Doctopic: Applied Mathematics ARTICLE IN PRESS YJMAA:22709

J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Optimal convergence analysis of the unilateral contact problem with and without Tresca friction conditions by the penalty method

Ibrahima Dione

Département de mathématiques et de statistique, Pavillon Vachon, 1045 Avenue de la médecine, Université Laval, Québec, G1V 0A6, Canada

A R T I C L E I N F O

Article history:
Received 8 February 2018
Available online xxxx
Submitted by R.G. Durán

Keywords:
A priori estimate
Finite elements
Linear elasticity
Penalty method
Unilateral contact
Tresca friction

A B S T R A C T

We study the linear finite element approximation of the elasticity equations with and without unilateral friction contact (of Tresca type) conditions in a polygonal or polyhedral domain. The unilateral contact condition is weakly imposed by the penalty method. We derive error estimates which depend on the penalty parameter ε and the mesh size h. In fact, under the $\boldsymbol{H}^{\frac{3}{2}+\nu}(\Omega), 0 < \nu \leq \frac{1}{2}$, regularity of the solution of the contact problems (with and without friction) and with the requirement $\varepsilon > h$, we prove a convergence rate of $\mathcal{O}\left(h^{\frac{1}{2}+\nu} + \varepsilon^{\frac{1}{2}+\nu}\right)$ in the energy norm. Therefore, if the penalty parameter is taken as $\varepsilon := ch^{\theta}$ where $0 < \theta \leq 1$, the convergence rate of $\mathcal{O}\left(h^{\theta\left(\frac{1}{2}+\nu\right)}\right)$ is obtained. In particular, we obtain an optimal linear convergence when ε behaves like h (i.e. $\theta = 1$) and $\nu = \frac{1}{2}$.

1. Introduction

Let $\Omega \subset \mathbb{R}^d, d = 2, 3$, be a polygonal or polyhedral domain representing the configuration of a linearly elastic body. The equilibrium equations satisfied by this body are defined by

$$\begin{aligned} -\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}) &= \boldsymbol{f}, && \text{in } \Omega, \\ \boldsymbol{\sigma}(\boldsymbol{u}) &= \boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), && \text{in } \Omega, \end{aligned} \tag{1}$$

where $\boldsymbol{u}$ is the displacement field, $\boldsymbol{f} \in \left(L^2(\Omega)\right)^d$ is a body force and $\boldsymbol{\sigma} := (\sigma_{ij})_{1\leq i,j\leq d}$ is the stress tensor field. The tensor $\boldsymbol{A}$ is the fourth order symmetric elasticity tensor satisfying the usual uniform ellipticity and boundedness properties, and $\boldsymbol{\varepsilon}$ represents the linearized strain tensor field defined by

E-mail address: ibrahima.dione.1@ulaval.ca.

https://doi.org/10.1016/j.jmaa.2018.11.023

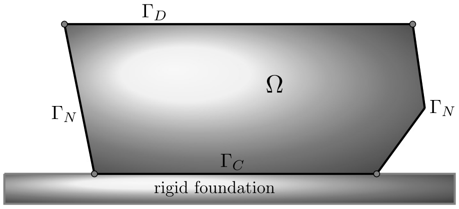

Fig. 1. A deformable body Ω in contact with a rigid foundation corresponding to a zero gap ($\ell = 0$).

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^t\right).$$

We assume that the polygonal or polyhedral boundary $\partial\Omega$ is partitioned into three non-overlapping parts Γ_D, Γ_N and Γ_C ($\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_C$) where $meas(\Gamma_D) > 0$ and $meas(\Gamma_C) > 0$. The contact boundary Γ_C is supposed to be a straight line segment when $d = 2$ (see Fig. 1) or a polygon when $d = 3$.

To problem (1), we associate the following Dirichlet and Neumann boundary conditions on Γ_D and Γ_N

$$\boldsymbol{u} = \mathbf{0}, \text{ on } \Gamma_D, \tag{2}$$

$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{g}, \text{ on } \Gamma_N, \tag{3}$$

where the vector $\boldsymbol{n}$ is the unit outward normal on $\partial\Omega$ and $\boldsymbol{g} \in \left(L^2(\Gamma_N)\right)^d$ is a surface load. The body is clamped on Γ_D for sake of simplicity and is subjected to a surface load on Γ_N. We adopt the following decomposition of the displacement field $\boldsymbol{v}$ and the density of the surface forces $\boldsymbol{\sigma}(\boldsymbol{v})\boldsymbol{n}$ into the normal and the tangential components

$$\boldsymbol{v} = (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n} + \boldsymbol{v_t} \text{ and } \boldsymbol{\sigma}(\boldsymbol{v})\boldsymbol{n} = \boldsymbol{\sigma}_n(\boldsymbol{v})\boldsymbol{n} + \boldsymbol{\sigma_t}(\boldsymbol{v}),$$

where $\boldsymbol{\sigma}_n(\boldsymbol{u})$ is the contact pressure.

We also add to the problem (1), the following nonlinear boundary conditions describing the unilateral contact on Γ_C

$$\boldsymbol{u}\cdot\boldsymbol{n} - \ell \le 0, \quad \boldsymbol{\sigma}_n(\boldsymbol{u}) \le 0, \quad \big(\boldsymbol{u}\cdot\boldsymbol{n} - \ell\big)\boldsymbol{\sigma}_n(\boldsymbol{u}) = 0, \text{ on } \Gamma_C, \tag{4}$$

where the gap function $\ell : \Gamma_C \longrightarrow \mathbb{R}^+$ is a continuous mapping that associates any point $\boldsymbol{s} \in \Gamma_C$ with its normal distance from the rigid foundation.

If we consider that there is no friction on Γ_C, the following additional condition completes that in (4)

$$\boldsymbol{\sigma_t}(\boldsymbol{u}) = \mathbf{0}, \text{ on } \Gamma_C. \tag{5}$$

Otherwise, if we consider friction of the Tresca type, the contact condition (4) is completed by

$$\begin{cases} |\boldsymbol{\sigma_t}(\boldsymbol{u})| \le g, & \text{if } \boldsymbol{u_t} = \mathbf{0} \\ \boldsymbol{\sigma_t}(\boldsymbol{u}) = -g\dfrac{\boldsymbol{u_t}}{|\boldsymbol{u_t}|}, & \text{if } \boldsymbol{u_t} \ne \mathbf{0} \end{cases} \quad \text{on } \Gamma_C, \tag{6}$$

where $g \in L^2(\Gamma_C)$ is a nonnegative function and $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^{d-1}$. We can easily verify that if $g = 0$, the boundary condition (6) recovers the condition (5).

This work is devoted to the finite element analysis of the following two contact problems by the penalty method:

- The frictionless contact problem defined by equations (1)–(4) and (5).
- The Tresca friction contact problem represented by equations (1)–(4) and (6).

Let us denote by $H^m(\Omega)$, $m \geq 0$, and $L^2(\Omega) := H^0(\Omega)$ the classical Sobolev spaces equipped with the usual norm $\|\cdot\|_{m,\Omega}$. Through the paper, bold letters like $\boldsymbol{u}$, $\boldsymbol{v}$ indicate vector or tensor valued quantities, while the capital ones represent functional sets. Thus, we define the following Sobolev spaces (see [1])

$$\boldsymbol{H}^m(\Omega) := (H^m(\Omega))^d, \quad \boldsymbol{L}^2(\Omega) := \left(L^2(\Omega)\right)^d.$$

We also introduce the set $\boldsymbol{K}$ of admissible displacements satisfying the non-interpenetration condition on the contact zone Γ_C

$$\boldsymbol{K} := \left\{\boldsymbol{v} \in \boldsymbol{V} : \boldsymbol{v}\cdot\boldsymbol{n} - \ell \leq 0 \;\text{ on }\; \Gamma_C\right\},$$

where

$$\boldsymbol{V} := \left\{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{v} = \boldsymbol{0} \;\text{ on }\; \Gamma_D\right\}.$$

On the other hand, we consider the bilinear and linear forms $A(\cdot,\cdot)$ and $F(\cdot)$, defined by

$$\begin{aligned} A(\boldsymbol{u},\boldsymbol{v}) &:= \int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \\ F(\boldsymbol{v}) &:= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\, dx + \int_{\Gamma_N} \boldsymbol{g}\cdot\boldsymbol{v}\, ds, \end{aligned}$$

for $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$. The bilinear form $A(\cdot,\cdot)$ is continuous and V-elliptical from $\boldsymbol{V}\times\boldsymbol{V}$ into $\mathbb{R}$, that is

$$m\|\boldsymbol{v}\|_{1,\Omega}^2 \leq A(\boldsymbol{v},\boldsymbol{v}), \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{7}$$

$$A(\boldsymbol{v},\boldsymbol{w}) \leq M\|\boldsymbol{v}\|_{1,\Omega}\|\boldsymbol{w}\|_{1,\Omega}, \forall\, \boldsymbol{v},\boldsymbol{w} \in \boldsymbol{V}, \tag{8}$$

where M and m are positive constants. We define the notation $[\,\cdot\,]^+$ for a scalar quantity $a \in \mathbb{R}$ by

$$[a]^+ := \begin{cases} a, & \text{if} \quad a > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let us remind the following two useful properties

$$a \leq [a]^+, \quad a[a]^+ = [a]^+[a]^+, \; \forall\, a \in \mathbb{R}, \tag{9}$$

from which the following monotonicity property is deduced (see [6], [7] and [8])

$$\left([a]^+ - [b]^+\right)(a-b) \geq \left([a]^+ - [b]^+\right)^2. \tag{10}$$

For $\alpha \in \mathbb{R}^+$, we define $\mathscr{B}(\boldsymbol{0},\alpha) \subset \mathbb{R}^{d-1}$ as the closed ball centered at the origin $\boldsymbol{0}$ and of radius α. We thus introduce the notation $[\,\cdot\,]_\alpha$ for the orthogonal projection onto $\mathscr{B}(\boldsymbol{0},\alpha)$, which stands for any $\boldsymbol{x} \in \mathbb{R}^{d-1}$ as follows

$$[\boldsymbol{x}]_\alpha := \begin{cases} \boldsymbol{x}, & \text{if } |\boldsymbol{x}| \leq \alpha, \\ \alpha\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}, & \text{if } |\boldsymbol{x}| > \alpha. \end{cases}$$

Please cite this article in press as: I. Dione, Optimal convergence analysis of the unilateral contact problem with and without Tresca friction conditions by the penalty method, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2018.11.023

We hence recall the following classical properties for projections, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{d-1}$

$$\left([\boldsymbol{x}]_\alpha - [\boldsymbol{y}]_\alpha \right) \cdot (\boldsymbol{x} - \boldsymbol{y}) \geq 0 \text{ and } \left| [\boldsymbol{x}]_\alpha - [\boldsymbol{y}]_\alpha \right| \leq |\boldsymbol{x} - \boldsymbol{y}|, \tag{11}$$

where the dot $\cdot$ is the Euclidean scalar product in $\mathbb{R}^{d-1}$.

