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Spectral properties of self-similar measures with product-form digit sets ☆

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ABSTRACT

In this paper, we consider the spectral properties of self-similar measures $\mu_{R,D}$ generated by the integer $R = N^q$ and the product-form digit set $D = \{0, 1, \dots, N-1\} \oplus N^p\{0, 1, \dots, N-1\}$, where the integers $q, p \geq 1$ and $N \geq 2$. We show that $\mu_{R,D}$ is a spectral measure if and only if $q \nmid p$.

1. Introduction

Let μ be a Borel probability measure with compact support K in $\mathbb{R}^n$. If there exists a countable set Λ such that $E_\Lambda := \{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ is an orthonormal basis for $L^2(\mu)$, then we say that μ is a *spectral measure*, Λ is a *spectrum* of μ, and (μ, Λ) is a *spectral pair*. Moreover, if K has positive Lebesgue measure and μ is the Lebesgue measure on K, then we say that K is a *spectral set*. The existence of spectral measures is a basic and important problem in harmonic analysis, and it has a long history.

Recall that the compact set K is said to be a *translational tile* if there exists a countable subset $\mathcal{J} \subset \mathbb{R}^n$ such that $K + \mathcal{J} = \mathbb{R}^n$ and $\{K + j : j \in \mathcal{J}\}$ is a disjoint family up to sets of Lebesgue zero. There is a large literature concerning the translational tiles (see [13], [14], [23–27] and the references therein). In 1974, Fuglede [12] mentioned an interesting conjecture connecting the spectral sets with the tiles in his seminal paper.

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Conjecture 1.1. *Let K be a compact subset of $\mathbb{R}^n$ with positive Lebesgue measure. Then K is a spectral set if and only if K is a translation tile.*

Although the conjecture has been proved to be false in both directions in dimension 3 and higher dimensions [10,18,22], it is still suggestive in the research of spectral measure theory. In dimension 1 and 2, the conjecture is still open in two directions.

Let $\{\phi_d(x)\}_{d\in D}$ be *iterated function system* (IFS) [16] defined by

$$\phi_d(x) = R^{-1}(x+d), \quad x \in \mathbb{R}^n, \quad d \in D, \tag{1.1}$$

where R is an $n \times n$ expanding real matrix (all the eigenvalues of R have moduli > 1), and D is a finite subset of $\mathbb{R}^n$ with the cardinality $|D|$. Then it arises a unique nonempty compact set $T := T(R, D)$, called an *attractor*, and a Borel probability measure $\mu := \mu_{R,D}$, called a *self-affine measure* and supported on T, satisfying

$$T = \cup_{d\in D}\, \phi_d(T), \quad \mu(\cdot) = \frac{1}{|D|}\sum_{d\in D} \mu \circ \phi_d^{-1}(\cdot). \tag{1.2}$$

In 1998, Jorgensen and Pedersen [17] gave the first example of singularly spectral measure, and showed that the k-th Cantor measure is a spectral measure if k is even. This surprising discovery received a lot of attention. Later on, many researchers concentrated their work on fractal measures and the construction of their spectrums (see [1–9], [11], [15], [19–21]).

Recently, Dai, He and Lai [5] studied the self-similar measure generated by the integer R and the consecutive digit set $D = \{0, 1, \dots, N-1\}$, and showed that it is a spectral measure provided that N divides R. Fu, He and Lau [11] considered the spectral properties of the self-similar measure with the modulo product-form (or product-form) digit set on $\mathbb{R}$, and showed that the self-similar set is a spectral set under some assumptions.

In this paper, we also consider the measure $\mu_{R,D}$ generated by a product-form digit set, and get the following theorem.

Theorem 1.2. *Let $R = N^q$ and $D = \{0, 1, \dots, N-1\} \oplus N^p\{0, 1, \dots, N-1\}$, where the integers $q, p \geq 1$ and $N \geq 2$. Then the self-similar measure $\mu_{R,D}$ defined by (1.2) is a spectral measure if and only if $q \nmid p$.*

Theorem 1.2 gives a sufficient and necessary condition for the self-similar measure $\mu_{R,D}$ to be a spectral measure when R and D are of the special forms. Specially, when $p = 1$, $\mu_{R,D}$ is a N^2-Bernoulli measure. In this case, $\mu_{R,D}$ is a spectral measure if and only if $R = N^2 k$ for some $k \in \mathbb{Z}$ [6]. Moreover, the spectrality of such measures has been completely considered in [4] and [5] by Dai et al.

In the last of this paper, we consider the case $R = |D|$, i.e., $q = 2$. Using Theorem 2.1 in [23], we show that $T(N^2, D)$ is a translational tile if and only if $2 \nmid p$, i.e., p is an odd number (see Proposition 3.5). This implies that Fuglede's conjecture is true in the special case.

Corollary 1.3. *Let $R = N^2$ and $D = \{0, 1, \dots, N-1\} \oplus N^p\{0, 1, \dots, N-1\}$, where the integers $p \geq 1$ and $N \geq 2$. Then the self-similar set $T(R, D)$ defined by (1.2) is a spectral set if and only if it is a translational tile.*

We arrange this paper as follows. In Section 2, we will recall some basic concepts, and give several propositions and lemmas, which we will need to prove our main results. In Section 3, we will give the proof of Theorem 1.2 and Corollary 1.3 in detail.

