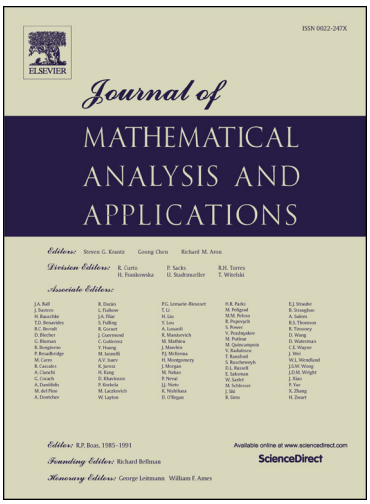
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Normal Structure and Orbital Fixed Point Conditions

W. A. Kirk and Naseer Shahzad

Abstract. In some sense all of the results discussed in this paper are an outgrowth of the concept of 'normal structure' introduced by Brodskii and Milman in 1948 and following related idea introduced by Belluce and Kirk in 1969. A mapping T of a metric space K into itself is said to have 'diminishing orbital diameters' if given any $x \in K$ it is the case that

$$\lim_{n\to\infty} diam\left(O\left(T^n\left(x\right)\right)\right) < diam\left(O\left(x\right)\right)$$

whenever $diam\left(O\left(x\right)\right) > 0$, where $O\left(x\right) = \left\{x, T\left(x\right), T^2\left(x\right), \cdots\right\}$. Among other things Belluce and Kirk observed that the assumption of diminishing orbital diameters on a mapping $T : K \to K$ is sufficient to assure that T has a fixed point if T is nonexpansive and if K is a weakly compact convex subset of a Banach space. It was later shown by Kirk that the convexity assumption on K can be dropped.

The purpose of this discussion is to illustrate how the above ideas are related to a number of similar ones that have been introduced more recently. Among these is the following concept introduced recently by Amini-Harandi, et al. A mapping $T : K \to K$ of a subset K of a Banach space is said to be *nonexpansive wrt orbits* if for all $x, y \in K$,

$$\left\|T\left(x\right) - T\left(y\right)\right\| \leq r_x\left(O\left(y\right)\right)$$

where

$$r_x\left(O\left(y\right)\right) = \sup\left\{\left\|x - u\right\| : u \in O\left(y\right)\right\}.$$

It is shown that if K is a weakly compact subset of a Banach space and if $T : K \to K$ is nonexpansive wrt orbits then the following condition always assures the existence of a fixed point for T : For each $x \in K$ with $T\left(x\right) \neq x$,

$$\inf_{m\in\mathbb{N}}\left\{\lim\sup_{n\to\infty}\left\|T^m\left(x\right) - T^n\left(x\right)\right\|\right\} < diam\left(O\left(x\right)\right).$$

Some open questions are also discussed.

1. Introduction

In this paper we discuss how certain classical metric fixed point theorems, and especially theorems involving nonexpansive mappings published over forty years ago, relate to some recently published results. As a point of departure we revisit results obtained in [**3**] and [**16**], and show how they are related to ideas that have emerged quite recently. The underlying setting is a Banach space, and the basic

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definitions are these: The *diameter* of a subset D of a Banach space $(X, \|\cdot\|)$ is denoted $diam\,(D)$; thus

$$diam\,(D) = \sup\{\|u - v\| : u, v \in D\}.$$

A closed convex subset K of a Banach space X is said to have *normal structure* ([**5**]) [resp., *weak normal structure*] if given any closed bounded [resp., weakly compact] convex subset H of K for which $diam\,(H) > 0$ there exists $u \in H$ such that

$$r_u(H) := \sup\{\|u - x\| : x \in H\} < diam\,(H).$$

(Thus normal structure and weak normal structure coincide if the underlying Banach space X is reflexive.) Also a Banach space is said to have the *fixed point property* [resp., *weak fixed point property*] for nonexpansive mappings if every nonexpansive mapping $T : K \to K$ has a fixed point for any bounded closed convex [resp., weakly compact convex] subset K of X. (Recall that a mapping $T : K \to K$ is said to be *nonexpansive* if for each $u, v \in K$, it is the case that $\|T(u) - T(v)\| \leq \|u - v\|$.)

2. Normal structure and related concepts

We refer to [**18**] for the historical development of many of the ideas discussed in this section. (See [**27**] for a more recent survey.) The following remarks are anchored in the following classical result. As is well-known, this result was proved simultaneously and independently by Browder [**6**] and Göhde [**11**] under the stronger assumption that the underlying space is uniformly convex.

THEOREM 2.1. (*Kirk* [**13**]) *Let K be a weakly compact convex subset of a Banach space, and suppose K has normal structure. Then every nonexpansive mapping $T : K \to K$ has a fixed point.*

The above theorem has motivated a number of related results involving weakening the normal structure assumption.

DEFINITION 2.1. A convex subset K of a Banach space X is said to have *normal structure relative to K* if given any bounded convex subset H of K for which $diam\,(H) > 0$ there exists $x \in K$ and $r < diam\,(H)$ such that $H \subseteq B(x; r)$, *where* $B(x; r) = \{y \in X : ||x - y|| < r\}$.

The above extension of normal structure was introduced in [**14**], and an example is given in that paper shows that it is properly weaker than the concept of normal structure. This weaker assumption yields a fixed point theorem for a class of mappings which lie strictly between the class of nonexpansive mappings and the Banach contractions. The following is Corollary 1 of [**14**]. (Recall that a mapping $T : M \to N$ of a metric space (M, d) into a metric space (N, d) is said to be *strictly contractive* if $d(T(x), T(y)) < d(x, y)$ for each $x, y \in M$ with $x \neq y$.)