The boundary conditions we have to deal with are thus characterized by the inequality $\boldsymbol{u} \cdot \boldsymbol{n} - \ell \leq 0$ on Γ_C. But this constraint is not favorable for computations, despite the fact that the variational inequality formulation obtained from these contact problems can be solved by various methods. We thus need techniques more favorable and one of the classical and most used method to circumvent this inequality constraint is the penalty method. The penalty technique is a classical method for the numerical treatment of constrained problems (see [12] and [14]). Unlike the Lagrange multiplier technique, the penalty method avoids to introduce a new variable. Moreover, it is more readily applicable in most numerical codes. Nevertheless, this method remains an approximation since the solution of the penalized problem is expected to coincide with the solution of the original problem when the penalty parameter ε is zero.

The present study is devoted to the application of the penalty method to these contact problems. If the finite element approximation space $\boldsymbol{V}_h$ is made of standard continuous and piecewise affine functions, it is expected, choosing the penalty parameter as $\varepsilon(h) = ch^\theta$ with a suitable value of θ, to obtain the following a priori error estimate between the displacement field $\boldsymbol{u}$ and the penalized finite element approximation solution $\boldsymbol{u}_{\varepsilon h}$

$$\|\boldsymbol{u} - \boldsymbol{u}_{\varepsilon h}\|_{1,\Omega} = \mathcal{O}(h). \tag{12}$$

To the best of our knowledge, in this case of polygonal or polyhedral domain, the best estimate is established by F. Chouly and P. Hild ([6], Theorem 3.2 and Theorem 4.2). They obtained, without a specific numerical integration scheme to treat the penalty term, the a priori error estimate for the frictionless problem (1)–(5)

$$\begin{aligned} &\|\boldsymbol{u} - \boldsymbol{u}_{\varepsilon h}\|_{1,\Omega} + \varepsilon^{\frac{1}{2}} \left\| \boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon} [\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n}]^+ \right\|_{0,\Gamma_C} \\ &\qquad \leq c \begin{cases} \left(h^{\frac{1}{2} + \frac{\nu}{2} + \nu^2} + h^\nu \varepsilon^{\frac{1}{2}} + h^{\nu - \frac{1}{2}} \varepsilon \right) \|\boldsymbol{u}\|_{\frac{3}{2}+\nu,\Omega}, \text{ if } 0 < \nu < \dfrac{1}{2}, \\ \left(h\, |ln(h)|^{\frac{1}{2}} + (h\varepsilon)^{\frac{1}{2}} + \varepsilon \right) \|\boldsymbol{u}\|_{2,\Omega}, \qquad \text{if } \nu = \dfrac{1}{2}. \end{cases} \end{aligned} \tag{13}$$

As for the Tresca friction problem (1)–(4) and (6), by fixing $\varepsilon = h$ and taking $d = 2$, they established the following result (which remains valid in the three dimensional case $d = 3$ using similar tools)

$$\begin{aligned} &\|\boldsymbol{u} - \boldsymbol{u}_{\varepsilon h}\|_{1,\Omega} + h^{\frac{1}{2}} \left(\left\| \boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon} [\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n}]^+ \right\|_{0,\Gamma_C} + \left\| \boldsymbol{\sigma}_{\boldsymbol{t}}(\boldsymbol{u}) + \frac{1}{\varepsilon} [\boldsymbol{u}_{\varepsilon h_{\boldsymbol{t}}}]_{\varepsilon g} \right\|_{0,\Gamma_C} \right) \\ &\qquad \leq c \begin{cases} h^{\frac{1}{2} + \frac{\nu}{2} + \nu^2} \|\boldsymbol{u}\|_{\frac{3}{2}+\nu,\Omega}, \text{ if } 0 < \nu < \dfrac{1}{2}, \\ h\, |ln(h)|^{\frac{1}{2}} \|\boldsymbol{u}\|_{2,\Omega}, \quad \text{if } \nu = \dfrac{1}{2}. \end{cases} \end{aligned} \tag{14}$$

These two estimates remain under optimal because of the first terms on the right hand side. These bounds were obtained by estimating the following contact term

$$\int_{\Gamma_C} \boldsymbol{\sigma}_n(\boldsymbol{u}) \left(\boldsymbol{\mathcal{I}}_h^1(\boldsymbol{u}) \cdot \boldsymbol{n} \right) ds, \tag{15}$$

where $\mathcal{I}_h^1$ is the Lagrange interpolation operator mapping onto $\boldsymbol{V}_h$. Recently, this contact term is optimally estimated by G. Drouet and P. Hild [10]. And taking again the approach of F. Chouly and P. Hild [6], an optimal estimate would be obtained for $\varepsilon = h$.

Otherwise in the limit case $h \to 0$ with $0 < \nu < \frac{1}{2}$, the a priori estimate (13) cannot provide a decreasing bound since one of the terms in its right hand side, namely the term $h^{\nu-\frac{1}{2}}\varepsilon$, diverges. This is why in Remark 3.6 in [6], the authors emphasized the penalty parameter to be chosen as follows $\varepsilon < ch^{\frac{1}{2}-\nu}$ in order to ensure the convergence to zero of the error term $\|\boldsymbol{u} - \boldsymbol{u}_{\varepsilon h}\|_{1,\Omega} + \varepsilon^{\frac{1}{2}} \left\| \boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n}\right]^+ \right\|_{0,\Gamma_C}$. Nevertheless in the limit case $h \to 0$ with $\nu = \frac{1}{2}$, this estimate is optimal in term of the penalty parameter ε in the sense that it gives the same convergence order than the continuous estimate obtained at Theorem 3.1 in [6] (which is recalled in the forthcoming analysis at Theorem 1, estimate (20)).

In the present work, we improve both a priori finite element estimates (13) and (14) by establishing for the frictionless unilateral problem the following result

$$\boxed{\|\boldsymbol{u} - \boldsymbol{u}_{\varepsilon h}\|_{1,\Omega} + \left(\varepsilon^{\frac{1}{2}} - ch^{\frac{1}{2}}\right) \left\| \boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell\right]^+ \right\|_{0,\Gamma_C} \leq c\left(h^{\frac{1}{2}+\nu} + \varepsilon^{\frac{1}{2}+\nu}\right) \|\boldsymbol{u}\|_{\frac{3}{2}+\nu,\Omega}. \quad (16)}$$

And for the Tresca friction unilateral problem, we obtain the following similar result

$$\boxed{\begin{aligned} &\|\boldsymbol{u} - \boldsymbol{u}_{\varepsilon h}\|_{1,\Omega} + \left(\varepsilon^{\frac{1}{2}} - ch^{\frac{1}{2}}\right)\left(\left\| \boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell\right]^+ \right\|_{0,\Gamma_C} + \left\| \boldsymbol{\sigma}_{\boldsymbol{t}}(\boldsymbol{u}) + \frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon h_{\boldsymbol{t}}}\right]_{\varepsilon g} \right\|_{0,\Gamma_C}\right) \\ &\leq c\left(h^{\frac{1}{2}+\nu} + \varepsilon^{\frac{1}{2}+\nu}\right) \|\boldsymbol{u}\|_{\frac{3}{2}+\nu,\Omega}. \quad (17) \end{aligned}}$$

The first advantage of the results (16) and (17) is that, the expected a priori optimal estimate (12) is achieved if the penalty parameter is chosen for example in the form $\varepsilon(h) = (c+1)^2 h$ by fixing $\nu = \frac{1}{2}$.

Another useful advantage is, the results (16) and (17) do not present a negative power in h for $0 < \nu \leq \frac{1}{2}$. Thus the warnings of Remark 3.6 in [6] are not necessary here and we do not need any additional conditions on the penalty parameter to ensure convergence of terms in the left hand side of estimates (16) and (17). Moreover in the limit case $h \to 0$ with $0 < \nu \leq \frac{1}{2}$, the estimates (16) and (17) are optimal in term of ε since the same estimates as Theorem 3.1 and Theorem 4.1 in [6] are recovered.

The analysis of the article is presented in four parts where to each of them, we distinguish the frictionless unilateral contact problem (defined by equations (1)–(5)) to the Tresca friction unilateral problem (given by equations (1)–(4) and equation (6)). We first present in Section 2, the weak formulations of the unilateral contact problems and their penalty weak formulation. Section 3 is devoted to the finite element approximations of the penalized formulations. We present in Section 4, the results of the paper by providing the a priori estimates of each problem. A general conclusion ends the article in Section 5 where we make a point of the different established results and evoke an extension to the quadratic case.

Throughout the paper, we assume that c denotes various positive constants which are independent of the discretization parameter h and the penalty parameter ε, and have different values depending on the context.

Please cite this article in press as: I. Dione, Optimal convergence analysis of the unilateral contact problem with and without Tresca friction conditions by the penalty method, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2018.11.023

2. Penalty weak formulations of the unilateral problem

2.1. The penalty formulation of the frictionless problem

The weak formulation of the frictionless unilateral problem (1)–(5) is defined by

$$\begin{cases} \text{Find } \boldsymbol{u} \in \boldsymbol{K} \text{ such that,} \\ A(\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}) \geq F(\boldsymbol{v}-\boldsymbol{u}), \; \forall \boldsymbol{v} \in \boldsymbol{K}. \end{cases} \tag{18}$$

The problem (18) admits a unique solution according to Stampacchia's theorem. The penalty weak formulation, derived from the variational inequality (18), reads [12]

$$\begin{cases} \text{Find } \boldsymbol{u}_\varepsilon \in \boldsymbol{V} \text{ such that,} \\ A(\boldsymbol{u}_\varepsilon, \boldsymbol{v}) + \dfrac{1}{\varepsilon} \displaystyle\int_{\Gamma_C} [\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ (\boldsymbol{v} \cdot \boldsymbol{n}) \, ds = F(\boldsymbol{v}), \; \forall \boldsymbol{v} \in \boldsymbol{V}. \end{cases} \tag{19}$$

Well-posedness and uniform convergence of the penalized problem (19) have already been studied by N. Kikuchi and Y. J. Song in [13] (Theorems 3.1 and 3.2), but also by N. Kikuchi and J. T. Oden in [12] (Theorems 3.15 and 6.6). Recently, F. Chouly and P. Hild ([6], Theorems 2.2 and 3.1) tackle again these questions in the context of linear finite elements where the $\boldsymbol{H}^{\frac{3}{2}+\nu}(\Omega), 0 < \nu \leq \frac{1}{2}$, regularity of the contact solution $\boldsymbol{u}$ is required. We take again the Theorems 3.1 established in [6] in order to adapt it to a non-zero gap $\ell \geq 0$ by establishing the following convergence theorem.