2. Preliminaries

We define the Fourier transform of a probability measure μ in $\mathbb{R}^n$ by

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{2\pi i\langle x,\, \xi\rangle} d\mu(x), \quad \xi \in \mathbb{R}^n.$$

Let ϕ_d, $\mu_{R,D}$ be defined by (1.1), (1.2) respectively. Then it follows from [9] or (1.2) that

$$\widehat{\mu}_{R,D}(\xi) = \prod_{n=1}^{\infty} m_D(R^{*-n}\xi), \quad \xi \in \mathbb{R}^n, \tag{2.1}$$

where R^* denotes the transposed conjugate of R, and

$$m_D(x) = \frac{1}{|D|}\sum_{d\in D} e^{2\pi i\langle d,x\rangle}, \quad x \in \mathbb{R}^n.$$

It follows from (2.1) that

$$\mathcal{Z}(\widehat{\mu}_{R,D}) = \cup_{n=1}^{\infty} R^{*n}(\mathcal{Z}(m_D)), \tag{2.2}$$

where $\mathcal{Z}(f)$ denotes the set of all the zeros of the function f. For $\lambda_1, \lambda_2 \in \mathbb{R}^n$,

$$\langle e^{2\pi i\langle\lambda_1,x\rangle}, e^{2\pi i\langle\lambda_2,x\rangle}\rangle_{L^2(\mu_{R,D})} = \int_{\mathbb{R}^n} e^{2\pi i\langle\lambda_1-\lambda_2,x\rangle} d\mu_{R,D}(x) = \widehat{\mu}_{R,D}(\lambda_1 - \lambda_2).$$

It is easy to see that for a countable subset $\Lambda \subset \mathbb{R}^n$, E_Λ is an orthonormal set in $L^2(\mu_{R,D})$ if and only if

$$(\Lambda - \Lambda) \setminus \{0\} \subset \mathcal{Z}(\widehat{\mu}_{R,D}). \tag{2.3}$$

Now, we consider the case in $\mathbb{R}$. Assume that the integers $p, q, L \geq 1$ and $N \geq 2$. When there is no confusion, we will take the same assumptions for p, q, L, N in this paper. Let the digit set

$$D = \{0, 1, \ldots, N-1\} \oplus N^pL\{0, 1, \ldots, N-1\}. \tag{2.4}$$

Then we can calculate the zeros of $m_D(x)$. Let $D_1 = \{0, 1, \ldots, N-1\}$, and let $D_2 = MD_1$ with $M = N^pL$. From $D = D_1 \oplus D_2$, it follows that

$$m_D(x) = m_{D_1}(x)m_{D_2}(x) = m_{D_1}(x)m_{D_1}(Mx).$$

Note that

$$m_{D_1}(x) = \frac{1}{N}\sum_{n=0}^{N-1} e^{2\pi inx} = \frac{1-e^{2\pi iNx}}{N(1-e^{2\pi ix})}, \quad x \notin \mathbb{N}.$$

It is easy to see that $\mathcal{Z}(m_{D_1}) = \{\frac{k}{N} : N \nmid k\}$. Since

$$\mathcal{Z}(m_D) = \mathcal{Z}(m_{D_1}) \cup \frac{1}{M}\mathcal{Z}(m_{D_1}),$$

we can get

$$\mathcal{Z}(m_D) = \{\frac{k}{N} : N \nmid k\} \cup \{\frac{k}{N^{p+1}L} : N \nmid k\}. \tag{2.5}$$

Let

$$\mathcal{C} := \frac{1}{N^{p+1}} \left(\{-1, 0, 1, \ldots, N-2\} \oplus N^p\{0, 1, \ldots, N-1\}\right). \tag{2.6}$$

Then the following proposition tells us that there exists a distance between the attractor $T(N^q, \mathcal{C})$ and the zeros of m_D.

Proposition 2.1. *Let D be defined by (2.4) with $L = 1$. If $q \nmid p$, then $m_D(x) \neq 0$ for each $x \in T(N^q, \mathcal{C})$.*

Proof. Let $C = C_1 \oplus N^p C_2$, where $C_1 = \{-1, 0, 1, \ldots, N-2\}$ and $C_2 = \{0, 1, \ldots, N-1\}$. The assumption $q \nmid p$ implies that there exist $m \in \mathbb{N}$ and $r \in [1, q-1] \cap \mathbb{N}$ such that $p = mq + r$. Then it follows from the left equality in (1.2) that

$$\begin{aligned} T(N^q, C) &= \sum_{n=1}^{\infty} N^{-qn} C = \sum_{n=1}^{\infty} N^{-qn} C_1 + \sum_{n=1}^{\infty} N^{p-qn} C_2 \\ &= \sum_{n=1}^{\infty} N^{-qn} C_1 + \sum_{j=1}^{m} N^{q(m-j)+r} C_2 + \sum_{j=m+1}^{\infty} N^{q(m-j)+r} C_2 \\ &= \sum_{j=0}^{m-1} N^{qj+r} C_2 + \sum_{n=1}^{\infty} N^{-qn}(C_1 + N^r C_2) := A + B, \end{aligned} \tag{2.7}$$

where we define $A = \emptyset$ if $m = 0$. Making use of the definition of C_1 and C_2, and noting that the integers $N, q \geq 2$, we can check that

$$\inf B = \sum_{n=1}^{\infty} \frac{-1}{N^{qn}} = \frac{-1}{N^q - 1} > -1,$$

$$\sup B \leq \sum_{n=1}^{\infty} \left(\frac{N-2}{N^{qn}} + \frac{N^{q-1}(N-1)}{N^{qn}}\right) = \frac{(N-1)N^{q-1} + N - 2}{N^q - 1} < 1.$$

This implies that the set B lies in the interval $(-1, 1)$.