THEOREM 2.2. *Let K be a weakly compact convex subset of a Banach space, suppose $T : K \to K$ is strictly contractive, and suppose K has normal structure relative to K. Then T has a (unique) fixed point.*

A closed subset K of a Banach space X has *uniform relative normal structure* if there exists a $c < 1$ such that, for every nonvoid bounded closed subset H of K, there exists a $z \in K$ satisfying (a) $H \subset B(z; cd)$, where $d = diam\,(H)$; (b) if, for

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some $y \in K$, $H \subset B(y; cd)$, then $\| z - y \| < cd$. The following theorem is due to P. Soardi [**28**]:

THEOREM 2.3. *Suppose that X is a dual Banach space and K is a weak* closed subset of X with uniform relative normal structure. Let $T : K \to K$ be a nonexpansive mapping which leaves invariant a weak* compact subset H of K. Then T has a fixed point in K.*

3. Diminishing orbital diameters and pointwise contractions

Let A be a subset of a metric space M and $T : A \to A$. For $x \in A$, let $O(x) = \{x, T(x), T^2(x), \cdots\}$, and for $n \geq 0$, let $O(T^n(x)) = \bigcup_{i=n}^{\infty} \{T^i(x)\}$. The set $O(x)$ is called the *orbit* of T at x. The sequence $\{diam(O(T^n(x)))\}$ is nondecreasing and has limit $r(x)$, called the *limiting orbital diameter* of T at x. (In general, this limit may be infinite, but typically the domain of T is a bounded set.)

DEFINITION 3.1 ([**3**]). Let T be a mapping of a metric space M into itself. If for each $x \in M$ it is the case that $diam(O(x)) < \infty$ and $r(x) < diam(O(x))$ when $diam(O(x)) > 0$, then T is said to have *diminishing orbital diameters* on M.

The following is the main result of [**3**].

THEOREM 3.1 ([**3**, Theorem 2]). *Let K be a bounded closed convex subset of a Banach space X, and let M be a weakly compact subset of X. If T is a nonexpansive mapping of K into K such that*

(i) *for each $x \in K$, $conv(O(x)) \cap M \neq \emptyset$, where $conv(O(x))$ denotes the convex hull of $O(x)$ and*

(ii) *T has diminishing orbital diameters;*

then T has a fixed point.

REMARK 3.1. *Under the assumptions of Theorem 3.1, for each $x \in K$,*

$$diam(O(x)) = \sup_{i \in \mathbb{N}} \left\| x - T^i(x) \right\|.$$

Since $r(x) < diam(O(x))$ it follows that

$$r(x) = \lim_{n \to \infty} diam(O(T^n(x))) < diam(O(x)).$$

Thus for $n \in \mathbb{N}$ sufficiently large,

$$\sup_{i > n} \left\| T^n(x) - T^i(x) \right\| < \sup_{i \in \mathbb{N}} \left\| x - T^i(x) \right\|.$$

Therefore if $r > r(x)$ then for $n \in \mathbb{N}$ sufficiently large, $diam(O(T^n(x))) \leq r$ and

$$O(T^n(x)) \subseteq B(T^n(x); r).$$

This idea is further refined in Section 5 (see Theorem 5.2).

Upon taking $M = K$ Theorem 3.1 reduces to the following.

COROLLARY 3.1 ([**3**, Corollary 2]). *If K is a convex weakly compact subset of a Banach space and if T is a nonexpansive mapping of K into K which has diminishing orbital diameters, then T has a fixed point.*

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Shortly after the publication of [**3**] it was noted that the convexity assumption on the domain K in the above corollary may be dropped. The following is the main result of [**16**].

THEOREM 3.2. *If K is a weakly compact subset of a Banach space and if T is a nonexpansive mapping of K into itself which has diminishing orbital diameters, then T has a fixed point.*

It was also noted in [**16**] that if T is assumed to be strictly contractive in the sense that $\|T(x) - T(y)\| < \|x - y\|$ for each $x, y \in K$, $x \neq y$, then a slightly more general result holds.

THEOREM 3.3 ([**16**]). *Suppose K is a weakly compact subset of a Banach space X and suppose $T : K \to K$ is strictly contractive. Suppose also that for each $x \in K$ there is an integer N (depending on X) such that T^N has diminishing orbital diameters. Then T has a fixed point.*

A special class of mappings which have 'diminishing orbital diameters' was also introduced in [**3**], and in many ways this restricted class seems more natural. These mappings have been studied in numerous papers where they have been called by various names, among them, 'generalized contractive mappings' in [**17**] and 'uniformly strictly contractive mappings' in [**15**]. In the present paper, as in [**19**], [**12**], [**26**], and elsewhere, we shall refer to them as *pointwise contractions*.

DEFINITION 3.2. Let M be a metric space. A mapping $T : M \to M$ is called a *pointwise contraction* if for each $x \in M$ there exists $\alpha(x) \in (0, 1)$ such that for each $y \in M$,

$$d(T(x), T(y)) \leq \alpha(x)\, d(x, y). \tag{3.1}$$

In contrast to the concept of diminishing orbital diameters, pointwise contractions seem to arise in a somewhat natural way. In fact the central result of [**17**] is the following: *Let A be a bounded open convex subset of a Banach space X and let $F : A \to X$ be continuously Fréchet differentiable. Then F is a pointwise contraction on A if and only if for each $x_0 \in A$, $\left\|F'_{x_0}\right\| < 1$ (where F'_{x_0} denotes the Fréchet derivative of F at x_0).*

As pointed out in [**3**], it is easy to see that if $T$is a pointwise contraction then it is both nonexpansive and has diminishing orbital diameters on M. Therefore the following is a very special case of Theorem 3.2.

