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Concentrating solutions for a magnetic Schrödinger equation with critical growth

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ABSTRACT

We deal with the following nonlinear Schrödinger equation with magnetic field and critical growth:

$$\begin{cases} \left(\frac{\varepsilon}{i}\nabla - A(x)\right)^2 u + V(x)u = f(|u|^2)u + |u|^{2^*-2}u \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N, \mathbb{C}), \end{cases}$$

where $\varepsilon > 0$ is a small parameter, $N \geq 3$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent, $A \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ is a magnetic vector potential, $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous positive potential having a local minimum and $f : \mathbb{R} \to \mathbb{R}$ is a superlinear continuous function with subcritical growth. Using penalization techniques and variational methods, we investigate the existence and concentration of nontrivial solutions for $\varepsilon > 0$ small enough.

1. Introduction

The aim of this paper is to study the existence and concentration of nontrivial complex-valued solutions for the following nonlinear Schrödinger equation with critical growth:

$$\begin{cases} \left(\frac{\varepsilon}{i}\nabla - A(x)\right)^2 u + V(x)u = f(|u|^2)u + |u|^{2^*-2}u \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N, \mathbb{C}), \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, $N \geq 3$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent, $A \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ is a magnetic vector potential, $V \in C(\mathbb{R}^N, \mathbb{R})$ is an electric potential and $f : \mathbb{R} \to \mathbb{R}$ is a subcritical nonlinearity. Here the magnetic Schrödinger operator is defined by

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$$\left(\frac{\varepsilon}{\imath}\nabla - A(x)\right)^2 u = -\varepsilon^2 \Delta u - \frac{2\varepsilon}{\imath} A \cdot \nabla u + |A|^2 u - \frac{\varepsilon}{\imath} u \operatorname{div} A.$$

When $N = 3$, the magnetic field B is the usual curl operator of A, while for higher dimensions $N \geq 4$ it is the 2-form given by $B_{i,j} := \partial_j A_k - \partial_k A_j$ with $1 \leq j, k \leq N$. The main driving force for studying (1.1) is related to the following time-dependent Schrödinger equation:

$$\imath\hbar \frac{\partial \Psi}{\partial t} = \left(\frac{\varepsilon}{\imath}\nabla - A(x)\right)^2 \Psi + U(z)\psi - g(x, |\Psi|^2)\Psi \text{ in } \mathbb{R}^N \times \mathbb{R}, \tag{1.2}$$

when we look for standing waves, that is, solutions having the form $\Psi(x,t) = u(x)e^{-\frac{\imath ct}{\hbar}}$, where $\hbar$ is the Planck's constant and $E \in \mathbb{R}$. In fact, it is clear that Ψ solves (1.2) if and only if u satisfies (1.1), with $\varepsilon = \hbar$, $V(x) = U(x) - E$ and $g(x, |u|^2)u = f(|u|^2)u + |u|^{2^*-2}u$. Equation (1.2) appears in quantum mechanics and describes the dynamics of a particle in a non-relativistic setting. It arises in different physical theories, e.g., the description of Bose-Einstein condensates and nonlinear optics; see [9,30] for more physical background. Indeed, the analysis of the existence and shape of standing wave solutions in the semiclassical regime, that is, as $\hbar \to 0$, is motivated by the fact that the transition from Quantum Mechanics to Classical Mechanics can be formally performed by sending the Planck's constant to zero.

In recent years, a great attention has been devoted to the nonlinear Schrödinger equations without the magnetic field (i.e. $A \equiv 0$) and for which several existence, multiplicity and qualitative property of standing wave solutions have been established, see [3,4,11,16,18,21–23,29] and the references therein. On the other hand, when we consider the case $A \neq 0$, the first result is probably due to Esteban and Lions [20] that used the concentration-compactness principle [27] and minimization arguments to establish the existence of a ground state solution in dimensions $N = 2$ or $N = 3$. Subsequently, Kurata [25] proved, via variational methods, that a subcritical magnetic Schrödinger equation has a least energy solution for any $\varepsilon > 0$, assuming a technical condition linking V and A. Chabrowski and Szulkin [13] applied minimax arguments to deduce the existence of nontrivial solutions for a critical magnetic Schrödinger equation when the potential V changes sign. In [14,15] the authors used Ljusternik-Schnirelmann theory to obtain multiple solutions for a subcritical magnetic Schrödinger equation assuming the following global condition on the potential V proposed by Rabinowitz [29]:

$$\inf_{x \in \mathbb{R}^N} V(x) < \liminf_{|x| \to \infty} V(x).$$

Alves et al. [6] combined penalization technique and Ljusternik-Schnirelmann theory to relate the number of solutions with the topology of the set where the potential attains its minimum value. They considered subcritical nonlinearities and assumed local conditions on V inspired by the following conditions introduced by del Pino and Felmer [18]:

(V_1) there exists $V_1 > 0$ such that $V_1 = \inf_{x \in \mathbb{R}^N} V(x)$,
(V_2) there exists a bounded open set $\Lambda \subset \mathbb{R}^N$ such that

$$0 < V_0 = \inf_{x \in \Lambda} V(x) < \min_{x \in \partial\Lambda} V(x).$$

We also mention the papers [5,8,10,17,19] for other interesting results related to (1.1).

Motivated by the above papers, in this work we focus our attention on the existence and concentration of nontrivial solutions for a magnetic Schrödinger equation with critical growth. Along the paper, we suppose that the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(t) = 0$ for $t \leq 0$ and satisfies the following conditions:

(f_1) $f(t) \to 0$ as $t \to 0^+$,

(f_2) there exist $q, \sigma \in (2, 2^*)$ and $\lambda > 0$ such that

$$f(t) \geq \lambda t^{\frac{q-2}{2}} \quad \forall t > 0, \quad \lim_{t\to\infty} \frac{f(t)}{t^{\frac{\sigma-2}{2}}} = 0,$$

where λ is such that
- $\lambda > 0$ if either $N \geq 4$, or $N = 3$ and $4 < q < 6$,
- λ is sufficiently large if $N = 3$ and $2 < q \leq 4$,

(f_3) there exists $\vartheta \in (2, \sigma)$ such that $0 < \frac{\vartheta}{2} F(t) \leq t f(t)$ for all $t > 0$, where $F(t) = \int_0^t f(s)\, ds$,

(f_4) the map $t \mapsto f(t)$ is increasing in $(0, \infty)$.

Our main result can be stated as follows:

Theorem 1.1. *Assume that (V_1)-(V_2) and (f_1)-(f_4) hold. Then, there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, problem (1.1) admits a nontrivial solution u_ε. Moreover, if $x_\varepsilon \in \mathbb{R}^N$ denotes a global maximum point of $|u_\varepsilon|$, then*

$$\lim_{\varepsilon\to 0} V(x_\varepsilon) = V_0.$$

The proof of Theorem 1.1 is obtained using suitable variational arguments. Since we do not have informations on the behavior of V at infinity, we use a variant of the penalization argument introduced in [18] which consists in modifying appropriately the nonlinearity outside the set Λ, solving a modified problem and then check that, for $\varepsilon > 0$ small enough, the solutions of the modified problem are indeed solutions of the original one. We emphasized that, even for the modified problem, it is rather tough to obtain compactness in view of the critical growth of the nonlinearity. Indeed, compared with the subcritical case considered in [6], a more careful analysis will be needed to overcome this obstacle. More precisely, after proving that the modified energy functional has a mountain pass geometry [7], we use a suitable truncated complex-value function inspired by Brezis and Nirenberg [12], which takes care of the presence of the magnetic field A, and the concentration-compactness principle of Lions [27], to verify that the Palais-Smale condition is regained below a suitable level related to the best constant of the Sobolev embedding $H^1(\mathbb{R}^N, \mathbb{R})$ into $L^{2^*}(\mathbb{R}^N, \mathbb{R})$; see Lemma 3.2 and Lemma 3.3. Finally, making use of the diamagnetic inequality [20] and Kato's inequality [24], we show that the solutions of the modified problem are solutions of the original one; see Lemma 3.6. To the best of our knowledge, this is the first time that the penalization argument is used to obtain the existence and concentration of solutions to (1.1) under local conditions (V_1)-(V_2).

The paper is organized as follows. In Section 2 we give the notations and collect some useful preliminary results. In Section 3 we introduce the modified problem and we prove the existence of a positive solution for it via mountain pass theorem [7]. In Section 4 we give the proof of Theorem 1.1.

2. Preliminaries

Let us denote by $\mathcal{B}_R(x)$ the ball of radius R and center at x. When $x = 0$, we write $\mathcal{B}_R = \mathcal{B}_R(0)$. Let $1 \leq r \leq \infty$ and $A \subset \mathbb{R}^N$. We denote by $|u|_{L^r(A)}$ the $L^r(A)$-norm of a function $u : \mathbb{R}^N \to \mathbb{R}$ belonging to $L^r(A)$, and by $|u|_q$ its $L^q(\mathbb{R}^N)$-norm. We define $\mathcal{D}^{1,2}(\mathbb{R}^N, \mathbb{R})$ as the closure of $C_c^\infty(\mathbb{R}^N, \mathbb{R})$ with respect to

$$|\nabla u|_2^2 = \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

Let us denote by $H^1(\mathbb{R}^N, \mathbb{R})$ the set of functions $u : \mathbb{R}^N \to \mathbb{R}$ such that $u, \nabla u \in L^2(\mathbb{R}^N, \mathbb{R})$ endowed with the usual norm

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$$\|u\|^2 = |\nabla u|_2^2 + |u|_2^2.$$

We remark (see [1]) that there exists a sharp constant $S_* > 0$ such that for any $u \in \mathcal{D}^{1,2}(\mathbb{R}^N, \mathbb{R})$

$$|u|_{2^*}^2 \leq S_*^{-1} |\nabla u|_2^2.$$

Moreover, $H^1(\mathbb{R}^N, \mathbb{R})$ is continuously embedded in $L^q(\mathbb{R}^N, \mathbb{R})$ for any $q \in [2, 2^*]$ and compactly in $L^q_{loc}(\mathbb{R}^N, \mathbb{R})$ for any $q \in [1, 2^*)$. We recall the following compactness-Lions type result [26]:

Lemma 2.1. *Let $N \geq 3$ and $r \in [2, 2^*)$. If $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^1(\mathbb{R}^N, \mathbb{R})$ and if*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int\limits_{\mathcal{B}_R(y)} |u_n|^r \, dx = 0, \tag{2.1}$$

where $R > 0$, then $u_n \to 0$ in $L^t(\mathbb{R}^N, \mathbb{R})$ for all $t \in (2, 2^*)$.

In what follows, we collect some useful estimates which will be needed to overcome the difficulty coming from the critical exponent.

For any $\varepsilon > 0$ and $y \in \mathbb{R}^N$, we consider the following family of instantons (see [31])

$$U_{\varepsilon,y}(x) = \varepsilon^{-\frac{(N-2)}{2}} U\left(\frac{x-y}{\varepsilon}\right), \quad U(x) = \frac{(N(N-2))^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}$$

where $U \in \mathcal{D}^{1,2}(\mathbb{R}^N, \mathbb{R})$ is a solution to

$$-\Delta U = U^{2^*-1} \quad \text{in } \mathbb{R}^N.$$

Moreover, we have $|\nabla U|_2^2 = |U|_{2^*}^{2^*} = S_*^{\frac{N}{2}}$.