Next, we will prove this proposition by contradiction. Suppose there exists $\xi_0 \in T(N^q, \mathcal{C})$ such that $m_D(\xi_0) = 0$. From $N^{p+1}T(N^q, \mathcal{C}) = T(N^q, C)$, it follows that $N^{p+1}\xi_0 \in T(N^q, C)$. Moreover, using (2.5), we can find an integer k with $N \nmid k$ such that $\xi_0 = \frac{k}{N}$ or $\frac{k}{N^{p+1}}$.

Case 1: $\xi_0 = \frac{k}{N}$. It follows from (2.7) that there exist $\xi \in B$ and a finite sequence $\{c_j\}_{j=0}^{m-1} \subset C_2$ such that

$$kN^p = N^{p+1}\xi_0 = \xi + \sum_{j=0}^{m-1} N^{qj+r} c_j.$$

Note that $N^{p+1}\xi_0, \sum_{j=0}^{m-1} N^{qj+r} c_j \in \mathbb{Z}$ and $\xi \in (-1, 1)$. Therefore the above equality holds if and only if $\xi = 0$. Since $p = mq + r$ with $1 \leq r \leq q - 1$, we have

$$k = \sum_{j=0}^{m-1} N^{(j-m)q} c_j \in [0, \frac{N-1}{N^q - 1}] \subseteq [0, 1).$$

This implies that $k = 0$, which contradicts with $N \nmid k$.

Case 2: $\xi_0 = \frac{k}{N^{p+1}}$. Similarly, it follows from (2.7) that there exist $\xi' \in B$ and a finite sequence $\{c'_j\}_{j=0}^{m-1} \subset C_2$ such that

$$k = N^{p+1}\xi_0 = \xi' + \sum_{j=0}^{m-1} N^{qj+r} c'_j.$$

By the similar way, we have $\xi' \in \mathbb{N}$, which implies that $\xi' = 0$. Hence it follows from the above equality that

$$k = \sum_{j=0}^{m-1} N^{qj+r} c'_j.$$

This contradicts with $N \nmid k$, since the integer $r \in [1, q-1]$. $\square$

Noting that $T(N^q, \mathcal{C})$ is a compact set, and making using of Proposition 2.1 and the finite covering theorem, we can find a small positive real number η such that $m_D(x) \neq 0$ for each x lying in $T_\eta := \{x : \operatorname{dist}(x, T(N^q, \mathcal{C})) \leq \eta\}$. Define

$$\beta := \inf\{|m_D(x)| : x \in T_\eta\}.$$

Then we can conclude that $\beta > 0$, since T_η is compact and $|m_D(x)|$ is continuous. For the Fourier transform $\widehat{\mu}_{N^q,D}(\xi)$, we have the following proposition.

Proposition 2.2. *Let D be defined by (2.4) with $L = 1$. If $q \nmid p$, then there exists $c > 0$ such that $|\widehat{\mu}_{N^q,D}(\xi)| \geq c$ for each $\xi \in T_\eta$.*

Proof. We can claim that $|m_D(N^{-nq}\xi)| \geq \beta$ for each positive integer n if ξ lies in T_η. In fact, if $\xi \in T_\eta$, then there exists $\xi_0 \in T(N^q, \mathcal{C})$ such that $\operatorname{dist}(\xi, \xi_0) \leq \eta$. Using the left equality in (1.2), and noting that $0 \in \mathcal{C}$, we can obtain $N^{-q}T(N^q, \mathcal{C}) \subset T(N^q, \mathcal{C})$. Therefore it follows that $N^{-q}\xi_0 \in T(N^q, \mathcal{C})$, which implies that

$$\operatorname{dist}(N^{-q}\xi, T(N^q, \mathcal{C})) \leq \operatorname{dist}(N^{-q}\xi, N^{-q}\xi_0) \leq N^{-q}\eta \leq \eta.$$

Hence we can obtain that $N^{-nq}\xi$ lies in T_η for each positive integer n by induction. By the definition of β, we complete the proof of our claim.

Since $|\widehat{\mu}_{N^q,D}(0)| = 1$ and $\widehat{\mu}_{N^q,D}(\xi)$ is continuous in $\mathbb{R}$, there exists $\rho > 0$ such that $|\widehat{\mu}_{N^q,D}(\xi)| \geq 1/2$ for each point in the closed disk $|\xi| \leq \rho$. Let

$$M = \sup\{|y| : y \in T_\eta\}.$$

Since T_η is a compact set, M is finite. Let n_0 be the least integer such that $N^{-n_0 q}M \leq \rho$. Therefore we can obtain that $|\widehat{\mu}_{N^q,D}(N^{-n_0 q}\xi)| \geq 1/2$, since $N^{-n_0 q}|\xi| \leq \rho$ for every point ξ contained in T_η. Thus it follows from (2.1) that

$$\begin{aligned} |\widehat{\mu}_{N^q,D}(\xi)| &= |\prod_{n=1}^{\infty} m_D(N^{-nq}\xi)| = |\prod_{n=1}^{n_0} m_D(N^{-nq}\xi)||\prod_{n=n_0+1}^{\infty} m_D(N^{-nq}\xi)| \\ &= |\hat{\mu}_{N^q,D}(N^{-n_0 q}\xi)| \prod_{n=1}^{n_0} |m_D(N^{-nq}\xi)| \geq \frac{\beta^{n_0}}{2}. \end{aligned}$$

Hence we complete the proof by choosing $c = \beta^{n_0}/2$. $\square$

Next, we will introduce two useful lemmas. Let Λ be a countable subset of $\mathbb{R}$, and let

$$Q_\Lambda(\xi) = \sum_{\lambda\in\Lambda} |\widehat{\mu}(\xi+\lambda)|, \quad \xi \in \mathbb{R}.$$

Then the following well-known lemma gives an important criterion for a measure μ to be a spectral measure.

