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Existence of least-energy sign-changing solutions for Schrödinger-Poisson system with critical growth

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ABSTRACT

In this paper, we study the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = |u|^4 u + \mu f(u), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $V(x)$ is a smooth function and $\mu, \lambda > 0$. Under suitable conditions on f, by using constraint variational method and the quantitative deformation lemma, if μ is large enough, we obtain a least-energy sign-changing (or nodal) solution u_λ to this problem for each $\lambda > 0$, and its energy is strictly larger than twice that of the ground state solutions. Moreover, we study the asymptotic behavior of u_λ as the parameter $\lambda \searrow 0$.

1. Introduction and main results

In the past decades, a great attention has been given to the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = f(x,u), & x \in \mathbb{R}^N, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $V(x)$ is a smooth function and $\lambda > 0$. System (1.1) derives from time-dependent Schrödinger-Poisson equation, which describes quantum (nonrelativistic) particles interacting with the electromagnetic field generated by the motion. For more details on the mathematical and physical background of the system (1.1), we refer the readers to the paper [10,30,31] and the references therein.

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Since so-called nonlocal term $\lambda\phi_u(x)u$ is involving, system (1.1) is called the nonlocal problems. The appearance of nonlocal term in the equations embodies its importance in many physical applications, but it brings some difficulties and challenges from a mathematical point of view. This fact makes the study of system (1.1) particularly interesting. Therefore, a lot of researches have been carried out in the past decades, such as those in [3,4,9,13–16,18,19,21,23,24,28,29,33,36,41,42] and the references therein.

Recently, some authors began to focus on sign-changing solutions of system (1.1) [1,2,9,12,20,22,25,26,32,34,35,37]. Especially, when $\mathbb{R}^3$ is replaced by a bounded domain with smooth boundary, Alves and Souto [1] studied the existence of least-energy sign-changing solution for system (1.1). Later, Alves, Souto and Soares [2] improved and generalized results obtained in [1] to whole space $\mathbb{R}^3$. In case of $f(x,u) = |u|^{p-1}u$, $p \in (3,5)$, combining constraint variational method with the Brouwer degree theory, Wang and Zhou [37] got the existence of sign-changing solution for system (1.1) for all $\lambda > 0$ and proved that if $V(x) \equiv 1$, the energy of least-energy sign-changing radial solution of (1.1) is strictly larger than the least energy for $\lambda > 0$ small. Shuai and Wang [32] noticed that the method used in [37] strongly depends on the fact that the nonlinearity is homogeneous, which cannot be applied to a more general nonlinearity $f(u)$ directly, they studied the system (1.1) in which $f(x,u)$ was replaced by $f(u)$. Via the constraint variational method and quantitative deformation lemma, they obtained the existence and asymptotic behavior of least-energy sign-changing solution for system (1.1). Later, under some more weaker assumptions on f (especially, Nehari type monotonicity condition been removed), Cheng and Tang [12] improved and generalized results obtained in [32]. However, to the best of our knowledge, few people have studied the existence and asymptotic behavior of sign-changing solutions for Schrödinger-Poisson system in case of critical growth except for the study in [43]. In [43], Zhong and Tang considered the following Schrödinger-Poisson system with critical growth

$$\begin{cases} -\Delta u + u + k(x)\phi u = |u|^4 u + \lambda f(x)u, & x \in \mathbb{R}^3, \\ -\Delta \phi = k(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

where k and f are nonnegative functions, $0 < \lambda < \lambda_1$ and λ_1 is the first eigenvalue of the problem $-\Delta u + u = \lambda f(x)u$ in $H^1(\mathbb{R}^3)$. Via the constraint variational method, they obtained the existence of least-energy sign-changing solution for system (1.2) and showed that its energy is strictly larger than two times of the least energy. However, if $k(x) \equiv 1$, their method seems invalid because their results depend on $k \in L^p(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ for some $p \in [2,\infty)$. Furthermore, they did not study the asymptotic behavior of sign-changing solution.

Motivated by the above references, in this article, we are interested in the existence of the least-energy sign-changing solution for the following Schrödinger-Poisson system with critical growth

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = |u|^4 u + \mu f(u), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $\mu, \lambda > 0$. As in [32,37], to make it easier to check the compactness, we always assume that $V \in C(\mathbb{R}^3, \mathbb{R}^+)$ and satisfies:

(V) $H \subset H^1(\mathbb{R}^3)$ such that, for $2 < p < 6$, the embedding $H \hookrightarrow L^p(\mathbb{R}^3)$ is compact, where H is given by

$$H = \begin{cases} H^1_r(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) : u(x) = u(|x|)\}, & \text{if } V(x) \text{ is a constant}, \\ \{u \in D^{1,2}(\mathbb{R}^3) : \int\limits_{\mathbb{R}^3} V(x)u^2 \mathrm{d}x < \infty\}, & \text{if } V(x) \text{ is not a constant}, \end{cases}$$

with the norm

$$\|u\|^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx.$$

As for the function f, we assume $f \in C^1(\mathbb{R}, \mathbb{R})$ and satisfies the following hypotheses:

(f_1) $\lim_{t\to 0} \frac{f(t)}{t^3} = 0$;
(f_2) There exists $q \in (4, 6)$ such that $\lim_{t\to\infty} \frac{f(t)}{t^{q-1}} = 0$;
(f_3) $\frac{f(t)}{|t|^3}$ is an increasing function of $t \in \mathbb{R}\backslash\{0\}$.

Before presenting our main results, we denote $L^p(\mathbb{R}^3)$ a Lebesgue space with the norm $|u|_p := (\int_{\mathbb{R}^3} |u|^p dx)^{\frac{1}{p}}$, $1 \le p < \infty$.

It is well known that, by the Lax-Milgram Theorem, for given $u \in H$, there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that $-\Delta\phi_u = u^2$. Furthermore,

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy. \tag{1.4}$$

Substituting (1.4) into system (1.3), we can rewrite system (1.3) as the following equivalent form

$$-\Delta u + V(x)u + \lambda\phi_u u = |u|^4 u + \mu f(u), \;\; x \in \mathbb{R}^3. \tag{1.5}$$

Therefore, the energy functional associated with system (1.3) is defined by

$$I_\lambda^\mu(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx + \frac{\lambda}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \mu\int_{\mathbb{R}^3} F(u)dx - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 dx, \forall u \in H.$$

Moreover, $I_\lambda^\mu(u)$ belongs to $C^1(H, \mathbb{R})$ and

$$\langle (I_\lambda^\mu)'(u), v\rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv)dx + \lambda\int_{\mathbb{R}^3} \phi_u uv dx - \mu\int_{\mathbb{R}^3} f(u)v dx - \int_{\mathbb{R}^3} |u|^4 uv dx$$

for any $u, v \in H$.

The solution of system (1.3) is the critical point of the functional $I_\lambda^\mu(u)$. If $u \in H$ is a solution of system (1.3) and $u^\pm \neq 0$, then u is a sign-changing solution of system (1.3), where $u^+ = \max\{u(x), 0\}$, $u^- = \min\{u(x), 0\}$.

In fact, there are some essential differences between $\lambda > 0$ and $\lambda = 0$ when we study the sign-changing solutions of system (1.3). Since these differences are obviously caused by the nonlocal term $\lambda\phi_u(x)u$, some good methods of seeking sign-changing solutions for local problems, for example [5–8,11,39,44], seems not be applicable to nonlocal problems. Therefore, as in [1,12,17,32,37,38], to overcome the difficulties and challenges stem from the nonlocal term, we borrow some ideas from [5]. Specifically, we first try to seek a minimizer of the energy functional I_λ^μ over the constraint $\mathcal{M}_\lambda^\mu = \{u \in H, u^\pm \neq 0 \text{ and } \langle (I_\lambda^\mu)'(u), u^+\rangle = \langle (I_\lambda^\mu)'(u), u^-\rangle = 0\}$, and then prove that the minimizer is a sign-changing solution of system (1.3).

Since system (1.3) involves critical exponent in the nonlinearity, it is rather difficult to show that $\inf_{u\in\mathcal{M}_\lambda^\mu} I_\lambda^\mu(u)$ is achieved in $\mathcal{M}_\lambda^\mu$ because of the lack of the compactness caused by the critical term. As we will see later, this problem prevents us from using the way as in [1,12,32,37]. So we need some new ideas to deal with this essential problem (see Lemma 2.2 and Lemma 2.3).

The main results can be stated as follows.

Theorem 1.1. *Suppose that* $(f_1)-(f_3)$ *are satisfied. Then, there exists* $\mu^\star > 0$ *such that for all* $\mu \geq \mu^\star$ *and each* $\lambda > 0$*, the system* (1.3) *has a least-energy sign-changing solution* u_λ*, which has precisely two nodal domains.*

Theorem 1.2. *Suppose that* $(f_1)-(f_3)$ *are satisfied. Then, there exists* $\mu^{\star\star} > 0$ *such that for all* $\mu \geq \mu^{\star\star}$ *and each* $\lambda > 0$*, the* $c^* > 0$ *is achieved and* $I^\mu_\lambda(u_\lambda) > 2c^*$*, where* $c^* = \inf_{u\in\mathcal{N}^\mu_\lambda} I^\mu_\lambda(u)$*,* $\mathcal{N}^\mu_\lambda = \{u \in H\setminus \{0\} | \langle (I^\mu_\lambda)'(u), u\rangle = 0\}$*, and* u_λ *is the least-energy sign-changing solution obtained in Theorem 1.1. In particular,* $c^* > 0$ *is achieved either by a positive or a negative function.*

Remark 1.1. Theorem 1.2 shows that the energy of any sign-changing solution for system (1.3) is strictly larger than two times of the least energy. If $\lambda = 0$, this property is called energy doubling by Weth in [39].