Let ψ be a $C^1(\mathbb{R}^N, \mathbb{R})$ function such that $\psi(x) = 1$ in $|x-y| \leq \frac{\delta}{2}$ and $\psi(x) = 0$ if $|x-y| > \delta$. Let $\tilde{w}_{\varepsilon,y} = \psi U_{\varepsilon,y}$ and $w_{\varepsilon,y} = \frac{\tilde{w}_{\varepsilon,y}}{|w_{\varepsilon,y}|_{2^*}}$. Then, arguing as in [12], we can prove that:

Lemma 2.2. *The following estimates for $w_{\varepsilon,y}$ hold true:*

$$|\nabla w_{\varepsilon,y}|_2^2 = S_* + O(\varepsilon^{N-2}),$$

$$|w_{\varepsilon,y}|_2^2 = \begin{cases} O(\varepsilon^2) & \text{if } N > 4, \\ O(\varepsilon^2 \log(\frac{1}{\varepsilon})) & \text{if } N = 4, \\ O(\varepsilon) & \text{if } N = 3, \end{cases}$$

and

$$|w_{\varepsilon,y}|_q^q \geq \begin{cases} O(\varepsilon^{N-\frac{(N-2)}{2}q}) & \text{if } q > \frac{N}{N-2}, \\ O(\varepsilon^{N-\frac{(N-2)}{2}q} \log(\frac{1}{\varepsilon})) & \text{if } q = \frac{N}{N-2}, \\ O(\varepsilon^{\frac{(N-2)q}{2}}) & \text{if } q < \frac{N}{N-2}. \end{cases}$$

In order to study (1.1), it is important to introduce the Hilbert space $H_\varepsilon = H_\varepsilon(\mathbb{R}^N, \mathbb{C})$ obtained by the closure of $C_c^\infty(\mathbb{R}^N, \mathbb{C})$ under the scalar product

$$(u, v)_\varepsilon = \Re\left(\int\limits_{\mathbb{R}^N} \nabla_\varepsilon u \overline{\nabla_\varepsilon v} + V(\varepsilon\, x) u \bar{v} \, dx\right),$$

where $\Re(w)$ denotes the real part of $w \in \mathbb{C}$, $\bar{w}$ is its conjugate, $\nabla_\varepsilon u = (D^1_\varepsilon u, \dots, D^N_\varepsilon u)$ and $D^j_\varepsilon u = \imath^{-1}\partial_j u - A_j(\varepsilon x)u$ for $j = 1, \dots, N$. The norm induced by this inner product is given by

$$\|u\|^2_\varepsilon = \int_{\mathbb{R}^N} |\nabla_\varepsilon u|^2 + V(\varepsilon x)|u|^2 dx.$$

As proved in [20], for any $u \in H_\varepsilon$ it holds the following diamagnetic inequality:

$$|\nabla|u|| \leq |\nabla_\varepsilon u|. \tag{2.2}$$

Consequently, if $u \in H_\varepsilon$ then $|u| \in H^1(\mathbb{R}^N, \mathbb{R})$. Moreover, the embedding $H_\varepsilon \subset L^q(\mathbb{R}^N, \mathbb{C})$ is continuous for all $q \in [2, 2^*]$ and $H_\varepsilon \subset L^q(\Lambda, \mathbb{C})$ is compact for all $q \in [1, 2^*)$. We also recall the following distributional Kato's inequality [24] (see also Theorem X.33 in [30]):

$$\Delta|u| \geq -\Re(sign(u)D^2_\varepsilon u) \quad \forall u \in H_\varepsilon, \tag{2.3}$$

where $D^2_\varepsilon = \sum_{j=1}^N (D^j_\varepsilon)^2$ and

$$sign(u) = \begin{cases} \frac{\bar{u}}{|u|} & \text{if } u \neq 0 \\ 0 & \text{if } u = 0. \end{cases}$$

3. Variational setting

Using the change of variable $x \mapsto \varepsilon x$, we can see that the study of (1.1) is equivalent to investigate the following problem

$$\begin{cases} \left(\frac{1}{\imath}\nabla - A_\varepsilon(x)\right)^2 u + V_\varepsilon(x)u = f(|u|^2)u + |u|^{2^*-2}u & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N, \mathbb{C}), \end{cases} \tag{3.1}$$

where $A_\varepsilon(x) = A(\varepsilon x)$ and $V_\varepsilon(x) = V(\varepsilon x)$. Now, we introduce a penalized function [18] which will be useful to obtain our results. First of all, without loss of generality, we may assume that

$$0 \in \Lambda \text{ and } V(0) = V_0 = \min_\Lambda V.$$

Take $K > \frac{\vartheta}{\vartheta - 2} > 1$ and $a > 0$ such that $f(a) + a^{\frac{2^*-2}{2}} = \frac{V_1}{K}$, and we define

$$\tilde{f}(t) = \begin{cases} f(t) + (t^+)^{\frac{2^*-2}{2}} & \text{if } t \leq a, \\ \frac{V_1}{K} & \text{if } t > a, \end{cases}$$

and

$$g(x,t) = \chi_\Lambda(x)(f(t) + (t^+)^{\frac{2^*-2}{2}}) + (1 - \chi_\Lambda(x))\tilde{f}(t) \quad \text{and} \quad G(x,t) = \int_0^t g(x,s)\, ds.$$

It is easy to check that g satisfies the following properties:

(g_1) $\lim\limits_{t \to 0} g(x,t) = 0$ uniformly with respect to $x \in \mathbb{R}^N$,

(g_2) $g(x,t) \leq f(t) + t^{\frac{2^*-2}{2}}$ for all $x \in \mathbb{R}^N$, $t > 0$,

(g_3) (i) $0 < \frac{\vartheta}{2} G(x,t) < g(x,t)t$ for all $x \in \Lambda$ and $t > 0$,
(ii) $0 \leq G(x,t) < g(x,t)t \leq \frac{V_1}{K} t$ for all $x \in \mathbb{R}^N \setminus \Lambda$ and $t > 0$,

(g_4) for each $x \in \Lambda$ the function $g(x,t)$ is increasing in $(0,\infty)$, and for each $x \in \mathbb{R}^N \setminus \Lambda$ the function $g(x,t)$ is increasing in $(0,a)$.

Then, we consider the following modified problem

$$\left(\frac{1}{\imath}\nabla - A_\varepsilon(x)\right)^2 u + V_\varepsilon(x) u = g(\varepsilon x, |u|^2) u \quad \text{in } \mathbb{R}^N. \tag{3.2}$$

In view of the definition of g, we look for weak solutions to (3.2) having the property

$$|u(x)| \leq \sqrt{a} \quad \forall x \in \mathbb{R}^N \setminus \Lambda_\varepsilon,$$

where $\Lambda_\varepsilon = \Lambda/\varepsilon$. In order to study (3.2), we seek the critical points of the following functional $\mathcal{J}_\varepsilon : H_\varepsilon \to \mathbb{R}$ defined as

$$\mathcal{J}_\varepsilon(u) = \frac{1}{2}\|u\|_\varepsilon^2 - \frac{1}{2}\int_{\mathbb{R}^N} G(\varepsilon x, |u|^2)\, dx.$$

By the growth assumptions on f and Sobolev embeddings for H_ε, it is easy to check that $\mathcal{J}_\varepsilon \in C^1(H_\varepsilon, \mathbb{R})$ and its differential is given by

$$\langle \mathcal{J}_\varepsilon'(u), v\rangle = \Re\left(\int_{\mathbb{R}^N} \nabla_\varepsilon u \overline{\nabla_\varepsilon v} + V_\varepsilon u\bar{v} dx - \int_{\mathbb{R}^N} g(\varepsilon x, |u|^2) u \bar{v}\, dx\right)$$

for any $u, v \in H_\varepsilon$. Let us note that $\mathcal{J}_\varepsilon$ possesses a mountain pass geometry [7]:

Lemma 3.1. *The functional $\mathcal{J}_\varepsilon$ satisfies the following properties:*

(a) there exist $\alpha, \rho > 0$ such that $\mathcal{J}_\varepsilon(u) \geq \alpha$ with $\|u\|_\varepsilon = \rho$;
(b) there exists $e \in \mathcal{W}_\varepsilon$ such that $\|e\|_\varepsilon > \rho$ and $\mathcal{J}_\varepsilon(e) < 0$.

Proof. (a) By (g_1), (g_2), (f_2), we can see that for any $\xi > 0$ there exists $C_\xi > 0$ such that

$$|g(x,t)| \leq \xi + C_\xi |t|^{\frac{2^*-2}{2}} \quad \text{for any } (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.3}$$

Therefore, we obtain that

$$\mathcal{J}_\varepsilon(u) \geq \frac{1}{2}\|u\|_\varepsilon^2 - \frac{1}{2}\int_{\mathbb{R}^N} G(\varepsilon x, |u|^2)\, dx \geq \frac{1}{2}\|u\|_\varepsilon^2 - \xi C\|u\|_\varepsilon^2 - C_\xi C \|u\|_\varepsilon^{2^*}.$$

Hence we can find $\alpha, \rho > 0$ such that $\mathcal{J}_\varepsilon(u) \geq \alpha$ with $\|u\|_\varepsilon = \rho$.

(b) By (g_3)-(i) we can see that for any $u \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$ such that $u \not\equiv 0$ and $\mathrm{supp}(u) \subset \Lambda$

$$\begin{aligned}\mathcal{J}_\varepsilon(\tau u) &\leq \frac{\tau^2}{2}\|u\|_\varepsilon^2 - \frac{1}{2}\int_{\Lambda_\varepsilon} G(\varepsilon x, |\tau u|^2)\, dx \\ &\leq \frac{\tau^2}{2}\|u\|_\varepsilon^2 - C_1 \tau^\vartheta \int_{\Lambda_\varepsilon} |u|^\vartheta\, dx + C_2 \quad \forall \tau > 0,\end{aligned} \tag{3.4}$$

for some positive constants C_1 and C_2. Since $\vartheta \in (2, 2^*)$, we get $\mathcal{J}_\varepsilon(\tau u) \to -\infty$ as $\tau \to +\infty$. $\square$

In view of Lemma 3.1, we can define the minimax level

$$c_\varepsilon = \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0,1]} \mathcal{J}_\varepsilon(\gamma(t)), \qquad \text{where} \qquad \Gamma_\varepsilon = \{v \in H_\varepsilon : \mathcal{J}_\varepsilon(0) = 0, \mathcal{J}_\varepsilon(\gamma(1)) \leq 0\}.$$

As in [32], we can use the equivalent characterization of c_ε more appropriate to our aim given by

$$c_\varepsilon = \inf_{u \in H_\varepsilon \setminus \{0\}} \max_{t \geq 0} \mathcal{J}_\varepsilon(tu).$$

In order to obtain the existence of a nontrivial solution to (3.2), we need to prove the next fundamental result.