Theorem 1. *Let $\Omega \subset \mathbb{R}^d, d = 2, 3,$ be a polygonal bounded domain, $\boldsymbol{u}$ and $\boldsymbol{u}_\varepsilon$ solutions of problems (18) and (19), respectively. If $\boldsymbol{u}$ belongs to $\boldsymbol{H}^{\frac{3}{2}+\nu}(\Omega)$ where $0 < \nu \leq \frac{1}{2}$, then we obtain the a priori estimate*

$$\|\boldsymbol{u} - \boldsymbol{u}_\varepsilon\|_{1,\Omega} + \varepsilon^{\frac{1}{2}} \left\| \boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon} \left[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell \right]^+ \right\|_{0,\Gamma_C} \leq c\varepsilon^{\frac{1}{2}+\nu} \|\boldsymbol{u}\|_{\frac{3}{2}+\nu,\Omega}, \tag{20}$$

where $c > 0$ is a constant, independent of the penalty parameter ε and the solution $\boldsymbol{u}$.

Proof. Multiplying the equation (1) by test functions $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ and integrating over Ω by taking into account Green formula and boundary conditions (2), (3) and (5), we obtain

$$A(\boldsymbol{u}, \boldsymbol{v}) - \int_{\Gamma_C} \boldsymbol{\sigma}_n(\boldsymbol{u})(\boldsymbol{v} \cdot \boldsymbol{n}) ds = F(\boldsymbol{v}). \tag{21}$$

If $\boldsymbol{u} \in \boldsymbol{V}$, the equation (21) may have no meaning because of lack of regularity (it has a meaning if the boundary integral is interpreted as a duality pairing on $H^{-\frac{1}{2}}(\Gamma_C) \times H^{\frac{1}{2}}(\Gamma_C)$). Then, we suppose $\boldsymbol{u} \in \boldsymbol{H}^{\frac{3}{2}+\nu}(\Omega)$ where $\nu > 0$ to justify this calculation by obtaining $\boldsymbol{\sigma}_n(\boldsymbol{u}) \in H^\nu(\Gamma_C)$.

Using the coercivity relation (7) of $A(\cdot, \cdot)$, the equation (21) and the Problem (19), we obtain

$$\begin{aligned} m\|\boldsymbol{u} - \boldsymbol{u}_\varepsilon\|^2_{1,\Omega} &\leq A(\boldsymbol{u} - \boldsymbol{u}_\varepsilon, \boldsymbol{u} - \boldsymbol{u}_\varepsilon) \\ &= A(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}_\varepsilon) - A(\boldsymbol{u}_\varepsilon, \boldsymbol{u} - \boldsymbol{u}_\varepsilon) \\ &= \int_{\Gamma_C} \left(\boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon} \left[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell \right]^+ \right) \left(\boldsymbol{u} \cdot \boldsymbol{n} - \boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} \right) ds \end{aligned}$$

$$= \int\limits_{\Gamma_C} \left(\boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon} [\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ \right) \Big((\boldsymbol{u} \cdot \boldsymbol{n} - \ell) - (\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell) \Big)\, d\boldsymbol{s}$$
$$= \int\limits_{\Gamma_C} \boldsymbol{\sigma}_n(\boldsymbol{u}) \big(\boldsymbol{u} \cdot \boldsymbol{n} - \ell \big)\, d\boldsymbol{s} + \int\limits_{\Gamma_C} \frac{1}{\varepsilon} [\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ \big(\boldsymbol{u} \cdot \boldsymbol{n} - \ell \big)\, d\boldsymbol{s}$$
$$- \int\limits_{\Gamma_C} \boldsymbol{\sigma}_n(\boldsymbol{u}) \big(\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell \big)\, d\boldsymbol{s} - \int\limits_{\Gamma_C} \frac{1}{\varepsilon} [\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ \big(\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell \big)\, d\boldsymbol{s}. \tag{22}$$

We first remark, thanks to the unilateral contact condition (4), the following two estimates

$$\begin{aligned} &\int\limits_{\Gamma_C} \boldsymbol{\sigma}_n(\boldsymbol{u}) \big(\boldsymbol{u} \cdot \boldsymbol{n} - \ell \big)\, d\boldsymbol{s} = 0 \\ &\int\limits_{\Gamma_C} \frac{1}{\varepsilon} [\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ \big(\boldsymbol{u} \cdot \boldsymbol{n} - \ell \big)\, d\boldsymbol{s} \le 0. \end{aligned} \tag{23}$$

And using the property (9) and the same condition (4), we obtain

$$\begin{aligned} -\int\limits_{\Gamma_C} \boldsymbol{\sigma}_n(\boldsymbol{u}) \big(\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell \big)\, d\boldsymbol{s} &\le -\int\limits_{\Gamma_C} \boldsymbol{\sigma}_n(\boldsymbol{u})\, [\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+\, d\boldsymbol{s} \\ -\int\limits_{\Gamma_C} \frac{1}{\varepsilon} [\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ \big(\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell \big)\, d\boldsymbol{s} &= -\int\limits_{\Gamma_C} \frac{1}{\varepsilon} [\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ [\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+\, d\boldsymbol{s}. \end{aligned} \tag{24}$$

Then taking into account the relations (23), (24) and the Young inequality, the estimate (22) becomes

$$\begin{aligned} m\|\boldsymbol{u} - \boldsymbol{u}_\varepsilon\|^2_{1,\Omega} &\le -\int\limits_{\Gamma_C} \left(\boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon} [\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ \right) [\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+\, d\boldsymbol{s} \\ &\le -\varepsilon \int\limits_{\Gamma_C} \left(\boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon} [\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ \right) \left(\boldsymbol{\sigma}_n(\boldsymbol{u}) - \boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon} [\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ \right) d\boldsymbol{s} \\ &\le -\varepsilon \left\| \boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon} [\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ \right\|^2_{0,\Gamma_C} + \varepsilon \int\limits_{\Gamma_C} \left(\boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon} [\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ \right) \boldsymbol{\sigma}_n(\boldsymbol{u})\, d\boldsymbol{s} \\ &\le -\varepsilon \left\| \boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon} [\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ \right\|^2_{0,\Gamma_C} + \varepsilon^\delta \left\| \boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon} [\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ \right\|_{-\nu,\Gamma_C} \varepsilon^{1-\delta} \|\boldsymbol{\sigma}_n(\boldsymbol{u})\|_{\nu,\Gamma_C} \\ &\le -\varepsilon \left\| \boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon} [\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ \right\|^2_{0,\Gamma_C} + \frac{\varepsilon^{2\delta}}{2\beta} \left\| \boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon} [\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ \right\|^2_{-\nu,\Gamma_C} \\ &+ \frac{\beta \varepsilon^{2-2\delta}}{2} \|\boldsymbol{\sigma}_n(\boldsymbol{u})\|^2_{\nu,\Gamma_C}, \end{aligned} \tag{25}$$

with $\delta \in [0, 1]$ and $\beta > 0$. Thus using the estimate (18) of Lemma 3.9 in [6], we obtain from estimate (25)

$$\begin{aligned} m\|\boldsymbol{u} - \boldsymbol{u}_\varepsilon\|^2_{1,\Omega} \le -\varepsilon \left(1 - c\frac{\varepsilon^{2(\delta+\nu)-1}}{\beta} \right) \left\| \boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon} [\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ \right\|^2_{0,\Gamma_C} + c\frac{\varepsilon^{2(\delta+\nu)-1}}{\beta} \|\boldsymbol{u} - \boldsymbol{u}_\varepsilon\|^2_{1,\Omega} \\ + \frac{\beta \varepsilon^{2-2\delta}}{2} \|\boldsymbol{\sigma}_n(\boldsymbol{u})\|^2_{\nu,\Gamma_C}. \end{aligned} \tag{26}$$

Thus from the equation $2(\delta+\nu)-1=0$, we choose the parameter δ as follows $\delta=\frac{1}{2}-\nu$. Moreover, making the same choice as in the proof of Theorem 3.1 [6] on the parameter $\beta=2c\max\left(1,m^{-1}\right)$ and taking into account the estimate $\|\boldsymbol{\sigma}_n(\boldsymbol{u})\|_{\nu,\Gamma_C}\leq c\|\boldsymbol{u}\|_{\frac{3}{2}+\nu,\Omega}$, we obtain from the inequality (26) the following estimate which establishes (20)

$$\left(m-\frac{1}{2\max\left(1,m^{-1}\right)}\right)\|\boldsymbol{u}-\boldsymbol{u}_\varepsilon\|_{1,\Omega}^2+\varepsilon\left(1-\frac{1}{2\max\left(1,m^{-1}\right)}\right)\left\|\boldsymbol{\sigma}_n(\boldsymbol{u})+\frac{1}{\varepsilon}\left[\boldsymbol{u}_\varepsilon\cdot\boldsymbol{n}-\ell\right]^+\right\|_{0,\Gamma_C}^2$$
$$\leq c\max\left(1,m^{-1}\right)\varepsilon^{1+2\nu}\|\boldsymbol{u}\|_{\frac{3}{2}+\nu,\Omega}^2. \quad \square$$

2.2. The penalty formulation of the Tresca friction unilateral problem

The weak formulation of the Tresca friction problem (1)–(4) and (6) is defined by the following second kind variational inequality problem

$$\begin{cases}\text{Find } \boldsymbol{u}\in\boldsymbol{K} \text{ such that,}\\ A(\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u})+\displaystyle\int_{\Gamma_C} g\big(|\boldsymbol{v}_{\boldsymbol{t}}|-|\boldsymbol{u}_{\boldsymbol{t}}|\big)\,ds\geq F(\boldsymbol{v}-\boldsymbol{u}),\ \forall\,\boldsymbol{v}\in\boldsymbol{K}.\end{cases} \tag{27}$$

According to [15], the formulation (27) admits a unique solution. Applying the penalty method to the formulation (27), we have the following penalty weak formulation

$$\begin{cases}\text{Find } \boldsymbol{u}_\varepsilon\in\boldsymbol{V} \text{ such that,}\\ A(\boldsymbol{u}_\varepsilon,\boldsymbol{v})+\dfrac{1}{\varepsilon}\displaystyle\int_{\Gamma_C}\left[\boldsymbol{u}_\varepsilon\cdot\boldsymbol{n}-\ell\right]^+(\boldsymbol{v}\cdot\boldsymbol{n})\,ds\\ \qquad +\dfrac{1}{\varepsilon}\displaystyle\int_{\Gamma_C}\left[\boldsymbol{u}_{\varepsilon\boldsymbol{t}}\right]_{\varepsilon g}\cdot\boldsymbol{v}_{\boldsymbol{t}}\,ds=F(\boldsymbol{v}),\ \forall\,\boldsymbol{v}\in\boldsymbol{V}.\end{cases} \tag{28}$$

The problem (28) admits a unique solution (since the proof of Theorem 2.2 in [6] still holds). The uniform convergence of its solution is established by F. Chouly and P. Hild ([6], Theorem 4.1). In the case of a gap not identical to zero (that is $\ell\geq 0$), we have the following theorem:

Theorem 2. *Suppose that* $\boldsymbol{u}$, *the solution of Problem* (27), *belongs to* $\boldsymbol{H}^{\frac{3}{2}+\nu}(\Omega)$ *with* $0<\nu\leq\frac{1}{2}$. *Let* $\boldsymbol{u}_\varepsilon$ *be the solution of Problem* (28). *We have the a priori estimate*

$$\|\boldsymbol{u}-\boldsymbol{u}_\varepsilon\|_{1,\Omega}+\varepsilon^{\frac{1}{2}}\left(\left\|\boldsymbol{\sigma}_n(\boldsymbol{u})+\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon\cdot\boldsymbol{n}-\ell]^+\right\|_{0,\Gamma_C}+\left\|\boldsymbol{\sigma}_{\boldsymbol{t}}(\boldsymbol{u})+\frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon\boldsymbol{t}}]_{\varepsilon g}\right\|_{0,\Gamma_C}\right)$$
$$\leq c\varepsilon^{\frac{1}{2}+\nu}\|\boldsymbol{u}\|_{\frac{3}{2}+\nu,\Omega}, \tag{29}$$

where $c>0$ *is a constant, independent of* ε *and* $\boldsymbol{u}$.

