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A new general asymptotic formula for the Mills' ratio of the skew-generalized normal distribution

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A B S T R A C T

In this paper, we establish some lower and upper bounds with explicit expressions for the Mills' ratio of the skew-generalized normal distribution. And for demonstrating the efficiency of our estimates, some numerical computations are provided.

1. Introduction

It is well known that Mills' ratio $R(x)$ is defined to be the normal probability beyond a certain point divided by the normal density at that point, that is,

$$R(x) = e^{x^2/2} \int_x^\infty e^{-t^2/2} dt.$$

For the purpose of computation, it has been studied for a long history. Many lower and upper estimates were obtained by varied mathematicians, see Birnbaum [5], Steck [13] and Lu and Li [9] for reference.

The inverse Gaussian distribution, also known as Wald distribution, is a two-parameter family of continuous probability distributions with support on $(0, \infty)$. And its density is

$$f(x; \alpha, \beta) = \frac{\alpha}{\sqrt{2\pi\beta}} x^{-\frac{3}{2}} e^{-\frac{(\alpha-\beta x)^2}{2\beta x}}, x > 0.$$

In Lu [8], a method of evaluating the Mills' ratio of the inverse Gaussian distribution had been studied.

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The skew-normal (SN) distribution, introduced by Azzalini [4], has been studied and generalized extensively. Its density is defined in the following form with parameter λ,

$$f(z;\lambda) = 2\phi(z)\Phi(\lambda z), z \in \mathbb{R}, \lambda \in \mathbb{R},$$

where $\phi(x)$ and $\Phi(x)$ are the standard normal density and distribution, respectively. We denote a random variable Z with the density by $Z \sim SN(\lambda)$. The parameter λ controls skewness. This distribution has been studied and generalized by many researchers, such as Sharafi and Behboodian [12], Fathi-Vajargha and Hasanalipour [6].

There has been a growing interest in the construction of flexible parametric distributions that exhibit skewness and kurtosis which is different from the normal distribution. For example, the Beta SN distribution introduced by Mameli and Musio [10], the skew-generalized normal (SGN) distribution considered by Arellano-Valle et al. [1], as well as the Beta SGN distribution in Oskouei [11].

Arellano-Valle et al. [1] considered a generalization of SN(λ) by the name of SGN distribution defined as

$$f(x;\lambda_1,\lambda_2) = 2\phi(x)\Phi\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right), x \in \mathbb{R},$$

where $\lambda_1 \in \mathbb{R}$, $\lambda_2 \geq 0$. This distribution is denoted by $X \sim SGN(\lambda_1, \lambda_2)$. In Oskouei [11], some properties of the SGN distribution were introduced. The SGN model has singularity problems for the Fisher's information matrix (see Arellano-Valle et al. [2] for more detail). Some particular cases are studied, such as the particular case $\lambda_2 = \lambda_1^2$ considered by Gomez et al. [7], which is called skew-curved normal (SCN), and the case $\lambda_2 = 1$ by Arrue et al. [3].

The Mills' ratio of the SGN distribution is defined as

$$I(x) = \frac{\int_x^\infty 2\phi(t)\Phi(\frac{\lambda_1 t}{\sqrt{1+\lambda_2 t^2}})dt}{2\phi(x)\Phi(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}})}, \tag{1.1}$$

with parameters $\lambda_1 \in \mathbb{R}, \lambda_2 \geq 0$. However, we find that there are very few papers on estimating the Mills' ratio of the SGN distribution in the literature. This is the motivation of our work in this paper.

We aim to establish some lower and upper bounds with explicit expressions for $I(x)$. We concern the asymptotic of $I(x)$ as $x \to +\infty$. And pairs of lower and upper estimates for $I(x)$ are obtained, which are asymptotic and sharp approximation. We also provide some conjectures about a series asymptotic expansion of $I(x)$.

The rest of this paper is arranged as follows. In Section 2, we provide some definitions and propositions that will help the functions associated with the SGN distribution. Based on these propositions, some lower and upper bounds for $I(x)$ are obtained. In Section 3, some numerical computations are listed to show the efficiency of our bounds. Some proofs are given in Section 4. In the last section, some conjectures are brought out.

2. The lower and upper bounds for Mills' ratio of the SGN distribution

Throughout the paper, we denote $f^{(i)}(x)$ as the ith order derivative of $f(x)$ and $f^{(0)}(x) = f(x)$.

The following definitions and corresponding propositions play important roles in the article.