Lemma 3.2. *There exists $v \in H_\varepsilon \setminus \{0\}$ such that*

$$\max_{t \geq 0} \mathcal{J}_\varepsilon(tv) < \frac{1}{N} S_*^{\frac{N}{2}}.$$

In particular, $c_\varepsilon < \frac{1}{N} S_*^{\frac{N}{2}}$.

Proof. Set $w_h = w_{h,0}$, with $w_{h,y}$ defined as in Section 2, and we consider $u_h(x) = e^{\imath \theta(x)} w_h(x)$, where $\theta(x) = -\sum_{j=1}^N A_j(0) x_j$. Then, $(A + \nabla\theta)(0) = 0$ and by the continuity of A at 0 we have $|(A + \nabla\theta)(x)|^2 < c$ for all $|x| < \delta_1$, with $\delta_1 > 0$ sufficiently small. Assume that $\mathrm{supp}(\psi) \subset B_{\frac{r}{\varepsilon}}$ where $r = \min\{\delta, \delta_1\}$ and $\delta > 0$ is such that $\mathcal{B}_\delta \subset \Lambda$. We note that

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla_\varepsilon u_h|^2 + V_\varepsilon |u_h|^2 dx &= \int_{\mathbb{R}^N} |\nabla w_h|^2 + w_h^2 |A_\varepsilon + \nabla\theta|^2 + V_\varepsilon w_h^2 \, dx \\ &\leq \int_{\mathbb{R}^N} |\nabla w_h|^2 + (c + |V|_{L^\infty(\Lambda)}) |w_h|^2 \, dx. \end{aligned} \tag{3.5}$$

Then, using (f_2), we can see that

$$\begin{aligned} \mathcal{J}_\varepsilon(tu_h) &= \frac{t^2}{2} \|u_h\|_\varepsilon^2 - \frac{1}{2} \int_{\mathbb{R}^N} F(t|u_h|^2) \, dx - \frac{t^{2^*}}{2^*} |u_h|_{2^*}^{2^*} \\ &\leq \frac{t^2}{2} \left(|\nabla_\varepsilon u_h|_2^2 + |V|_{L^\infty(\Lambda)} |u_h|_2^2 \right) - \lambda t^q |u_h|_q^q - \frac{t^{2^*}}{2^*} |u_h|_{2^*}^{2^*} \to -\infty \text{ as } t \to \infty, \end{aligned}$$

so there exists $t_h > 0$ such that

$$\mathcal{J}_\varepsilon(t_h u_h) = \max_{t \geq 0} \mathcal{J}_\varepsilon(tu_h).$$

Let us show that there exist $A, B > 0$ such that

$$A \leq t_h \leq B \qquad \text{for } h > 0 \text{ sufficiently small.} \tag{3.6}$$

Since $\langle \mathcal{J}'_\varepsilon(t_h u_h), u_h \rangle = 0$, we deduce that

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$$\|u_h\|_\varepsilon^2 = \int_{\mathbb{R}^N} f(|t_h u_h|^2)|u_h|^2\,dx + (t_h)^{2^*-2}|u_h|_{2^*}^{2^*}. \tag{3.7}$$

If $t_{h_n} \to \infty$ as $h_n \to 0$, by (3.7) it follows that

$$\|u_{h_n}\|_\varepsilon^2 \geq (t_{h_n})^{2^*-2}|u_{h_n}|_{2^*}^{2^*},$$

which gives a contradiction in view of $2^* > 2$ and Lemma 2.2.

Now, assume that there exists $t'_{h_n} \to 0$ as $h_n \to 0$. From (f_1), (f_2) and (V_1) we can see that for any $\xi > 0$ there exists $C_\xi > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^N} f(|t'_{h_n} u_{h_n}|^2)|u_{h_n}|^2\,dx &\leq \xi|u_{h_n}|_2^2 + C_\xi(t'_{h_n})^{2^*-2}|u_{h_n}|_{2^*}^{2^*} \\ &\leq \frac{\xi}{V_1}\|u_{h_n}\|_\varepsilon^2 + C_\xi(t'_{h_n})^{2^*-2}|u_{h_n}|_{2^*}^{2^*}.\end{aligned} \tag{3.8}$$

Choosing $\xi = \frac{V_1}{2}$, and using (3.7) and (3.8), we obtain

$$\frac{1}{2}\|u_{h_n}\|_\varepsilon^2 \leq C_\xi(t'_{h_n})^{2^*-2}|u_{h_n}|_{2^*}^{2^*} + (t'_{h_n})^{2^*-2}|u_{h_n}|_{2^*}^{2^*}$$

which leads to an absurd. Therefore, (3.6) holds true.

Now, we note that for $C, D > 0$ it holds

$$\frac{t^2}{2}C - \frac{t^{2^*}}{2^*}D \leq \frac{1}{N}\left(\frac{C}{D^{\frac{N-2}{N}}}\right)^{\frac{N}{2}} \qquad \text{for all } t \geq 0.$$

Thus, using (3.5) and (3.6), we get

$$\begin{aligned}\mathcal{J}_\varepsilon(t_h u_h) &\leq \frac{t_h^2}{2}\|u_h\|_\varepsilon^2 - \lambda t_h^q|u_h|_q^q - \frac{t_h^{2^*}}{2^*}|u_h|_{2^*}^{2^*} \\ &\leq \frac{1}{N}\left(\frac{|\nabla w_h|_2^2 + (c + |V|_{L^\infty(\Lambda)})|w_h|_2^2}{|w_h|_{2^*}^2}\right)^{\frac{N}{2}} - \lambda A^q|w_h|_q^q.\end{aligned}$$

In the light of the following elementary inequality

$$(a+b)^r \leq a^r + r(a+b)^{r-1}b \qquad \text{for all } a, b > 0, r \geq 1,$$

and gathering the estimates in Lemma 2.2, we can deduce that

$$\mathcal{J}_\varepsilon(t_h u_h) \leq \begin{cases} \frac{1}{N}S_*^{\frac{N}{2}} + O(h^{N-2}) + O(h^2) - \lambda A^q|w_h|_q^q & \text{if } N > 4, \\ \frac{1}{N}S_*^{\frac{N}{2}} + O\left(h^2\left(1 + \log(1/h)\right)\right) - \lambda A^q|w_h|_q^q & \text{if } N = 4, \\ \frac{1}{N}S_*^{\frac{N}{2}} + O(h) - \lambda A^q|w_h|_q^q & \text{if } N = 3. \end{cases}$$

At this point we distinguish several cases. Let $N > 4$. Then $q > 2 > \frac{N}{N-2}$ and using Lemma 2.2 we have

$$\mathcal{J}_\varepsilon(t_h u_h) \leq \frac{1}{N}S_*^{\frac{N}{2}} + O(h^{N-2}) + O(h^2) - O(h^{N-\frac{(N-2)}{2}q}).$$

Now

$$N-\frac{(N-2)}{2}q<2<N-2,$$

because of $q>2$ and $N>4$, so we can infer that

$$\mathcal{J}_\varepsilon(t_h u_h)<\frac{1}{N}S_*^{\frac{N}{2}},$$

provided that $h>0$ is sufficiently small.

Assume that $N=4$. Thus, $q>2=\frac{N}{N-2}$ and in view of Lemma 2.2 we obtain

$$\mathcal{J}_\varepsilon(t_h u_h)\leq\frac{1}{N}S_*^{\frac{N}{2}}+O\left(h^2\left(1+\log(1/h)\right)\right)-O(h^{4-q}),$$

and observing that $2<q<2^*=4$ yields

$$\lim_{h\to 0}\frac{h^{4-q}}{h^2\left(1+\log(1/h)\right)}=\infty,$$

we get the conclusion for h small enough.

Finally, we consider the case $N=3$. Suppose that $4<q<6$. Then,

$$q>4>1=\frac{N}{N-2}$$

from which

$$\mathcal{J}_\varepsilon(t_h u_h)\leq\frac{1}{N}S_*^{\frac{N}{2}}+O(h)-O(h^{3-\frac{q}{2}}).$$

Using the fact that $4<q<6$ implies that $0<3-\frac{q}{2}<1$, we have for $h>0$ small enough

$$\mathcal{J}_\varepsilon(t_h u_h)<\frac{1}{N}S_*^{\frac{N}{2}}.$$

Now, we assume that $N=3$ and $2<q<4$.

When $2<q<3$ then, for $h>0$ small, it holds

$$\mathcal{J}_\varepsilon(t_h u_h)\leq\frac{1}{N}S_*^{\frac{N}{2}}+O(h)-\lambda O(h^{\frac{q}{2}})$$

and noting that $\frac{q}{2}>1$, we can take $\lambda=h^{-\mu}$, with $\mu>\frac{q-2}{2}$, to get the thesis for $h>0$ small.

If $q=3$, then

$$\mathcal{J}_\varepsilon(t_h u_h)\leq\frac{1}{N}S_*^{\frac{N}{2}}+O(h)-\lambda O(h^{\frac{3}{2}}|\log(h)|)$$

and taking $\lambda=h^{-\mu}$, with $\mu>\frac{1}{2}$, we can deduce the thesis.

When $3<q\leq 4$, we have

$$\mathcal{J}_\varepsilon(t_h u_h)\leq\frac{1}{N}S_*^{\frac{N}{2}}+O(h)-\lambda O(h^{3-\frac{q}{2}})$$

and choosing $\lambda=h^{-\mu}$, with $\mu>2-\frac{q}{2}$, we have again the desired estimate. □

In the next lemma, we prove that $\mathcal{J}_\varepsilon$ satisfies a local compactness condition. More precisely:

Lemma 3.3. *Let $c \in \mathbb{R}$ be such that $c < \frac{1}{N} S_*^{\frac{N}{2}}$. Then $\mathcal{J}_\varepsilon$ fulfills the Palais-Smale condition at the level c.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a $(PS)_c$ sequence at the level $c < \frac{1}{N} S_*^{\frac{N}{2}}$, that is

$$\mathcal{J}_\varepsilon(u_n) \to c \quad \text{and} \quad \mathcal{J}'_\varepsilon(u_n) \to 0 \text{ in } H^*_\varepsilon.$$

In view of (g_3), we can deduce that

$$\begin{aligned}
c + o_n(1)\|u_n\|_\varepsilon &= \mathcal{J}_\varepsilon(u_n) - \frac{1}{\vartheta}\langle \mathcal{J}'_\varepsilon(u_n), u_n\rangle \\
&= \left(\frac{\vartheta-2}{2\vartheta}\right)\|u_n\|^2_\varepsilon + \frac{1}{\vartheta}\int\limits_{\mathbb{R}^N\setminus\Lambda_\varepsilon} [g(\varepsilon\, x, |u_n|^2)|u_n|^2 - \frac{\vartheta}{2}G(\varepsilon\, x, |u_n|^2)]\, dx \\
&\quad + \frac{1}{\vartheta}\int\limits_{\Lambda_\varepsilon}[f(|u_n|^2)|u_n|^2 - \frac{\vartheta}{2}F(|u_n|^2)]\, dx + \left(\frac{1}{\vartheta} - \frac{1}{2^*}\right)\int\limits_{\Lambda_\varepsilon}|u_n|^{2^*}\, dx \\
&\geq \left(\frac{\vartheta-2}{2\vartheta}\right)\|u_n\|^2_\varepsilon + \frac{1}{\vartheta}\int\limits_{\mathbb{R}^N\setminus\Lambda_\varepsilon} [g(\varepsilon\, x, |u_n|^2)|u_n|^2 - \frac{\vartheta}{2}G(\varepsilon\, x, |u_n|^2)]\, dx \\
&\geq \left(\frac{\vartheta-2}{2\vartheta}\right)\|u_n\|^2_\varepsilon - \left(\frac{\vartheta-2}{2\vartheta}\right)\frac{1}{K}\int\limits_{\mathbb{R}^N\setminus\Lambda_\varepsilon} V(\varepsilon\, x)|u_n|^2 dx \\
&\geq \left(\frac{\vartheta-2}{2\vartheta}\right)\left(1-\frac{1}{K}\right)\|u_n\|^2_\varepsilon.
\end{aligned}$$

