonempty) set of solutions of (D), and assume that $v(D) = 0$. The following hold.*

(a) If $v = 0$ is the unique solution to the dual problem (D) then we must have $C_1 \cap C_2 \neq \emptyset$. Conversely, if $C_1 \cap C_2 \neq \emptyset$ then the dual problem has for unique solution $v = 0$.

(b) The dual problem (D) has multiple solutions if and only if $C_1 \cap C_2 = \emptyset$. In this situation, every nonzero dual solution induces a possibly improper separation of the sets.

Proof. First, the set of solutions of (D) is not empty by Proposition 3.7. Let us prove part (a). If $v = 0$ is a solution of (D), the optimal value is $d(C_1, C_2) = 0$. By part 2 of the proof of Proposition 3.8, we are in the case in which $\Phi(1) = 0$ and the infimal convolution $(\sigma_{C_1} \square \sigma_{C_2})(\cdot)$ is lsc at 0. As in the proof of case 2.1 in Proposition 3.8, this is equivalent to having $C_1 \cap C_2 \neq \emptyset$. Conversely, if $v(D) = 0$ we are in the case $\Phi(1) = 0$. By Corollary 3.9 and the fact that $C_1 \cap C_2 \neq \emptyset$ we deduce that $(\sigma_{C_1} \square \sigma_{C_2})(\cdot)$ is lsc at 0, and by case 2.1 this implies that (D) has $v = 0$ as its unique solution.
For proving part (b) we use Proposition 3.8 again. Indeed, note that the case in which there are multiple solutions of (D) is the case in which $(\sigma_{C_1} \square \sigma_{C_2})(\cdot)$ is not lsc at 0, and this implies that $C_1 \cap C_2 = \emptyset$ by the proof of part 2.2 in Proposition 3.8. Conversely, if $C_1 \cap C_2 = \emptyset$, since $v(D) = 0$ we must be in case 2.2. The proof of case 2.2 shows that the set $\operatorname{Argmax}_{\|v\| \leq 1} -\sigma_{C_1}(v) - \sigma_{C_2}(-v)$ cannot be only $v = 0$.

The last assertion in (b) follows from the proof of part 2.2 in Proposition 3.8. This completes the proof. □

Remark 3.11. As mentioned above, the critical case $\Phi(1) = 0 = v(D)$, considered in Case 2 of Proposition 3.8, cannot be analyzed using only primal information. Namely, in this case we may have either a consistent or an inconsistent case. An infinite dimensional example of two disjoint closed affine sets which cannot be properly separated can be found in [2, Example 3.41]. Since the sets are disjoint, by Corollary 3.9, these examples correspond to the case in which the corresponding infimal convolution is not lsc at 0. By Corollary 3.10, the set of dual solutions is not a singleton.

3.6 Consistency of CFP and the primal solutions

In this section, we assume that problem (P) has a solution, and we study all possible cases for the set $C_1 \cap C_2$ in terms of the location of the solutions. Recall, that, for a closed convex set $C \subseteq H$, the subdifferential of the distance function d_C is given by (see [6])

$$\partial d_C(x) = \begin{cases} 0 & \text{if } x \in \operatorname{int} C, \\ N_C(x) \cap B & \text{if } x \in \operatorname{Bd} C, \\ \dfrac{x - P_C(x)}{\|x - P_C(x)\|} & \text{if } x \notin C, \end{cases}$$

where $P_C(x)$ is the metric projection of x onto C.

The following lemma characterizes the set of solutions of (P) for both the consistent and inconsistent case.

Lemma 3.12. *Given C_1, C_2 two sets, we always have.*

(a) $\inf_{x \in H} \|P_{C_1}(x) - P_{C_2}(x)\| = d(C_1, C_2)$.

(b) The set of solutions of (P) is the set

$$\operatorname{sol}(P) = \{x \in H \;:\; d(C_1, C_2) = \|P_{C_1}(x) - P_{C_2}(x)\|\}.$$

Proof. The proof of (a) follows easily from the definitions. Indeed, it is clear that

$$\inf_{x \in H} \|P_{C_1}(x) - P_{C_2}(x)\| \geq d(C_1, C_2)$$

On the other hand, we have, for all $y \in H$,

$$\inf_{x \in H} \|P_{C_1}(x) - P_{C_2}(x)\| \leq \|P_{C_1}(y) - P_{C_2}(y)\| \leq \|P_{C_1}(y) - y\| + \|y - P_{C_2}(y)\| = d_{C_1}(y) + d_{C_2}(y).$$

Taking infimum on the right hand side we deduce that

$$\inf_{x\in H} \|P_{C_1}(x) - P_{C_2}(x)\| \le \inf_{y\in H} d_{C_1}(y) + d_{C_2}(y) = d(C_1, C_2),$$

as wanted. We proceed to prove part (b). We will show that both sets coincide, and this clearly covers the case in which either set is are empty. Denote by E the set in the right-hand side. We start by showing that, if (P) has solutions, then all solutions belong to E, and thus E is not empty. Fix x a solution of (P).