Proof. The proof is established by taking again the different steps of Theorem 4.1 in [6] and by following the ideas of the proof of Theorem 1 where a positive gap is used. The Theorem 4.1 in [6] has been proved in the two dimensional case but it also remains valid in dimension three as specified by its authors. $\square$

3. Finite element approximations of the penalty formulations

For the construction of the finite element approximations, we mesh the polyhedral domain $\Omega \in \mathbb{R}^d, d = 2, 3$, by a finite set of d-simplices $\top$

$$\overline{\Omega} := \bigcup_{\top \in \mathcal{T}_h} \top, \tag{30}$$

such that the intersection between two elements, when non-empty, is assumed to be a vertex, an edge or a face of both elements. We denote by $\mathcal{T}_h$ this finite set of d-simplices which is assumed to be regular, that is there exists $\rho \geq 0$ such that for any $\top \in \mathcal{T}_h$,

$$\frac{h_\top}{\rho_\top} \leq \rho, \tag{31}$$

where $\rho_\top$ denotes the radius of the inscribed ball in $\top$ and where $h_\top$ is the diameter of the element $\top$. We denote by h the mesh size defined as the largest diameter of all elements $\top$

$$h := \max_{\top \in \mathcal{T}_h} h_\top. \tag{32}$$

We thus define the family of finite-dimensional spaces V_h, composed of continuous and piecewise affine functions, as follows

$$V_h := \Big\{ v_h \in \mathscr{C}^0(\overline{\Omega});\ v_{h|\top} \in \mathscr{P}_1(\top),\ \forall \top \in \mathcal{T}_h,\ v_h = 0 \text{ on } \Gamma_D \Big\},$$

where $\mathscr{P}_1(\top)$ is the set of polynomials of degree less or equal to 1 on $\top$. From the discrete space V_h, we introduce the vector space $\boldsymbol{V}_h := (V_h)^d$ and the following space of traces on Γ_C

$$W_h(\Gamma_C) := \Big\{ \mu_h \in \mathscr{C}^0(\overline{\Gamma}_C);\ \exists\, v_h \in V_h,\ v_{h|\Gamma_C} = \mu_h \Big\}.$$

We assume that the endpoints or the border (for the three dimensional space) of the boundary part Γ_C belong to Γ_N, and the mesh on Γ_C induced by $\mathcal{T}_h$ is quasi-uniform. This implies, in the sense of Blamble et al. [4], the locally quasi-uniformity of the mesh on Γ_C.

Let us recall the stability and the interpolations properties of the $L^2(\Gamma_C)$-projection operator onto $W_h(\Gamma_C)$, denoted by $\mathcal{P}^h : L^2(\Gamma_C) \longrightarrow W_h(\Gamma_C)$. These properties, of which proofs can be found in [2] and [4], are stated in the following lemma.

Lemma 1. *Suppose that the mesh associated to $W_h(\Gamma_C)$ is locally quasi-uniform. For all $s \in [0, 1]$ and all $v \in H^s(\Gamma_C)$, we have the stability estimate*

$$\left\| \mathcal{P}^h(v) \right\|_{s,\Gamma_C} \leq c \|v\|_{s,\Gamma_C}. \tag{33}$$

The following interpolation estimate also holds:

$$\left\| v - \mathcal{P}^h(v) \right\|_{0,\Gamma_C} \leq c h^s \|v\|_{s,\Gamma_C}, \tag{34}$$

for all $v \in H^s(\Gamma_C)$, where the constant $c > 0$ in both cases is independent of v and the mesh size h.

We recall another lemma proven in [3] (see also [9]), which concerns the existence of a discrete bounded lifting from the contact boundary Γ_C to the domain Ω.

Lemma 2. *Suppose that the mesh on the contact boundary* Γ_C *is quasi-uniform. There exist an extension operator* $\mathcal{R}^h : W_h(\Gamma_C) \longrightarrow V_h$ *and* $c > 0$*, such that*

$$\mathcal{R}^h(\mu_h)_{|\Gamma_C} = \mu_h, \quad \left\|\mathcal{R}^h(\mu_h)\right\|_{1,\Omega} \leq c\|\mu_h\|_{\frac{1}{2},\Gamma_C}, \forall\ \mu_h \in W_h(\Gamma_C). \tag{35}$$

Remark 1. Let us denote by $\boldsymbol{\mathcal{P}}^h$ and $\boldsymbol{\mathcal{R}}^h$ the vector forms of the operators $\mathcal{P}^h$ and $\mathcal{R}^h$ respectively, defined for any vector $\boldsymbol{w} := (w_i)_{1\leq i\leq d}$ as follows

$$\boldsymbol{\mathcal{P}}^h(\boldsymbol{w}) := \left(\mathcal{P}^h(w_i)\right)_{1\leq i\leq d} \text{ and } \boldsymbol{\mathcal{R}}^h(\boldsymbol{w}) := \left(\mathcal{R}^h(w_i)\right)_{1\leq i\leq d}. \tag{36}$$

It is an easy task to see that the operators $\boldsymbol{\mathcal{P}}^h$ and $\boldsymbol{\mathcal{R}}^h$ also satisfy stability and interpolation properties (33), (35) and (34).

3.1. Finite element approximations of the frictionless unilateral problem

The finite element approximation of the penalized problem (19) is defined by

$$\begin{cases} \text{Given } \varepsilon > 0, \text{ find } \boldsymbol{u}_{\varepsilon h} \in \boldsymbol{V}_h \text{ such that,} \\ A(\boldsymbol{u}_{\varepsilon h}, \boldsymbol{v}_h) + \dfrac{1}{\varepsilon}\displaystyle\int_{\Gamma_C} [\boldsymbol{u}_{\varepsilon h}\cdot\boldsymbol{n} - \ell]^+ (\boldsymbol{v}_h\cdot\boldsymbol{n})\, ds = F(\boldsymbol{v}_h), \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h. \end{cases} \tag{37}$$

It can easily be verified that the nonlinear operator $B_h : \boldsymbol{V}_h \longrightarrow \boldsymbol{V}_h$, defined by

$$(B_h\boldsymbol{v}_h, \boldsymbol{w}_h) := A(\boldsymbol{v}_h, \boldsymbol{w}_h) + \frac{1}{\varepsilon}\int_{\Gamma_C} [\boldsymbol{v}_h\cdot\boldsymbol{n} - \ell]^+ (\boldsymbol{w}_h\cdot\boldsymbol{n})\, ds, \forall\, \boldsymbol{v}_h, \boldsymbol{w}_h \in \boldsymbol{V}_h, \tag{38}$$

is hemicontinuous and $\boldsymbol{V}_h$-elliptic (see [6], Theorem 2.2). Hence applying the Corollary 15 in [5], we conclude that the operator B_h is one to one. Thus, the existence and the uniqueness of the solution of the discrete penalty problem (37) are assured.

Unlike the Nitsche method (see [7] and [8]), the penalty method is not consistent in the sense that the solution to the continuous constrained problem (18) does not satisfy conditions corresponding to the discrete penalty problem (37). In order to be consistent with the discrete scheme (37), we focus on the a priori analysis between the continuous penalty formulation (19) and the discrete penalty problem (37).

Remark 2. The penalty formulation (19) is consistent with the finite element problem (37) in the sense that the solution $\boldsymbol{u}_\varepsilon$ of the Problem (19) also satisfies the following equation for test functions $\boldsymbol{v}_h$ in $\boldsymbol{V}_h$

$$A(\boldsymbol{u}_\varepsilon, \boldsymbol{v}_h) + \frac{1}{\varepsilon}\int_{\Gamma_C} [\boldsymbol{u}_\varepsilon\cdot\boldsymbol{n} - \ell]^+ (\boldsymbol{v}_h\cdot\boldsymbol{n})\, ds = F(\boldsymbol{v}_h), \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{39}$$

The uniform convergence of the penalized solution $\boldsymbol{u}_\varepsilon$ being ensured from the estimate (20), one can focus exclusively on the convergence of the penalty finite element approximation solution $\boldsymbol{u}_{\varepsilon h}$ towards $\boldsymbol{u}_\varepsilon$, and not towards the solution of the constrained problem (18). This approach is the main difference between this study and most of works on the penalty method of the unilateral contact problem (see for instance [13], [12] and [6]).

Please cite this article in press as: I. Dione, Optimal convergence analysis of the unilateral contact problem with and without Tresca friction conditions by the penalty method, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2018.11.023

3.2. Finite element approximations of the Tresca friction unilateral problem

The finite element approximation problem, issued from the penalized problem (28), is defined by

$$\begin{cases} \text{Given } \varepsilon > 0, \text{ find } \boldsymbol{u}_{\varepsilon h} \in \boldsymbol{V}_h \text{ such that,} \\ A(\boldsymbol{u}_{\varepsilon h}, \boldsymbol{v}_h) + \dfrac{1}{\varepsilon} \displaystyle\int_{\Gamma_C} [\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+ (\boldsymbol{v}_h \cdot \boldsymbol{n}) \, ds \\ \qquad\qquad + \dfrac{1}{\varepsilon} \displaystyle\int_{\Gamma_C} [\boldsymbol{u}_{\varepsilon h_{\boldsymbol{t}}}]_{\varepsilon g} \cdot \boldsymbol{v}_{h_{\boldsymbol{t}}} \, ds = F(\boldsymbol{v}_h), \, \forall \, \boldsymbol{v}_h \in \boldsymbol{V}_h. \end{cases} \tag{40}$$

The existence and the uniqueness of the solution of the discrete problem (40) are obtained similarly to the problem (28). The consistency of the solution $\boldsymbol{u}_\varepsilon$ with respect to the discrete problem (40), described in the following remark, is very useful in the forthcoming analysis.