Theorem 1.3. *Suppose that* $(f_1)-(f_3)$ *are satisfied. Then, there exists* $\mu^{\star\star\star} > 0$ *such that for all* $\mu \geq \mu^{\star\star\star}$*, for any sequence* $\{\lambda_n\}$ *with* $\lambda_n \to 0$ *as* $n \to \infty$*, there exists a subsequence, still denoted by* $\{\lambda_n\}$*, such that* u_{λ_n} *converges to* u_0 *weakly in* H *as* $n \to \infty$*, where* u_{λ_n} *is a least-energy sign-changing solution of system* (1.3) *with* $\lambda = \lambda_n$ *and* u_0 *is a least-energy sign-changing solution to the following problem*

$$-\Delta u + V(x)u = |u|^4 u + \mu f(u), \quad x \in \mathbb{R}^3. \tag{1.6}$$

Remark 1.2. It is noticed that, although our paper and [43] both study the critical problem, the method used in our paper to overcome difficulties COMING from the critical term is different from the one used in [43].

2. Technical lemmas

Proposition 2.1 *([13,28,41]). For any* $u \in H$*, we have*

- (i) *there exists* $C > 0$ *such that* $\int_{\mathbb{R}^3} \phi_u u^2 dx \leq C\|u\|^4 \ \forall u \in H$*;*
- (ii) $\phi_u \geq 0$*,* $\forall u \in H$*;*
- (iii) $\phi_{tu} = t^2\phi_u$*,* $\forall t > 0$ *and* $u \in H$*;*
- (iv) *if* $u_n \rightharpoonup u$ *in* H*, then* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $D^{1,2}(\mathbb{R}^3)$*.*

Now, fixed $u \in H$ with $u^\pm \neq 0$, we define function $\psi_u : [0,\infty) \times [0,\infty) \to \mathbb{R}$ and mapping $W_u : [0,\infty) \times [0,\infty) \to \mathbb{R}^2$ by

$$\psi_u(s,t) = I^\mu_\lambda(su^+ + tu^-),$$
$$W_u(s,t) = (\langle (I^\mu_\lambda)'(su^+ + tu^-), su^+\rangle, \langle (I^\mu_\lambda)'(su^+ + tu^-), tu^-\rangle).$$

Lemma 2.1. *Assume that* (f_1)*-*(f_3) *hold, if* $u \in H$ *with* $u^\pm \neq 0$*, then* ψ_u *has the following properties:*

- *(i) The pair* (s,t) *is a critical point of* ψ_u *with* $s,t > 0$ *if and only if* $su^+ + tu^- \in \mathcal{M}^\mu_\lambda$*;*
- *(ii) The function* ψ_u *has a unique critical point* (s_u, t_u) *on* $(0,\infty)\times(0,\infty)$*, which is also the unique maximum point of* ψ_u *on* $[0,\infty) \times [0,\infty)$*; Furthermore, if* $\langle (I^\mu_\lambda)'(u), u^\pm\rangle \leq 0$*, then* $0 < s_u, t_u \leq 1$*.*

Proof. *(i)* By definition of ψ_u, we have that

$$\nabla\psi_u(s,t) = (\langle (I^\mu_\lambda)'(su^+ + tu^-), u^+\rangle, \langle (I^\mu_\lambda)'(su^+ + tu^-), u^-\rangle)$$

$$= (\frac{1}{s}\langle (I_\lambda^\mu)'(su^+ + tu^-), su^+\rangle, \frac{1}{t}\langle (I_\lambda^\mu)'(su^+ + tu^-), tu^-\rangle).$$

Thus, item *(i)* holds.

(ii) Firstly, we prove the existence of s_u and t_u.

From (f_1) and (f_2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(t)| \le \varepsilon|t| + C_\varepsilon|t|^{q-1}, \text{ for all } t \in \mathbb{R}. \tag{2.1}$$

Then, by Sobolev embedding theorem, we have that

$$\begin{aligned}\langle (I_\lambda^\mu)'(su^+ + tu^-), su^+\rangle &\ge s^2\|u^+\|^2 - s^6\int_{\mathbb{R}^3}|u^+|^6dx - \mu\varepsilon s^2\int_{\mathbb{R}^3}|u^+|^2dx - \mu C_\varepsilon s^q\int_{\mathbb{R}^3}|u^+|^q dx \\ &\ge s^2\|u^+\|^2 - C_1 s^6\|u^+\|^6 - \mu\varepsilon C_2 s^2\|u^+\|^2 - \mu C_\varepsilon C_3 s^q\|u^+\|^q \\ &\ge (1-\mu\varepsilon C_4)s^2\|u^+\|^2 - C_4 s^6\|u^+\|^6 - \mu C_4 s^q\|u^+\|^q.\end{aligned}$$

Choosing $\varepsilon > 0$ such that $(1-\mu\varepsilon C_4) > 0$. Since $q > 4$, we have that $\langle (I_\lambda^\mu)'(su^+ + tu^-), su^+\rangle > 0$, for s small enough and all $t \ge 0$.

Similarly, we obtain that $\langle (I_\lambda^\mu)'(su^+ + tu^-), tu^-\rangle > 0$, for t small enough and all $s \ge 0$.

Therefore, there is $\delta_1 > 0$ such that

$$\langle (I_\lambda^\mu)'(\delta_1 u^+ + tu^-), \delta_1 u^+\rangle > 0, \langle (I_\lambda^\mu)'(su^+ + \delta_1 u^-), \delta_1 u^-\rangle > 0, \text{ for all } s,\ t \ge 0. \tag{2.2}$$

On the other hand, by (f_1) and (f_3), we have that

$$f(t)t > 0, t \ne 0;\ F(t) \ge 0, t \in \mathbb{R}. \tag{2.3}$$

Therefore, choose $s = \delta_2' > \delta_1$, if $t \in [\delta_1, \delta_2']$ and δ_2' is large enough, it follows that

$$\begin{aligned}\langle (I_\lambda^\mu)'(\delta_2' u^+ + tu^-), \delta_2' u^+\rangle &\le (\delta_2')^2\|u^+\|^2 + (\delta_2')^4\lambda\int_{\mathbb{R}^3}\phi_{u^+}|u^+|^2 + (\delta_2')^4\lambda\int_{\mathbb{R}^3}\phi_{u^-}|u^+|^2 \\ &\quad - (\delta_2')^6\int_{\mathbb{R}^3}|u^+|^6dx \le 0.\end{aligned}$$

Similarly, we have that

$$\begin{aligned}\langle (I_\lambda^\mu)'(su^+ + tu^-), tu^-\rangle &\le t^2\|u^-\|^2 + t^4\lambda\int_{\mathbb{R}^3}\phi_{u^-}|u^-|^2dx + s^2t^2\lambda\int_{\mathbb{R}^3}\phi_{u^+}|u^-|^2dx \\ &\quad - t^6\int_{\mathbb{R}^3}|u^-|^6dx.\end{aligned}$$

Let $\delta_2 > \delta_2'$ be large enough, we can obtain that

$$\langle (I_\lambda^\mu)'(\delta_2 u^+ + tu^-), \delta_2 u^+\rangle < 0, \langle (I_\lambda^\mu)'(su^+ + \delta_2 u^-), \delta_2 u^-\rangle < 0, \text{ for all } s,\ t \in [\delta_1, \delta_2]. \tag{2.4}$$

Combining (2.2), (2.4) with Miranda's Theorem [27], there exists $(s_u, t_u) \in (0,\infty)\times(0,\infty)$ such that $W_u(s,t) = (0,0)$, i.e., $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda^\mu$.

Secondly, we prove the uniqueness of the pair (s_u, t_u).

Case 1. $u \in \mathcal{M}_\lambda^\mu$.

If $u \in \mathcal{M}_\lambda^\mu$, we have that

$$\|u^\pm\|^2 + \lambda \int_{\mathbb{R}^3} \phi_{u^\pm}|u^\pm|^2 dx + \lambda \int_{\mathbb{R}^3} \phi_{u^\mp}|u^\pm|^2 dx = \int_{\mathbb{R}^3} |u^\pm|^6 dx + \mu \int_{\mathbb{R}^3} f(u^\pm)u^\pm dx. \tag{2.5}$$

Now, we show that $(s_u, t_u) = (1, 1)$ is the unique pair of numbers such that $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda^\mu$. Let (s_0, t_0) be a pair of numbers such that $s_0 u^+ + t_0 u^- \in \mathcal{M}_\lambda^\mu$ with $0 < s_0 \le t_0$. So, one has that

$$s_0^2\|u^+\|^2 + s_0^4\lambda \int_{\mathbb{R}^3} \phi_{u^+}|u^+|^2 dx + s_0^2 t_0^2 \lambda \int_{\mathbb{R}^3} \phi_{u^-}|u^+|^2 dx = s_0^6 \int_{\mathbb{R}^3} |u^+|^6 dx + \mu \int_{\mathbb{R}^3} f(s_0 u^+) s_0 u^+ dx, \tag{2.6}$$

$$t_0^2\|u^-\|^2 + t_0^4\lambda \int_{\mathbb{R}^3} \phi_{u^-}|u^-|^2 dx + s_0^2 t_0^2 \lambda \int_{\mathbb{R}^3} \phi_{u^+}|u^-|^2 dx = t_0^6 \int_{\mathbb{R}^3} |u^-|^6 dx + \mu \int_{\mathbb{R}^3} f(t_0 u^-) t_0 u^- dx. \tag{2.7}$$

Thanks to $0 < s_0 \le t_0$ and (2.7), we have that

$$\frac{\|u^-\|^2}{t_0^2} + \lambda \int_{\mathbb{R}^3} \phi_{u^-}|u^-|^2 dx + \lambda \int_{\mathbb{R}^3} \phi_{u^+}|u^-|^2 dx \ge t_0^2 \int_{\mathbb{R}^3} |u^-|^6 dx + \mu \int_{\mathbb{R}^3} [\frac{f(t_0 u^-)}{(t_0 u^-)^3}](u^-)^4 dx. \tag{2.8}$$

Combining (2.5) with (2.8), we obtain that

$$(\frac{1}{t_0^2} - 1)\|u^-\|^2 \ge (t_0^2 - 1) \int_{\mathbb{R}^3} |u^-|^6 dx + \mu \int_{\mathbb{R}^3} [\frac{f(t_0 u^-)}{(t_0 u^-)^3} - \frac{f(u^-)}{(u^-)^3}](u^-)^4 dx.$$

If $t_0 > 1$, the left side of above inequality is negative, which is absurd because the right side is positive by condition (f_3). Therefore, we obtain that $0 < s_0 \le t_0 \le 1$.