Since $\vartheta > 2$ and $K > 1$, we can conclude that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in H_ε. Moreover, from (2.2), we know that $\{|u_n|\}_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N, \mathbb{R})$. Taking into account $\langle \mathcal{J}'_\varepsilon(u_n), u_n\rangle = o_n(1)$, we can see that

$$\|u_n\|^2_\varepsilon = \int\limits_{\mathbb{R}^N} g(\varepsilon\, x, |u_n|^2)|u_n|^2\, dx + o_n(1). \tag{3.9}$$

On the other hand, it is easy to check that

$$\begin{aligned}
&(u_n, \varphi)_\varepsilon \to (u, \varphi)_\varepsilon, \\
&\Re\left(\int\limits_{\mathbb{R}^N} g(\varepsilon\, x, |u_n|^2)u_n\bar{\varphi}\, dx\right) \to \Re\left(\int\limits_{\mathbb{R}^N} g(\varepsilon\, x, |u|^2)u\bar{\varphi}\, dx\right),
\end{aligned}$$

for all $\varphi \in C^\infty_c(\mathbb{R}^N, \mathbb{C})$, which together with $\mathcal{J}'_\varepsilon(u_n) \to 0$ and the density of $C^\infty_c(\mathbb{R}^N, \mathbb{C})$ in H_ε, implies that $\langle \mathcal{J}'_\varepsilon(u), \varphi\rangle = 0$ for all $\varphi \in H_\varepsilon$. In particular,

$$\|u\|^2_\varepsilon = \int\limits_{\mathbb{R}^N} g(\varepsilon\, x, |u|^2)|u|^2\, dx. \tag{3.10}$$

Hence, in view of (3.9) and (3.10), it is enough to show that

$$\int\limits_{\mathbb{R}^N} g(\varepsilon\, x, |u_n|^2)|u_n|^2\, dx = \int\limits_{\mathbb{R}^N} g(\varepsilon\, x, |u|^2)|u|^2\, dx + o_n(1) \tag{3.11}$$

to deduce that $\|u_n\|_\varepsilon^2 \to \|u\|_\varepsilon^2$ which together with the fact that H_ε is a Hilbert space yields $u_n \to u$ in H_ε. In order to achieve our goal, we first prove that for all $\eta > 0$ there exists $R = R_\eta > 0$ such that

$$\limsup_{n\to\infty} \int\limits_{\mathcal{B}_R^c} |\nabla_\varepsilon u_n|^2 + V_\varepsilon |u_n|^2 \, dx < \eta. \tag{3.12}$$

Let us consider the function $\eta_R \in C^\infty(\mathbb{R}^N)$ defined as

$$\eta_R(x) = \begin{cases} 0 & \text{if } x \in \mathcal{B}_R, \\ 1 & \text{if } x \notin \mathcal{B}_{2R}, \end{cases}$$

and $|\nabla \eta_R|_\infty \leq C/R$. Take $R > 0$ sufficiently large such that $\Lambda_\varepsilon \subset \mathcal{B}_R$. Since $\langle \mathcal{J}_\varepsilon'(u_n), \eta_R u_n\rangle = o_n(1)$ and

$$\overline{\nabla_\varepsilon(u_n \eta_R)} = \imath \bar{u}_n \nabla \eta_R + \eta_R \overline{\nabla_\varepsilon u_n},$$

we can use (g_3) to get

$$\int\limits_{\mathbb{R}^N} (|\nabla_\varepsilon u_n|^2 + V_\varepsilon |u_n|^2)\eta_R \, dx \leq \frac{1}{K} \int\limits_{\mathbb{R}^N} V_\varepsilon |u_n|^2 \eta_R \, dx + \Re\left(\int\limits_{\mathbb{R}^N} -\imath \bar{u}_n \nabla_\varepsilon u_n \nabla \eta_R \, dx \right) + o_n(1).$$

By the Hölder inequality and the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in H_ε we deduce that

$$\begin{aligned} \int\limits_{\mathbb{R}^N} |\nabla_\varepsilon u_n|^2 \eta_R \, dx + \left(1 - \frac{1}{K}\right) \int\limits_{\mathbb{R}^N} V_\varepsilon |u_n|^2 \eta_R \, dx &\leq |\bar{u}_n|_2 |\nabla_\varepsilon u_n|_2 |\nabla \eta_R|_\infty + o_n(1) \\ &\leq \frac{C}{R} + o_n(1), \end{aligned}$$

from which

$$\lim_{R\to\infty} \limsup_{n\to\infty} \int\limits_{\mathcal{B}_R^c} |\nabla_\varepsilon u_n|^2 + V_\varepsilon |u_n|^2 \, dx = 0,$$

that is (3.12) holds true.

Using (3.12), (g_2), (f_1), (f_2) and the Sobolev embeddings, we obtain that for n large enough,

$$\int\limits_{\mathbb{R}^N \setminus \mathcal{B}_R} g(\varepsilon x, |u_n|^2)|u_n|^2 \, dx \leq C(\eta + \eta^{\frac{q}{2}} + \eta^{\frac{2^*}{2}}). \tag{3.13}$$

On the other hand, taking R large enough, we can suppose that

$$\int\limits_{\mathbb{R}^N \setminus \mathcal{B}_R} g(\varepsilon x, |u|^2)|u|^2 \, dx < \eta.$$

The above expression and (3.13) imply that for n large

$$\left| \int\limits_{\mathbb{R}^N \setminus \mathcal{B}_R} g(\varepsilon x, |u_n|^2)|u_n|^2 - \int\limits_{\mathbb{R}^N \setminus \mathcal{B}_R} g(\varepsilon x, |u|^2)|u|^2 \, dx \right| < C\eta. \tag{3.14}$$

Now, we note that, in view of the definition of g, there holds

$$g(\varepsilon x, |u_n|^2)|u_n|^2 \leq f(|u_n|^2)|u_n|^2 + a^{2^*} + \frac{V_1}{K}|u_n|^2 \text{ in } \mathbb{R}^N \setminus \Lambda_\varepsilon.$$

Since the set $\mathcal{B}_R \cap (\mathbb{R}^N \setminus \Lambda_\varepsilon)$ is bounded, we can use the above estimate, (f_1), (f_2) and the dominated convergence theorem to conclude that

$$\int_{\mathcal{B}_R \cap (\mathbb{R}^N \setminus \Lambda_\varepsilon)} g(\varepsilon x, |u_n|^2)|u_n|^2 = \int_{\mathcal{B}_R \cap (\mathbb{R}^N \setminus \Lambda_\varepsilon)} g(\varepsilon x, |u|^2)|u|^2 \, dx + o_n(1). \tag{3.15}$$

In what follows, we aim to show that

$$\int_{\Lambda_\varepsilon} |u_n|^{2^*} \, dx = \int_{\Lambda_\varepsilon} |u|^{2^*} \, dx + o_n(1). \tag{3.16}$$

Indeed, if we assume that (3.16) holds true, we can use (g_2), (f_1), (f_2) and the dominated convergence theorem to deduce that

$$\int_{\Lambda_\varepsilon} g(\varepsilon x, |u_n|^2)|u_n|^2 = \int_{\Lambda_\varepsilon} g(\varepsilon x, |u|^2)|u|^2 \, dx + o_n(1). \tag{3.17}$$

Therefore, (3.11) is a direct consequence of the above expression, (3.15) and (3.17).

Now, we show the validity of (3.16). Using the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in H_ε and (2.2), we may assume that

$$|\nabla |u_n||^2 \rightharpoonup \mu \text{ and } |u_n|^{2^*} \rightharpoonup \nu \tag{3.18}$$

in the sense of measures. Moreover, in view of (2.2) and (3.12), $\{|u_n|\}_{n\in\mathbb{N}}$ is a tight sequence in $H^1(\mathbb{R}^N, \mathbb{R})$, so, using the concentration compactness principle [27], we obtain an at most countable index set I, sequences $\{x_i\}_{i\in I} \subset \mathbb{R}^N$, $\{\mu_i\}_{i\in I}$, $\{\nu_i\}_{i\in I}$ such that

$$\mu \geq |\nabla |u||^2 + \sum_{i\in I} \mu_i \delta_{x_i}, \quad \nu = |u|^{2^*} + \sum_{i\in I} \nu_i \delta_{x_i} \text{ and } S_* \nu_i^{2/2^*} \leq \mu_i \quad \forall i \in I. \tag{3.19}$$

Thus, it is enough to prove that $\{x_i\}_{i\in I} \cap \Lambda_\varepsilon = \emptyset$. Suppose by contradiction that there exists $i \in I$ such that $x_i \in \Lambda_\varepsilon$. For any $\rho > 0$, we define $\psi_\rho(x) = \psi((x - x_i)/\rho)$, where $\psi \in C_c^\infty(\mathbb{R}^N, [0, 1])$, $\psi = 1$ in $\mathcal{B}_1$, $\psi = 0$ in $\mathcal{B}_2^c$ and $|\nabla \psi|_\infty \leq 2$. We may assume that ρ is chosen in such a way that $supp(\psi_\rho) \subset \Lambda_\varepsilon$. Since $\{\psi_\rho u_n\}_{n\in\mathbb{N}}$ is bounded in H_ε and $\langle \mathcal{J}'_\varepsilon(u_n), \psi_\rho u_n \rangle = o_n(1)$, we can use (2.2) to see that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla |u_n||^2 \psi_\rho \, dx &\leq \int_{\mathbb{R}^N} |\nabla_\varepsilon u_n|^2 \psi_\rho \, dx \\
&\leq -\Re \left(\int_{\mathbb{R}^N} \imath \bar{u}_n \nabla_\varepsilon u_n \nabla \psi_\rho \, dx \right) + \int_{\mathbb{R}^N} f(|u_n|^2)|u_n|^2 \psi_\rho \, dx \\
&\quad + \int_{\mathbb{R}^N} \psi_\rho |u_n|^{2^*} \, dx + o_n(1).
\end{aligned} \tag{3.20}$$

Since f has subcritical growth and ψ_ρ has compact support, we get

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} f(|u_n|^2)|u_n|^2\psi_\rho\,dx = \lim_{\rho\to 0}\int_{\mathbb{R}^N} f(|u|^2)|u|^2\psi_\rho\,dx = 0. \tag{3.21}$$