THEOREM 3.4. *Let K be a weakly compact subset of a Banach space, and suppose $T : K \to K$ is a pointwise contraction. Then T has a (unique) fixed point $z \in K$. Moreover, the sequence $\{T^n(x)\}$ converges to this fixed point for each $x \in K$.*

Uniqueness of the fixed point in Theorem 3.4 is immediate from the fact that pointwise contractions are strictly contractive. Also, in view of Condition (3.1), it is equally obvious that the sequence $\{T^n(x)\}$ converges to this fixed point for each $x \in K$. (Mappings satisfying the conclusion of Theorem 3.4 are often referred to as Picard operators.)

The following is a trivial extension of Theorem 3.4.

THEOREM 3.5. *Let K be a weakly compact subset of a Banach space, and suppose $T : K \to K$ has the property that T^N is a pointwise contraction for some positive integer N. Then T has a fixed point.*

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PROOF. By Theorem 3.4 T^N has a *unique* fixed point $x \in K$. However

$$T^N \circ T(x) = T \circ T^N(x) = T(x),$$

so it must be the case that $T(x) = x$. □

4. Diametrical contractions

The preceding ideas are closely related to a much more recent concept. The following definition was introduced by Hong-Kun Xu in [**29**].

DEFINITION 4.1. Let (K, d) be a metric space. A mapping $T : K \to K$ is said to be *diametrically contractive if*

$$diam(T(A)) < diam(A)$$

for all closed subsets A of K for which $0 < diam(A) < \infty$.

The following is one of the principal results of [**29**].

THEOREM 4.1 ([**29**, Theorem 2.3]). *Let K be a weakly compact subset of a Banach space and let $T : K \to K$ be a diametrically contractive mapping. Then T has a fixed point.*

(Mappings of this type have been studied more recently in w-distance spaces (see [**10**]) where they are called *diametrally contractive*.)

Suppose $T : K \to K$ is diametrically contractive. Then taking $A = \{x, y\}$ in Definition 4.1 yields

$$d(T(x), T(y)) < d(x, y)$$

for all $x, y \in K$ such that $x \neq y$. Thus a diametrically contractive mapping is always strictly contractive and hence nonexpansive. Now suppose for $x \in K$ it is the case that $0 < diam(O(x)) < \infty$. Then if T is diametrically contractive,

$$diam(T(O(x))) < diam(O(x)).$$

However $T(O(x)) = O(T(x))$, and therefore

$$r(x) = \lim_{n\to\infty} diam(O(T^n(x))) \leq diam(O(T(x))).$$

It follows that $r(x) < diam(O(x))$. Since this is true for each $x \in K$, T also has diminishing orbital diameters on (bounded) K.

Therefore Theorem 4.1 follows directly from Theorem 3.2. Also under the assumptions of Theorem 4.1 the fixed point of T must be unique. This need not be the case under the assumptions of Theorem 3.2. Therefore it is easy to give trivial examples of nonexpansive mappings that have diminishing orbital diameters and yet fail to be diametrically contractive. (For example let T be a retraction of the unit ball in $\mathbb{R}^2$ onto its intersection with the x-axis.) Thus Theorem 3.2 properly contains Theorem 4.1.

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5. Mappings nonexpansive wrt orbits

We continue this discussion by showing that the assumptions on T in Corollary 3.1 may be significantly weakened. The orbital condition on T can be be relaxed, and it also suffices to assume only that T is 'nonexpansive with respect to (wrt) orbits'. This is a metric concept introduced in 2010 by Nicolae in CAT(0) spaces [**26**, cf., Theorem 5.1] and studied further in 2016 by Amini-Harandi et al. in [**1**].

DEFINITION 5.1. Let C be a subset of a Banach space X. A mapping $T : C \to C$ is said to be *nonexpansive wrt orbits* if for all $x, y \in C$,

$$\|T(x) - T(y)\| \leq r_x(O(y))$$

where

$$r_x(O(y)) = \sup\{\|x - u\| : u \in O(y)\}.$$

If T is nonexpansive then $\|T(x) - T(y)\| \leq \|x - y\| \leq r_x(O(y))$ so it follows that nonexpansive mappings, and especially pointwise contractions, are always nonexpansive wrt orbits. On the other hand, mappings that are nonexpansive wrt orbits need not even be continuous.

The central result of [**1**] is the following characterization:

THEOREM 5.1. *A Banach space X has weak normal structure if and only if X has the fixed point property for mappings which are nonexpansive wrt orbits.*

We now prove an analog of Theorem 3.2 for such mappings. This appears to be a new result.

THEOREM 5.2. *Let D be a weakly compact subset of a Banach space X, suppose $T : D \to D$ is nonexpansive wrt orbits, and suppose for each $x \in D$ with $x \neq T(x)$,*

$$\inf_{m \in \mathbb{N}} \left\{ \limsup_{n \to \infty} \|T^m(x) - T^n(x)\| \right\} < diam(O(x)). \tag{5.1}$$

Then T has a fixed point in D.