Similarly, by (2.6) and $0 < s_0 \le t_0$, we get

$$(\frac{1}{s_0^2} - 1)\|u^+\|^2 \le (s_0^2 - 1) \int_{\mathbb{R}^3} |u^+|^6 dx + \mu \int_{\mathbb{R}^3} [\frac{f(s_0 u^+)}{(s_0 u^+)^3} - \frac{f(u^+)}{(u^+)^3}](u^+)^4 dx.$$

In view of (f_3), we have that $s_0 \ge 1$. Consequently, $s_0 = t_0 = 1$.

Case 2. $u \notin \mathcal{M}_\lambda^\mu$.

Suppose that there exist (s_1, t_1), (s_2, t_2) such that $u_1 = s_1 u^+ + t_1 u^- \in \mathcal{M}_\lambda^\mu$, $u_2 = s_2 u^+ + t_2 u^- \in \mathcal{M}_\lambda^\mu$. So,

$$u_2 = (\frac{s_2}{s_1}) s_1 u^+ + (\frac{t_2}{t_1}) t_1 u^- = (\frac{s_2}{s_1}) u_1^+ + (\frac{t_2}{t_1}) u_1^- \in \mathcal{M}_\lambda^\mu.$$

Thanks to $u_1 \in \mathcal{M}_\lambda^\mu$, we get that $\frac{s_2}{s_1} = \frac{t_2}{t_1} = 1$. Hence, $s_1 = s_2$, $t_1 = t_2$.

Next, we prove that (s_u, t_u) is the unique maximum point of ψ_u on $[0, \infty) \times [0, \infty)$.

In fact, by (2.3), we have that

$$\begin{aligned}\psi_u(s,t) &= I_\lambda^\mu(su^+ + tu^-) \\ &= \frac{s^2}{2}\|u^+\|^2 + \frac{t^2}{2}\|u^-\|^2 + \frac{s^4}{4}\lambda \int_{\mathbb{R}^3} \phi_{u^+}|u^+|^2 dx + \frac{t^4}{4}\lambda \int_{\mathbb{R}^3} \phi_{u^-}|u^-|^2 dx\end{aligned}$$

$$
\begin{aligned}
&+\frac{s^2t^2}{4}\lambda\int\limits_{\mathbb{R}^3}\phi_{u^-}|u^+|^2dx+\frac{s^2t^2}{4}\lambda\int\limits_{\mathbb{R}^3}\phi_{u^+}|u^-|^2dx-\mu\int\limits_{\mathbb{R}^3}F(su^+)dx\\
&-\mu\int\limits_{\mathbb{R}^3}F(tu^-)dx-\frac{s^6}{6}\int\limits_{\mathbb{R}^3}|u^+|^6dx-\frac{t^6}{6}\int\limits_{\mathbb{R}^3}|u^-|^6dx\\
&\le\frac{s^2}{2}\|u^+\|^2+\frac{t^2}{2}\|u^-\|^2+\frac{s^4}{4}\lambda\int\limits_{\mathbb{R}^3}\phi_{u^+}|u^+|^2dx+\frac{t^4}{4}\lambda\int\limits_{\mathbb{R}^3}\phi_{u^-}|u^-|^2dx\\
&+\frac{s^2t^2}{4}\lambda\int\limits_{\mathbb{R}^3}\phi_{u^-}|u^+|^2dx+\frac{s^2t^2}{4}\lambda\int\limits_{\mathbb{R}^3}\phi_{u^+}|u^-|^2dx-\frac{s^6}{6}\int\limits_{\mathbb{R}^3}|u^+|^6dx-\frac{t^6}{6}\int\limits_{\mathbb{R}^3}|u^-|^6dx,
\end{aligned}
$$

which implies that $\lim_{|(s,t)|\to\infty}\psi_u(s,t)=-\infty$.

Hence, (s_u,t_u) is the unique critical point of ψ_u in $[0,\infty)\times[0,\infty)$. So it is sufficient to check that a maximum point cannot be achieved on the boundary of $[0,\infty)\times[0,\infty)$. By contradiction, we suppose that $(0,t_0)$ is a maximum point of ψ_u with $t_0\ge 0$. Then, we have that

$$
\begin{aligned}
\psi_u(s,t_0)=&\frac{s^2}{2}||u^+||^2+\frac{s^4}{4}\lambda\int\limits_{\mathbb{R}^3}\phi_{u^+}|u^+|^2dx-\frac{s^6}{6}\int\limits_{\mathbb{R}^3}|u^+|^6dx-\int\limits_{\mathbb{R}^3}F(su^+)dx\\
&+\frac{t_0^2}{2}||u^-||+\frac{t_0^4}{4}\lambda\int\limits_{\mathbb{R}^3}\phi_{u^-}|u^-|^2dx-\frac{t_0^6}{6}\int\limits_{\mathbb{R}^3}|u^+|^6dx-\int\limits_{\mathbb{R}^3}F(t_0u^-)dx\\
&+\frac{s^2t_0^2}{4}\lambda\int\limits_{\mathbb{R}^3}\phi_{u^-}|u^+|^2dx+\frac{s^2t_0^2}{4}\lambda\int\limits_{\mathbb{R}^3}\phi_{u^+}|u^-|^2dx.
\end{aligned}
$$

Therefore, it is obvious that

$$
\begin{aligned}
(\psi_u)'_s(s,t_0)=&\,s||u^+||^2+s^3\lambda\int\limits_{\mathbb{R}^3}\phi_{u^+}|u^+|^2dx+\frac{st_0^2}{2}\lambda\int\limits_{\mathbb{R}^3}\phi_{u^-}|u^+|^2dx+\frac{st_0^2}{2}\lambda\int\limits_{\mathbb{R}^3}\phi_{u^+}|u^-|^2dx\\
&-s^5\int\limits_{\mathbb{R}^3}|u^+|^6dx-\int\limits_{\mathbb{R}^3}f(su^+)u^+dx>0,\ \text{if } s \text{ is small enough.}
\end{aligned}
$$

That is, ψ_u is an increasing function with respect to s if s is small enough. This yields the contradiction. Similarly, ψ_u can not achieve its global maximum on $(s,0)$ with $s\ge 0$.

Lastly, we prove that $0<s_u,t_u\le 1$ if $\langle (I_\lambda^\mu)'(u),u^\pm\rangle\le 0$.

Suppose $s_u\ge t_u>0$. By $s_uu^++t_uu^-\in\mathcal{M}_\lambda^\mu$, we have

$$
\begin{aligned}
&s_u^2\|u^+\|^2+s_u^4\lambda\int\limits_{\mathbb{R}^3}\phi_{u^+}|u^+|^2dx+s_u^4\lambda\int\limits_{\mathbb{R}^3}\phi_{u^-}|u^+|^2dx\\
&\ge s_u^2\|u^+\|^2+s_u^4\lambda\int\limits_{\mathbb{R}^3}\phi_{u^+}|u^+|^2dx+s_u^2t_u^2\lambda\int\limits_{\mathbb{R}^3}\phi_{u^-}|u^+|^2dx\\
&=s_u^6\int\limits_{\mathbb{R}^3}|u^+|^6dx+\mu\int\limits_{\mathbb{R}^3}f(s_uu^+)s_uu^+dx. \qquad (2.9)
\end{aligned}
$$

On the other hand, by $\langle (I_\lambda^\mu)'(u), u^+\rangle \le 0$, we have

$$\|u^+\|^2 + \lambda \int_{\mathbb{R}^3} \phi_{u^+}|u^+|^2 dx + \lambda \int_{\mathbb{R}^3} \phi_{u^-}|u^+|^2 dx \le \int_{\mathbb{R}^3} |u^+|^6 dx + \mu \int_{\mathbb{R}^3} f(u^+)u^+ dx. \tag{2.10}$$

So, according to (2.9) and (2.10), we have that

$$(\frac{1}{s_u^2} - 1)\|u^+\|^2 \ge (s_u^2 - 1)\int_{\mathbb{R}^3} |u^+|^6 dx + \mu \int_{\mathbb{R}^3} [\frac{f(s_u u^+)}{(s_u u^+)^3} - \frac{f(u^+)}{(u^+)^3}](u^+)^4 dx.$$

Thanks to condition (f_3), we conclude that $s_u \le 1$. Thus, we have that $0 < s_u, t_u \le 1$. □

Lemma 2.2. *Let $c_\lambda^\mu = \inf_{u\in\mathcal{M}_\lambda^\mu} I_\lambda^\mu(u)$, then we have that $\lim_{\mu\to\infty} c_\lambda^\mu = 0$.*

Proof. For any $u \in \mathcal{M}_\lambda^\mu$, we have

$$\|u^\pm\|^2 + \lambda \int_{\mathbb{R}^3} \phi_{u^\pm}|u^\pm|^2 dx + \lambda \int_{\mathbb{R}^3} \phi_{u^\mp}|u^\pm|^2 dx = \mu \int_{\mathbb{R}^3} f(u^\pm)u^\pm dx + \int_{\mathbb{R}^3} |u^\pm|^6 dx.$$

Then, by (2.1) and Sobolev inequalities, we have that

$$\|u^\pm\|^2 \le \mu \int_{\mathbb{R}^3} f(u^\pm)u^\pm dx + \int_{\mathbb{R}^3} |u^\pm|^6 dx \le \mu\varepsilon C_1\|u^\pm\|^2 + \mu C_2\|u^\pm\|^q + C_3\|u^\pm\|^6.$$

Thus, we get $(1 - \mu\varepsilon C_1)\|u^\pm\|^2 \le \mu C_2\|u^\pm\|^q + C_3\|u^\pm\|^6$.