Applying the Hölder inequality, and using the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in H_ε, the strong convergence of $\{|u_n|\}_{n\in\mathbb{N}}$ in $L^2_{loc}(\mathbb{R}^N,\mathbb{R})$, $|u|\in L^{2^*}(\mathbb{R}^N,\mathbb{R})$, $|\nabla\psi_\rho|\leq C\rho^{-1}$ and $|\mathcal{B}(x_i,2\rho)|\sim\rho^N$ we have

$$\begin{aligned}\limsup_{n\to\infty}\left|\int_{\mathbb{R}^N} \imath\bar{u}_n\nabla_\varepsilon u_n\nabla\psi_\rho\,dx\right| &\leq \limsup_{n\to\infty}\left(\int_{\mathcal{B}_{2\rho}(x_i)}|u_n|^2|\nabla\psi_\rho|^2\,dx\right)^{1/2}|\nabla_\varepsilon u_n|_2\\ &\leq C\left(\int_{\mathcal{B}_{2\rho}(x_i)}|u|^2|\nabla\psi_\rho|^2\,dx\right)^{1/2}\\ &\leq C\left(\int_{\mathcal{B}_{2\rho}(x_i)}|u|^{2^*}\,dx\right)^{1/2^*}\to 0 \text{ as } \rho\to 0,\end{aligned}$$

which gives

$$\lim_{\rho\to 0}\limsup_{n\to\infty}\left|\int_{\mathbb{R}^N}\imath\bar{u}_n\nabla_\varepsilon u_n\nabla\psi_\rho\,dx\right| = 0. \tag{3.22}$$

Then, taking into account (3.18), (3.20), (3.21) and (3.22), we can conclude that $\nu_i\geq\mu_i$. This combined with the last statement in (3.19) yields that

$$\nu_i\geq S_*^{N/2}. \tag{3.23}$$

Now, in the light (g_3) and (f_4), we obtain

$$\begin{aligned}c &= \mathcal{J}_\varepsilon(u_n)-\frac{1}{2}\langle\mathcal{J}'_\varepsilon(u_n),u_n\rangle+o_n(1)\\ &= \frac{1}{2}\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}[g(\varepsilon\,x,|u_n|^2)|u_n|^2-G(\varepsilon\,x,|u_n|^2)]\,dx\\ &\quad+\frac{1}{2}\int_{\Lambda_\varepsilon}[f(|u_n|^2)|u_n|^2-F(|u_n|^2)]\,dx+\frac{1}{N}\int_{\Lambda_\varepsilon}|u_n|^{2^*}\,dx+o_n(1)\\ &\geq\frac{1}{N}\int_{\Lambda_\varepsilon}|u_n|^{2^*}\,dx+o_n(1)\\ &\geq\frac{1}{N}\int_{\Lambda_\varepsilon}|u_n|^{2^*}\psi_\rho\,dx+o_n(1).\end{aligned}$$

Taking the limit and using (3.19) and (3.23) we get

$$c\geq\frac{1}{N}\sum_{\{i\in I: x_i\in\Lambda_\varepsilon\}}\psi_\rho(x_i)\,\nu_i=\frac{1}{N}\sum_{\{i\in I: x_i\in\Lambda_\varepsilon\}}\nu_i\geq\frac{1}{N}S_*^{\frac{N}{2}}$$

which does not make sense. This ends the proof of (3.16). □

In view of Lemma 3.1, Lemma 3.2 and Lemma 3.3, we can apply the mountain pass theorem [7] to deduce that for all $\varepsilon > 0$ there exists $u_\varepsilon \in H_\varepsilon$ such that

$$\mathcal{J}_\varepsilon(u_\varepsilon) = c_\varepsilon \text{ and } \mathcal{J}'_\varepsilon(u_\varepsilon) = 0. \tag{3.24}$$

Now, we deal with the following autonomous problem, with $\mu > 0$,

$$\begin{cases} -\Delta u + \mu u = f(|u|^2)u + |u|^{2^*-2}u & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N, \mathbb{R}), \quad u > 0 \text{ in } \mathbb{R}^N. \end{cases} \tag{3.25}$$

The Euler-Lagrange functional associated with (3.25) is given by

$$\mathcal{I}_\mu(u) = \frac{1}{2}(|\nabla u|_2^2 + \mu |u|_2^2) - \frac{1}{2}\int_{\mathbb{R}^N} F(|u|^2)\, dx - \frac{1}{2^*}\int_{\mathbb{R}^N} (u^+)^{2^*}\, dx.$$

Let us denote by $\mathbb{X}_\mu$ the Sobolev space $H^1(\mathbb{R}^N, \mathbb{R})$ endowed with the norm

$$\|u\|_\mu^2 = |\nabla u|_2^2 + \mu |u|_2^2.$$

The Nehari manifold associated with $\mathcal{I}_\mu$ is given by

$$\mathcal{N}_\mu = \{u \in \mathbb{X}_\mu \setminus \{0\} : \langle \mathcal{I}'_\mu(u), u\rangle = 0\}.$$

It is standard to check that $\mathcal{I}_\mu$ has a mountain pass geometry and we denote by c_μ its mountain pass level. Moreover, arguing as in [32], we can show that

$$c_\mu = \inf_{\mathcal{N}_\mu} \mathcal{I}_\mu = \inf_{u \in \mathbb{X}_\mu \setminus \{0\}} \max_{t \geq 0} \mathcal{I}(tu).$$

As proved in [2], we know that

Theorem 3.1. *For all $\mu > 0$, problem (3.25) admits a positive ground state solution $u_0 \in \mathbb{X}_\mu$. Moreover, $c_\mu < \frac{1}{N} S_*^{\frac{N}{2}}$.*

Next, we establish a very useful relation between c_ε and c_{V_0}:

Lemma 3.4. *It holds $\limsup_{\varepsilon \to 0} c_\varepsilon \leq c_{V_0}$.*

Proof. For any $\varepsilon > 0$, we set $u_\varepsilon(x) = \psi_\varepsilon(x) u_0(x) e^{\imath \theta(x)}$, where u_0 is a positive ground state of (3.25) whose existence is guaranteed by Theorem 3.1, and $\psi_\varepsilon(x) = \psi(\varepsilon x)$ with $\psi \in C_c^\infty(\mathbb{R}^N, \mathbb{R})$, $\psi \in [0, 1]$, $\psi(x) = 1$ if $|x| \leq \frac{1}{2}$ and $\psi(x) = 0$ if $|x| \geq 1$. For simplicity, we assume that $\mathrm{supp}(\psi) \subset \mathcal{B}_1 \subset \Lambda$. Arguing as in Lemma 3.2 in [14], we can see that

$$\|u_\varepsilon\|_\varepsilon^2 \to \|u_0\|_{V_0}^2 \text{ as } \varepsilon \to 0. \tag{3.26}$$

Now, for each $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$\mathcal{J}_\varepsilon(t_\varepsilon u_\varepsilon) = \max_{t \geq 0} \mathcal{J}_\varepsilon(t u_\varepsilon).$$

Hence, $\langle \mathcal{J}'_\varepsilon(t_\varepsilon u_\varepsilon), u_\varepsilon\rangle = 0$ and this implies that

$$\|u_\varepsilon\|_\varepsilon^2 = \int\limits_{\mathbb{R}^N} f(|t_\varepsilon u_\varepsilon|^2)|u_\varepsilon|^2\, dx + t_\varepsilon^{2^*-2} \int\limits_{\mathbb{R}^N} |u_\varepsilon|^{2^*}\, dx. \tag{3.27}$$

In view of (f_1)-(f_3), (3.26), (3.27), it is easy to see that $t_\varepsilon \to t_0 > 0$ as $\varepsilon \to 0$. Then, taking the limit as $\varepsilon \to 0$ in (3.27) and using (3.26) we get

$$\|u_0\|_{V_0}^2 = \int\limits_{\mathbb{R}^N} f(|t_0 u_0|^2)|u_0|^2\, dx + t_0^{2^*-2} \int\limits_{\mathbb{R}^N} |u_0|^{2^*}\, dx.$$

From $u_0 \in \mathcal{N}_{V_0}$ and assumption (f_4) we deduce that $t_0 = 1$.

Therefore,

$$c_\varepsilon \le \max_{t\ge 0} \mathcal{J}_\varepsilon(tu_\varepsilon) = \mathcal{J}_\varepsilon(t_\varepsilon u_\varepsilon)$$

which implies that

$$\limsup_{\varepsilon\to 0} c_\varepsilon \le \mathcal{I}_{V_0}(u_0) = c_{V_0}. \quad \square$$

Now, we prove the following useful lemma.

Lemma 3.5. *There exist $R, \beta, \varepsilon^* > 0$ and $\{y_\varepsilon\} \subset \mathbb{R}^N$ such that*

$$\int\limits_{\mathcal{B}_R(y_\varepsilon)} |u_\varepsilon|^2 dx \ge \beta, \quad \forall\, \varepsilon \in (0, \varepsilon^*). \tag{3.28}$$

Proof. Firstly, using (3.24) and (g_1), (g_2), we can note that there is $\gamma > 0$ (independent of ε) such that

$$\|u_\varepsilon\|_\varepsilon \ge \gamma > 0 \quad \forall\, \varepsilon > 0. \tag{3.29}$$

Now, we show that for any sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0, \infty)$ with $\varepsilon_n \to 0$, the limit below

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int\limits_{\mathcal{B}_r(y)} |u_{\varepsilon_n}|^2 dx = 0$$

does not hold for any $r > 0$. Otherwise, if it holds for some $r > 0$, we can use Lemma 2.1 to see that $|u_{\varepsilon_n}| \to 0$ in $L^q(\mathbb{R}^N, \mathbb{R})$ for all $q \in (2, 2^*)$. In particular, by (f_1) and (f_2) it follows that

$$\int\limits_{\mathbb{R}^N} F(|u_{\varepsilon_n}|^2) dx = \int\limits_{\mathbb{R}^N} f(|u_{\varepsilon_n}|^2)|u_{\varepsilon_n}|^2 dx = o_n(1).$$

This implies that

$$\frac{1}{2}\int\limits_{\mathbb{R}^N} G(\varepsilon_n\, x, |u_{\varepsilon_n}|^2) dx \le \frac{1}{2^*} \int\limits_{\Lambda_{\varepsilon_n} \cup \{|u_{\varepsilon_n}|^2 \le a\}} |u_{\varepsilon_n}|^{2^*} dx + \frac{V_1}{2K} \int\limits_{\Lambda_{\varepsilon_n}^c \cap \{|u_{\varepsilon_n}|^2 > a\}} |u_{\varepsilon_n}|^2 dx + o_n(1) \tag{3.30}$$

and

$$\int\limits_{\mathbb{R}^N} g(\varepsilon_n\, x, |u_{\varepsilon_n}|^2)|u_{\varepsilon_n}|^2 dx = \int\limits_{\Lambda_{\varepsilon_n} \cup \{|u_{\varepsilon_n}|^2 \le a\}} |u_{\varepsilon_n}|^{2^*} dx + \frac{V_1}{K} \int\limits_{\Lambda_{\varepsilon_n}^c \cap \{|u_{\varepsilon_n}|^2 > a\}} |u_{\varepsilon_n}|^2 dx + o_n(1), \tag{3.31}$$

where we used the notation $\Lambda^c_{\varepsilon_n} = \mathbb{R}^N \setminus \Lambda_{\varepsilon_n}$. Taking into account $\langle \mathcal{J}'_{\varepsilon_n}(u_{\varepsilon_n}), u_{\varepsilon_n}\rangle = 0$ and (3.31), we deduce that

$$\|u_{\varepsilon_n}\|^2_{\varepsilon_n} - \frac{V_1}{K} \int\limits_{\Lambda^c_{\varepsilon_n} \cap \{|u_{\varepsilon_n}|^2 > a\}} |u_{\varepsilon_n}|^2 dx = \int\limits_{\Lambda_{\varepsilon_n} \cup \{|u_{\varepsilon_n}|^2 \le a\}} |u_{\varepsilon_n}|^{2^*} dx + o_n(1). \tag{3.32}$$