If $x \notin C_1 \cup C_2$ then by Proposition 3.7 and the expression for the subdifferential of the distance function, there exists a nonzero $v \in H$ such that

$$v = \frac{x - P_{C_1}(x)}{\|x - P_{C_1}(x)\|} = \frac{-x + P_{C_2}(x)}{\|x - P_{C_2}(x)\|}.$$

It is a matter of simple algebra to show that in this case x is a convex combination of $P_{C_1}(x)$ and $P_{C_2}(x)$, and this yields

$$d(C_1, C_2) = d(x, C_1) + d(x, C_2) = \|x - P_{C_1}(x)\| + \|x - P_{C_2}(x)\| = \|P_{C_1}(x) - P_{C_2}(x)\|,$$

where we used the fact that $x \in [P_{C_1}(x), P_{C_2}(x)]$ in the third equality. The last expression implies that $x \in E$.

Assume now that $x \in C_1 \cup C_2$, say, $x \in C_1$. In this case, we have $x = P_{C_1}(x)$ and hence

$$d(C_1, C_2) = d(x, C_1) + d(x, C_2) = 0 + \|x - P_{C_2}(x)\| = \|P_{C_1}(x) - P_{C_2}(x)\|.$$

Therefore, $x \in E$ also in this case.

Conversely, assume that $x \in E$. We can write

$$\begin{aligned} d^2(C_1, C_2) &= d^2(x, C_1) + d^2(x, C_2) + 2d(x, C_1)d(x, C_2) \\ &= \|P_{C_1}(x) - P_{C_2}(x)\|^2 \\ &= \|P_{C_1}(x) - x + x - P_{C_2}(x)\|^2 \\ &= \|P_{C_1}(x) - x\|^2 + \|x - P_{C_2}(x)\|^2 + 2\langle P_{C_1}(x) - x, x - P_{C_2}(x)\rangle, \end{aligned}$$

where we used the fact that $x \in E$ in the second equality. The above expression simplifies to

$$\|P_{C_1}(x) - x\| \, \|x - P_{C_2}(x)\| = \langle P_{C_1}(x) - x, x - P_{C_2}(x)\rangle,$$

which can only happen when there exists $t > 0$ such that $P_{C_1}(x) - x = t(x - P_{C_2}(x))$. It is again a matter of simple algebra to deduce from the latter fact that x is a convex combination of $P_{C_1}(x)$ and $P_{C_2}(x)$. Altogether, we have

$$d(C_1, C_2) = \|P_{C_1}(x) - P_{C_2}(x)\| = \|x - P_{C_1}(x)\| + \|x - P_{C_2}(x)\| = d(x, C_1) + d(x, C_2),$$

where we used the fact that $x \in E$ in the first equality, and the fact that $x \in [P_{C_1}(x), P_{C_2}(x)]$ in the second equality. The expression above implies that the set of solutions of (P) is not empty and x solves (P). The proof of the lemma is complete. $\square$

As mentioned above, if $d(C_1, C_2) > 0$ the CFP is inconsistent. We consider next the critical case $d(C_1, C_2) = 0$. The following corollary combines Corollary 3.9 and Lemma 3.12.

Corollary 3.13. *Assume that $d(C_1, C_2) = 0$. In this situation, the following statements are equivalent.*

(i) *(P) has no solutions.*

(ii) $0 \in \mathrm{cl}(C_1 - C_2) \setminus (C_1 - C_2)$.

(iii) *$\sigma_{C_1} \Box \sigma_{C_1}$ is not lsc at* 0.

(iv) $C_1 \cap C_2 = \emptyset$

(v) $\{0\} \times \mathbb{R}_{--} \cap \operatorname{epi}(\sigma_{C_1} \Box \sigma_{C_2}) \neq \emptyset$.

Proof. The equivalence between (i) and (ii) follows from Lemma 3.12. Indeed, the lemma shows that (P) has solutions if and only if $0 \in (C_1 - C_2)$. The equivalence between (ii) and (iv) is trivial. The equivalence between (iii), (iv) and (v) follows directly from Corollary 3.9. $\square$

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