Lemma 2.3. *[17] Let μ be a Borel probability measure with compact support in $\mathbb{R}$. Then*
(i) E_Λ is an orthonormal set in $L^2(\mu)$ if and only if $Q_\Lambda(\xi) \le 1$ for each $\xi \in \mathbb{R}$;
(ii) E_Λ is an orthonormal basis of $L^2(\mu)$ if and only if $Q_\Lambda(\xi) = 1$ for each $\xi \in \mathbb{R}$;
(iii) $Q_\Lambda(\xi)$ is an entire function if E_Λ is an orthonormal set in $L^2(\mu)$.

Using Lemma 2.3, we can obtain the following lemma, which can be used to show that a measure is a non-spectral measure.

Lemma 2.4. *[6] Let $\mu = \mu_1 * \mu_2$ be the convolution of two probability measures μ_i, $i = 1, 2$, and they are not Dirac measures. If E_Λ is an orthonormal set in $L^2(\mu_1)$, then E_Λ is also an orthonormal set in $L^2(\mu)$, but Λ cannot be a spectrum of μ.*

In the last, we will end this section by introducing something useful about the translational tile. Recall that D is said to be *a tile digit set* if $T(R, D)$ is a translational tile. It is a challenge to characterize the structure of all tile digit sets for a given matrix. In the following, we will introduce a criterion for it.

Lemma 2.5. *[23] For the expanding matrix $R \in M_n(\mathbb{Z})$ and the digit set D with $|D| = |\det(R)|$, the following three conditions are equivalent:*
(i) D is a tile digit set of R;
(ii) for each $k \ge 1$, the set $\sum_{n=1}^{k} R^n D$ contains $|\det(R)|^k$ distinct elements;
(iii) for each $s \in \mathbb{Z}^n \setminus \{0\}$, there exists a nonnegative integer $k = k(s)$ such that s is the zero of the function

$$h_k(x) := \frac{1}{|D|} \sum_{d\in D} e^{2\pi i<R^{-k}d,x>}. \tag{2.8}$$

3. Proof of the main results

In the first, we will prove the necessity of Theorem 1.2 in more general case.

Theorem 3.1. *Let D be defined by (2.4) with $N \nmid L$. If $q \mid p$, then $\mu_{N^q,D}$ is a non-spectral measure.*

Proof. It follows from the assumption that there exists a positive integer s such that $p = sq$. Divide the digit D into $D_1 \oplus D_2$, where $D_1 = \{0, 1, \dots, N-1\}$ and $D_2 = N^p L D_1$. Then it follows from (2.2) and (2.5) that

$$\mathcal{Z}(\widehat{\mu}_{N^q,D_1}) = \cup_{n=1}^{\infty} N^{nq}\{\frac{k}{N} : k \in \mathbb{Z}, N \nmid k\}, \quad \mathcal{Z}(\widehat{\mu}_{N^q,D_2}) = \cup_{n=1}^{\infty} N^{(n-s)q}\{\frac{k}{NL} : k \in \mathbb{Z}, N \nmid k\}.$$

Note that $\mu_{N^q,D} = \mu_{N^q,D_1} * \mu_{N^q,D_2}$, which implies that $\mathcal{Z}(\widehat{\mu}_{N^q,D}) = \mathcal{Z}(\widehat{\mu}_{N^q,D_1}) \cup \mathcal{Z}(\widehat{\mu}_{N^q,D_2})$. Then we will divide the proof into two cases.

Case 1: $\gcd(N, L) = 1$. For each point x lying in $\mathcal{Z}(\widehat{\mu}_{N^q,D_1})$, there exist a positive integer n and an integer k such that $x = N^{nq} \cdot \frac{k}{N}$ and $N \nmid k$. From $\gcd(N, L) = 1$, it follows that $N \nmid kL$. Thus we see that $x = N^{nq} \cdot \frac{kL}{NL} \in \mathcal{Z}(\widehat{\mu}_{N^q,D_2})$. This implies that $\mathcal{Z}(\widehat{\mu}_{N^q,D_1}) \subset \mathcal{Z}(\widehat{\mu}_{N^q,D_2})$, and hence $\mathcal{Z}(\widehat{\mu}_{N^q,D}) = \mathcal{Z}(\widehat{\mu}_{N^q,D_2})$.

Therefore it follows from (2.3) that E_Λ is an orthonormal set in $L^2(\mu_{N^q,D})$ if and only if E_Λ is an orthonormal set in $L^2(\mu_{N^q,D_2})$. Then $\mu_{N^q,D}$ is a non-spectral measure by Lemma 2.4.

Case 2: $\gcd(N,L)=d>1$. Let $N=N'd$ and $L=L'd$, where $\gcd(N',L')=1$. Let

$$D_1=\{0,1,\dots,d-1\}\oplus d\{0,1,\dots,N'-1\}:=D_1^1\oplus D_1^2.$$

Then it is easy to see that

$$\mathcal{Z}(\widehat{\mu}_{N^q,D_1^1})=\cup_{n=1}^{\infty}N^{nq}\{\frac{k}{d}:k\in\mathbb{Z},d\nmid k\},\quad \mathcal{Z}(\widehat{\mu}_{N^q,D_1^2})=\cup_{n=1}^{\infty}N^{nq}\{\frac{k}{dN'}:k\in\mathbb{Z},N'\nmid k\}.$$