Let $\ell \ge 0$ be such that

$$\|u_{\varepsilon_n}\|^2_{\varepsilon_n} - \frac{V_1}{K} \int\limits_{\Lambda^c_{\varepsilon_n} \cap \{|u_{\varepsilon_n}|^2 > a\}} |u_{\varepsilon_n}|^2 dx \to \ell.$$

It is easy to see that $\ell > 0$, otherwise $u_{\varepsilon_n} \to 0$ in H_{ε_n} and this is impossible in view of (3.29). It follows from (3.32) that

$$\int\limits_{\Lambda_{\varepsilon_n} \cup \{|u_{\varepsilon_n}|^2 \le a\}} |u_{\varepsilon_n}|^{2^*} dx \to \ell.$$

Using $\mathcal{J}_{\varepsilon_n}(u_{\varepsilon_n}) - \frac{1}{2^*}\langle \mathcal{J}'_{\varepsilon}(u_{\varepsilon_n}), u_{\varepsilon_n}\rangle = c_{\varepsilon_n}$, (3.30) and (3.31) we can see that $\ell \le N \liminf_{n\to\infty} c_{\varepsilon_n}$. Now, by the definition of S_*, (V_1) and (2.2) we obtain that

$$\begin{aligned}
&\|u_{\varepsilon_n}\|^2_{\varepsilon_n} - \frac{V_1}{K} \int\limits_{\Lambda^c_{\varepsilon_n} \cap \{|u_{\varepsilon_n}|^2 > a\}} |u_{\varepsilon_n}|^2 dx \\
&\ge \int\limits_{\mathbb{R}^N} |\nabla_{\varepsilon_n} u_{\varepsilon_n}|^2 dx + V_1\left(1 - \frac{1}{K}\right) \int\limits_{\Lambda^c_{\varepsilon_n} \cap \{|u_{\varepsilon_n}|^2 > a\}} |u_{\varepsilon_n}|^2 dx \\
&\ge \int\limits_{\mathbb{R}^N} |\nabla |u_{\varepsilon_n}||^2 dx \\
&\ge S_* \left(\int\limits_{\mathbb{R}^N} |u_{\varepsilon_n}|^{2^*} dx \right)^{\frac{2}{2^*}} \\
&\ge S_* \left(\int\limits_{\Lambda_{\varepsilon_n} \cup \{|u_{\varepsilon_n}|^2 \le a\}} |u_{\varepsilon_n}|^{2^*} dx \right)^{\frac{2}{2^*}},
\end{aligned}$$

that is

$$\|u_{\varepsilon_n}\|^2_{\varepsilon_n} - \frac{V_1}{K} \int\limits_{\Lambda^c_{\varepsilon_n} \cap \{|u_{\varepsilon_n}|^2 > a\}} |u_{\varepsilon_n}|^2 dx \ge S_* \left(\int\limits_{\Lambda_{\varepsilon_n} \cup \{|u_{\varepsilon_n}|^2 \le a\}} |u_{\varepsilon_n}|^{2^*} dx \right)^{\frac{2}{2^*}}$$

and taking the limit as $n \to \infty$ we can infer that $\ell \ge S_* \ell^{\frac{2}{2^*}}$. Then we can deduce that

$$\liminf_{n\to\infty} c_{\varepsilon_n} \ge \frac{1}{N} S_*^{\frac{N}{2}}$$

which contradicts Lemma 3.4. □

We conclude this section by proving the following compactness result which will be fundamental for showing that the solutions of the modified problem are solutions of the original problem.

Lemma 3.6. *Let $\varepsilon_n \to 0^+$ and $\{u_n\}_{n\in\mathbb{N}} \subset H_{\varepsilon_n}$ be such that $\mathcal{J}_{\varepsilon_n}(u_n) = c_{\varepsilon_n}$ and $\mathcal{J}'_{\varepsilon_n}(u_n) = 0$. Then there exists $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that the translated sequence*

$$\tilde{u}_n(x) := |u_n|(x + \tilde{y}_n)$$

has a subsequence which converges in $H^1(\mathbb{R}^N, \mathbb{R})$. Moreover, up to a subsequence, $\{y_n\}_{n\in\mathbb{N}} := \{\varepsilon_n \tilde{y}_n\}_{n\in\mathbb{N}}$ is such that $y_n \to y_0$ for some $y_0 \in \Lambda$ such that $V(y_0) = V_0$.

Proof. Taking into account $\mathcal{J}_{\varepsilon_n}(u_n) = c_{\varepsilon_n}$, $\langle \mathcal{J}'_{\varepsilon_n}(u_n), u_n\rangle = 0$ and Lemma 3.4, we can argue as in the proof of Lemma 3.3 to deduce that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in H_{ε_n}. Therefore, proceeding as in Lemma 3.5, we can find a sequence $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and constants $R, \alpha > 0$ such that

$$\liminf_{n\to\infty} \int_{\mathcal{B}_R(\tilde{y}_n)} |u_n|^2 dx \geq \alpha.$$

Set $\tilde{u}_n(x) := |u_n|(x + \tilde{y}_n)$. Then, by (2.2), it follows that $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N, \mathbb{R})$, and we may assume that

$$\tilde{u}_n \rightharpoonup \tilde{u} \text{ weakly in } H^1(\mathbb{R}^N, \mathbb{R}). \tag{3.33}$$

Moreover, $\tilde{u} \neq 0$ in view of

$$\int_{\mathcal{B}_R} |\tilde{u}|^2 dx \geq \alpha. \tag{3.34}$$

Now, we set $y_n := \varepsilon_n \tilde{y}_n$. Let us begin by proving that $\{y_n\}_{n\in\mathbb{N}}$ is bounded. To this end, it is enough to show the following claim:

Claim 1. $\lim_{n\to\infty} dist(y_n, \overline{\Lambda}) = 0$.

Indeed, if the claim does not hold, there exists $\delta > 0$ and a subsequence of $\{y_n\}_{n\in\mathbb{N}}$, still denoted by itself, such that

$$dist(y_n, \overline{\Lambda}) \geq \delta \quad \forall n \in \mathbb{N}.$$

Then we can find $r > 0$ such that $\mathcal{B}_r(y_n) \subset \Lambda^c$ for all $n \in \mathbb{N}$. Since $\tilde{u} \geq 0$ and $C_c^\infty(\mathbb{R}^N, \mathbb{R})$ is dense in $H^1(\mathbb{R}^N, \mathbb{R})$ (see [1]), we can find a sequence $\{\psi_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^N, \mathbb{R})$ such that $\psi_j \geq 0$ and $\psi_j \to \tilde{u}$ in $H^1(\mathbb{R}^N, \mathbb{R})$.

Now, thanks to Kato's inequality (2.3), we can note that $\tilde{u}_n$ satisfies (in weak sense)

$$-\Delta \tilde{u}_n + V(\varepsilon_n x + \varepsilon_n \tilde{y}_n)\tilde{u}_n \leq g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, |\tilde{u}_n|^2)\tilde{u}_n \text{ in } \mathbb{R}^N. \tag{3.35}$$

Then, fixed $j \in \mathbb{N}$ and taking ψ_j as test function in (3.35), we get

$$\int_{\mathbb{R}^N} \nabla \tilde{u}_n \nabla \psi_j \, dx + \int_{\mathbb{R}^N} V(\varepsilon_n x + \varepsilon_n \tilde{y}_n)\tilde{u}_n \psi_j \, dx \leq \int_{\mathbb{R}^N} g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, |\tilde{u}_n|^2)\tilde{u}_n \psi_j \, dx. \tag{3.36}$$

Since $\tilde{u}_n, \psi_j \geq 0$ and using (g_2), $\mathcal{B}_r(y_n) \subset \Lambda^c$ and that $g(x,t) = \tilde{f}(t) \leq \frac{V_1}{K}$ for $(x,t) \in \Lambda^c \times \mathbb{R}$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, |\tilde{u}_n|^2)\tilde{u}_n\psi_j \, dx &= \int_{\mathcal{B}_{r/\varepsilon_n}} g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, |\tilde{u}_n|^2)\tilde{u}_n\psi_j \, dx \\ &+ \int_{\mathbb{R}^N \setminus \mathcal{B}_{r/\varepsilon_n}} g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, |\tilde{u}_n|^2)\tilde{u}_n\psi_j \, dx \\ &\leq \frac{V_1}{K} \int_{\mathcal{B}_{r/\varepsilon_n}} \tilde{u}_n\psi_j \, dx + \int_{\mathbb{R}^N \setminus \mathcal{B}_{r/\varepsilon_n}} \left(f(|\tilde{u}_n|^2)\tilde{u}_n\psi_j + \tilde{u}_n^{2^*-1}\psi_j \right) dx\end{aligned}$$

which together with (3.36) implies that

$$\int_{\mathbb{R}^N} \nabla\tilde{u}_n \nabla\psi_j \, dx + \mu_0 \int_{\mathbb{R}^N} \tilde{u}_n\psi_j \, dx \leq \int_{\mathbb{R}^N \setminus \mathcal{B}_{r/\varepsilon_n}} \left(f(|\tilde{u}_n|^2)\tilde{u}_n\psi_j + \tilde{u}_n^{2^*-1}\psi_j \right) dx \tag{3.37}$$

where $\mu_0 = V_1(1 - \frac{1}{K})$. By (3.33), ψ_j has compact support in $\mathbb{R}^N$ and $\varepsilon_n \to 0$ we can deduce that as $n \to \infty$

$$\int_{\mathbb{R}^N} \nabla\tilde{u}_n \nabla\psi_j \, dx \to \int_{\mathbb{R}^N} \nabla\tilde{u} \nabla\psi_j \, dx$$

and

$$\int_{\mathbb{R}^N \setminus \mathcal{B}_{r/\varepsilon_n}} \left(f(|\tilde{u}_n|^2)\tilde{u}_n\psi_j + \tilde{u}_n^{2^*-1}\psi_j \right) dx \to 0.$$

The above limits and (3.37) yield

$$\int_{\mathbb{R}^N} \nabla\tilde{u} \nabla\psi_j \, dx + \mu_0 \int_{\mathbb{R}^N} \tilde{u}\psi_j \, dx \leq 0$$

and taking the limit as $j \to \infty$ we obtain that

$$\|\tilde{u}\|^2_{\mu_0} = |\nabla\tilde{u}|_2^2 + \mu_0|\tilde{u}|_2^2 \leq 0$$

which contradicts (3.34). Hence, there exists a subsequence of $\{y_n\}_{n\in\mathbb{N}}$ such that $y_n \to y_0 \in \overline{\Lambda}$.