Definition 2.1. Let $a > 0$ be a constant. A non-negative function $P(x)$ is said to be the nth order completely monotonic function on (a, ∞), denoted by (n)-cmf, if there exists an integer $n \geq 1$, such that

$$(-1)^k P^{(k)}(x) \geq 0$$

holds for any integer $0 \le k \le n$ when $x \in (a, \infty)$.

Definition 2.2. Let $a > 0$ be a constant. A non-positive function $Q(x)$ is said to be the nth order Bernstein function on (a, ∞), denoted by (n)-bnf, if there exists an integer $n \ge 1$, such that

$$(-1)^k Q^{(k)}(x) \le 0$$

holds for any integer $0 \le k \le n$ when $x \in (a, \infty)$.

Definition 2.3. Let $a > 0$ be a constant. A function $G(x)$ is said to be the nth order logarithmically completely monotonic function on (a, ∞), if $\ln G(x)$ is (n)-cmf on (a, ∞).

Remark 2.1. A nth order logarithmically completely monotonic function $G(x)$ is also (n)-cmf.

In fact, we can get more detailed results related to those definitions without proofs.

Proposition 2.1. *If $P(x)$ is (n)-cmf and $Q(x)$ is (n)-bnf on (a, ∞), for any integer $0 \le k \le n/2$, it is easy to get that*

$$P^{(2k-1)}(x) \le 0, Q^{(2k)}(x) \le 0,$$
$$P^{(2k)}(x) \ge 0, Q^{(2k-1)}(x) \ge 0.$$

In addition, $P^{(2k-1)}(x)$ is $(n+1\text{-}2k)$-bnf, and $Q^{(2k)}(x)$ is $(n\text{-}2k)$-bnf, while $Q^{(2k-1)}(x)$ is $(n+1\text{-}2k)$-cmf, and $P^{(2k)}(x)$ is $(n\text{-}2k)$-cmf.

Proposition 2.2. *If $P(x)$ and $Q(x)$ are both (n)-cmf on (a, ∞), then $F(x) = P(x)Q(x)$ is (n)-cmf on (a, ∞).*

Proposition 2.3. *If $P(x)$ is (n)-cmf and $Q(x)$ is (n)-bnf on (a, ∞), then $F(x) = P(x)Q(x)$ is (n)-bnf on (a, ∞).*

Proposition 2.4. *If $P(x)$ and $Q(x)$ are both (n)-bnf on (a, ∞), then $F(x) = P(x)Q(x)$ is (n)-cmf on (a, ∞).*

Because $\Phi(-\lambda_1 x) = 1 - \Phi(\lambda_1 x)$, we only consider the case in which $\lambda_1 > 0$. And when $\lambda_2 = 0$, the special SGN distribution is also the SN distribution, and $\frac{\lambda_1 t}{\sqrt{1+\lambda_2 t^2}} = \lambda_1 t$. We focus on the case extra.

Denote

$$\theta_x = \frac{\lambda_1 x}{\sqrt{1 + \lambda_2 x^2}}, P_u(x) = e^{-ux}, Q_u(x) = \frac{\Phi(\theta_{x+u})}{\Phi(\theta_x)}.$$

And concern the case that $x > 0$.

It's easy to obtain that

$$\theta_x^{(1)} = \frac{\lambda_1}{(1 + \lambda_2 x^2)^{3/2}} > 0, \ \theta_x^{(2)} = \frac{-3\lambda_1 \lambda_2 x}{(1 + \lambda_2 x^2)^{5/2}} < 0$$

for $x > 0$.

On the one hand, we give an asymptotic estimate for $I(x)$, via L-hospital law, that is,

$$\lim_{x\to\infty} \frac{\int_x^\infty 2\phi(t)\Phi(\frac{\lambda_1 t}{\sqrt{1+\lambda_2 t^2}})dt}{2\phi(x)\Phi(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}})} = \lim_{x\to\infty} \frac{\int_x^\infty \phi(t)\Phi(\theta_t)dt}{\phi(x)\Phi(\theta_x)} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

On the other hand, using variable substitution for (1.1), we can obtain

$$\begin{aligned} I(x) =& \frac{\int_x^\infty 2\phi(t)\Phi(\frac{\lambda_1 t}{\sqrt{1+\lambda_2 t^2}})dt}{2\phi(x)\Phi(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}})} = \frac{e^{x^2/2}}{\Phi(\theta_x)}\int_x^\infty \Phi(\theta_t)e^{-t^2/2}dt \\ =& \int_0^\infty e^{-u^2/2}e^{-ux}\frac{\Phi(\theta_{x+u})}{\Phi(\theta_x)}du = \int_0^\infty e^{-u^2/2}P_u(x)Q_u(x)du. \end{aligned} \tag{2.1}$$

The following propositions are useful in the proof of our main results.