PROOF. This proof is patterned after the proof of Theorem 5 in [**16**]. Let D_1 be a subset of D which is minimal with respect to being nonempty, weakly compact, and invariant under T. Let $x \in D_1$ and suppose $T(x) \neq x$. Then $d_x := diam(O(x)) > 0$, and by (5.1)

$$\exists\, m, n \in \mathbb{N}, \text{ and } r \in (0, d_x) \text{ such that } O(T^n(x)) \subseteq B(T^m(x); r).$$

Let $\varepsilon > 0$ and let

$$S_\varepsilon = \left\{ y \in X : T^i(x) \in B(y; r + \varepsilon) \text{ for almost all } i \right\}.$$

Let

$$S = \bigcap_{\varepsilon > 0} S_\varepsilon.$$

Since $T^m(x) \in S$, $S \neq \emptyset$.

Claim 1. *S is closed.*

Suppose $\{z_n\} \subset S$ with $\lim_{n\to\infty} z_n = z$, and let $\varepsilon > 0$. Then there exists $n \in \mathbb{N}$ such that $\|z - z_n\| < \varepsilon/2$. Since $z_n \in S_{\varepsilon/2}$ there exists i_0 such that $i \geq i_0 \Rightarrow \left\|z_n - T^i(x)\right\| \leq r + \varepsilon/2$. Thus $i \geq i_0 \Rightarrow$

$$\left\|z - T^i(x)\right\| \leq \|z - z_n\| + \left\|z_n - T^i(x)\right\| \leq r + \varepsilon.$$

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Thus $z \in S_\varepsilon$ for each $\varepsilon > 0$, so $z \in S$; hence S is closed.

Claim 2. *S is convex.*

If $y_1, y_2 \in S$ and $\varepsilon > 0$, then for almost all i :

$$\left\|y_1 - T^i(x)\right\| \leq r + \varepsilon/2 \text{ and } \left\|y_2 - T^i(x)\right\| \leq r + \varepsilon/2.$$

Since

$$\begin{aligned} \left\|\frac{y_1 + y_2}{2} - T^i(x)\right\| &\leq \frac{1}{2}[\left\|y_1 - T^i(x)\right\| + \left\|y_2 - T^i(x)\right\|] \\ &\leq r + \varepsilon, \end{aligned}$$

it follows that $\dfrac{y_1 + y_2}{2} \in S_\varepsilon$ for each $\varepsilon > 0$; hence $\dfrac{y_1 + y_2}{2} \in S$ and it follows from this that S is convex.

Therefore, as a bounded closed convex subset of a Banach space, S is weakly closed. It follows that $C = S \cap D_1$ is weakly compact. Also C is nonempty since $T^m(x) \in C$. Now let $y \in C$. Then $\left\|y - T^i(x)\right\| \leq r + \varepsilon$ for almost all i, say for $i \geq N$. Since T is nonexpansive wrt orbits,

$$\left\|T(y) - T^{i+1}(x)\right\| \leq r_y\left(O\left(T^i(x)\right)\right) = \sup\left\{\|y - u\| : u \in O\left(T^i(x)\right)\right\} \leq r + \varepsilon$$

for $i \geq N$. Thus $\left\|T(y) - T^i(x)\right\| \leq r + \varepsilon$ for almost all i and therefore $T : C \to C$. It now follows from minimality of D_1 that $C = D_1$.

Now let

$$W = \left(\bigcap_{x \in D_1} B(x; r)\right) \cap D_1 = \{y \in D_1 : D_1 \subseteq B(y; r)\}.$$

Then any weak subsequential limit point of $(T^n(x))$ lies in W so W is nonempty, and clearly W is weakly compact.

If $y \in W$ and $x \in D_1$ then $\|y - x\| \leq r$, i.e., $D_1 \subseteq B(y; r)$ for each $y \in W$. Also $O(x) \subseteq D_1$. It follows that $r_y(O(x)) \leq r$ and since T is nonexpansive wrt orbits,

$$\|T(y) - T(x)\| \leq r_y(O(x)) \leq r.$$

Therefore $T(D_1) \subseteq B(T(y); r)$ for each $y \in W$. But this implies

$$\overline{T(D_1)}^w \subseteq B(T(y); r)$$

where $\overline{T(D_1)}^w$ denotes the weak closure of $T(D_1)$. Since $\overline{T(D_1)}^w = D_1$ by minimality of D_1 we conclude that $T(y) \in W$; hence $T : W \to W$. Again by minimality of D_1, it follows that $W = D_1$. But $diam(W) \leq r < diam(D_1)$ – a contradiction. □

The preceding theorem has the following 'pointwise' analog.

COROLLARY 5.1. *Suppose D be a weakly compact subset of a Banach space X, suppose $T : D \to D$ is nonexpansive wrt orbits, and suppose for each $x \in D$ there exist $\alpha(x) \in (0, 1)$ and $k \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ with $n \geq k$,*

$$\|T(x) - T^n(x)\| \leq \alpha(x)\left\|x - T^{n-1}(x)\right\|. \tag{5.2}$$

Then T has a fixed point in D.

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Condition (5.2) implies Condition (5.1) upon taking $m = 1$ in Condition (5.1). In this case if $diam\,(O\,(x)) > 0$, then $O\,(T(x)) \subseteq B\,(T(x); r)$ where

$$r = \alpha\,(x)\,\left\|x - T^{n-1}\,(x)\right\| \leq \alpha\,(x) \sup_i \left\|x - T^i\,(x)\right\| < diam\,(O\,(x))\,.$$

Also Condition (5.2) does not imply that T has diminishing orbital diameters since it does not preclude the possibility that $diam\,(O\,(T^n\,(x))) = diam\,(O\,(x))\,.$

The following extension of Theorem 3.4 is another immediate consequence of Theorem 5.2.