Claim 2. $y_0 \in \Lambda$.

Using (g_2) and (3.36) we can see that

$$\int_{\mathbb{R}^N} \nabla\tilde{u}_n \nabla\psi_j \, dx + \int_{\mathbb{R}^N} V(\varepsilon_n x + \varepsilon_n \tilde{y}_n)\tilde{u}_n\psi_j \, dx \leq \int_{\mathbb{R}^N} (f(|\tilde{u}_n|^2)\tilde{u}_n + \tilde{u}_n^{2^*-1})\psi_j \, dx.$$

Letting $n \to \infty$ we get

$$\int_{\mathbb{R}^N} \nabla\tilde{u} \nabla\psi_j \, dx + \int_{\mathbb{R}^N} V(y_0)\tilde{u}\psi_j \, dx \leq \int_{\mathbb{R}^N} (f(|\tilde{u}|^2)\tilde{u} + \tilde{u}^{2^*-1})\psi_j \, dx,$$

and passing to the limit as $j \to \infty$ we obtain

$$\int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 \, dx + \int_{\mathbb{R}^N} V(y_0)|\tilde{u}|^2 \, dx \leq \int_{\mathbb{R}^N} f(|\tilde{u}|^2)|\tilde{u}|^2 + |\tilde{u}|^{2^*} \, dx.$$

Accordingly, we can find $\tau \in (0,1)$ such that $\tau\tilde{u} \in \mathcal{N}_{V(y_0)}$. Hence, denoting by $c_{V(y_0)}$ the mountain pass level associated with $\mathcal{I}_{V(y_0)}$, and using (2.2) and Lemma 3.4, we have

$$c_{V(y_0)} \leq \mathcal{I}_{V(y_0)}(\tau u) \leq \liminf_{n\to\infty} \mathcal{J}_{\varepsilon_n}(u_{\varepsilon_n}) = \liminf_{n\to\infty} c_{\varepsilon_n} \leq c_{V_0}$$

from which we deduce that $V(y_0) \leq V(0) = V_0$. Since $V_0 = \inf_{\overline{\Lambda}} V$, we can infer that $V(y_0) = V_0$. By (V_2), it follows that $y_0 \notin \partial\Lambda$, that is $y_0 \in \Lambda$.

Claim 3. *$\tilde{u}_n \to \tilde{u}$ in $H^1(\mathbb{R}^N, \mathbb{R})$ as $n \to \infty$.*

Put

$$\tilde{\Lambda}_n = \frac{\Lambda - \varepsilon_n \tilde{y}_n}{\varepsilon_n},$$

and define

$$\begin{aligned} &\tilde{\chi}^1_n(x) = \begin{cases} 1 & \text{if } x \in \tilde{\Lambda}_n, \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \tilde{\Lambda}_n, \end{cases} \\ &\tilde{\chi}^2_n(x) = 1 - \tilde{\chi}^1_n(x). \end{aligned}$$

Now, we introduce the following functions for all $x \in \mathbb{R}^N$

$$\begin{aligned}
h^1_n(x) &= \left(\frac{1}{2} - \frac{1}{\vartheta}\right) V(\varepsilon_n x + \varepsilon_n \tilde{y}_n)|\tilde{u}_n(x)|^2 \tilde{\chi}^1_n(x) \\
h^1(x) &= \left(\frac{1}{2} - \frac{1}{\vartheta}\right) V(y_0)|\tilde{u}(x)|^2 \\
h^2_n(x) &= \left[\left(\frac{1}{2} - \frac{1}{\vartheta}\right) V(\varepsilon_n x + \varepsilon_n \tilde{y}_n)|\tilde{u}_n(x)|^2 + \frac{1}{\vartheta} g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, |\tilde{u}_n(x)|^2)|\tilde{u}_n(x)|^2 \right. \\
&\left. -\frac{1}{2} G(\varepsilon_n x + \varepsilon_n \tilde{y}_n, |\tilde{u}_n(x)|^2)\right] \tilde{\chi}^2_n(x) \geq \left(\left(\frac{1}{2} - \frac{1}{\vartheta}\right) - \frac{1}{K}\right) V(\varepsilon_n x + \varepsilon_n \tilde{y}_n)|\tilde{u}_n(x)|^2 \tilde{\chi}^2_n(x) \\
h^3_n(x) &= \left(\frac{1}{\vartheta} g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, |\tilde{u}_n(x)|^2)|\tilde{u}_n(x)|^2 - \frac{1}{2} G(\varepsilon_n x + \varepsilon_n \tilde{y}_n, |\tilde{u}_n(x)|^2)\right) \tilde{\chi}^1_n(x) \\
&= \left[\frac{1}{\vartheta}\left(f(|\tilde{u}_n(x)|^2)|\tilde{u}_n(x)|^2 + |\tilde{u}_n(x)|^{2^*}\right) - \left(\frac{1}{2} F(|\tilde{u}_n(x)|^2) + \frac{1}{2^*}|\tilde{u}_n(x)|^{2^*}\right)\right] \tilde{\chi}^1_n(x) \\
h^3(x) &= \frac{1}{\vartheta}\left(f(|\tilde{u}(x)|^2)|\tilde{u}(x)|^2 + |\tilde{u}(x)|^{2^*}\right) - \left(\frac{1}{2} F(|\tilde{u}(x)|^2) + \frac{1}{2^*}|\tilde{u}(x)|^{2^*}\right).
\end{aligned}$$

In view of (f_3) and (g_3), we can observe that all the above functions are nonnegative. Moreover, using (3.33) and Claim 2, we can see that

$$\begin{aligned} &\tilde{u}_n(x) \to \tilde{u}(x) \quad \text{a.e. } x \in \mathbb{R}^N, \\ &\varepsilon_n \tilde{y}_n \to y_0 \in \Lambda, \end{aligned}$$

which imply that

$$\tilde{\chi}_n^1(x) \to 1,\ h_n^1(x) \to h^1(x),\ h_n^2(x) \to 0 \ \text{ and } \ h_n^3(x) \to h^3(x) \ \text{ a.e. } x \in \mathbb{R}^N.$$

Thus, by Fatou's Lemma, Lemma 3.4 and (2.2) we get

$$\begin{aligned}
c_{V_0} &\geq \limsup_{n\to\infty} c_{\varepsilon_n} = \limsup_{n\to\infty} \left(\mathcal{J}_{\varepsilon_n}(u_n) - \frac{1}{\vartheta}\langle \mathcal{J}'_{\varepsilon_n}(u_n), u_n\rangle \right) \\
&\geq \limsup_{n\to\infty} \left[\left(\frac{1}{2} - \frac{1}{\vartheta}\right) |\nabla \tilde{u}_n|_2^2 + \int_{\mathbb{R}^N} (h_n^1 + h_n^2 + h_n^3)\, dx \right] \\
&\geq \liminf_{n\to\infty} \left[\left(\frac{1}{2} - \frac{1}{\vartheta}\right) |\nabla \tilde{u}_n|_2^2 + \int_{\mathbb{R}^N} (h_n^1 + h_n^2 + h_n^3)\, dx \right] \\
&\geq \left(\frac{1}{2} - \frac{1}{\vartheta}\right) |\nabla \tilde{u}|_2^2 + \int_{\mathbb{R}^N} (h^1 + h^3)\, dx \geq c_{V_0},
\end{aligned}$$

which yields

$$\lim_{n\to\infty} |\nabla \tilde{u}_n|_2^2 = |\nabla \tilde{u}|_2^2 \tag{3.38}$$

and

$$h_n^1 \to h^1,\ h_n^2 \to 0 \ \text{ and } \ h_n^3 \to h^3 \ \text{ in } \ L^1(\mathbb{R}^N, \mathbb{R}).$$

Therefore,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V(\varepsilon_n x + \varepsilon_n \tilde{y}_n)|\tilde{u}_n|^2\, dx = \int_{\mathbb{R}^N} V(y_0)|\tilde{u}|^2\, dx,$$

from which we deduce that

$$\lim_{n\to\infty} |\tilde{u}_n|_2^2 = |\tilde{u}|_2^2. \tag{3.39}$$

Putting together (3.38) and (3.39) and using the fact that $H^1(\mathbb{R}^N, \mathbb{R})$ is a Hilbert space, we obtain

$$\|\tilde{u}_n - \tilde{u}\|^2 = \|\tilde{u}_n\|^2 - \|\tilde{u}\|^2 + o_n(1) = o_n(1).$$

This fact ends the proof of lemma. $\square$

4. Proof of Theorem 1.1

This last section is devoted to the proof of Theorem 1.1. Firstly, we prove the following result:

Lemma 4.1. *Let $\varepsilon_n \to 0$ and $u_n \in H_{\varepsilon_n}$ be a mountain pass solution to (3.2). Then, up to a subsequence, $v_n := |u_n|(\cdot + \tilde{y}_n) \in L^\infty(\mathbb{R}^N, \mathbb{R})$, where $\{\tilde{y}_n\}_{n\in\mathbb{N}}$ is defined as in Lemma 3.6, and there exists $C > 0$ such that*

$$|v_n|_\infty \leq C \quad \textit{for all } n \in \mathbb{N}.$$

Moreover, $v_n(x) \to 0$ as $|x| \to \infty$ uniformly in $n \in \mathbb{N}$.

Proof. The proof of this result can be obtained arguing as in Lemma 4.1 in [6]. Alternatively, we can use Kato's inequality (2.3) and (V_1) to see that v_n satisfies

$$-\Delta v_n + V_1 v_n \leq f(v_n^2)v_n + v_n^{2^*-1} \text{ in } \mathbb{R}^N.$$

In view of assumptions (f_1) and (f_2) and using a Moser iteration argument [28] (see Proposition 2.2 in [13]), we can prove that $v_n \in L^\infty(\mathbb{R}^N, \mathbb{R})$ and $|v_n|_\infty \leq C$ for all $n \in \mathbb{N}$, for some $C > 0$ independent of n (we use the fact that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in H_{ε_n}). In particular, arguing as in Lemma 5 in [25] or Proposition 2.2 in [13], we can see that $v_n(x)$ decays at zero (in exponential way) as $|x| \to \infty$ uniformly in $n \in \mathbb{N}$. □

Now, we are ready to give the proof of the main result of this work.