Choosing ε small enough such that $(1 - \mu\varepsilon C_1) > 0$, since $q > 4$, there exists $\rho > 0$ such that $\|u^\pm\| \ge \rho$ for all $u \in \mathcal{M}_\lambda^\mu$.

On the other hand, for any $u \in \mathcal{M}_\lambda^\mu$, it is obvious that $\langle (I_\lambda^\mu)'(u), u\rangle = 0$. Thanks to (f_3), we obtain that

$$H(t) := f(t)t - 4F(t) \ge 0, \tag{2.11}$$

and $H(t)$ is increasing when $t > 0$ and decreasing when $t < 0$.

Then, we get

$$\begin{aligned} I_\lambda^\mu(u) &= I_\lambda^\mu(u) - \frac{1}{4}\langle (I_\lambda^\mu)'(u), u\rangle \\ &= \frac{1}{4}\|u\|^2 + \frac{1}{12}\int_{\mathbb{R}^3} |u|^6 dx + \frac{\mu}{4}\int_{\mathbb{R}^3} [f(u)u - 4F(u)]dx \ge \frac{1}{4}\|u\|^2, \text{ for any } u \in \mathcal{M}_\lambda^\mu. \end{aligned}$$

From above discussions, we have that $I_\lambda^\mu(u) > 0$, for all $u \in \mathcal{M}_\lambda^\mu$. So, I_λ^μ is bounded below on $\mathcal{M}_\lambda^\mu$, that is, $c_\lambda^\mu = \inf_{u\in\mathcal{M}_\lambda^\mu} I_\lambda^\mu(u)$ is well-defined.

Let $u \in H$ with $u^\pm \ne 0$ be fixed. According to Lemma 2.1, for each $\mu > 0$, there exist $s_\mu, u_\mu > 0$ such that $s_\mu u^+ + t_\mu u^- \in \mathcal{M}_\lambda^\mu$.

Therefore, by (2.3) and Proposition 2.1, we get

$$\begin{aligned} 0 \le c^\mu &= \inf_{u\in\mathcal{M}_\lambda^\mu} I_\lambda^\mu(u) \le I_\lambda^\mu(s_\mu u^+ + t_\mu u^-) \\ &\le \frac{1}{2}\|su^+ + tu^-\|^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3} \phi_{su^+ + tu^-}|su^+ + tu^-|^2 dx \end{aligned}$$

$$\leq s_\mu^2\|u^+\|^2 + t_\mu^2\|u^-\|^2 + \frac{\lambda C s_\mu^4}{2}\|u^+\|^4 + \frac{\lambda C t_\mu^4}{2}\|u^-\|^4.$$

To our end, we just prove that $s_\mu \to 0$ and $t_\mu \to 0$, as $\mu \to \infty$.

Let $\Phi_u = \{(s_\mu, t_\mu) \in [0,\infty) \times [0,\infty) : W_u(s_\mu, t_\mu) = (0,0), \mu > 0\}$, where W_u is defined as in Lemma 2.1. Then, we have that

$$\begin{aligned} s_\mu^6 \int_{\mathbb{R}^3} |u^+|^6 dx + t_\mu^6 \int_{\mathbb{R}^3} |u^-|^6 dx &\leq s_\mu^6 \int_{\mathbb{R}^3} |u^+|^6 dx + t_\mu^6 \int_{\mathbb{R}^3} |u^-|^6 dx \\ &\quad + \mu \int_{\mathbb{R}^3} f(s_\mu u^+) s_\mu u^+ dx + \mu \int_{\mathbb{R}^3} f(t_\mu u^-) t_\mu u^- dx \\ &= \|su^+ + tu^-\|^2 + \lambda \int_{\mathbb{R}^3} \phi_{su^+ + tu^-} |su^+ + tu^-|^2 dx \\ &\leq 2s_\mu^2\|u^+\|^2 + 2t_\mu^2\|u^-\|^2 + 2\lambda C s_\mu^4 \|u^+\|^4 + 2\lambda C t_\mu^4 \|u^-\|^4. \end{aligned}$$

Therefore, Φ_u is bounded.

Let $\{\mu_n\} \subset (0,\infty)$ be such that $\mu_n \to \infty$ as $n \to \infty$. Then, there exist s_0 and t_0 such that, up to a subsequence, $(s_{\mu_n}, t_{\mu_n}) \to (s_0, t_0)$ as $n \to \infty$.

We claim $s_0 = t_0 = 0$. Suppose, by contradiction, that $s_0 > 0$ or $t_0 > 0$. Thanks to $s_{\mu_n}u^+ + t_{\mu_n}u^- \in \mathcal{M}_\lambda^{\mu_n}$, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} &\|s_{\mu_n}u^+ + t_{\mu_n}u^-\|^2 + \lambda \int_{\mathbb{R}^3} \phi_{s_{\mu_n}u^+ + t_{\mu_n}u^-} |s_{\mu_n}u^+ + t_{\mu_n}u^-|^2 dx \\ &= \int_{\mathbb{R}^3} |s_{\mu_n}u^+ + t_{\mu_n}u^-|^6 dx + \mu_n \int_{\mathbb{R}^3} f(s_{\mu_n}u^+ + t_{\mu_n}u^-)(s_{\mu_n}u^+ + t_{\mu_n}u^-) dx. \end{aligned} \tag{2.12}$$

According to $s_{\mu_n}u^+ \to s_0 u^+$ and $t_{\mu_n}u^- \to t_0 u^-$ in H, (2.1) and (2.3), we have that

$$\int_{\mathbb{R}^3} f(s_{\mu_n}u^+ + t_{\mu_n}u^-)(s_{\mu_n}u^+ + t_{\mu_n}u^-) dx \to \int_{\mathbb{R}^3} f(s_0 u^+ + t_0 u^-)(s_0 u^+ + t_0 u^-) dx > 0, \text{ as } n \to \infty.$$

So, it follows from $\mu_n \to \infty$ as $n \to \infty$ and $\{s_{\mu_n}u^+ + t_{\mu_n}u^-\}$ is bounded in H that we have a contradiction with the equality (2.12).

Hence, $s_0 = t_0 = 0$. That is, $\lim_{\mu\to\infty} c_\lambda^\mu = 0$. □

Lemma 2.3. *There exists $\mu^\star > 0$ such that for all $\mu \geq \mu^\star$, the infimum c_λ^μ is achieved.*

Proof. According to definition of c_λ^μ, there is a sequence $\{u_n\} \subset \mathcal{M}_\lambda^\mu$ such that $\lim_{n\to\infty} I_\lambda^\mu(u_n) = c_\lambda^\mu$.

Obviously, $\{u_n\}$ is bounded in H. Then, up to a subsequence, still denoted by $\{u_n\}$, there exists $u \in H$ such that $u_n \rightharpoonup u$.

Since the embedding $H \hookrightarrow L^p(\mathbb{R}^3)$ is compact, for all $p \in (2,6)$, we have

$$u_n \to u \ \text{ in } L^p(\mathbb{R}^3), u_n(x) \to u(x) \ \text{ a.e. } x \in \mathbb{R}^3.$$

So,

$$u_n^\pm \rightharpoonup u^\pm \ \text{ in } H, u_n^\pm \to u^\pm \ \text{ in } L^p(\mathbb{R}^3), u_n^\pm(x) \to u^\pm(x) \ \text{ a.e. } x \in \mathbb{R}^3.$$

Denote $\beta := \frac{1}{3}S^{\frac{3}{2}}$, where

$$S := \inf_{u\in H\setminus\{0\}} \frac{\|u\|^2}{(\int_{\mathbb{R}^3}|u|^6dx)^{\frac{1}{3}}}.$$

According to Lemma 2.2, there is $\mu^\star > 0$ such that $c_\lambda^\mu < \beta$ for all $\mu \geq \mu^\star$.

Fix $\mu \geq \mu^\star$, it follows from Lemma 2.1 that $I_\lambda^\mu(su_n^+ + tu_n^-) \leq I_\lambda^\mu(u_n)$ for all $s, t \geq 0$.