COROLLARY 5.2. *Let K be a weakly compact subset of a Banach space, and suppose $T : K \to K$ is a nonexpansive mapping. Suppose also that for each $x \in K$ such that $x \neq T\,(x)$ there exists $\alpha\,(x) \in (0, 1)$ such that*

$$\|T\,(x) - T\,(y)\| \leq \alpha\,(x)\,\|x - y\| \text{ for each } y \in K.$$

Then T has at least one fixed point.

6. Characterizations of normal structure

It is well known that the presence of 'normal structure' in a weakly compact convex set K is sufficient to assure the existence of a fixed point for an arbitrary nonexpansive mapping of K into K. As a result this concept has attracted independent interest in Banach space geometry. The characterization of normal structure given by Theorem 5.1 is a recent reminder of this fact. In this section we state some known results and give selected proofs. Further details may be found in the cited literature.

It is in this seminal paper by Brodskii and Milman [**5**] that one finds the first nontrivial characterization of the concept. Let K be a bounded subset of a Banach space X. A point $x_0 \in K$ is a *nondiametral point* of K if

$$\sup\{\|x_0 - x\| : x \in K\} < diam\,(K)\,.$$

A non-constant bounded sequence $\{x_n\}$ of points in a Banach space X is said to be a *diametral sequence* if

$$\lim_{n\to\infty} dist\,(x_{n+1}, conv\,(\{x_1, \cdots, x_n\})) = diam\,(\{x_1, x_2, \cdots\})\,,$$

where $dist\,(x, K) = \inf\{||x - y|| : y \in K\}$. The following is Brodskii and Milman's original characterization. For the convenience of the reader we include the proof in the Appendix.

THEOREM 6.1 ([**5**]). *A bounded convex subset K of a Banach space has normal structure if and only if it does not contain a diametral sequence.*

Subsequently there have been a number of characterizations of normal structure, many of which follow from the Brodskii-Milman characterization. In particular see Proposition 1 of Landes [**20**] for an early comprehensive survey.

Another early variant of the concept of normal structure was introduced by Baillon and Schöneberg in [**2**]. A subset K of a Banach space X is said to have *asymptotic normal structure* if for each bounded closed convex subset H of K for which $diam\,(H) > 0$ and each sequence $\{x_n\}$ in H satisfying $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$ there is a point $x \in K$ such that $\limsup_{n\to\infty} \|x_n - x\| < diam\,(H)$. The following is the central result of [**2**].

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THEOREM 6.2. *If K is a weakly compact convex set which possesses asymptotic normal structure in the Baillon-Schöneberg sense, then every nonexpansive mapping $T : K \to K$ has a fixed point.*

However Baillon and Schöneberg were not the first to use the term 'asymptotic normal structure'. We turn now to an earlier (and more common) meaning of the term. For this we need some additional notation. For a nonempty subset W of a Banach space and a decreasing net $\{W_\alpha\}_{\alpha\in A}$ of nonempty bounded subsets of W (A is a directed set), let

$$\begin{aligned} r_\alpha(x) &= \sup\{\|x-y\| : y \in W_\alpha\} \text{ for each } x \in W; \\ r(x) &= \inf\{r_\alpha(x) : \alpha \in A\}; \\ r &= \inf\{r(x) : x \in W\}; \\ \mathfrak{C}(\{W_\alpha\}, W) &= \{x \in W : r(x) = r\}. \end{aligned}$$

The number r is called the *asymptotic radius* of $\{W_\alpha\}$ in W and the set $\mathfrak{C}(\{W_\alpha\}, W)$ is called the *asymptotic center* of $\{W_\alpha\}$ in W.

DEFINITION 6.1. A convex subset K of a Banach space has *asymptotic normal structure* (in the sense of T. C. Lim [**21**]) if, given any bounded convex subset W of K which contains more than one point and given any decreasing net $\{W_\alpha\}_{\alpha\in A}$ of nonempty subsets of W, $\mathfrak{C}(\{W_\alpha\}, W)$ is a proper subset of W. (This takes account of the possibility that if $\mathfrak{C}(\{W_\alpha\}, W) = \emptyset$ then $\mathfrak{C}(\{W_\alpha\}, W)$ is a proper subset of W.)

The following characterization of normal structure due to T. C. Lim [**21**] is a very important application of the Brodskii-Milman characterization. This serves to illustrate the subtle nature of the concept. Again we include further details in the Appendix.

THEOREM 6.3. *A bounded convex subset K of a Banach space has normal structure if and only if it has asymptotic normal structure.*

Quite recently another orbital condition was introduced by E. Llorens-Fuster in [**22**]. These mappings also properly include the nonexpansive mappings.

DEFINITION 6.2 ([**22**]). Let X be a Banach space and $C \subseteq X$. A mapping $T : C \to C$ is said to be *orbitally nonexpansive* if for every nonempty closed convex T-invariant subset D of C there exists $x_0 \in D$ such that for every $x \in D$,

$$\limsup_{n\to\infty} \|T^n(x_0) - T(x)\| \le \limsup_{n\to\infty} \|T^n(x_0) - x\|.$$

The so-called L-type mappings of [**23**] are among the examples of orbitally nonexpansive mappings discussed in [**22**]. Among other things, it is shown there that the original existence theorem for nonexpansive mappings extends to the class of orbitally nonexpansive mappings. Specifically:

THEOREM 6.4. *Let K be a bounded closed convex subset of a reflexive Banach space X, and suppose K has normal structure. Then every orbitally nonexpansive mapping $T : K \to K$ has a fixed point.*

The proof of Theorem 6.4 is based on yet another characterization of normal structure, one due to J. Bogin [**4**] which is easily derived from Lim's asymptotic characterization. The following is Corollary 1 of [**4**].