Proposition 2.5. *With $\lambda_2 > 0$, for any fixed integer $n \geq 1$, there exists a positive constant x_n such that*

$$g(x) = \frac{\lambda_1^2 x}{(1+\lambda_2 x^2)^2} + \frac{3\lambda_2 x}{1+\lambda_2 x^2} \tag{2.2}$$

is (n)-cmf on (x_n, ∞), where x_n depends on n (see more details for x_n in Section 4).

Proposition 2.6. *With $\lambda_2 > 0$, for any fixed integer $n \geq 1$, $Q_u(x) = \frac{\Phi(\theta_{x+u})}{\Phi(\theta_x)}$ is (n)-cmf on (x_n, ∞).*

Proposition 2.7. *When $\lambda_2 = 0$, for any fixed integer $n \geq 1$,*

$$Q_u(x) = \frac{\Phi(\theta_{x+u})}{\Phi(\theta_x)} = \frac{\Phi(\lambda_1(x+u))}{\Phi(\lambda_1 x)}$$

is (n)-cmf on (x_n, ∞).

Using Proposition 2.6 and Proposition 2.7, we provide some lower and upper bounds for $I(x)$ as follows.

Theorem 2.1. *For any fixed integer $n \geq 1$, there exists a positive constant x_n such that when $x \in (x_n, \infty)$, $I(x)$ is (n)-cmf, namely*

$$(-1)^k I^{(k)}(x) \geq 0$$

holds for any integer $0 \leq k \leq n$.

Besides, the lower and upper bounds for $I(x)$ are as follows,

$$\frac{M_{2k}(x)}{m_{2k}(x)} < I(x) < \frac{M_{2k-1}(x)}{m_{2k-1}(x)} \tag{2.3}$$

for any integer $1 \leq k \leq n/2$, where

$$\begin{cases} m_1(x) = x - \frac{\phi(\theta_x)}{\Phi(\theta_x)}\theta_x^{(1)}, \\ M_1(x) = 1, \end{cases} \quad \begin{cases} m_k(x) = m_{k-1}^{(1)}(x) + m_{k-1}(x)m_1(x), \\ M_k(x) = M_{k-1}^{(1)}(x) + m_{k-1}(x)M_1(x). \end{cases} \tag{2.4}$$

The following corollary are brought out to simplify the calculation of (2.4).

Corollary 2.1. *When $\lambda_2 > 0$, for any fixed integer $n \geq 1$, there exists a positive constant x_n such that when $x \in (x_n, \infty)$,*

$$\frac{\tilde{M}_{2k}(x)}{\tilde{m}_{2k}(x)} < I(x) < \frac{\bar{M}_{2k-1}(x)}{\bar{m}_{2k-1}(x)} \tag{2.5}$$

holds for any integer $1 \le k \le n/2$, *where*

$$\begin{cases} \tilde{m}_1(x) = x, \\ \tilde{M}_1(x) = 1, \end{cases} \quad \begin{cases} \tilde{m}_k(x) = \tilde{m}_{k-1}^{(1)}(x) + \tilde{m}_{k-1}(x)\tilde{m}_1(x), \\ \tilde{M}_k(x) = \tilde{M}_{k-1}^{(1)}(x) + \tilde{m}_{k-1}(x)\tilde{M}_1(x), \end{cases} \tag{2.6}$$

$$\begin{cases} \bar{m}_1(x) = x - \frac{2\lambda_1}{(\lambda_2)^{3/2}x^3}, \\ \bar{M}_1(x) = 1, \end{cases} \quad \begin{cases} \bar{m}_k(x) = \bar{m}_{k-1}^{(1)}(x) + \bar{m}_{k-1}(x)\bar{m}_1(x), \\ \bar{M}_k(x) = \bar{M}_{k-1}^{(1)}(x) + \bar{m}_{k-1}(x)\bar{M}_1(x). \end{cases} \tag{2.7}$$

When $\lambda_2 = 0$, *(2.6) and (2.7) still hold, but* $\bar{m}_1(x)$ *and* $\bar{M}_1(x)$ *are replaced by*

$$\begin{aligned} \bar{m}_1(x) =& x - 2\lambda_1, \\ \bar{M}_1(x) =& 1. \end{aligned}$$