Therefore, by using Brezis-Lieb Lemma, Fatou's Lemma and Hardy-Littlewood-Sobolev inequality, we have that

$$\begin{aligned}
\liminf_{n\to\infty} I_\lambda^\mu(su_n^+ + tu_n^-) \geq{}& \frac{s^2}{2}\lim_{n\to\infty}(\|u_n^+ - u^+\|^2 + \|u^+\|^2) + \frac{t^2}{2}\lim_{n\to\infty}(\|u_n^- - u^-\|^2 + \|u^-\|^2)\\
&+ \frac{\lambda s^4}{4}\liminf_{n\to\infty}\int_{\mathbb{R}^3}\phi_{u_n^+}|u_n^+|^2dx + \frac{\lambda t^4}{4}\liminf_{n\to\infty}\int_{\mathbb{R}^3}\phi_{u_n^-}|u_n^-|^2dx\\
&- \frac{s^6}{6}\lim_{n\to\infty}(|u_n^+ - u^+|_6^6 + |u^+|_6^6) - \frac{t^6}{6}\lim_{n\to\infty}(|u_n^- - u^-|_6^6 + |u^-|_6^6)\\
&- \mu\int_{\mathbb{R}^3}F(su^+)dx - \mu\int_{\mathbb{R}^3}F(tu^-)dx\\
&+ \frac{\lambda s^2t^2}{4}\liminf_{n\to\infty}\int_{\mathbb{R}^3}\phi_{u_n^+}|u_n^-|^2dx + \frac{\lambda s^2t^2}{4}\liminf_{n\to\infty}\int_{\mathbb{R}^3}\phi_{u_n^-}|u_n^+|^2dx\\
&\geq I_\lambda^\mu(su^+ + tu^-) + \frac{s^2}{2}\lim_{n\to\infty}\|u_n^+ - u^+\|^2 + \frac{t^2}{2}\lim_{n\to\infty}\|u_n^- - u^-\|^2\\
&- \frac{s^6}{6}\lim_{n\to\infty}|u_n^+ - u^+|_6^6 - \frac{t^6}{6}\lim_{n\to\infty}|u_n^- - u^-|_6^6\\
&= I_\lambda^\mu(su^+ + tu^-) + \frac{s^2}{2}A_1 - \frac{s^6}{6}B_1 + \frac{t^2}{2}A_2 - \frac{t^6}{6}B_2,
\end{aligned}$$

where

$$A_1 = \lim_{n\to\infty}\|u_n^+ - u^+\|^2, A_2 = \lim_{n\to\infty}\|u_n^- - u^-\|^2, B_1 = \lim_{n\to\infty}|u_n^+ - u^+|_6^6, B_2 = \lim_{n\to\infty}|u_n^- - u^-|_6^6.$$

So, we have that

$$I_\lambda^\mu(su^+ + tu^-) + \frac{s^2}{2}A_1 - \frac{s^6}{6}B_1 + \frac{t^2}{2}A_2 - \frac{t^6}{6}B_2 \leq c_\lambda^\mu, \text{ for all } s \geq 0 \text{ and } t \geq 0. \tag{2.13}$$

Firstly, we prove that $u^\pm \neq 0$.

Since the situation $u^- \neq 0$ is analogous, we just prove $u^+ \neq 0$. By contradiction, we suppose $u^+ = 0$.

Case 1: $B_1 = 0$.

If $A_1 = 0$, that is, $u_n^+ \to u^+$ in H. In view of discussions in Lemma 2.1, we obtain $\|u^+\| > 0$, which contradicts our supposition. If $A_1 > 0$, it follows from (2.13) that $\frac{s^2}{2}A_1 \leq c_\lambda^\mu$ for all $s \geq 0$, which is absurd. Anyway, we have a contradiction.

Case 2: $B_1 > 0$.

According to definition of S, we have that $\beta = \frac{1}{3}S^{\frac{3}{2}} \leq \frac{1}{3}(\frac{A_1}{(B_1)^{\frac{1}{3}}})^{\frac{3}{2}}$.

It is easy to see that $\frac{1}{3}(\frac{A_1}{(B_1)^{\frac{1}{3}}})^{\frac{3}{2}} = \max_{s\geq 0}\{\frac{s^2}{2}A_1 - \frac{s^6}{6}B_1\}$. So, thanks to $c_\lambda^\mu < \beta$ and (2.13), we have that

$$\beta \leq \max_{s\geq 0}\{\frac{s^2}{2}A_1 - \frac{s^6}{6}B_1\} < \beta,$$

which is a contradiction.

From above discussions, we have that $u^{\pm} \neq 0$.

Secondly, we prove that $B_1 = B_2 = 0$.

Since the situation $B_2 = 0$ is analogous, we only prove $B_1 = 0$. By contradiction, we suppose that $B_1 > 0$.

Case 1: $B_2 > 0$.

Let $\widetilde{s}$ and $\widetilde{t}$ satisfy

$$\frac{\widetilde{s}^2}{2}A_1 - \frac{\widetilde{s}^6}{6}B_1 = \max_{s\geq 0}\{\frac{s^2}{2}A_1 - \frac{s^6}{6}B_1\}, \frac{a\widetilde{t}^2}{2}A_2 - \frac{\widetilde{t}^6}{6}B_2 = \max_{t\geq 0}\{\frac{t^2}{2}A_2 - \frac{t^6}{6}B_2\}.$$

According to $[0, \widetilde{s}] \times [0, \widetilde{t}]$ is compact and ψ_u is continuous, there exists $(s_u, t_u) \in [0, \widetilde{s}] \times [0, \widetilde{t}]$ such that $\psi_u(s_u, t_u) = \max_{(s,t)\in[0,\widetilde{s}]\times[0,\widetilde{t}]} \psi_u(s, t)$.

In the following, we prove that $(s_u, t_u) \in (0, \widetilde{s}) \times (0, \widetilde{t})$.

If t is small enough, we have that

$$\psi_u(s, 0) = I^{\mu}_{\lambda}(su^+) < I^{\mu}_{\lambda}(su^+) + I^{\mu}_{\lambda}(tu^-) \leq I^{\mu}_{\lambda}(su^+ + tu^-) = \psi_u(s, t), \text{ for all } s \in [0, \widetilde{s}].$$

So, there exists $t_0 \in [0, \widetilde{t}]$ such that $\psi_u(s, 0) \leq \psi_u(s, t_0)$, for all $s \in [0, \widetilde{s}]$.

That is, any point of $(s, 0)$ with $0 \leq s \leq \widetilde{s}$ is not the maximizer of ψ_u. Hence $(s_u, t_u) \notin [0, \widetilde{s}] \times \{0\}$. Similarly, we have that $(s_u, t_u) \notin \{0\} \times [0, \widetilde{t}]$.

On the other hand, it is easy to see that

$$\frac{s^2}{2}A_1 - \frac{s^6}{6}B_1 > 0 \text{ for all } s \in (0, \widetilde{s}], \frac{t^2}{2}A_2 - \frac{t^6}{6}B_2 > 0 \text{ for all } t \in (0, \widetilde{t}]. \tag{2.14}$$

Then,

$$\beta \leq \frac{\widetilde{s}^2}{2}A_1 - \frac{\widetilde{s}^6}{6}B_1 + \frac{t^2}{2}A_2 - \frac{t^6}{6}B_2 \text{ for } t \in [0, \widetilde{t}], \beta \leq \frac{a\widetilde{t}^2}{2}A_2 - \frac{\widetilde{t}^6}{6}B_2 + \frac{s^2}{2}A_1 - \frac{s^6}{6}B_1 \text{ for } s \in [0, \widetilde{s}].$$

Therefore, according to (2.13), we have that $\psi_u(s, \widetilde{t}) \leq 0$, $\psi_u(\widetilde{s}, t) \leq 0$ for all $s \in [0, \widetilde{s}]$ and all $t \in [0, \widetilde{t}]$. So, $(s_u, t_u) \notin \{\widetilde{s}\} \times [0, \widetilde{t}]$ and $(s_u, t_u) \notin \times[0, \widetilde{s}] \times \{\widetilde{t}\}$.

At last, we get that $(s_u, t_u) \in (0, \widetilde{s}) \times (0, \widetilde{t})$. By Lemma 2.1, (s_u, t_u) is a critical point of ψ_u. Hence, $s_u u^+ + t_u u^- \in \mathcal{M}^{\mu}_{\lambda}$.

So, combining (2.13) with (2.14), we have that

$$c^{\mu}_{\lambda} \geq I^{\mu}_{\lambda}(s_u u^+ + t_u u^-) + \frac{s_u^2}{2}A_1 - \frac{s_u^6}{6}B_1 + \frac{at_u^2}{2}A_2 - \frac{t_u^6}{6}B_2 > I^{\mu}_{\lambda}(s_u u^+ + t_u u^-) \geq c^{\mu}_{\lambda}.$$

That is, we have a contradiction.

Case 2: $B_2 = 0$.

In this case, we can maximize in $[0, \widetilde{s}] \times [0, \infty)$. Indeed, it is possible to show that there exists $t_0 \in [0, \infty)$ such that $I^{\mu}_{\lambda}(su^+ + tu^-) \leq 0$, for all $(s, t) \in [0, \widetilde{s}] \times [t_0, \infty)$. Hence, there is $(s_u, t_u) \in [0, \widetilde{s}] \times [0, \infty)$ such that $\psi_u(s_u, t_u) = \max_{(s,t)\in[0,\widetilde{s}]\times[0,\infty)} \psi_u(s, t)$.

In the following, we prove that $(s_u, t_u) \in (0, \widetilde{s}) \times (0, \infty)$.

It is noticed that $\psi_u(s, 0) < \psi_u(s, t)$ for $s \in [0, \widetilde{s}]$ and t is small enough, so we conclude that $(s_u, t_u) \notin [0, \widetilde{s}] \times \{0\}$.

Meantime, $\psi_u(0, t) < \psi_u(s, t)$ for $t \in [0, \infty)$ and s is small enough, then we have $(s_u, t_u) \notin \{0\} \times [0, \infty)$.

On the other hand, it is obvious that $\beta \leq \frac{a\widetilde{s}^2}{2}A_1 - \frac{\widetilde{s}^6}{6}B_1 + \frac{t^2}{2}A_2$, for all $t \in [0, \infty)$.

Hence, we have that $\psi_u(\widetilde{s}, t) \leq 0$ for all $t \in [0, \infty)$. Thus, $(s_u, t_u) \notin \{\widetilde{s}\} \times [0, \infty)$. And so $(s_u, t_u) \in (0, \widetilde{s}) \times (0, \infty)$. That is, (s_u, t_u) is an inner maximizer of ψ_u in $[0, \widetilde{s}) \times [0, \infty)$. So, $s_u u^+ + t_u u^- \in \mathcal{M}^{\mu}_{\lambda}$.

Therefore, according to (2.14), we have that

$$c_\lambda^\mu \geq I_\lambda^\mu(s_u u^+ + t_u u^-) + \frac{s_u^2}{2} A_1 - \frac{s_u^6}{6} B_1 + \frac{t_u^2}{2} A_2 > I_\lambda^\mu(s_u u^+ + t_u u^-) \geq c_\lambda^\mu.$$

That is, we have a contradiction. Therefore, we deduce that $B_1 = B_2 = 0$.