Remark 3. The penalty formulation (28) is consistent with the finite element approximation problem (40) in the sense that the solution $\boldsymbol{u}_\varepsilon$ of the Problem (28) also satisfies the following weak formulation

$$\begin{aligned} A(\boldsymbol{u}_\varepsilon, \boldsymbol{v}_h) &+ \frac{1}{\varepsilon} \int_{\Gamma_C} [\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ (\boldsymbol{v}_h \cdot \boldsymbol{n}) \, ds \\ &+ \frac{1}{\varepsilon} \int_{\Gamma_C} [\boldsymbol{u}_{\varepsilon_{\boldsymbol{t}}}]_{\varepsilon g} \cdot \boldsymbol{v}_{h_{\boldsymbol{t}}} \, ds = F(\boldsymbol{v}_h), \, \forall \, \boldsymbol{v}_h \in \boldsymbol{V}_h. \end{aligned} \tag{41}$$

We will focus exclusively on the convergence of the penalty finite element approximation solution $\boldsymbol{u}_{\varepsilon h}$ towards $\boldsymbol{u}_\varepsilon$, due to this consistency but also thanks to the uniform convergence result (29).

4. A priori estimate in terms of the penalty parameter ε and the mesh size h

4.1. A priori estimate of the frictionless unilateral contact problem

We first present the estimate of the error $\left(\boldsymbol{u}_\varepsilon - \boldsymbol{u}_{\varepsilon h}, \, \frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right)$ in terms of the interpolation estimate error and the penalty approximation error. And thanks to the consistency of the estimate in Theorem 1 and the triangle inequality, we establish the second a priori estimate of the term $\left(\boldsymbol{u} - \boldsymbol{u}_{\varepsilon h}, \, \boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right)$.

Theorem 3. *Let $\Omega \subset \mathbb{R}^d, d = 2, 3$, be a polygonal bounded domain, $\boldsymbol{u}_\varepsilon$ and $\boldsymbol{u}_{\varepsilon h}$ solutions of Problems (19) and (37), respectively. Then for any $\varepsilon > 0$ and $h > 0$, we have the following a priori estimate*

$$\begin{aligned} \|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_{\varepsilon h}\|_{1,\Omega} + \left(\varepsilon^{\frac{1}{2}} - ch^{\frac{1}{2}}\right) &\left\| \frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+ \right\|_{0,\Gamma_C} \\ &\leq c\left(\|\boldsymbol{u} - \boldsymbol{v}_h\|_{1,\Omega} + \|\boldsymbol{u} - \boldsymbol{u}_\varepsilon\|_{1,\Omega}\right), \end{aligned}$$

for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and where $c > 0$ is a constant independent of $\boldsymbol{u}$, h and ε.

Proof. Taking as test functions $\boldsymbol{v}_h - \boldsymbol{u}_{\varepsilon h}$ where $\boldsymbol{v}_h \in \boldsymbol{V}_h$, we obtain from Problem (37) the following

$$A(\boldsymbol{u}_{\varepsilon h}, \boldsymbol{v}_h - \boldsymbol{u}_{\varepsilon h}) + \frac{1}{\varepsilon} \int_{\Gamma_C} [\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+ (\boldsymbol{v}_h \cdot \boldsymbol{n} - \boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n})\, ds = F(\boldsymbol{v}_h - \boldsymbol{u}_{\varepsilon h}), \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{42}$$

Similarly, considering the same test functions and taking into account of the consistency described at (39), we obtain from Problem (19) the following equation

$$A(\boldsymbol{u}_{\varepsilon}, \boldsymbol{v}_h - \boldsymbol{u}_{\varepsilon h}) + \frac{1}{\varepsilon} \int_{\Gamma_C} [\boldsymbol{u}_{\varepsilon} \cdot \boldsymbol{n} - \ell]^+ (\boldsymbol{v}_h \cdot \boldsymbol{n} - \boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n})\, ds = F(\boldsymbol{v}_h - \boldsymbol{u}_{\varepsilon h}), \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{43}$$

Due to the continuity and ellipticity of the bilinear form $A(\cdot,\cdot)$ described in (7) and (8), we have

$$\begin{aligned} m\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_{\varepsilon h}\|_{1,\Omega}^2 &\le A(\boldsymbol{u}_\varepsilon - \boldsymbol{u}_{\varepsilon h}, \boldsymbol{u}_\varepsilon - \boldsymbol{u}_{\varepsilon h}) \\ &\le A(\boldsymbol{u}_\varepsilon - \boldsymbol{u}_{\varepsilon h}, \boldsymbol{u}_\varepsilon - \boldsymbol{v}_h) + A(\boldsymbol{u}_\varepsilon - \boldsymbol{u}_{\varepsilon h}, \boldsymbol{v}_h - \boldsymbol{u}_{\varepsilon h}) \\ &\le M\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_{\varepsilon h}\|_{1,\Omega}\|\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h\|_{1,\Omega} + A(\boldsymbol{u}_\varepsilon, \boldsymbol{v}_h - \boldsymbol{u}_{\varepsilon h}) - A(\boldsymbol{u}_{\varepsilon h}, \boldsymbol{v}_h - \boldsymbol{u}_{\varepsilon h}), \end{aligned} \tag{44}$$

for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$. Then, by Young's inequality and the triangle inequality, the estimate (44) becomes

$$\begin{aligned} m\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_{\varepsilon h}\|_{1,\Omega}^2 &\le \frac{1}{2\alpha}\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_{\varepsilon h}\|_{1,\Omega}^2 + \frac{\alpha M^2}{2}\|\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h\|_{1,\Omega}^2 + A(\boldsymbol{u}_\varepsilon, \boldsymbol{v}_h - \boldsymbol{u}_{\varepsilon h}) - A(\boldsymbol{u}_{\varepsilon h}, \boldsymbol{v}_h - \boldsymbol{u}_{\varepsilon h}) \\ &\le \frac{1}{2\alpha}\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_{\varepsilon h}\|_{1,\Omega}^2 + \alpha M^2 \left(\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|_{1,\Omega}^2 + \|\boldsymbol{u} - \boldsymbol{v}_h\|_{1,\Omega}^2\right) + A(\boldsymbol{u}_\varepsilon, \boldsymbol{v}_h - \boldsymbol{u}_{\varepsilon h}) \\ &\quad - A(\boldsymbol{u}_{\varepsilon h}, \boldsymbol{v}_h - \boldsymbol{u}_{\varepsilon h}), \end{aligned} \tag{45}$$

for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$. For the estimation of the last two terms to the right hand side of the inequality (45), equations (42) and (43) are first used to obtain

$$\begin{aligned} A(\boldsymbol{u}_\varepsilon, \boldsymbol{v}_h - \boldsymbol{u}_{\varepsilon h}) - A(\boldsymbol{u}_{\varepsilon h}, \boldsymbol{v}_h - \boldsymbol{u}_{\varepsilon h}) &= \int_{\Gamma_C} \left(\frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+\right)(\boldsymbol{v}_h \cdot \boldsymbol{n} - \boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n})\, ds \\ &= \int_{\Gamma_C} \left(\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right)(\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \boldsymbol{u}_\varepsilon \cdot \boldsymbol{n})\, ds \\ &\quad + \int_{\Gamma_C} \left(\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right)(\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \boldsymbol{v}_h \cdot \boldsymbol{n})\, ds. \end{aligned} \tag{46}$$

Using the monotonicity property (10), we estimate the first term in the right hand side of the equality (46) as follows

$$\begin{aligned} &\int_{\Gamma_C} \left(\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right)(\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \boldsymbol{u}_\varepsilon \cdot \boldsymbol{n})\, ds \\ &\qquad = -\varepsilon \int_{\Gamma_C} \left(\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right)\left(\frac{1}{\varepsilon}(\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell) - \frac{1}{\varepsilon}(\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell)\right) ds \\ &\qquad \le -\varepsilon \left\|\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right\|_{0,\Gamma_C}^2. \end{aligned} \tag{47}$$

For the second term in the right hand side of the equality (46), we use the notations in (36), the equality in (35) and the Cauchy–Schwartz inequality to obtain

$$\begin{aligned}
&\int\limits_{\Gamma_C} \left(\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right)(\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \boldsymbol{v}_h \cdot \boldsymbol{n})ds \\
&\qquad = \int\limits_{\Gamma_C} \left(\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right) \left((\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h)\cdot \boldsymbol{n} - \boldsymbol{\mathcal{P}}^h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h)\cdot \boldsymbol{n}\right) ds \\
&\qquad + \int\limits_{\Gamma_C} \left(\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right) \boldsymbol{\mathcal{P}}^h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h)\cdot \boldsymbol{n}\, ds \qquad (48)\\
&\qquad \le \left\|\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right\|_{0,\Gamma_C} \left\|\left((\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h) - \boldsymbol{\mathcal{P}}^h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h)\right)\cdot \boldsymbol{n}\right\|_{0,\Gamma_C} \\
&\qquad + \int\limits_{\Gamma_C} \left(\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right) \boldsymbol{\mathcal{R}}^h\left(\boldsymbol{\mathcal{P}}^h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h)\right)\cdot \boldsymbol{n}\, ds.
\end{aligned}$$

We estimate the first term of the right hand side of the inequality (48) using the interpolation (34), the Young's inequality and the continuity of the trace operator

$$\begin{aligned}
&\left\|\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right\|_{0,\Gamma_C} \left\|\left((\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h) - \boldsymbol{\mathcal{P}}^h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h)\right)\cdot \boldsymbol{n}\right\|_{0,\Gamma_C} \\
&\qquad \le ch^{\frac{1}{2}} \left\|\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right\|_{0,\Gamma_C} \|\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h\|_{\frac{1}{2},\Gamma_C} \\
&\qquad \le ch \left\|\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right\|^2_{0,\Gamma_C} + c\,\|\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h\|^2_{1,\Omega} \qquad (49)\\
&\qquad \le ch \left\|\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right\|^2_{0,\Gamma_C} + c\left(\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|^2_{1,\Omega} + \|\boldsymbol{u} - \boldsymbol{v}_h\|^2_{1,\Omega}\right).
\end{aligned}$$