Remark 2.2. High-order derivatives of $\frac{\phi(\theta_x)}{\Phi(\theta_x)}\theta_x^{(1)}$ causes the computation for (2.4) too complex. To deal with this problem, Corollary 2.1 enlarges the range of $\frac{\phi(\theta_x)}{\Phi(\theta_x)}\theta_x^{(1)}$, and makes $\tilde{m}_k(x)$, $\tilde{M}_k(x)$, $\bar{m}_k(x)$ and $\bar{M}_k(x)$ in the form of polynomials. Although the estimate for $I(x)$ in Corollary 2.1 may be not as sharp as Theorem 2.1, the estimate in Corollary 2.1 is also good enough. More details are provided in Section 3.

3. Application and numerical studies

The general representations of $\frac{M_k(x)}{m_k(x)}$, $\frac{\tilde{M}_{2k}(x)}{\tilde{m}_{2k}(x)}$ and $\frac{\bar{M}_{2k-1}(x)}{\bar{m}_{2k-1}(x)}$ are not provided in Theorem 2.1 and Corollary 2.1. However, through the computations in this section, we find that $\frac{M_2(x)}{m_2(x)}$, $\frac{M_3(x)}{m_3(x)}$, $\frac{\tilde{M}_2(x)}{\tilde{m}_2(x)}$ and $\frac{\bar{M}_3(x)}{\bar{m}_3(x)}$ are sharp enough as the lower and upper estimates for $I(x)$ respectively when $x > 10$.

Denote $\frac{M_k(x)}{m_k(x)}$ by $I_k(x)$, $k = 1, 2, \cdots, 6$, and we list $I_k(x)$ in the form of polynomials as follows,

$$\begin{aligned} I_1(x) =& \frac{1}{x} + \frac{d_1}{x^5} + O\left(\frac{1}{x^7}\right), \\ I_2(x) =& \frac{1}{x} - \frac{1}{x^3} + \frac{1+d_1}{x^5} + O\left(\frac{1}{x^7}\right), \\ I_3(x) =& \frac{1}{x} - \frac{1}{x^3} + \frac{3+d_1}{x^5} + \frac{d_2+6}{x^7} + O\left(\frac{1}{x^9}\right), \\ I_4(x) =& \frac{1}{x} - \frac{1}{x^3} + \frac{3+d_1}{x^5} + \frac{d_2}{x^7} + \frac{d_3}{x^9} + O\left(\frac{1}{x^{11}}\right), \\ I_5(x) =& \frac{1}{x} - \frac{1}{x^3} + \frac{3+d_1}{x^5} + \frac{d_2}{x^7} + \frac{d_3+z_1}{x^9} + \frac{d_4+z_2}{x^{11}} + O\left(\frac{1}{x^{13}}\right), \\ I_6(x) =& \frac{1}{x} - \frac{1}{x^3} + \frac{3+d_1}{x^5} + \frac{d_2}{x^7} + \frac{d_3+z_1}{x^9} + \frac{d_4}{x^{11}} + O\left(\frac{1}{x^{13}}\right), \end{aligned} \tag{3.1}$$

where d_1, d_2, d_3, d_4, and z_1, z_2 are all positive constants. When $\lambda_2 > 0$, those constants are comprised of $\Phi(\frac{\lambda_1}{\sqrt{\lambda_2}})$ and π. And when $\lambda_2 = 0$, $\Phi(\frac{\lambda_1}{\sqrt{\lambda_2}})$ is replaced by 1.

Denote $r_k(x) = I_k(x) - I_{k+1}(x)$, $1 \le k \le 5$, which is the error of the upper and lower bounds, that is,

$$\begin{aligned} r_1(x) =& \frac{1}{x^3} + O\left(\frac{1}{x^5}\right), r_2(x) = -\frac{2}{x^5} + O\left(\frac{1}{x^7}\right), r_3(x) = \frac{6}{x^7} + O\left(\frac{1}{x^9}\right), \\ r_4(x) =& -\frac{z_1}{x^9} + O\left(\frac{1}{x^{11}}\right), r_5(x) = \frac{z_2}{x^{11}} + O\left(\frac{1}{x^{13}}\right). \end{aligned} \tag{3.2}$$

On the one hand, we can see from (3.2) that as $x \to \infty$, $r_k(x)$ converges to 0 faster than $r_{k-1}(x)$, meaning that $I_{2k}(x)$ and $I_{2k-1}(x)$, as a pair of lower and upper bounds for $I(x)$, has better estimate than the former ones, namely

$$I_1(x) - I_2(x) > I_3(x) - I_4(x) > I_5(x) - I_6(x)$$

for x large enough.