Proof of Theorem 1.1. We begin by proving that there exists $\tilde{\varepsilon}_0 > 0$ such that for any $\varepsilon \in (0, \tilde{\varepsilon}_0)$ and any solution $u_\varepsilon \in H_\varepsilon$ of (3.2), we have

$$|u_\varepsilon|_{L^\infty(\mathbb{R}^N \setminus \Lambda_\varepsilon)} < \sqrt{a}. \tag{4.1}$$

Assume by contradiction that for some subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that $\varepsilon_n \to 0$, we can find $u_{\varepsilon_n} \in H_{\varepsilon_n}$ such that $\mathcal{J}_{\varepsilon_n}(u_{\varepsilon_n}) = c_{\varepsilon_n}$, $\mathcal{J}'_{\varepsilon_n}(u_{\varepsilon_n}) = 0$ and

$$|u_{\varepsilon_n}|_{L^\infty(\mathbb{R}^N \setminus \Lambda_{\varepsilon_n})} \geq \sqrt{a}. \tag{4.2}$$

In view of Lemma 3.6, there is $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $\tilde{u}_n = |u_{\varepsilon_n}|(\cdot + \tilde{y}_n) \to \tilde{u}$ in $H^1(\mathbb{R}^N, \mathbb{R})$ and $\varepsilon_n \tilde{y}_n \to y_0$ for some $y_0 \in \Lambda$ such that $V(y_0) = V_0$.

Now, if we choose $r > 0$ such that $\mathcal{B}_r(y_0) \subset \mathcal{B}_{2r}(y_0) \subset \Lambda$, we can see that $\mathcal{B}_{\frac{r}{\varepsilon_n}}(\frac{y_0}{\varepsilon_n}) \subset \Lambda_{\varepsilon_n}$. Then, for any $y \in \mathcal{B}_{\frac{r}{\varepsilon_n}}(\tilde{y}_n)$ it holds

$$\left|y - \frac{y_0}{\varepsilon_n}\right| \leq |y - \tilde{y}_n| + \left|\tilde{y}_n - \frac{y_0}{\varepsilon_n}\right| < \frac{1}{\varepsilon_n}(r + o_n(1)) < \frac{2r}{\varepsilon_n} \text{ for } n \text{ sufficiently large.}$$

Hence,

$$\mathbb{R}^N \setminus \Lambda_{\varepsilon_n} \subset \mathbb{R}^N \setminus \mathcal{B}_{\frac{r}{\varepsilon_n}}(\tilde{y}_n) \tag{4.3}$$

for any n big enough. Using Lemma 4.1, we can see that

$$\tilde{u}_n(x) \to 0 \text{ as } |x| \to \infty \tag{4.4}$$

uniformly in $n \in \mathbb{N}$. Therefore there exists $R > 0$ such that

$$\tilde{u}_n(x) < \sqrt{a} \text{ for } |x| \geq R, n \in \mathbb{N}.$$

Consequently, $|u_{\varepsilon_n}(x)| < \sqrt{a}$ for any $x \in \mathbb{R}^N \setminus \mathcal{B}_R(\tilde{y}_n)$ and $n \in \mathbb{N}$. On the other hand, (4.3) implies that there exists $\nu \in \mathbb{N}$ such that for any $n \geq \nu$ we have

$$\mathbb{R}^N \setminus \Lambda_{\varepsilon_n} \subset \mathbb{R}^N \setminus \mathcal{B}_{\frac{r}{\varepsilon_n}}(\tilde{y}_n) \subset \mathbb{R}^N \setminus \mathcal{B}_R(\tilde{y}_n),$$

which yields $|u_{\varepsilon_n}(x)| < \sqrt{a}$ for any $x \in \mathbb{R}^N \setminus \Lambda_{\varepsilon_n}$ and $n \geq \nu$, and this contradicts (4.2). Now, since $u_\varepsilon \in H_\varepsilon$ satisfies (4.1), it follows from the definition of g that u_ε is a solution of (3.1). Since $\hat{u}(x) = u(x/\varepsilon)$ is a solution to (1.1), we can conclude that (1.1) has a nontrivial solution. Finally, we study the behavior of the maximum points of solutions to problem (1.1). Take $\varepsilon_n \to 0$ and consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subset H_{\varepsilon_n}$ of solutions to (3.1) as above. Let us observe that (g_1) implies that we can find $\gamma > 0$ such that

$$g(\varepsilon\, x, t^2)t^2 \leq \frac{V_1}{K} t^2 \quad \text{for any } x \in \mathbb{R}^N, t \leq \gamma. \tag{4.5}$$

Arguing as before, we can find $R > 0$ such that

$$|u_n|_{L^\infty(\mathbb{R}^N \setminus \mathcal{B}_R(\tilde{y}_n))} < \gamma. \tag{4.6}$$

Moreover, up to extract a subsequence, we may assume that

$$|u_n|_{L^\infty(\mathcal{B}_R(\tilde{y}_n))} \geq \gamma. \tag{4.7}$$

Indeed, if (4.7) does not hold, in view of (4.6) we can see that $|u_n|_\infty < \gamma$. Then, using $\langle \mathcal{J}'_{\varepsilon_n}(u_n), u_n\rangle = 0$ and (4.5) we can infer

$$\|u_n\|^2_{\varepsilon_n} = \int_{\mathbb{R}^N} g(\varepsilon_n\, x, |u_n|^2)|u_n|^2\, dx \leq \frac{V_1}{K} \int_{\mathbb{R}^N} |u_n|^2\, dx$$

which yields $\|u_n\|_{\varepsilon_n} = 0$, and this is impossible. Hence, (4.7) is satisfied.

Taking into account (4.6) and (4.7), we can deduce that the maximum points $p_n \in \mathbb{R}^N$ of u_n belong to $\mathcal{B}_R(\tilde{y}_n)$. Therefore $p_n = \tilde{y}_n + q_n$, for some $q_n \in \mathcal{B}_R$. Consequently, $\eta_{\varepsilon_n} = \varepsilon_n\, \tilde{y}_n + \varepsilon_n\, q_n$ is the maximum point of $|\hat{u}_n|(x) = |u_n|(x/\varepsilon_n)$. Since $|q_n| < R$ for any $n \in \mathbb{N}$ and $\varepsilon_n\, \tilde{y}_n \to y_0$, by the continuity of V we can deduce that

$$\lim_{n\to\infty} V(\eta_{\varepsilon_n}) = V(y_0) = V_0.$$

This ends the proof of Theorem 1.1. □

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References

[1] R. Adams, Sobolev Spaces, Pure and Applied Mathematics, vol. 65, Academic Press, New York-London, 1975, xviii+268 pp.
[2] C.O. Alves, D.C. de Morais Filho, M.A.S. Souto, Radially symmetric solutions for a class of critical exponent elliptic problems in $\mathbb{R}^N$, Electron. J. Differential Equations (07) (1996), 12 pp.
[3] C.O. Alves, J.M. do Ó, M.A.S. Souto, Local mountain-passes for a class of elliptic problems in $\mathbb{R}^N$ involving critical growth, Nonlinear Anal. 46 (2001) 495–510.
[4] C.O. Alves, G.M. Figueiredo, Multiplicity of positive solutions for a quasilinear problem in $\mathbb{R}^N$ via penalization method, Adv. Nonlinear Stud. 5 (4) (2005) 551–572.
[5] C.O. Alves, G.M. Figueiredo, Multiple solutions for a semilinear elliptic equation with critical growth and magnetic field, Milan J. Math. 82 (2) (2014) 389–405.
[6] C.O. Alves, G.M. Figueiredo, M.F. Furtado, Multiple solutions for a nonlinear Schrödinger equation with magnetic fields, Comm. Partial Differential Equations 36 (2011) 1565–1586.
[7] A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973) 349–381.
[8] G. Arioli, A. Szulkin, A semilinear Schrödinger equation in the presence of a magnetic field, Arch. Ration. Mech. Anal. 170 (2003) 277–295.
[9] J. Avron, I. Herbst, B. Simon, Schrödinger operators with magnetic fields. I. General interactions, Duke Math. J. 45 (1978) 847–883.

[10] S. Barile, A multiplicity result for singular NLS equations with magnetic potentials, Nonlinear Anal. 68 (11) (2008) 3525–3540.
[11] T. Bartsch, Z.X. Wang, Existence and multiplicity results for some superlinear elliptic problems in $\mathbb{R}^N$, Comm. Partial Differential Equations 20 (1995) 1725–1741.
[12] H. Brézis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math. 36 (4) (1983) 437–477.
[13] J. Chabrowski, A. Szulkin, On the Schrödinger equation involving a critical Sobolev exponent and magnetic field, Topol. Methods Nonlinear Anal. 25 (1) (2005) 3–21.
[14] S. Cingolani, Semiclassical stationary states of nonlinear Schrödinger equations with an external magnetic field, J. Differential Equations 188 (2003) 52–79.
[15] S. Cingolani, M. Clapp, Intertwining semiclassical bound states to a nonlinear magnetic Schrödinger equation, Nonlinearity 22 (9) (2009) 2309–2331.
[16] S. Cingolani, M. Lazzo, Multiple positive solutions to nonlinear Schrödinger equations with competing potential functions, J. Differential Equations 160 (2000) 118–138.
[17] S. Cingolani, S. Secchi, Semiclassical limit for nonlinear Schrödinger equations with electromagnetic fields, J. Math. Anal. Appl. 275 (2002) 108–130.
[18] M. Del Pino, P.L. Felmer, Local mountain pass for semilinear elliptic problems in unbounded domains, Calc. Var. Partial Differential Equations 4 (1996) 121–137.
[19] Y. Ding, X. Liu, Semiclassical solutions of Schrödinger equations with magnetic fields and critical nonlinearities, Manuscripta Math. 140 (1–2) (2013) 51–82.
[20] M. Esteban, P.L. Lions, Stationary solutions of nonlinear Schrödinger equations with an external magnetic field, in: Partial Differential Equations and the Calculus of Variations, vol. I, in: Progr. Nonlinear Differential Equations Appl., vol. 1, Birkhäuser Boston, Boston, MA, 1989, pp. 401–449.
[21] G.M. Figueiredo, M. Furtado, Positive solutions for a quasilinear Schrödinger equation with critical growth, J. Dynam. Differential Equations 24 (1) (2012) 13–28.
[22] A. Floer, A. Weinstein, Nonspreading wave packets fort the cubic Schrödinger equation with a bounded potential, J. Funct. Anal. 69 (1986) 397–408.
[23] L. Jeanjean, K. Tanaka, Singularly perturbed elliptic problems with superlinear or asymptotically linear nonlinearities, Calc. Var. 21 (2004) 287–318.
[24] T. Kato, Schrödinger operators with singular potentials, in: Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces, Jerusalem, 1972, Israel J. Math. 13 (1973) 135–148.
[25] K. Kurata, Existence and semi-classical limit of the least energy solution to a nonlinear Schrödinger equation with electromagnetic fields, Nonlinear Anal. 41 (2000) 763–778.
[26] P.L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. II, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (4) (1984) 223–283.
[27] P.L. Lions, The concentration-compactness principle in the calculus of variations. The limit case. Part I, Rev. Mat. Iberoam. 1 (1) (1985) 145–201.
[28] J. Moser, A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations, Comm. Pure Appl. Math. 13 (1960) 457–468.
[29] P. Rabinowitz, On a class of nonlinear Schrödinger equations, Z. Angew. Math. Phys. 43 (2) (1992) 270–291.
[30] M. Reed, B. Simon, Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness, Academic Press, New York-London, 1975, xv+361 pp.
[31] G. Talenti, Best constants in Sobolev inequality, Ann. Mat. 110 (1976) 353–372.
[32] M. Willem, Minimax Theorems, Birkhäuser, 1996.

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