For every point x contained in $\mathcal{Z}(\widehat{\mu}_{N^q,D_1^2})$, there exist a positive integer n and an integer k such that $x=N^{nq}\cdot\frac{k}{dN'}$ and $N'\nmid k$. Hence it follows from $\gcd(N',L')=1$ that $N'\nmid kL'$ and $N\nmid kL$. This implies that $x=N^{nq}\cdot\frac{kL}{NL}\in\mathcal{Z}(\widehat{\mu}_{N^q,D_2})$. Hence we obtain that

$$\mathcal{Z}(\widehat{\mu}_{N^q,D_1^2})\subset\mathcal{Z}(\widehat{\mu}_{N^q,D_2}).$$

Let $\mu_1=\mu_{N^q,D_1^2}$, and let $\mu_2=\mu_{N^q,D_1^1}*\mu_{N^q,D_2}$. Then it is easy to check that $\mu_{N^q,D}=\mu_1*\mu_2$ and $\mathcal{Z}(\widehat{\mu}_1)\subset\mathcal{Z}(\widehat{\mu}_2)$. This implies that $\mathcal{Z}(\widehat{\mu}_{N^q,D})=\mathcal{Z}(\widehat{\mu}_2)$. Using Lemma 2.4 again, we can show that $\mu_{N^q,D}$ is also a non-spectral measure in this case. $\square$

In the second, we will prove the sufficiency of Theorem 1.2. In the following, we assume that $q\nmid p$. Let D be defined by (2.4) with $L=1$, and let $\mu_n=\delta_{N^{-q}D}*\cdots*\delta_{N^{-nq}D}$. Then μ_n converges weakly to $\mu_{N^q,D}$, and it follows that

$$\widehat{\mu}_n(\xi)=\prod_{k=1}^{n}m_D(N^{-kq}\xi),\quad \xi\in\mathbb{R}.$$

Let $\mathcal{C}$ be defined by (2.6), and define

$$\Lambda_n=N^q\mathcal{C}+N^{2q}\mathcal{C}+\cdots+N^{nq}\mathcal{C},\quad \Lambda=\cup_{n=1}^{\infty}\Lambda_n. \tag{3.1}$$

Theorem 3.2. *Let μ_n and Λ_n are defined in the above. Then Λ_n is a spectrum of μ_n if $q\nmid p$.*

Proof. Let $T_n=\sum_{k=1}^{n}N^{-kq}D$. Then the finite set T_n is the support of the measure μ_n, and hence the dimension of $L^2(\mu_n)$ equals to the cardinality of T_n. Moreover, we can claim that the cardinality of T_n equals to the cardinality of Λ_n. In fact, we can find a bijective mapping f from T_n to Λ_n. For each given point x lying in T_n, there exist $d_1,\dots,d_n\in D$ such that $x=\sum_{k=1}^{n}N^{-qk}d_k$. Then there exist $d_{k,1},d_{k,2}\in D_1:=\{0,1,\dots,N-1\}$ such that $d_k=d_{k,1}+N^pd_{k,2}$ for $1\le k\le n$. We define the value of the point x by

$$f(x)=\frac{1}{N^{p+1}}\sum_{k=1}^{n}N^{qk}(d_{n+1-k,1}-1+N^pd_{n+1-k,1}).$$

Then we will show that the map f is well defined. Let x' be contained in T_n, and hence there exist $d'_1,\dots,d'_n\in D$ such that $x'=\sum_{k=1}^{n}N^{-qk}d'_k$. Then there exist $d'_{k,1},d'_{k,2}\in D_1$ such that $d'_k=d'_{k,1}+N^pd'_{k,2}$ for $1\le k\le n$. Thus it follows that

$$\frac{N^{p+1}}{N^{(n+1)q}}(f(x)-f(x'))=\sum_{k=1}^{n}N^{-qk}(d_{k,1}+N^pd_{k,2})-\sum_{k=1}^{n}N^{-qk}(d'_{k,1}+N^pd'_{k,2})=x-x'.$$

This implies that f can be well defined in T_n, and f is injective. Noting that $N^{p+1}\mathcal{C} = (D_1 - 1) \oplus N^p D_1$, we can easily see that f is a bijective mapping from T_n to Λ_n.

Therefore we only need to prove that E_{Λ_n} is an orthogonal set of $L^2(\mu_n)$, i.e., $\Lambda_n - \Lambda_n \in \mathcal{Z}(\widehat{\mu}_n) \cup \{0\}$. Let σ, σ' be two different points lying in Λ_n. Then there exist $i_k, i'_k \in D_1 - 1$, $j_k, j'_k \in D_1$ for $1 \le k \le n$ such that

$$\sigma = \frac{1}{N^{p+1}} \sum_{k=1}^{n} N^{kq}(i_k + N^p j_k), \quad \sigma' = \frac{1}{N^{p+1}} \sum_{k=1}^{n} N^{kq}(i'_k + N^p j'_k).$$

Then it follows that

$$\sigma - \sigma' = \sum_{k=1}^{n} N^{kq} \frac{i_k - i'_k}{N^{p+1}} + \sum_{m=1}^{n} N^{mq} \frac{j_m - j'_m}{N}.$$

By calculations, we have

$$\mathcal{Z}(\widehat{\mu}_n) = \cup_{k=1}^{n} N^{qk}\{\frac{k}{N} : k \in \mathbb{Z}, N \nmid k\} \cup \cup_{k=1}^{n} N^{qk}\{\frac{k}{N^{p+1}} : k \in \mathbb{Z}, N \nmid k\}. \tag{3.2}$$

Then we can give the proof into three cases.