Lastly, we prove that c_λ^μ is achieved.

For $u^\pm \neq 0$, according to Lemma 2.1, there exist $s_u, t_u > 0$ such that $\widetilde{u} := s_u u^+ + t_u u^- \in \mathcal{M}_\lambda^\mu$. Furthermore, it is easy to see that $\langle (I_\lambda^\mu)'(u), u^\pm \rangle \leq 0$. So, we have that $0 < s_u, t_u \leq 1$.

Since $u_n \in \mathcal{M}_\lambda^\mu$, thanks to (2.11), $B_1 = B_2 = 0$ and the norm in H is lower semicontinuous, we have that

$$\begin{aligned}
c_\lambda^\mu &\leq I_\lambda^\mu(\widetilde{u})) - \frac{1}{4}\langle (I_\lambda^\mu)'(\widetilde{u}), \widetilde{u}\rangle \\
&= \frac{1}{4}\|\widetilde{u}\|^2 + \frac{1}{12}|\widetilde{u}|_6^6 + \frac{\mu}{4}\int_{\mathbb{R}^3} [f(\widetilde{u})\widetilde{u} - 4F(\widetilde{u})]dx \\
&= \frac{1}{4}(\|s_u u^+\|^2 + \|t_u u^-\|^2) + \frac{1}{12}(|s_u u^+|_6^6 + |t_u u^-|_6^6) \\
&+ \frac{\mu}{4}\int_{\mathbb{R}^3} [f(s_u u^+)(s_u u^+) - 4F(s_u u^+)]dx + \frac{\mu}{4}\int_{\mathbb{R}^3} [f(t_u u^-)(t_u u^-) - 4F(t_u u^-)]dx \\
&\leq \frac{1}{4}\|u\|^2 + \frac{1}{12}|u|_6^6 + \frac{\mu}{4}\int_{\mathbb{R}^3} [f(u)u - 4F(u)]dx \\
&\leq \liminf_{n\to\infty} [I_\lambda^\mu(u_n) - \frac{1}{4}\langle (I_\lambda^\mu)'(u_n), u_n\rangle] \\
&= \liminf_{n\to\infty} I_\lambda^\mu(u_n) = c_\lambda^\mu.
\end{aligned}$$

Therefore, we conclude that $s_u = t_u = 1$, and c_λ^μ is achieved by $u_\lambda := u^+ + u^- \in \mathcal{M}_\lambda^\mu$. □

3. The proof of main results

Proof of Theorem 1.1. In fact, thanks to Lemma 2.3, we just prove that the minimizer u_λ for c_λ^μ is indeed a sign-changing solution of system (1.3).

Since $u_\lambda \in \mathcal{M}_\lambda^\mu$, according to Lemma 2.1, we have that

$$I_\lambda^\mu(su_\lambda^+ + tu_\lambda^-) < I_\lambda^\mu(u_\lambda^+ + u_\lambda^-) = c_\lambda^\mu, \text{ for } (s,t) \in (\mathbb{R}_+ \times \mathbb{R}_+)\backslash(1,1). \tag{3.1}$$

If $(I_\lambda^\mu)'(u_\lambda) \neq 0$, then there exist $\delta > 0$ and $\theta > 0$ such that $\|(I_\lambda^\mu)'(v)\| \geq \theta$, for all $\|v - u_\lambda\| \leq 3\delta$.

Choose $\sigma \in (0, \min\{1/2, \frac{\delta}{\sqrt{2}\|u_\lambda\|}\})$. Let $D := (1-\sigma, 1+\sigma) \times (1-\sigma, 1+\sigma)$ and $g(s,t) = su_\lambda^+ + tu_\lambda^-$, $(s,t) \in D$. In view of (3.1), it is easy to see that

$$\overline{c}_\lambda^\mu := \max_{\partial D} I \circ g < c_\lambda^\mu. \tag{3.2}$$

Let $\varepsilon := min\{(c_\lambda^\mu - \overline{c}_\lambda^\mu)/2, \theta\delta/8\}$ and $S_\delta := B(u_\lambda, \delta)$, according to Lemma 2.3 in [40], there exists a deformation $\eta \in C([0,1] \times H, H)$ such that

(a) $\eta(1, v) = v$ if $v \notin (I_\lambda^\mu)^{-1}([c_\lambda^\mu - 2\varepsilon, c_\lambda^\mu + 2\varepsilon] \cap S_{2\delta})$;
(b) $\eta(1, (I_\lambda^\mu)^{c_\lambda^\mu + \varepsilon} \cap S_\delta) \subset (I_\lambda^\mu)^{c_\lambda^\mu - \varepsilon}$;
(c) $I_\lambda^\mu(\eta(1, v)) \leq I_\lambda^\mu(v)$ for all $v \in H$.

Firstly, we need to prove that

$$\max_{(s,t)\in\bar{D}} I_\lambda^\mu(\eta(1,g(s,t))) < c_\lambda^\mu. \tag{3.3}$$

In fact, it follows from Lemma 2.1 that $I_\lambda^\mu(g(s,t)) \le c_\lambda^\mu < c_\lambda^\mu + \varepsilon$. That is, $g(s,t) \in (I_\lambda^\mu)^{c_\lambda^\mu+\varepsilon}$. On the other hand, we have

$$\begin{aligned}\|g(s,t)-u_\lambda\|^2 &= \|(s-1)u_\lambda^+ + (t-1)u_\lambda^-\|\\ &\le 2((s-1)^2\|u_\lambda^+\|^2 + (t-1)^2\|u_\lambda^-\|^2) \le 2\sigma\|u_\lambda\|^2 < \delta^2,\end{aligned}$$

which shows that $g(s,t)\in S_\delta$ for all $(s,t)\in\bar{D}$.

Therefore, according to (b), we obtain $I_\lambda^\mu(\eta(1,g(s,t))) < c_\lambda^\mu - \varepsilon$. Hence, (3.3) holds.

In the following, we prove that $\eta(1,g(D))\cap\mathcal{M}_\lambda^\mu \neq \varnothing$, which contradicts the definition of c_λ^μ. Let $h(s,t) := \eta(1,g(s,t))$ and

$$\begin{aligned}\Psi_0(s,t) &:= (\langle (I_\lambda^\mu)'(g(s,t)), u_\lambda^+\rangle, \langle (I_\lambda^\mu)'(g(s,t)), u_\lambda^-\rangle)\\ &:= (\varphi_u^1(s,t), \varphi_u^2(s,t)),\\ \Psi_1(s,t) &:= (\frac{1}{s}\langle (I_\lambda^\mu)'(h(s,t)), (h(s,t))^+\rangle, \frac{1}{t}\langle (I_\lambda^\mu)'(h(s,t)), (h(s,t))^-\rangle).\end{aligned}$$

By direct calculation, we have that

$$\begin{aligned}\frac{\partial\varphi_u^1(s,t)}{\partial s}|_{(1,1)} &= \|u_\lambda^+\|^2 + 3\lambda\int_{\mathbb{R}^3}\phi_{u_\lambda^+}|u_\lambda^+|^2dx + \lambda\int_{\mathbb{R}^3}\phi_{u_\lambda^-}|u_\lambda^+|^2dx\\ &\quad - 5|u_\lambda^+|_6^6 - \mu\int_{\mathbb{R}^3} f'(u_\lambda^+)(u_\lambda^+)^2dx,\\ \frac{\partial\varphi_u^1(s,t)}{\partial t}|_{(1,1)} &= 2\lambda\int_{\mathbb{R}^3}\phi_{u_\lambda^-}|u_\lambda^+|^2dx, \quad \frac{\partial\varphi_u^2(s,t)}{\partial s}|_{(1,1)} = 2\lambda\int_{\mathbb{R}^3}\phi_{u_\lambda^+}|u_\lambda^-|^2dx,\\ \frac{\partial\varphi_u^2(s,t)}{\partial t}|_{(1,1)} &= \|u_\lambda^-\|^2 + 3\lambda\int_{\mathbb{R}^3}\phi_{u_\lambda^-}|u_\lambda^-|^2dx + \lambda\int_{\mathbb{R}^3}\phi_{u_\lambda^+}|u_\lambda^-|^2dx\\ &\quad - 5|u_\lambda^-|_6^6 - \mu\int_{\mathbb{R}^3} f'(u_\lambda^-)(u_\lambda^-)^2dx.\end{aligned}$$

Let

$$M = \begin{bmatrix} \frac{\partial\varphi_u^1(s,t)}{\partial s}|_{(1,1)} & \frac{\partial\varphi_u^2(s,t)}{\partial s}|_{(1,1)} \\ \frac{\partial\varphi_u^1(s,t)}{\partial t}|_{(1,1)} & \frac{\partial\varphi_u^2(s,t)}{\partial t}|_{(1,1)} \end{bmatrix}.$$

By (f_3), we conclude that $f'(s)s^2 - 3f(s)s > 0$ for $s\neq 0$.

Then, since $u_\lambda\in\mathcal{M}_\lambda^\mu$, we have that

$$\det M = [2\|u_\lambda^+\|^2 + 2|u_\lambda^+|_6^6 + 2\lambda\int_{\mathbb{R}^3}\phi_{u_\lambda^-}|u_\lambda^+|^2dx + \mu\int_{\mathbb{R}^3}[f'(u_\lambda^+)(u_\lambda^+)^2 - 3f(u_\lambda^+)u_\lambda^+]dx]$$

$$\times [2\|u_\lambda^-\|^2 + 2|u_\lambda^-|_6^6 + 2\lambda \int\limits_{\mathbb{R}^3} \phi_{u_\lambda^+}|u_\lambda^-|^2 dx + \mu \int\limits_{\mathbb{R}^3} [f'(u_\lambda^-)(u_\lambda^-)^2 - 3f(u_\lambda^-)u_\lambda^-]dx]$$

$$- 4\lambda^2 \int\limits_{\mathbb{R}^3} \phi_{u_\lambda^+}|u_\lambda^-|^2 dx \int\limits_{\mathbb{R}^3} \phi_{u_\lambda^-}|u_\lambda^+|^2 dx > 0.$$

Since $\Psi_0(s,t)$ is a C^1 function and $(1,1)$ is the unique isolated zero point of Ψ_0, by using the degree theory, we deduce that $\deg(\Psi_0, D, 0) = 1$.