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PROPOSITION 6.1. *A convex subset K of a Banach space has normal structure if and only if the following property holds:*

For each non-constant bounded sequence $\{x_n\}$ in K, the function

$$g(x) = \limsup_{n\to\infty} \|x_n - x\|$$

is not constant on $conv(\{x_n\})$.

PROOF. ([**4**]) Suppose K has normal structure. Let $W = conv(\{x_n\})$ where $\{x_n\}$ is a bounded non-constant sequence in K. Then the sets $W_k = \{x_n : n \geq k\}$, $k \in \mathbb{N}$, form a decreasing net of subsets of W. Moreover

$$g(x) = \limsup_{n\to\infty} \|x_n - x\| = r(x).$$

Since normal structure implies asymptotic normal structure, $r(x)$, hence $g(x)$, is not constant on W.

Now suppose K satisfies the condition of the Proposition and let $\{x_n\}$ be any non-constant bounded sequence in K. Then it must be the case that there exists a point $x \in W := conv(\{x_n\})$ for which

$$\limsup_{n\to\infty} \|x - x_n\| < diam(\{x_1, x_2, \cdots\}).$$

Thus for n sufficiently large, $x \in conv(\{x_1, \cdots, x_{n-1}\})$ and it follows that

$$\limsup_{n\to\infty} dist(x_n, conv(\{x_1, \cdots, x_{n-1}\})) \leq \limsup_{n\to\infty} \|x - x_n\| < diam(\{x_1, x_2, \cdots\}).$$

Therefore $\{x_n\}$ is a nondiametral sequence. Hence K does not contain a diametral sequence and thus has normal structure by the Brodskii-Milman characterization. □

PROBLEM 1. *The question of whether Theorem 5.2 holds for orbitally nonexpansive mappings in the Llorens-Fuster sense is left open.*

Several other geometric conditions which are related to normal structure have recently been studied, among them are ones related to the notion of diametral completeness.

DEFINITION 6.3 (see [**24**]). A subset A of a Banach space X is said to be *diametrically complete* if for each $x \in X \backslash A$,

$$diam(A \cup \{x\}) > diam(A).$$

For a subset A of a Banach space X and $x \in X$, let

$$r_x(A) = \sup\{\|x - y\| : y \in A\}.$$

DEFINITION 6.4. A subset A of a Banach space X is said to be a *set of constant radius* if for every $x \in \partial A$ (the boundary of A),

$$r_x(A) = diam(A).$$

In [**24**] Maluta and Papini extend an earlier result of Moreno, Papini, and Phelps (see [**25**, Prop. 2.2]) as follows:

PROPOSITION 6.2. *The class of diametrically complete sets and that of sets of constant radius in a Banach space X coincide if X has normal structure.*

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PROBLEM 2. *Maluta and Papini leave open the interesting question of whether the coincidence of these two classes of spaces actually provides another characterization of spaces with normal structure.*

7. Proximal normal structure

The preceding results suggest a fundamental problem, namely: Does the characterization of normal structure in terms of the fixed point property for mappings which are nonexpansive wrt orbits extend to weak proximal normal structure? The concept of proximal normal structure was introduced by Eldred, et al. in [**8**]. To describe this problem in more detail we need some definitions and notation. We say that a pair (A, B) of subsets of a given Banach space satisfies a given property if each of the sets A and B have that property. Thus (A, B) is said to be convex if both A and B are convex, $(A, B) \subseteq (C, D)$ if $A \subseteq C$ and $B \subseteq D$, etc. Also we adopt the following notations:

$$dist\,(A, B) = \inf\{\|x - y\| : x \in A, y \in B\},$$

$$\delta\,(A, B) = \sup\{\|x - y\| : x \in A, y \in B\}.$$

DEFINITION 7.1. *A pair (A, B) of subsets of a Banach space X is said to be a proximal pair if for each $x \in A$, $x' \in B$ there exist $y \in A$, $y' \in B$ such that*

$$\|x - y'\| = \|x' - y\| = dist\,(A, B).$$

A nonempty convex pair (A, B) in X is said to have proximal normal structure if for every closed bounded convex proximal pair $(K_1, K_2) \subset (A, B)$ for which $dist\,(K_1, K_2) = dist\,(A, B)$ and $\delta\,(K_1, K_2) > dist\,(K_1, K_2)$ there exists $(x, y) \in K_1 \times K_2$ such that

$$r_x\,(K_2) < \delta\,(K_1, K_2) \text{ and } r_y\,(K_1) < \delta\,(K_1, K_2).$$

A Banach space X is said to have proximal normal structure if every convex pair (A, B) in X has proximal normal structure.

Also, a mapping $T : (A, B) \to (B, A)$ is said to be *relatively nonexpansive* if $T(A) \subseteq B$, $T(B) \subseteq A$, and

$$\|T(x) - T(y)\| \leq \|x - y\| \text{ for } x \in A, y \in B.$$

Finally, $(x, y) \in (A, B)$ is said to be a *best proximity pair* for $T : (A, B) \to (A, B)$ if

$$\|x - T(x)\| = \|y - T(y)\| = dist\,(A, B).$$

The following result is Theorem 2.1 of [**8**].