Case 1: $i_k = i'_k$ for all $1 \le k \le n$. Let $k_1 = \min\{k : j_k \ne j'_k, 1 \le k \le n\}$. Then k_1 can be well defined, and it follows that

$$\sigma - \sigma' = \sum_{k=k_1}^{n} N^{kq} \frac{j_k - j'_k}{N} = N^{k_1 q}\Big(\frac{j_{k_1} - j'_{k_1}}{N} + \sum_{k=k_1+1}^{n} N^{(k-k_1)q-1}(j_k - j'_k)\Big).$$

It is easy to check that $j_{k_1} - j'_{k_1} \notin N\mathbb{Z}$ and $N^{(k-k_1)q-1}(j_k - j'_k) \in \mathbb{Z}$ for $k \ge k_1 + 1$. Hence it follows from (3.2) that $\sigma - \sigma' \in \mathcal{Z}(\widehat{\mu}_n)$.

Case 2: $j_k = j'_k$ for all $1 \le k \le n$. Let $k_2 = \min\{k : i_k \ne i'_k, 1 \le k \le n\}$. Then k_2 can be well defined, and it follows that

$$\sigma - \sigma' = \sum_{k=k_2}^{n} N^{kq} \frac{i_k - i'_k}{N^{p+1}} = N^{k_2 q} \cdot N^{-(p+1)}\big(i_{k_2} - i'_{k_2} + \sum_{k=k_2+1}^{n} N^{(k-k_2)q}(i_k - i'_k)\big).$$

It is easy to check that $i_{k_2} - i'_{k_2} \notin N\mathbb{Z}$ and $N^{(k-k_2)q}(i_k - i'_k) \in N\mathbb{Z}$ for $k \ge k_2 + 1$. Hence it follows from (3.2) that $\sigma - \sigma' \in \mathcal{Z}(\widehat{\mu}_n)$.

Case 3: $i_k \ne i'_k$ and $j_l \ne j'_l$ for two integers $k, l \in [1, n]$. Let $k_1 = \min\{k : i_k \ne i'_k, 1 \le k \le n\}$, and $k_2 = \min\{k : j_k \ne j'_k, 1 \le k \le n\}$. Then it follows that

$$\sigma - \sigma' = \sum_{k=k_1}^{n} N^{kq} \frac{i_k - i'_k}{N^{p+1}} + \sum_{k=k_2}^{n} N^{kq} \frac{j_k - j'_k}{N}. \tag{3.3}$$

Let $p = mq + r$ with the integer $r \in [1, q-1]$. We will divide this case into two subcases.

Subcase I: $k_1 > m + k_2$. It follows from (3.3) that

$$\sigma - \sigma' = N^{k_2 q}\left(\frac{j_{k_2} - j'_{k_2}}{N} + \sum_{k=k_1}^{n} N^{(k-k_2-m)q-(r+1)}(i_k - i'_k) + \sum_{k=k_2+1}^{n} N^{(k-k_2)q-1}(j_k - j'_k)\right).$$

Noting that $j_{k_2} - j'_{k_2} \notin N\mathbb{Z}$, and using (3.2), we can obtain that $\sigma - \sigma' \in \mathcal{Z}(\widehat{\mu}_n)$.

Subcase II: $k_1 \le m + k_2$. It follows from (3.3) that

$$\sigma - \sigma' = N^{k_1 q} \cdot \frac{1}{N^{p+1}} \left(i_{k_1} - i'_{k_1} + \sum_{k=k_1+1}^{n} N^{(k-k_1)q}(i_k - i'_k) + \sum_{k=k_2}^{n} N^{(k-k_1)q+p}(j_k - j'_k) \right).$$

For $k \ge k_2$, it follows that

$$N^{(k-k_1)q+p} = N^{(k-k_1+m)q+r} = N^r \cdot N^{(m+k-k_1)q} \in N\mathbb{Z}.$$

Noting that $i_{k_1} - i'_{k_1} \notin N\mathbb{Z}$, and using (3.2), we conclude that $\sigma - \sigma' \in \mathcal{Z}(\widehat{\mu}_n)$. $\square$

In the similar way, it is easy to check that E_Λ is also an orthonormal set of $L^2(\mu_{N^q,D})$. Moreover, we will show that E_Λ is an orthonormal basis of $L^2(\mu_{N^q,D})$ in the following theorem.

Theorem 3.3. *Let D be defined by (2.4) with $L = 1$. If $q \nmid p$, then $\mu_{N^q,D}$ is a spectral measure.*

Proof. Let Λ_n and Λ be defined by (3.1). Then we will show that Λ is a spectral of $\mu_{N^q,D}$. Let

$$Q_n(\xi) = \sum_{\lambda\in\Lambda_n} |\widehat{\mu}_{N^q,D}(\xi + \lambda)|^2, \quad Q(\xi) = \sum_{\lambda\in\Lambda} |\widehat{\mu}_{N^q,D}(\xi + \lambda)|^2.$$

For every $n \ge 1$, making use of (2.1) and the definition of μ_n, we have the following equality:

$$\begin{aligned} Q_{2n}(\xi) &= Q_n(\xi) + \sum_{\lambda\in\Lambda_{2n}\setminus\Lambda_n} |\widehat{\mu}_{N^q,D}(\xi + \lambda)|^2 \\ &= Q_n(\xi) + \sum_{\lambda\in\Lambda_{2n}\setminus\Lambda_n} |\widehat{\mu}_{2n}(\xi + \lambda)|^2 |\widehat{\mu}_{N^q,D}(\frac{\xi + \lambda}{N^{2nq}})|^2. \end{aligned}$$

Next, we will prove that $Q(\xi) \equiv 1$. Since Q is an entire function, we only need to determine the value of $Q(\xi)$ for $|\xi| \le \eta$ (< 1), where η is the same as Proposition 2.2. Obviously, $N^{-2nq}\lambda \in T(N^q, \mathcal{C})$, if $\lambda \in \Lambda_{2n}$. This implies that for $|\xi| \le \eta$,

$$\operatorname{dist}(\frac{\xi + \lambda}{N^{2nq}}, T(N^q, \mathcal{C})) \le \frac{|\xi|}{N^{2nq}} \le \eta.$$