So, combining (3.2) with (a), we obtain $g(s,t) = h(s,t)$ on ∂D. Consequently, we obtain $\deg(\Psi_1, D, 0) = 1$. That is, $\Psi_1(s_0,t_0) = 0$ for some $(s_0,t_0) \in D$, so that $\eta(1, g(s_0,t_0)) = h(s_0,t_0) \in \mathcal{M}_\lambda^\mu$. By (3.3), we have a contradiction. Therefore, we conclude that u_λ is a sign-changing solution for system (1.3).

Finally, we prove that u_λ has exactly two nodal domains. To this end, we assume by contradiction that $u_\lambda = u_1 + u_2 + u_3$ with

$$u_i \neq 0, u_1 \geq 0, u_2 \leq 0 \text{ and } \operatorname{suppt}(u_i) \cap \operatorname{suppt}(u_j) = \emptyset, \text{ for } i \neq j,\ i,j = 1,2,3$$

and $\langle (I_\lambda^\mu)'(u_\lambda), u_i\rangle = 0$, for $i = 1,2,3$.

Setting $v := u_1 + u_2$, we have that $v^+ = u_1$ and $v^- = u_2$, i.e., $v^\pm \neq 0$. Then, there exists a unique pair (s_v, t_v) of positive numbers such that $s_v u_1 + t_v u_2 \in \mathcal{M}_\lambda^\mu$. So, $I_\lambda^\mu(s_v u_1 + t_v u_2) \geq c_\lambda^\mu$.

Moreover, since $\langle (I_\lambda^\mu)'(u_\lambda), u_i\rangle = 0$, we obtain $\langle (I_\lambda^\mu)'(v), v^\pm\rangle < 0$.

So, according to Lemma 2.1, we have that $(s_v, t_v) \in (0,1] \times (0,1]$.

On the other hand, we have

$$0 = \frac{1}{4}\langle (I_\lambda^\mu)'(u_\lambda), u_3\rangle = \frac{1}{4}\|u_3\|^2 + \frac{\lambda}{4}\int\limits_{\mathbb{R}^3} \phi_{u_1}|u_3|^2 dx + \frac{\lambda}{4}\int\limits_{\mathbb{R}^3} \phi_{u_2}|u_3|^2 dx + \frac{\lambda}{4}\int\limits_{\mathbb{R}^3} \phi_{u_3}|u_3|^2 dx$$
$$- \frac{\lambda}{4}\int\limits_{\mathbb{R}^3} |u_3|^6 dx - \frac{\mu}{4}\int\limits_{\mathbb{R}^3} f(u_3)u_3 dx < I_\lambda^\mu(u_3) + \frac{\lambda}{4}\int\limits_{\mathbb{R}^3} \phi_{u_1}|u_3|^2 dx + \frac{\lambda}{4}\int\limits_{\mathbb{R}^3} \phi_{u_2}|u_3|^2 dx.$$

So, by (2.11), we have that

$$\begin{aligned}
c_\lambda^\mu &\leq I_\lambda^\mu(s_v u_1 + t_v u_2) = I_\lambda^\mu(s_v u_1 + t_v u_2) - \frac{1}{4}\langle (I_\lambda^\mu)'(s_v u_1 + t_v u_2), (s_v u_1 + t_v u_2)\rangle \\
&= \frac{1}{4}(\|s_v u_1\|^2 + \|t_v u_2\|^2) + \frac{\mu}{4}\int\limits_{\mathbb{R}^3} [f(s_v u_1)(s_v u_1) - 4F(s_v u_1)]dx \\
&+ \frac{\mu}{4}\int\limits_{\mathbb{R}^3} [f(t_v u_2)(t_v u_2) - 4F(t_v u_2)]dx + \frac{s_v^6}{12}\int\limits_{\mathbb{R}^3} |u_1|^6 dx + \frac{t_v^6}{12}\int\limits_{\mathbb{R}^3} |u_2|^6 dx \\
&\leq \frac{1}{4}(\|u_1\|^2 + \|u_2\|^2) + \frac{\mu}{4}\int\limits_{\mathbb{R}^3} [f(u_1)u_1 - 4F(u_1)]dx \\
&+ \frac{\mu}{4}\int\limits_{\mathbb{R}^3} [f(u_2)u_2 - 4F(u_2)]dx + \frac{1}{12}\int\limits_{\mathbb{R}^3} |u_1|^6 dx + \frac{1}{12}\int\limits_{\mathbb{R}^3} |u_2|^6 dx \\
&= I_\lambda^\mu(u_1 + u_2) - \frac{1}{4}\langle (I_\lambda^\mu)'(u_1 + u_2), (u_1 + u_2)\rangle \\
&< I_\lambda^\mu(u_1) + I_\lambda^\mu(u_2) + I_\lambda^\mu(u_3) + \frac{\lambda}{4}\int\limits_{\mathbb{R}^3} \phi_{u_2+u_3}|u_1|^2 dx + \frac{\lambda}{4}\int\limits_{\mathbb{R}^3} \phi_{u_1+u_3}|u_2|^2 dx + \frac{\lambda}{4}\int\limits_{\mathbb{R}^3} \phi_{u_1+u_2}|u_3|^2 dx \\
&= I_\lambda^\mu(u_\lambda) = c_\lambda^\mu,
\end{aligned}$$

which is a contradiction, this is, $u_3 = 0$ and u_λ has exactly two nodal domains. $\square$

By Theorem 1.1, we obtain a least-energy sign-changing solution u_λ of system (1.3). Next, we prove that the energy of u_λ is strictly larger than two times the least energy.

Proof of Theorem 1.2. Similar to the proof of Lemma 2.3, there exists $\mu_1^\star > 0$ such that for all $\mu \ge \mu_1^\star$ and for each $\lambda > 0$, there is $v_\lambda \in \mathcal{N}_\lambda^\mu$ such that $I_\lambda^\mu(v_\lambda) = c^* > 0$. By standard arguments, the critical points of the functional I_λ^μ on $\mathcal{N}_\lambda^\mu$ are critical points of I_λ^μ in H. So, we obtain that $(I_\lambda^\mu)'(v_\lambda) = 0$. That is, v_λ is a ground state solution of system (1.3).

For all $\mu \ge \mu^\star$, according to Theorem 1.1, for each $\lambda > 0$, we know that the system (1.3) has a least-energy sign-changing solution u_λ which changes sign only once.

Let $\mu^{\star\star} = \max\{\mu^\star, \mu_1^\star\}$. Suppose that $u_\lambda = u^+ + u^-$. As the proof of Lemma 2.1, there exist $s_{u^+}, t_{u^-} \in (0,1)$ such that $s_{u^+}u^+ \in \mathcal{N}_\mu, t_{u^-}u^- \in \mathcal{N}_\lambda^\mu$.

Therefore, in view of Lemma 2.1, we have that

$$2c^* \le I_\lambda^\mu(s_{u^+}u^+) + I_\lambda^\mu(t_{u^-}u^-) \le I_\lambda^\mu(s_{u^+}u^+ + t_{u^-}u^-) < I_\lambda^\mu(u^+ + u^-) = c_\lambda^\mu,$$

which shows that $I_\lambda^\mu(u_\lambda) > 2c^*$ and $c^* > 0$ cannot be achieved by a sign-changing function in H. $\square$

Lastly, we shall analyze the asymptotic behavior of u_λ as $\lambda \to 0$. In the following, we regard $\lambda > 0$ as a parameter in system (1.3).

Proof of Theorem 1.3. For any $\lambda > 0$, let $u_\lambda \in H$ be the least-energy sign-changing solution of system (1.3) obtained in Theorem 1.1. We shall proceed through several claims to complete the proof.

Claim 1. If $\lambda_n \to 0$ as $n \to \infty$, then $\{u_{\lambda_n}\}$ is bounded in H.

Choose a nonzero function $\eta \in C_c^\infty(\mathbb{R}^3)$ with $\eta^\pm \ne 0$. Similar to discussion as in Lemma 2.1, for any $\lambda \in [0,1]$, there exists a pair positive numbers (μ_1, μ_2) independent of λ, such that $\langle (I_\lambda^\mu)'(\mu_1\eta^+ + \mu_2\eta^-), \mu_1\eta^+\rangle < 0$, $\langle (I_\lambda^\mu)'(\mu_1\eta^+ + \mu_2\eta^-), \mu_2\eta^-\rangle < 0$.

Hence, according to Lemma 2.1, for any $\lambda \in [0,1]$, there is a unique pair $(s_\eta(\lambda), t_\eta(\lambda)) \in (0,1] \times (0,1]$ such that $\overline{\eta} := s_\eta(\lambda)\mu_1\eta^+ + t_\eta(\lambda)\mu_2\eta^- \in \mathcal{M}_\lambda^\mu$.