THEOREM 7.1. *Let (K_1, K_2) be weakly compact convex pair in a Banach space and suppose (K_1, K_2) has proximal normal structure. Then every relatively nonexpansive mapping $T : (K_1, K_2) \to (K_2, K_1)$ has a best proximity pair in (K_1, K_2).*

REMARK 7.1. It is easy to see that if $A \cap B \neq \emptyset$ in Definition 7.1 then (A, B) is a proximal pair $\Leftrightarrow A = B$. Therefore if (K_1, K_2) has proximal normal structure and $K_1 \cap K_2 \neq \emptyset$ then $K_1 = K_2$ and K_1 has normal structure. In view of this, Theorem 7.1 actually contains Theorem 2.1.

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In light of the theorem of Amini-Harandi, et al. (Theorem 5.1), one question is obvious. We say that $T : (A, B) \to (A, B)$ is *relatively nonexpansive wrt orbits* if $T(A) \subseteq B$, $T(B) \subseteq A$, and for all $(x, y) \in (A, B)$,

$$\|T(x) - T(y)\| \leq r_x(O(y)).$$

PROBLEM 3. *Can the assumption that T is relatively nonexpansive in Theorem 7.1 be replaced by the assumption that T is relatively nonexpansive wrt orbits?*

The argument of [**1**] showing that the weak fixed point property for mappings which are nonexpansive wrt orbits implies weak normal structure sufficies to assure that the fixed point property for mappings which are relatively nonexpansive wrt orbits implies weak proximal normal structure. Therefore an affirmative answer to the above problem would provide a characterization of proximal normal structure analogous to that given for normal structure by Theorem 5.1.

After completing a preliminary version of this manuscript the authors have learned of very recent papers, by Dutta and Veeramani [**7**] and Gabeleh [**9**], which contain several new and interesting observations about proximal normal structure. In particular, each contains a proximal normal structure analogue of the Brodskii and Milman characterization of normal structure. Also, among other things, it is shown in [**7**] that if a convex pair (A, B) in a Banach space has proximal normal structure it is not necessary that both A and B have normal structure. The question of whether the pair (A, B) has proximal normal structure if *both* A and B have normal structure is left open.

8. Appendix

PROOF OF THEOREM 6.1. This is based on Brodskii and Milman's original proof which appears in [**5**]. Throughout we assume $diam(K) = d > 0$.

($\Leftarrow$) Suppose K does not have normal structure. We show that this implies that K has a diametral sequence (thus, if K does not contain a diametral sequence then K has normal structure). Let $x_1 \in K$ and choose ε so that $d > \varepsilon > 0$. For each $n \in \mathbb{N}$ choose x_{n+1} so that

$$\|x_{n+1} - y_{n-1}\| > d - \frac{\varepsilon}{n^2} \text{ where } y_{n-1} = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

Now let $x \in conv(\{x_1, \cdots, x_n\})$. Then there exist α_j, $j = 1, \cdots, n$ such that

$$x = \sum_{j=1}^{n} \alpha_j x_j \text{ where } \alpha_j \geq 0 \text{ and } \sum_{j=1}^{n} \alpha_j = 1.$$

Suppose $\alpha_p = \max\{\alpha_1, \cdots, \alpha_n\}$. Then

$$y_{n-1} = \frac{1}{n\alpha_p} x + \sum_{j=1}^{n} \left(\frac{1}{n} - \frac{\alpha_j}{n\alpha_p} \right) x_j,$$

where

$$\frac{1}{n\alpha_p} + \sum_{j=1}^{n} \left(\frac{1}{n} - \frac{\alpha_j}{n\alpha_p} \right) = 1 \text{ and } \frac{1}{n} - \frac{\alpha_j}{n\alpha_p} \geq 0, \quad j = 1, 2, \cdots, n.$$

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Therefore

$$\begin{aligned} d-\frac{\varepsilon}{n^2} &< \|y_{n-1}-x_{n+1}\| \\ &\leq \frac{1}{n\alpha_p}\|x-x_{n+1}\|+\sum_{j\neq p}\left(\frac{1}{n}-\frac{\alpha_j}{n\alpha_p}\right)\|x_j-x_{n+1}\| \\ &\leq \frac{1}{n\alpha_p}\|x-x_{n+1}\|+\left(1-\frac{1}{n\alpha_p}\right)d. \end{aligned}$$

This implies

$$\|x-x_{n+1}\|>\left(\frac{d}{n\alpha_p}-\frac{\varepsilon}{n^2}\right)n\alpha_p=d-\frac{\varepsilon\alpha_p}{n}\geq d-\frac{\varepsilon}{n},$$

and it follows that $d_n > d-\dfrac{\varepsilon}{n}$ where $d_n = dist\left(x_{n+1}, conv\left(\{x_1,\cdots,x_n\}\right)\right)$. Hence $\lim_{n\to\infty} d_n = d$ proving that $\{x_n\}$ is a diametral sequence.

($\Rightarrow$) Now suppose K has normal structure and suppose K contains a diametral sequence $\{x_n\}$. Let $H_n = conv\left(\{x_1,\cdots,x_n\}\right)$ and let $H = conv\left(\{x_1,x_2,\cdots\}\right)$. Since $\{x_n\}$ is non-constant, $diam\,(H) > 0$, and if $u \in H$ then there exists $N \in \mathbb{N}$ such that $u \in H_n$ for all $n \geq N$. It follows that

$$\|x_{n+1}-u\| \geq dist\left(x_{n+1}, H_n\right) \to diam\,(H) \text{ as } n\to\infty.$$

Therefore u cannot be a nondiametral point of H and this contradicts the assumption that K has normal structure. It follows that if K has normal structure then K does not contain a diametral sequence. □

Proof of Theorem 6.3 (Outline). One implication is immediate. If K has asymptotic normal structure then by taking $W_\alpha \equiv W$ it follows that K has normal structure.