From Proposition 2.2 and Theorem 3.2, it follows that

$$\begin{aligned} Q_{2n}(\xi) &\ge Q_n(\xi) + c^2 \sum_{\lambda\in\Lambda_{2n}\setminus\Lambda_n} |\widehat{\mu}_{2n}(\xi + \lambda)|^2 \\ &= Q_n(\xi) + c^2(1 - \sum_{\lambda\in\Lambda_n} |\widehat{\mu}_{2n}(\xi + \lambda)|^2). \end{aligned} \tag{3.4}$$

For every point λ contained in Λ_n, it follows that

$$N^{-2nq}|\xi + \lambda| \le N^{-2nq} \left(1 + \frac{(N^p + 1)(N - 1)(N^q + N^{2q} + \cdots + N^{nq})}{N^{p+1}} \right) < N^{-(n-1)q}.$$

Define $l_n = \min_{|\xi|\le N^{-nq}} |\widehat{\mu}_{N^q,D}(\xi)|$ for $n \in \mathbb{N}$. Then it follows that $\lim_{n\to\infty} l_n = 1$. Hence we can obtain that

$$|\widehat{\mu}_{N^q,D}(\xi+\lambda)| = |\widehat{\mu}_{2n}(\xi+\lambda)||\widehat{\mu}_{N^q,D}(\frac{\xi+\lambda}{N^{2nq}})| \geq l_{n-1}|\widehat{\mu}_{2n}(\xi+\lambda)|. \tag{3.5}$$

By (3.4) and (3.5), we get

$$Q_{2n}(\xi) \geq Q_n(\xi) + c^2(1 - \frac{1}{l_{n-1}}Q_n(\xi)).$$

Letting $n \to \infty$, we can obtain that $Q(\xi) \geq Q(\xi)+c^2(1-Q(\xi))$. Since E_Λ is an orthonormal set of $L^2(\mu_{N^q,D})$, it holds that $Q(\xi) \leq 1$ for every point $\xi \in \mathbb{R}$. This implies that $Q(\xi) \equiv 1$ for $|\xi| \leq \eta$. Therefore we complete the proof of this theorem. □

Remark 3.4. Let $R = N^q$, and let D be defined by (2.4). If $\gcd(L, N) = 1$, we conjecture that Theorem 3.3 is also true. However, if $\gcd(L, N) > 1$, then $\mu_{R,D}$ maybe a non-spectral measure even though $q \nmid p$. For example, let $N = 4$, $p = 5$ and $q = L = 2$. Although $2 \nmid 5$, we can claim that the measure $\mu_{N^q,D}$ is a non-spectral measure. In fact, D and R can be rewritten as $D = \{0, 1\} \oplus 2\{0, 1\} \oplus 2^{11}\{0, 1\} \oplus 2^{12}\{0, 1\}$ and $R = 2^4$. Hence it follows that

$$\mu_{R,D} = \mu_{2^4,\{0,1\}} * \mu_{2^4,2\{0,1\}} * \mu_{2^4,2^{11}\{0,1\}} * \mu_{2^4,2^{12}\{0,1\}} := \mu_1 * \mu_2 * \mu_3 * \mu_4.$$

By (2.5), we have

$$\mathcal{Z}(\widehat{\mu}_1) = \cup_{n=1}^{\infty} 2^{4n}\{\frac{k}{2} : k \in 2\mathbb{Z}+1\}, \quad \mathcal{Z}(\widehat{\mu}_4) = \cup_{n=1}^{\infty} 2^{4(n-3)}\{\frac{k}{2} : k \in 2\mathbb{Z}+1\}.$$

Then it is easy to check that $\mathcal{Z}(\widehat{\mu}_1) \subset \mathcal{Z}(\widehat{\mu}_4) \subset \mathcal{Z}(\widehat{\mu}_5)$, where $\mu_5 := \mu_2 * \mu_3 * \mu_4$. Using $\widehat{\mu_1 * \mu_5} = \widehat{\mu}_1 \cdot \widehat{\mu}_5$, we can see that $\mathcal{Z}(\widehat{\mu}_{R,D}) = \mathcal{Z}(\widehat{\mu}_5)$. Let E_Λ be an orthonormal set in $L^2(\mu_{R,D})$. Then it follows from (2.3) that E_Λ must be an orthonormal set in $L^2(\mu_5)$. Noting that $\mu_{R,D} = \mu_1 * \mu_5$, and making use of Lemma 2.4, Λ cannot be a spectrum of $\mu_{R,D}$, i.e., E_Λ cannot be an orthonormal basis of $L^2(\mu_{R,D})$. Since E_Λ is arbitrary, our claim is true.

In the last, we will prove the following proposition to complete the proof of Corollary 1.3.