Thus, for any $\lambda \in [0,1]$, in view of (2.1), we have

$$\begin{aligned}
I_\lambda^\mu(u_\lambda) &\le I_\lambda^\mu(\overline{\eta}) = I_\lambda^\mu(\overline{\eta}) - \frac{1}{4}\langle (I_\lambda^\mu)'(\overline{\eta}), \overline{\eta}\rangle \\
&= \frac{1}{4}\|\overline{\eta}\|^2 + \frac{\mu}{4}\int_{\mathbb{R}^3}[f(\overline{\eta})\overline{\eta} - 4F(\overline{\eta})]dx + \frac{1}{12}\int_{\mathbb{R}^3}|\overline{\eta}|^6 dx \\
&\le \frac{1}{4}\|\overline{\eta}\|^2 + \frac{\mu}{4}\int_{\mathbb{R}^3}(C_1\overline{\eta}^2 + C_2\overline{\eta}^q)dx + \frac{1}{12}\int_{\mathbb{R}^3}|\overline{\eta}|^6 dx \\
&\le \frac{1}{4}\|\overline{\eta}\|^2 + \frac{\mu}{4}\int_{\mathbb{R}^3}(C_1{\mu_1}^2|\eta^+|^2 + C_1{\mu_2}^2|\eta^-|^2)dx + \frac{\mu}{4}\int_{\mathbb{R}^3}(C_2{\mu_1}^q|\eta^+|^q + C_2{\mu_2}^q|\eta^-|^q)dx \\
&+ \frac{{\mu_1}^6}{12}\int_{\mathbb{R}^3}|\eta^+|^6 dx + \frac{{\mu_2}^6}{12}\int_{\mathbb{R}^3}|\eta^-|^6 dx := C^*,
\end{aligned}$$

where $C^* > 0$ is a constant independent of λ. So, let $n \to \infty$, it follows that

$$C^* + 1 \ge I_{\lambda_n}^\mu(u_{\lambda_n}) = I_{\lambda_n}^\mu(u_{\lambda_n}) - \frac{1}{4}\langle (I_{\lambda_n}^\mu)'(u_{\lambda_n}), u_{\lambda_n}\rangle \ge \frac{1}{4}\|u_{\lambda_n}\|^2,$$

which implies that $\{u_{\lambda_n}\}$ is bounded in H.

Claim 2. System (1.6) possesses one sign-changing solution u_0.

Since $\{u_{\lambda_n}\}$ is bounded in H, according to Claim 1, going if necessary to a subsequence, there exists $u_0 \in H$ such that

$$u_{\lambda_n} \rightharpoonup u_0 \text{ in } H,\ u_{\lambda_n} \to u_0 \text{ in } L^p(\mathbb{R}^3) \text{ for } p \in (2,6),\ u_{\lambda_n} \to u_0 \text{ a.e. in } \mathbb{R}^3. \tag{3.4}$$

On the other hand, thanks to $\{u_{\lambda_n}\}$ is a weak solution of system (1.3) with $\lambda = \lambda_n$, we have that

$$\int_{\mathbb{R}^3} (\nabla u_{\lambda_n} \cdot \nabla v + V(x) u_{\lambda_n} v)dx + \lambda_n \int_{\mathbb{R}^3} \phi_{u_{\lambda_n}} u_{\lambda_n} v dx - \mu \int_{\mathbb{R}^3} f(u_{\lambda_n}) v dx - \int_{\mathbb{R}^3} |u_{\lambda_n}|^4 u_{\lambda_n} v dx = 0, \tag{3.5}$$

for any $v \in C_c^\infty(\mathbb{R}^3)$.

Combining (3.4), (3.5) with Claim 1, we have that

$$a \int_{\mathbb{R}^3} (\nabla u_0 \cdot \nabla v + V(x) u_0 v)dx - \mu \int_{\mathbb{R}^3} f(u_0) v dx - \int_{\mathbb{R}^3} |u_0|^4 u_0 v dx = 0,$$

for any $v \in C_c^\infty(\mathbb{R}^3)$. That is, u_0 is a solution of system (1.6).

We claim that $u_0^\pm \neq 0$. In fact, since $u_{\lambda_n} \in \mathcal{M}^\mu_{\lambda_n}$, we have that

$$\|u^\pm_{\lambda_n}\|^2 + \lambda_n \int_{\mathbb{R}^3} \phi_{u^\pm_{\lambda_n}} |u^\pm_{\lambda_n}|^2 dx + \lambda_n \int_{\mathbb{R}^3} \phi_{u^\mp_{\lambda_n}} |u^\pm_{\lambda_n}|^2 dx = \int_{\mathbb{R}^3} |u^\pm_{\lambda_n}|^6 dx + \mu \int_{\mathbb{R}^3} f(u^\pm_{\lambda_n}) u^\pm_{\lambda_n} dx.$$

So, by Claim 1, there exists $\mu_2^\star > 0$ such that, for all $\mu \geq \mu_2^\star$, we have that

$$\rho \leq \|u^\pm_{\lambda_n}\|^2 \leq \int_{\mathbb{R}^3} |u^\pm_{\lambda_n}|^6 dx + \mu \int_{\mathbb{R}^3} f(u^\pm_{\lambda_n}) u^\pm_{\lambda_n} dx \leq 2\mu \int_{\mathbb{R}^3} f(u^\pm_{\lambda_n}) u^\pm_{\lambda_n} dx.$$

Then, we have that $0 < \int_{\mathbb{R}^3} f(u_0^\pm) u_0^\pm dx$.

Since u_0 is a solution of system (1.6), we have that

$$\|u_0^\pm\|^2 = \mu \int_{\mathbb{R}^3} f(x, u_0^\pm) u_0^\pm dx + \int_{\mathbb{R}^3} |u_0^\pm|^6 dx \geq \mu \int_{\mathbb{R}^3} f(u_0^\pm) u_0^\pm dx > 0.$$

Therefore, $u_0^\pm \neq 0$.

Claim 3. System (1.6) possesses a least-energy sign-changing solution v_0. Furthermore, there exists a unique pair $(s_{\lambda_n}, t_{\lambda_n}) \in [0,\infty) \times [0,\infty)$ such that $s_{\lambda_n} v_0^+ + t_{\lambda_n} v_0^- \in \mathcal{M}^\mu_{\lambda_n}$ and $(s_{\lambda_n}, t_{\lambda_n}) \to (1,1)$ as $n \to \infty$.

By a similar argument to the proof of Theorem 1.1, there exists $\mu_3^\star > 0$ such that, for all $\mu \geq \mu_3^\star$, we obtain that system (1.6) possesses a least-energy sign-changing solution v_0, where $I_0^\mu(v_0) = c_0^\mu$ and $(I_0^\mu)'(v_0) = 0$.

Let $\mu^{\star\star\star} = \max\{\mu^\star, \mu_2^\star, \mu_3^\star\}$. Hence, by Lemma 2.1, it is easy to see that there exists a unique pair $(s_{\lambda_n}, t_{\lambda_n}) \in (0,\infty) \times (0,\infty)$ such that $s_{\lambda_n} v_0^+ + t_{\lambda_n} v_0^- \in \mathcal{M}^\mu_{\lambda_n}$. Then, we have

$$\begin{aligned} &s^2_{\lambda_n} \|v_0^+\|^2 + \lambda_n s^4_{\lambda_n} \int_{\mathbb{R}^3} \phi_{v_0^+} |v_0^+|^2 dx + \lambda_n s^2_{\lambda_n} t^2_{\lambda_n} \int_{\mathbb{R}^3} \phi_{v_0^-} |v_0^+|^2 \\ &= s^6_{\lambda_n} \int_{\mathbb{R}^3} |v_0^+|^6 dx + \mu \int_{\mathbb{R}^3} f(s_{\lambda_n} v_0^+) s_{\lambda_n} v_0^+ dx, \end{aligned} \tag{3.6}$$

$$
\begin{aligned}
&t_{\lambda_n}^2\|v_0^-\|^2+\lambda_n t_{\lambda_n}^4\int_{\mathbb{R}^3}\phi_{v_0^-}|v_0^-|^2+\lambda_n s_{\lambda_n}^2 t_{\lambda_n}^2\int_{\mathbb{R}^3}\phi_{v_0^+}|v_0^-|^2\\
&=t_{\lambda_n}^6\int_{\mathbb{R}^3}|v_0^-|^6dx+\mu\int_{\mathbb{R}^3}f(t_{\lambda_n}v_0^-)t_{\lambda_n}v_0^-dx.
\end{aligned}
\tag{3.7}
$$

According to (f_3) and $\lambda_n\to 0$ as $n\to\infty$, $\{s_{\lambda_n}\}$ and $\{t_{\lambda_n}\}$ are bounded. Up to a subsequence, suppose that $s_{\lambda_n}\to s_0$ and $t_{\lambda_n}\to t_0$, then it follows from (3.6) and (3.7) that

$$
s_0^2\|v_0^+\|^2=s_0^6\int_{\mathbb{R}^3}|v_0^+|^6dx+\mu\int_{\mathbb{R}^3}f(s_0v_0^+)s_0v_0^+dx,\tag{3.8}
$$

$$
t_0^2\|v_0^-\|^2=t_0^6\int_{\mathbb{R}^3}|v_0^-|^6dx+\mu\int_{\mathbb{R}^3}f(t_0v_0^-)t_0v_0^-dx.\tag{3.9}
$$

Thanks to v_0 is a sign-changing solution of system (1.6), we get

$$
\|v_0^\pm\|^2=\int_{\mathbb{R}^3}|v_0^\pm|^6dx+\mu\int_{\mathbb{R}^3}f(v_0^\pm)v_0^\pm dx.\tag{3.10}
$$

Hence, in view of (3.8)-(3.10), we can easily obtain that $(s_0,t_0)=(1,1)$.

Now, we can prove u_0 is a least-energy sign-changing solution of system (1.6). According to Lemma 2.1, we have

$$
I_0^\mu(v_0)\le I_0^\mu(u_0)=\lim_{n\to\infty}I_{\lambda_n}^\mu(u_{\lambda_n})\le\lim_{n\to\infty}I_{\lambda_n}^\mu(s_{\lambda_n}v_0^++t_{\lambda_n}v_0^-)=I_0^\mu(v_0^++v_0^-)=I_0^\mu(v_0).
$$

Hence, the proof of Theorem 1.3 is completed. □

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