On the other hand, using the consistence equation (39), the problem (37) and stability properties in (33) and (35), we estimate the last term of the right hand side of the inequality (48) as follows

$$\begin{aligned}
&\int\limits_{\Gamma_C} \left(\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right)\boldsymbol{\mathcal{R}}^h\left(\boldsymbol{\mathcal{P}}^h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h)\right)\cdot \boldsymbol{n}\, ds \\
&\qquad = A\left(\boldsymbol{u}_\varepsilon - \boldsymbol{u}_{\varepsilon h}, \boldsymbol{\mathcal{R}}^h\left(\boldsymbol{\mathcal{P}}^h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h)\right)\right) \\
&\qquad \le M\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_{\varepsilon h}\|_{1,\Omega} \left\|\boldsymbol{\mathcal{R}}^h\left(\boldsymbol{\mathcal{P}}^h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h)\right)\right\|_{1,\Omega} \qquad (50)\\
&\qquad \le cM\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_{\varepsilon h}\|_{1,\Omega} \left\|\boldsymbol{\mathcal{P}}^h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h)\right\|_{\frac{1}{2},\Gamma_C} \\
&\qquad \le cM\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_{\varepsilon h}\|_{1,\Omega}\left(\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|_{1,\Omega} + \|\boldsymbol{u} - \boldsymbol{v}_h\|_{1,\Omega}\right) \\
&\qquad \le \frac{1}{\alpha}\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_{\varepsilon h}\|^2_{1,\Omega} + \frac{\alpha\,(cM)^2}{2}\left(\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|^2_{1,\Omega} + \|\boldsymbol{u} - \boldsymbol{v}_h\|^2_{1,\Omega}\right).
\end{aligned}$$

Thus taking into account estimates (49) and (50), we rewrite the estimate (48) as follows

$$\begin{aligned}
&\int\limits_{\Gamma_C} \left(\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right)(\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \boldsymbol{v}_h \cdot \boldsymbol{n})ds \\
&\qquad \le \frac{1}{\alpha}\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_{\varepsilon h}\|^2_{1,\Omega} + c\left(1 + \frac{\alpha M^2}{2}\right)\left(\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|^2_{1,\Omega} + \|\boldsymbol{u} - \boldsymbol{v}_h\|^2_{1,\Omega}\right) \qquad (51)\\
&\qquad + ch\left\|\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right\|^2_{0,\Gamma_C}.
\end{aligned}$$

Combining the inequalities (47) and (51), we rewrite the estimate in (46) as follows

$$A(\boldsymbol{u}_\varepsilon, \boldsymbol{v}_h - \boldsymbol{u}_{\varepsilon h}) - A(\boldsymbol{u}_{\varepsilon h}, \boldsymbol{v}_h - \boldsymbol{u}_{\varepsilon h}) \leq \frac{1}{\alpha}\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_{\varepsilon h}\|^2_{1,\Omega} + (ch - \varepsilon)\left\|\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right\|^2_{0,\Gamma_C} + c\left(1 + \frac{\alpha M^2}{2}\right)\left(\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|^2_{1,\Omega} + \|\boldsymbol{u} - \boldsymbol{v}_h\|^2_{1,\Omega}\right). \tag{52}$$

Finally, rewriting the estimate (45) by taking into account the inequality (52), we establish the desired result

$$\left(m - \frac{3}{2\alpha}\right)\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_{\varepsilon h}\|^2_{1,\Omega} + \varepsilon\left\|\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right\|^2_{0,\Gamma_C} \leq c\left(1 + \frac{3\alpha M^2}{2}\right)\left(\|\boldsymbol{u} - \boldsymbol{v}_h\|^2_{1,\Omega} + \|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|^2_{1,\Omega}\right) + ch\left\|\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right\|^2_{0,\Gamma_C},$$

for $\alpha > \frac{3}{2m}$, where $c > 0$ is independent of $\boldsymbol{u}$, the penalty parameter ε and the mesh size h. □

Theorem 4. *Under assumptions of Theorem 3, if the solution $\boldsymbol{u}$ of Problem (18) belongs to $\boldsymbol{H}^{\frac{3}{2}+\nu}(\Omega)$ with $0 < \nu \leq \frac{1}{2}$, then for $\varepsilon > 0$ and $h > 0$, the following estimate holds*

$$\|\boldsymbol{u} - \boldsymbol{u}_{\varepsilon h}\|_{1,\Omega} + \left(\varepsilon^{\frac{1}{2}} - ch^{\frac{1}{2}}\right)\left\|\boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right\|_{0,\Gamma_C} \leq c\left(h^{\frac{1}{2}+\nu} + \varepsilon^{\frac{1}{2}+\nu}\right)\|\boldsymbol{u}\|_{\frac{3}{2}+\nu,\Omega}, \tag{53}$$

where $c > 0$ is a constant independent of $\boldsymbol{u}$, the penalty parameter ε and the mesh size h.

Proof. Since $\boldsymbol{v}_h \in \boldsymbol{V}_h$, then let us make the choice $\boldsymbol{v}_h := \boldsymbol{\mathcal{I}}^1_h(\boldsymbol{u})$ where $\boldsymbol{\mathcal{I}}^1_h$ is the Lagrange interpolation operator mapping onto $\boldsymbol{V}_h$. The standard Lagrange interpolation estimate in the $\boldsymbol{H}^1(\Omega)$ norm stands for (see [11])

$$\|\boldsymbol{u} - \boldsymbol{\mathcal{I}}^1_h(\boldsymbol{u})\|_{1,\Omega} \leq ch^{\frac{1}{2}+\nu}\|\boldsymbol{u}\|_{\frac{3}{2}+\nu,\Omega}, \tag{54}$$

where $-\frac{1}{2} < \nu \leq \frac{1}{2}$.

From Theorem 3, we prove Theorem 4 by taking into account estimates (20), (54), the triangle inequality and the penalty parameter chosen as $\varepsilon > h$. □

Remark 4. An important point to emphasize is, throughout the proofs leading to the estimation (53) we do not face estimating the contact term in (15). The sub-optimal estimate of this contact term explains the quasi-optimal estimate obtained in [6], even if recurrently its optimal estimate was established [10].

Remark 5. In order to determine the induced convergence rate at the estimate (53), we take the penalty parameter ε as a function of the size of the mesh, that is, we define it as follows:

$$\varepsilon(h) := ch^\theta,$$

where c and θ are positive constants to fix.

- If the penalty parameter behaves like the size of the mesh h, i.e. $\varepsilon(h) := (c+1)^2 h$, we obtain from Theorem 4 the following a priori estimate

$$\|\boldsymbol{u} - \boldsymbol{u}_{\varepsilon h}\|_{1,\Omega} + h^{\frac{1}{2}} \left\| \boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+ \right\|_{0,\Gamma_C} \leq ch^{\frac{1}{2}+\nu} \|\boldsymbol{u}\|_{\frac{3}{2}+\nu,\Omega}. \tag{55}$$

  The main consequence of the result (55) is the $\mathcal{O}\left(h^{\frac{1}{2}+\nu}\right)$ convergence rate obtained under the $\boldsymbol{H}^{\frac{3}{2}+\nu}(\Omega), 0 < \nu \leq \frac{1}{2}$, regularity and without any additional assumptions on the contact boundary Γ_C. We thus obtain the same optimal error estimate than the strong finite element approximation of the variational inequality (18) established by G. Drouet and P. Hild in [10]. By the Nitsche method, this optimal result was also obtained by F. Chouly, P. Hild and Y. Renard (see [7] and [8]).
- On the other hand, if the penalty parameter is defined by $\varepsilon(h) := c^2 h^\theta$, $0 < \theta < 1$, a sub-optimal estimate of $\mathcal{O}\left(h^{\theta(\frac{1}{2}+\nu)}\right)$ convergence rate is obtained from Theorem 4

$$\|\boldsymbol{u} - \boldsymbol{u}_{\varepsilon h}\|_{1,\Omega} + ch^{\frac{\theta}{2}} \left(1 - h^{\frac{1-\theta}{2}}\right) \left\| \boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+ \right\|_{0,\Gamma_C} \leq ch^{\theta(\frac{1}{2}+\nu)} \|\boldsymbol{u}\|_{\frac{3}{2}+\nu,\Omega}. \tag{56}$$

  Knowing that $\left(1 - h^{\frac{1-\theta}{2}}\right) \longrightarrow 1$ if $h \longrightarrow 0$.
- If the continuous estimate in (20) remains true for a more regular solution, namely for $\boldsymbol{u} \in \boldsymbol{H}^{\frac{3}{2}+\nu}(\Omega)$ with $0 < \nu \leq \frac{3}{2}$, then the result obtained in Theorem 4 will be valid and optimal for quadratic finite elements. Indeed, the steps of the proof of Theorem 3 do not directly use the current finite element. The latter only intervened in the estimation (54) which remains valid in the quadratic case.

4.2. A priori estimate of the Tresca friction unilateral contact problem

Similarly to the frictionless problem, we first focus on the convergence of the penalty finite element approximation solution $\left(\boldsymbol{u}_{\varepsilon h},\ \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+,\ \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h_{\boldsymbol{t}}}]_{\varepsilon g}\right)$ towards the penalized continuous solution $\left(\boldsymbol{u}_{\varepsilon},\ \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon} \cdot \boldsymbol{n} - \ell]^+,\ \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon_{\boldsymbol{t}}}]_{\varepsilon g}\right)$.

Theorem 5. *Let $\Omega \subset \mathbb{R}^d, d = 2, 3$, be a polygonal bounded domain, $\boldsymbol{u}_\varepsilon$ and $\boldsymbol{u}_{\varepsilon h}$ solutions of Problems (28) and (40), respectively. Then for $\varepsilon > 0$ and $h > 0$, we have the a priori estimate*

$$\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_{\varepsilon h}\|_{1,\Omega} + \left(\varepsilon^{\frac{1}{2}} - ch^{\frac{1}{2}}\right) \left(\left\| \frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+ \right\|_{0,\Gamma_C} + \left\| \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon_{\boldsymbol{t}}}]_{\varepsilon g} - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h_{\boldsymbol{t}}}]_{\varepsilon g} \right\|_{0,\Gamma_C} \right)$$
$$\leq c\left(\|\boldsymbol{u} - \boldsymbol{v}_h\|_{1,\Omega} + \|\boldsymbol{u} - \boldsymbol{u}_\varepsilon\|_{1,\Omega}\right)$$

for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and where $c > 0$ is a constant independent of $\boldsymbol{u}$, h and ε.