Proposition 3.5. *Let $R = N^2$, and let D be defined by (2.4) with $\gcd(L, N) = 1$. Then $T(R, D)$ is a translational tile if and only if $2 \nmid p$.*

Proof. We will use Lemma 2.5 to prove this proposition. Firstly, we will prove the sufficiency. Assume that $2 \nmid p$. By (2.8), we only need to show that for any $s \in \mathbb{Z} \setminus \{0\}$, there exists an integer k such that $h_k(s) := m_D(\frac{s}{N^{2k}}) = 0$, i.e.,

$$\frac{s}{N^{2k}} \in \mathcal{Z}(m_D) := \{\frac{m}{N} : N \nmid m\} \cup \{\frac{m}{N^{p+1}L} : N \nmid m\}.$$

In fact, there exist nonnegative integers n and s' such that $s = N^n s'$ with $N \nmid s'$. If $n \in 2\mathbb{N}+1$, we can choose $k = \frac{n+1}{2}$. Then it follows that $\frac{s}{N^{2k}} = \frac{s'}{N} \in \mathcal{Z}(m_D)$. If $n \in 2\mathbb{N}$, we can choose $k = \frac{n+p+1}{2}$. Noting that $2 \nmid p$ and $\gcd(L, N) = 1$, we obtain that $\frac{s}{N^{2k}} = \frac{s'L}{N^{p+1}L} \in \mathcal{Z}(m_D)$. Therefore we complete the proof the sufficiency.

Secondly, we will prove the necessity. Assume that $2 \mid p$. Let $s_0 = N^2$. Then we only need to prove that $\frac{s_0}{N^{2k}}$ is not contained in $\mathcal{Z}(m_D)$ for each integer $k \geq 1$. Otherwise, there exist integers $k_0 \geq 1$ and m with $N \nmid m$ such that

$$\frac{s_0}{N^{2k_0}} = \frac{m}{N} \quad \text{or} \quad \frac{s_0}{N^{2k_0}} = \frac{m}{N^{p+1}L}.$$

Obviously, the above two equalities can not hold since $p \in 2\mathbb{Z}$. This shows that D is not a tile digit set of R if $2 \mid p$. $\square$

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References

[1] L.X. An, X.G. He, A class of spectral Moran measures, J. Funct. Anal. 266 (2014) 343–354.
[2] L.X. An, X.G. He, K.S. Lau, Spectrality of a class of infinite convolutions, Adv. Math. 283 (2015) 362–376.
[3] X.R. Dai, When does a Bernoulli convolution admit a spectrum? Adv. Math. 231 (2012) 1681–1693.
[4] X.R. Dai, Spectra of Cantor measures, Math. Ann. 366 (2016) 1621–1647.
[5] X.R. Dai, X.G. He, C.K. Lai, Spectral structure of Cantor measures with consecutive digits, Adv. Math. 242 (2013) 187–208.
[6] X.R. Dai, X.G. He, K.S. Lau, On spectral N-Bernoulli measures, Adv. Math. 259 (2014) 511–531.
[7] Q.R. Deng, Sierpinski-type spectral self-similar measures, J. Funct. Anal. 270 (2016) 4426–4442.
[8] D. Dutkay, J. Haussermann, C.K. Lai, Hadamard triples generate self-affine spectral measure, arXiv:1506.01503 [.org/pdf].
[9] D. Dutkay, P. Jorgensen, Analysis of orthogonality and of orbits in affine iterated function systems, Math. Z. 256 (2007) 801–823.
[10] B. Farkas, M. Matolcsi, P. Mora, On Fuglede's conjecture and the existence of universal spectra, J. Fourier Anal. Appl. 12 (2006) 483–494.
[11] X.Y. Fu, X.G. He, K.S. Lau, Spectrality of self-similar tiles, Constr. Approx. 42 (2015) 519–541.
[12] B. Fuglede, Commuting self-adjoint partial differential operators and a group theoretic problem, J. Funct. Anal. 16 (1974) 101–121.
[13] K. Gröchenig, A. Haas, Self-similar lattice tilings, J. Fourier Anal. Appl. 1 (1994) 131–170.
[14] X.G. He, I. Kirat, K.S. Lau, Height reducing property of polynomials and self-affine tiles, Geom. Dedicata 152 (2011) 153–164.
[15] T.Y. Hu, K.S. Lau, Spectral property of the Bernoulli convolutions, Adv. Math. 219 (2008) 554–567.
[16] J. Hutchinson, Fractals and self-similarity, Indiana Univ. Math. J. 30 (1981) 713–747.
[17] P. Jorgenson, S. Pederson, Dense analytic subspaces in fractal L^2-spaces, J. Anal. Math. 75 (1998) 185–228.
[18] M. Kolountzakis, M. Matolcsi, Tiles with no spectra, Forum Math. 18 (2006) 519–528.
[19] I. Łaba, Y. Wang, On spectral Cantor measures, J. Funct. Anal. 193 (2002) 409–420.
[20] J.L. Li, Non-spectral problem for a class of planar self-affine measures, J. Funct. Anal. 255 (2008) 3125–3148.
[21] J.L. Li, Spectra of a class of self-affine measures, J. Funct. Anal. 260 (2011) 1086–1095.
[22] T. Tao, Fuglede's conjecture is false in 5 or higher dimensions, Math. Res. Lett. 11 (2004) 251–258.
[23] J. Lagarias, Y. Wang, Self-affine tiles in $\mathbb{R}^n$, Adv. Math. 121 (1996) 21–49.
[24] J. Lagarias, Y. Wang, Integral self-affine tiles in $\mathbb{R}^n$: I. Standard and nonstandard digit sets, J. Lond. Math. Soc. 54 (1996) 161–179.
[25] J. Lagarias, Y. Wang, Integral self-affine tiles in $\mathbb{R}^n$: II. Lattice tilings, J. Fourier Anal. Appl. 3 (1997) 83–102.
[26] C.K. Lai, K.S. Lau, H. Rao, Spectral structure of digit sets of self-similar tiles in $\mathbb{R}^1$, Trans. Amer. Math. Soc. 365 (2013) 3831–3850.
[27] C.K. Lai, K.S. Lau, H. Rao, Classification of tile digit sets as product-forms, Trans. Amer. Math. Soc. 369 (2017) 623–644.