The proof of the converse involves a rather delicate construction. Now suppose that K does not have asymptotic normal structure. Then there exists a bounded convex subset W of K with $diam\,(W) > 0$ and a decreasing net $\{W_\alpha\}$ of nonempty subsets of W such that $r(x) \equiv r$ for every $x \in W$. We show that this implies the existence of a diametral sequence $\{x_n\}$ in W. This in turn will imply that K does not have normal structure and hence, *ipso facto*, that normal structure of K implies asymptotic normal structure of K. The following technical lemma is crucial to our approach. This is Observation (1) of Landes [**20**]. The details are straightforward.

Lemma 8.1. *Let $\{x_n\}$ be a bounded sequence in a Banach space with $\bar{x}_n = \dfrac{x_1+\cdots+x_n}{n}$. Suppose $x = \sum_{i=1}^n \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{k=1}^n \lambda_k = 1$. Then*

$$\|x_{n+1}-x\| \geq r-\lambda n\left(r-\|x_{n+1}-\bar{x}_n\|\right),$$

where $diam\left(\{x_n\}\right) = r$, and $\lambda = \max_{i\leq n}\lambda_i$.

To see that Theorem 6.3 follows from the above observation, define a diametral sequence $\{x_n\}$ in W as follows. Let $\{\varepsilon_n\}$ be a sequence of positive numbers such that $n(n+1)\varepsilon_n \to 0$ as $n\to\infty$. Select $x_1 \in W$. Since $r(x_1) = \inf\{r_\alpha(x_1) : \alpha \in A\} = r$ there exist $\alpha \in A$ and $x_2 \in W_\alpha$ such that

$$\|x_1-x_2\| \geq r-\varepsilon_1.$$

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By moving x_2 toward x_1 on the segment joining x_1 and x_2 if necessary, we may further assume that

$$\|x_1 - x_2\| \leq r.$$

Now let

$$\bar{x}_2 = \frac{x_1 + x_2}{2}.$$

Since $r(x_i) = r$ for $i = 1, 2$ there exist $\alpha_i \in A$ for $i = 1, 2$ such that $r_{\alpha_i}(x_i) \leq r + \varepsilon_2$ for $i = 1, 2$. Since $\{W_\alpha\}$ is a decreasing net, $W_\alpha \subseteq W_{\alpha_1} \cap W_{\alpha_2}$ for some α; thus $r_\alpha(x_i) \leq r + \varepsilon_2$ for $i = 1, 2$. Moreover there exists $z_0 \in W_\alpha$ such that

$$\|\bar{x}_2 - z_0\| \geq r - \varepsilon_2.$$

If $\|z_0 - x_1\| \leq r$ take $z_1 = z_0$. Otherwise take z_1 on the segment joining x_1 and z_0 so that $\|x_1 - z_1\| = r$. In this case, because $r_\alpha(x_i) \leq r + \varepsilon_2$, we now have $\|z_1 - z_0\| \leq \varepsilon_2$ and therefore (in either case)

$$\begin{aligned} \|z_1 - \bar{x}_2\| &\geq \|z_0 - \bar{x}_2\| - \|z_1 - z_0\| \\ &\geq r - \varepsilon_2 - \varepsilon_2 = r - 2\varepsilon_2. \end{aligned}$$

If $\|z_1 - x_2\| \leq r$ take $z_2 = z_1$. Otherwise take z_2 on the segment joining x_2 and z_1 so that

$$\|z_2 - x_2\| = r.$$

Now take $x_3 = z_2$. Since $\|x_2 - x_1\| \leq r$, $\|x_2 - z_0\| \leq r$, and z_1 is on the segment joining x_1 and z_0 it follows that $\|x_2 - z_1\| \leq r + \varepsilon_2$. Thus $z_2 = z_1$ or $\|z_1 - z_2\| \leq \varepsilon_2$, and in either case

$$\begin{aligned} \|z_2 - \bar{x}_2\| &\geq \|z_1 - \bar{x}_2\| - \|z_1 - z_2\| \\ &\geq \|z_1 - \bar{x}_2\| - \varepsilon_2. \end{aligned}$$

Therefore

$$\begin{aligned} \|x_3 - \bar{x}_2\| &= \|z_2 - \bar{x}_2\| \\ &\geq \|z_1 - \bar{x}_2\| - \varepsilon_2 \\ &\geq r - 3\varepsilon_2. \end{aligned}$$

Proceeding in this way step by step it is possible to define a sequence $\{x_n\}$ in W which has the property that

$$\|x_{n+1} - \bar{x}_n\| \geq r - (n+1)\varepsilon_n.$$

In conjunction with Lemma 8.1 this implies

$$\begin{aligned} \|x_{n+1} - x\| &\geq r - \lambda n\left(r - \|x_{n+1} - \bar{x}_n\|\right) \\ &\geq r - \lambda n(n+1)\varepsilon_n. \end{aligned}$$

Since this is true for each $x \in conv(\{x_1, \cdots, x_n\})$,

$$dist(x_{n+1}, conv(\{x_1, \cdots, x_n\})) \geq r - \lambda n(n+1)\varepsilon_n.$$

Therefore

$$\lim_{n\to\infty} dist(x_{n+1}, conv(\{x_1, \cdots, x_n\})) \geq \lim_{n\to\infty} (r - \lambda n(n+1)\varepsilon_n) = r,$$

and thus $\{x_n\}$ is diametral. Theorem 6.3 now follows from Theorem 6.1.

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