Proof. This proof is established by following the same steps as those of the proof of Theorem 3. First of all by the same arguments as those used to establish inequality (45), we obtain the following similar inequality

$$\begin{aligned} m\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_{\varepsilon h}\|^2_{1,\Omega} \leq \frac{1}{2\alpha}\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_{\varepsilon h}\|^2_{1,\Omega} + \alpha M^2\left(\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|^2_{1,\Omega} + \|\boldsymbol{u} - \boldsymbol{v}_h\|^2_{1,\Omega}\right) + A(\boldsymbol{u}_\varepsilon, \boldsymbol{v}_h - \boldsymbol{u}_{\varepsilon h}) \\ - A(\boldsymbol{u}_{\varepsilon h}, \boldsymbol{v}_h - \boldsymbol{u}_{\varepsilon h}), \end{aligned} \tag{57}$$

Please cite this article in press as: I. Dione, Optimal convergence analysis of the unilateral contact problem with and without Tresca friction conditions by the penalty method, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2018.11.023

for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$. Instead of the equality (46), taking into account equations (40) and (41) by introducing the terms $\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n}$ and $\boldsymbol{u}_{\varepsilon_{\boldsymbol{t}}}$, we replace the last two terms to the right hand side of the inequality (57) as follows

$$\begin{aligned}
A(\boldsymbol{u}_\varepsilon, \boldsymbol{v}_h - \boldsymbol{u}_{\varepsilon h}) - A(\boldsymbol{u}_{\varepsilon h}, \boldsymbol{v}_h - \boldsymbol{u}_{\varepsilon h}) &= \int_{\Gamma_C} \left(\frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+\right)\left(\boldsymbol{v}_h \cdot \boldsymbol{n} - \boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n}\right) ds \\
&+ \int_{\Gamma_C} \left(\frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon h_{\boldsymbol{t}}}\right]_{\varepsilon g} - \frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon_{\boldsymbol{t}}}\right]_{\varepsilon g}\right) \cdot \left(\boldsymbol{v}_{h_{\boldsymbol{t}}} - \boldsymbol{u}_{\varepsilon h_{\boldsymbol{t}}}\right) ds \\
&= \int_{\Gamma_C} \left(\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right)\left(\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \boldsymbol{u}_\varepsilon \cdot \boldsymbol{n}\right) ds \\
&+ \int_{\Gamma_C} \left(\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right)\left(\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \boldsymbol{v}_h \cdot \boldsymbol{n}\right) ds \\
&+ \int_{\Gamma_C} \left(\frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon_{\boldsymbol{t}}}\right]_{\varepsilon g} - \frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon h_{\boldsymbol{t}}}\right]_{\varepsilon g}\right) \cdot \left(\boldsymbol{u}_{\varepsilon h_{\boldsymbol{t}}} - \boldsymbol{u}_{\varepsilon_{\boldsymbol{t}}}\right) ds \\
&+ \int_{\Gamma_C} \left(\frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon_{\boldsymbol{t}}}\right]_{\varepsilon g} - \frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon h_{\boldsymbol{t}}}\right]_{\varepsilon g}\right) \cdot \left(\boldsymbol{u}_{\varepsilon_{\boldsymbol{t}}} - \boldsymbol{v}_{h_{\boldsymbol{t}}}\right) ds.
\end{aligned} \tag{58}$$

In order to make easier the estimate of the terms in the right hand side of the equality (58), let us denote

$$\begin{aligned}
T_1 &:= \int_{\Gamma_C} \left(\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right)\left(\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \boldsymbol{u}_\varepsilon \cdot \boldsymbol{n}\right) ds, \\
T_2 &:= \int_{\Gamma_C} \left(\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right)\left(\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \boldsymbol{v}_h \cdot \boldsymbol{n}\right) ds, \\
T_3 &:= \int_{\Gamma_C} \left(\frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon_{\boldsymbol{t}}}\right]_{\varepsilon g} - \frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon h_{\boldsymbol{t}}}\right]_{\varepsilon g}\right) \cdot \left(\boldsymbol{u}_{\varepsilon h_{\boldsymbol{t}}} - \boldsymbol{u}_{\varepsilon_{\boldsymbol{t}}}\right) ds, \\
T_4 &:= \int_{\Gamma_C} \left(\frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon_{\boldsymbol{t}}}\right]_{\varepsilon g} - \frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon h_{\boldsymbol{t}}}\right]_{\varepsilon g}\right) \cdot \left(\boldsymbol{u}_{\varepsilon_{\boldsymbol{t}}} - \boldsymbol{v}_{h_{\boldsymbol{t}}}\right) ds.
\end{aligned}$$

We already have estimated the term T_1 in (47) and obtained

$$T_1 \leq -\varepsilon \left\| \frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+ \right\|_{0,\Gamma_C}^2 . \tag{59}$$

Using the inequalities in (11) under the following form

$$\left([\boldsymbol{v}_{\boldsymbol{t}}]_{\varepsilon g} - [\boldsymbol{w}_{\boldsymbol{t}}]_{\varepsilon g}\right) \cdot (\boldsymbol{v}_{\boldsymbol{t}} - \boldsymbol{w}_{\boldsymbol{t}}) \geq 0 \quad \text{and} \quad \left|[\boldsymbol{v}_{\boldsymbol{t}}]_{\varepsilon g} - [\boldsymbol{w}_{\boldsymbol{t}}]_{\varepsilon g}\right| \leq |\boldsymbol{v}_{\boldsymbol{t}} - \boldsymbol{w}_{\boldsymbol{t}}|,$$

on Γ_C for all $\boldsymbol{v}$, $\boldsymbol{w}$ in $\boldsymbol{V}$, we estimate the term T_3 as follows

$$\begin{aligned}
T_3 &= -\varepsilon \int_{\Gamma_C} \left(\frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon_{\boldsymbol{t}}}\right]_{\varepsilon g} - \frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon h_{\boldsymbol{t}}}\right]_{\varepsilon g}\right) \cdot \left(\frac{1}{\varepsilon}\left(\boldsymbol{u}_{\varepsilon_{\boldsymbol{t}}}\right) - \frac{1}{\varepsilon}\left(\boldsymbol{u}_{\varepsilon h_{\boldsymbol{t}}}\right)\right) ds \\
&\leq -\varepsilon \left\| \frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon_{\boldsymbol{t}}}\right]_{\varepsilon g} - \frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon h_{\boldsymbol{t}}}\right]_{\varepsilon g} \right\|_{0,\Gamma_C}^2 .
\end{aligned} \tag{60}$$

For the remaining terms T_2 and T_4, we use the notations in (36), the equality in (35) and the Cauchy–Schwartz inequality

$$\begin{aligned}
T_2 + T_4 = &\int_{\Gamma_C} \left(\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right) \Big((\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h) \cdot \boldsymbol{n} - \boldsymbol{\mathcal{P}}^h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h) \cdot \boldsymbol{n}\Big)\, ds \\
&+ \int_{\Gamma_C} \left(\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right) \boldsymbol{\mathcal{P}}^h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h) \cdot \boldsymbol{n}\, ds \\
&+ \int_{\Gamma_C} \left(\frac{1}{\varepsilon}\,[\boldsymbol{u}_{\varepsilon \boldsymbol{t}}]_{\varepsilon g} - \frac{1}{\varepsilon}\,[\boldsymbol{u}_{\varepsilon h \boldsymbol{t}}]_{\varepsilon g}\right) \cdot \Big((\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h)_{\boldsymbol{t}} - \boldsymbol{\mathcal{P}}^h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h)_{\boldsymbol{t}}\Big)\, ds \\
&+ \int_{\Gamma_C} \left(\frac{1}{\varepsilon}\,[\boldsymbol{u}_{\varepsilon \boldsymbol{t}}]_{\varepsilon g} - \frac{1}{\varepsilon}\,[\boldsymbol{u}_{\varepsilon h \boldsymbol{t}}]_{\varepsilon g}\right) \cdot \boldsymbol{\mathcal{P}}^h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h)_{\boldsymbol{t}}\, ds \\
\leq &\left\|\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right\|_{0,\Gamma_C} \left\|\Big((\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h) - \boldsymbol{\mathcal{P}}^h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h)\Big) \cdot \boldsymbol{n}\right\|_{0,\Gamma_C} \\
&+ \int_{\Gamma_C} \left(\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right) \boldsymbol{\mathcal{R}}^h\left(\boldsymbol{\mathcal{P}}^h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h)\right) \cdot \boldsymbol{n}\, ds \\
&+ \left\|\frac{1}{\varepsilon}\,[\boldsymbol{u}_{\varepsilon \boldsymbol{t}}]_{\varepsilon g} - \frac{1}{\varepsilon}\,[\boldsymbol{u}_{\varepsilon h \boldsymbol{t}}]_{\varepsilon g}\right\|_{0,\Gamma_C} \left\|\Big((\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h) - \boldsymbol{\mathcal{P}}^h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h)\Big)_{\boldsymbol{t}}\right\|_{0,\Gamma_C} \\
&+ \int_{\Gamma_C} \left(\frac{1}{\varepsilon}\,[\boldsymbol{u}_{\varepsilon \boldsymbol{t}}]_{\varepsilon g} - \frac{1}{\varepsilon}\,[\boldsymbol{u}_{\varepsilon h \boldsymbol{t}}]_{\varepsilon g}\right) \cdot \boldsymbol{\mathcal{R}}^h\left(\boldsymbol{\mathcal{P}}^h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h)\right)_{\boldsymbol{t}}\, ds.
\end{aligned} \tag{61}$$

By the same arguments as those used in estimation (49), we obtain the estimate of the first and the third terms of the right hand side of the inequality (61)

$$\begin{aligned}
&\left\|\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right\|_{0,\Gamma_C} \left\|\Big((\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h) - \boldsymbol{\mathcal{P}}^h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h)\Big) \cdot \boldsymbol{n}\right\|_{0,\Gamma_C} \\
&\qquad + \left\|\frac{1}{\varepsilon}\,[\boldsymbol{u}_{\varepsilon \boldsymbol{t}}]_{\varepsilon g} - \frac{1}{\varepsilon}\,[\boldsymbol{u}_{\varepsilon h \boldsymbol{t}}]_{\varepsilon g}\right\|_{0,\Gamma_C} \left\|\Big((\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h) - \boldsymbol{\mathcal{P}}^h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h)\Big)_{\boldsymbol{t}}\right\|_{0,\Gamma_C} \\
&\qquad \leq ch\left(\left\|\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right\|^2_{0,\Gamma_C} + \left\|\frac{1}{\varepsilon}\,[\boldsymbol{u}_{\varepsilon \boldsymbol{t}}]_{\varepsilon g} - \frac{1}{\varepsilon}\,[\boldsymbol{u}_{\varepsilon h \boldsymbol{t}}]_{\varepsilon g}\right\|^2_{0,\Gamma_C}\right) \\
&\qquad + c\Big(\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|^2_{1,\Omega} + \|\boldsymbol{u} - \boldsymbol{v}_h\|^2_{1,\Omega}\Big).
\end{aligned} \tag{62}$$

And by the same process as in estimate (50), we also have the estimate of the second and the fourth terms of the right hand side of inequality (61)

$$\begin{aligned}
&\int_{\Gamma_C} \Big(\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\Big) \boldsymbol{\mathcal{R}}^h\left(\boldsymbol{\mathcal{P}}^h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h)\right) \cdot \boldsymbol{n}\, ds \\
&\qquad + \int_{\Gamma_C} \left(\frac{1}{\varepsilon}\,[\boldsymbol{u}_{\varepsilon \boldsymbol{t}}]_{\varepsilon g} - \frac{1}{\varepsilon}\,[\boldsymbol{u}_{\varepsilon h \boldsymbol{t}}]_{\varepsilon g}\right) \cdot \boldsymbol{\mathcal{R}}^h\left(\boldsymbol{\mathcal{P}}^h(\boldsymbol{u}_\varepsilon - \boldsymbol{v}_h)\right)_{\boldsymbol{t}}\, ds \\
&\qquad \leq \frac{1}{\alpha}\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_{\varepsilon h}\|^2_{1,\Omega} + \frac{\alpha(cM)^2}{2}\Big(\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|^2_{1,\Omega} + \|\boldsymbol{u} - \boldsymbol{v}_h\|^2_{1,\Omega}\Big).
\end{aligned} \tag{63}$$

Finally, the sum $T_2 + T_4$ is estimated as follows

$$T_2 + T_4 \leq \frac{1}{\alpha}\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_{\varepsilon h}\|^2_{1,\Omega} + c\left(1 + \frac{\alpha M^2}{2}\right)\left(\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|^2_{1,\Omega} + \|\boldsymbol{u} - \boldsymbol{v}_h\|^2_{1,\Omega}\right) + ch\left(\left\|\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right\|^2_{0,\Gamma_C} + \left\|\frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon \boldsymbol{t}}\right]_{\varepsilon g} - \frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon h_{\boldsymbol{t}}}\right]_{\varepsilon g}\right\|^2_{0,\Gamma_C}\right). \tag{64}$$

Rewriting the estimate (57) by taking into account the inequalities (59), (60) and (64), we establish the desired result from the following

$$\begin{aligned}\left(m - \frac{3}{2\alpha}\right)\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_{\varepsilon h}\|^2_{1,\Omega} + \varepsilon\left(\left\|\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right\|^2_{0,\Gamma_C} + \left\|\frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon \boldsymbol{t}}\right]_{\varepsilon g} - \frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon h_{\boldsymbol{t}}}\right]_{\varepsilon g}\right\|^2_{0,\Gamma_C}\right) \\ \leq ch\left(\left\|\frac{1}{\varepsilon}[\boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} - \ell]^+ - \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right\|^2_{0,\Gamma_C} + \left\|\frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon \boldsymbol{t}}\right]_{\varepsilon g} - \frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon h_{\boldsymbol{t}}}\right]_{\varepsilon g}\right\|^2_{0,\Gamma_C}\right) \\ + c\left(1 + \frac{3\alpha M^2}{2}\right)\left(\|\boldsymbol{u} - \boldsymbol{v}_h\|^2_{1,\Omega} + \|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|^2_{1,\Omega}\right),\end{aligned}$$

for $\alpha > \frac{3}{2m}$, where $c > 0$ is independent of $\boldsymbol{u}$, the penalty parameter ε and the mesh size h. $\square$

Thus, we now focus on the convergence of the approximation solution $\left(\boldsymbol{u}_{\varepsilon h},\ \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+,\ \frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon h_{\boldsymbol{t}}}\right]_{\varepsilon g}\right)$ towards the solution $(\boldsymbol{u},\ \boldsymbol{\sigma}_n(\boldsymbol{u}),\ \boldsymbol{\sigma}_{\boldsymbol{t}}(\boldsymbol{u}))$ by combining the Theorem 2 and the Theorem 5.

Theorem 6. *Under assumptions of Theorem 5, if the solution $\boldsymbol{u}$ of Problem (27) belongs to $\boldsymbol{H}^{\frac{3}{2}+\nu}(\Omega)$ with $0 < \nu \leq \frac{1}{2}$, then for $\varepsilon > 0$ and $h > 0$, the following estimate holds*

$$\begin{aligned}\|\boldsymbol{u} - \boldsymbol{u}_{\varepsilon h}\|_{1,\Omega} + \left(\varepsilon^{\frac{1}{2}} - ch^{\frac{1}{2}}\right)\left(\left\|\boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right\|_{0,\Gamma_C} + \left\|\boldsymbol{\sigma}_{\boldsymbol{t}}(\boldsymbol{u}) + \frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon h_{\boldsymbol{t}}}\right]_{\varepsilon g}\right\|_{0,\Gamma_C}\right) \\ \leq c\left(h^{\frac{1}{2}+\nu} + \varepsilon^{\frac{1}{2}+\nu}\right)\|\boldsymbol{u}\|_{\frac{3}{2}+\nu,\Omega},\end{aligned}$$

where $c > 0$ is a constant independent of $\boldsymbol{u}$, the penalty parameter ε and the mesh size h.

Proof. We establish the Theorem 6 from the Theorem 5, the Theorem 2, and by taking into account the interpolation estimate (54) and the triangle inequality. $\square$

Remark 6. A choice of the penalty parameter as a function of the size of the mesh is necessary in order to obtain the induced convergence rate in the Theorem 6.

- If ε behaves like the mesh size h, that is $\varepsilon(h) := (c+1)^2 h$, we obtain the a priori estimate

$$\|\boldsymbol{u} - \boldsymbol{u}_{\varepsilon h}\|_{1,\Omega} + h^{\frac{1}{2}}\left(\left\|\boldsymbol{\sigma}_n(\boldsymbol{u}) + \frac{1}{\varepsilon}[\boldsymbol{u}_{\varepsilon h} \cdot \boldsymbol{n} - \ell]^+\right\|_{0,\Gamma_C} + \left\|\boldsymbol{\sigma}_{\boldsymbol{t}}(\boldsymbol{u}) + \frac{1}{\varepsilon}\left[\boldsymbol{u}_{\varepsilon h_{\boldsymbol{t}}}\right]_{\varepsilon g}\right\|_{0,\Gamma_C}\right) \leq ch^{\frac{1}{2}+\nu}\|\boldsymbol{u}\|_{\frac{3}{2}+\nu,\Omega}.$$

Thus the $\mathcal{O}\left(h^{\frac{1}{2}+\nu}\right)$ convergence rate is obtained under the regularity $\boldsymbol{H}^{\frac{3}{2}+\nu}(\Omega)$, $0 < \nu \leq \frac{1}{2}$. In particular if $\nu = \frac{1}{2}$, we obtain an optimal linear error estimate.

- Otherwise a sub-optimal choice of the penalty parameter, namely $\varepsilon = c^2 h^\theta$ with $0 < \theta < 1$, provides the sub-optimal convergence rate of $\mathcal{O}\left(h^{\theta(\frac{1}{2}+\nu)}\right), 0 < \nu \leq \frac{1}{2}$.
- If the continuous estimate (29) remains true for a more regular solution, that is for $\boldsymbol{u} \in \boldsymbol{H}^{\frac{3}{2}+\nu}(\Omega)$ with $0 < \nu \leq \frac{3}{2}$, then the result obtained in Theorem 6 (as well as the Theorem 5) would also be valid and optimal for quadratic finite elements (since the standard Lagrange interpolation estimate (54) remains valid).

5. General conclusion

Through this work, we have analyzed the application of the penalty method to the unilateral contact problem with or without Tresca friction. The approach developed here, focusing on the finite element approximation of the continuous penalized problems (19) and (28), is the main difference between this work and most of works on the subject. This approach has spared us an intrinsic difficulty of the unilateral contact problem, that is to have to estimate the contact term in (15) when considering the direct finite element approximation of the variational inequalities (18) or (27).

We establish general a priori estimates depending on the penalty parameter ε and the mesh size h. A similar behavior of these two parameters, that is to say $\varepsilon = ch$, provides an optimal estimation of the unilateral contact problem with the following requirement $\varepsilon > h$.

We finally note that the theorems established throughout the paper (mainly Theorem 3 and Theorem 5) remain valid for quadratic finis elements, and a generalization of results (20) and (29) for $0 < \nu \leq \frac{3}{2}$ are sufficient to obtain optimal results.

References

[1] R.A. Adams, Sobolev Spaces, Pure Appl. Math., vol. 65, Academic Press, New York, London, 1973.
[2] C. Bernardi, Y. Maday, A.T. Patera, A new nonconforming approach to domain decomposition: the mortar element method, in: Nonlinear Partial Differential Equations and Their Applications, in: Collège de France Seminar, vol. XI, 1994, pp. 13–51.
[3] P.E. Bjorstad, O.B. Widlund, Iterative methods for the solution of elliptic problems on regions partitioned into substructures, SIAM J. Numer. Anal. 23 (1986) 1097–1120.
[4] J.H. Bramble, J.E. Pasciak, O. Steinbach, On the stability of the L^2 projection in $H^1(\Omega)$, Math. Comp. 71 (2001) 147–156.
[5] H. Brezis, Équations et inéquations non linéaires dans les espaces vectoriels en dualité, Ann. Inst. Fourier (Grenoble) 18 (1968) 115–175.
[6] F. Chouly, P. Hild, On convergence of the penalty method for unilateral contact problems, Appl. Numer. Math. 65 (2013) 27–40.
[7] F. Chouly, P. Hild, A Nitsche-based method for unilateral contact problems: numerical analysis, SIAM J. Numer. Anal. 51 (2013) 1295–1307.
[8] F. Chouly, P. Hild, Y. Renard, Symmetric and non-symmetric variants of Nitsche's method for contact problems in elasticity: theory and numerical experiments, Math. Comp. 84 (2014) 1089–1112.
[9] V. Dominguez, F.J. Sayas, Stability of discrete liftings, C. R. Math. Acad. Sci. Paris (2003) 805–808.
[10] G. Drouet, P. Hild, Optimal convergence for discrete variational inequality modelling Signorini contact in 2D and 3D without additional assumptions on the unknown contact set, SIAM J. Numer. Anal. 53 (2015) 1488–1507.
[11] A. Ern, J.L. Guermond, Theory and Practice of Finite Elements, Applied Mathematical Science, vol. 159, Springer-Verlag, New York, 2004.
[12] N. Kikuchi, J.T. Oden, Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods, Society for Industrial and Applied Mathematics, 1988.
[13] N. Kikuchi, Y.J. Song, Penalty-finite element approximation of a class of unilateral problems in linear elasticity, Quart. Appl. Math. XXXIX (1981) 1.
[14] Z.C. Li, Combined Methods for Elliptic Equations with Singularities, Interfaces and Infinities, Kluwer Academic Publishers, 1998.
[15] J.-L. Lions, G. Stampacchia, Variational inequalities, Comm. Pure Appl. Math. XX (1967) 493–519.

Please cite this article in press as: I. Dione, Optimal convergence analysis of the unilateral contact problem with and without Tresca friction conditions by the penalty method, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2018.11.023