On the other hand, from (3.1), we can see that

$$I_2(x) < I_4(x) < I_6(x) < I_5(x) < I_3(x) < I_1(x) \tag{3.3}$$

for x large enough.

We also give the computations of Corollary 2.1, and compare the results with Theorem 2.1.

Denote $I_k^*(x)$, $k = 1, 2, \cdots, 6$, as the upper and lower bounds for $I(x)$ from Corollary 2.1, that is,

$$\begin{aligned}
I_1^*(x) &= \frac{\bar{M}_1(x)}{\bar{m}_1(x)} = \frac{1}{x} + \frac{c_1}{x^5} + O\left(\frac{1}{x^9}\right), \\
I_2^*(x) &= \frac{\tilde{M}_2(x)}{\tilde{m}_2(x)} = \frac{1}{x} - \frac{1}{x^3} + \frac{1}{x^5} - \frac{1}{x^7} + \frac{1}{x^9} + O\left(\frac{1}{x^{11}}\right), \\
I_3^*(x) &= \frac{\bar{M}_3(x)}{\bar{m}_3(x)} = \frac{1}{x} - \frac{1}{x^3} + \frac{c_1+3}{x^5} - \frac{6c_1+9}{x^7} + O\left(\frac{1}{x^9}\right), \\
I_4^*(x) &= \frac{\tilde{M}_4(x)}{\tilde{m}_4(x)} = \frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5} - \frac{15}{x^7} + \frac{81}{x^9} + O\left(\frac{1}{x^9}\right), \\
I_5^*(x) &= \frac{\bar{M}_5(x)}{\bar{m}_5(x)} = \frac{1}{x} - \frac{1}{x^3} + \frac{3+c_1}{x^5} - \frac{15+6c_1}{x^7} + O\left(\frac{1}{x^9}\right), \\
I_6^*(x) &= \frac{\tilde{M}_6(x)}{\tilde{m}_6(x)} = \frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5} - \frac{15}{x^7} + \frac{105}{x^9} + O\left(\frac{1}{x^{11}}\right),
\end{aligned} \tag{3.4}$$

when $\lambda_2 > 0$, where $c_1 = \frac{2\lambda_1}{\lambda_2^{3/2}}$.

And we calculate their errors $r_k^*(x)$ as follows,

$$\begin{aligned}
r_1^*(x) &= \frac{\bar{M}_1(x)}{\bar{m}_1(x)} - \frac{\tilde{M}_2(x)}{\tilde{m}_2(x)} = \frac{1}{x^3} + \frac{c_1-1}{x^5} + \frac{1}{x^7} + O\left(\frac{1}{x^9}\right), \\
r_2^*(x) &= \frac{\tilde{M}_2(x)}{\tilde{m}_2(x)} - \frac{\bar{M}_3(x)}{\bar{m}_3(x)} = -\frac{c_1+2}{x^5} + \frac{6\lambda_1+8}{x^7} + O\left(\frac{1}{x^9}\right), \\
r_3^*(x) &= \frac{\bar{M}_3(x)}{\bar{m}_3(x)} - \frac{\tilde{M}_4(x)}{\tilde{m}_4(x)} = \frac{c_1}{x^5} + \frac{6-6c_1}{x^7} + O\left(\frac{1}{x^9}\right), \\
r_4^*(x) &= \frac{\tilde{M}_4(x)}{\tilde{m}_4(x)} - \frac{\bar{M}_5(x)}{\bar{m}_5(x)} = -\frac{c_1}{x^5} + \frac{6c_1}{x^7} + O\left(\frac{1}{x^9}\right), \\
r_5^*(x) &= \frac{\bar{M}_5(x)}{\bar{m}_5(x)} - \frac{\tilde{M}_6(x)}{\tilde{m}_6(x)} = \frac{c_1}{x^5} - \frac{6c_1}{x^7} + O\left(\frac{1}{x^9}\right).
\end{aligned} \tag{3.5}$$

Table 1
Simulations for $r_k(x)$, $1 \le k \le 4$, when $\lambda_1 = 20$, $\lambda_2 = 0$.

x	$r_1(x)$	$r_2(x)$	$r_3(x)$	$r_4(x)$
50	7.99680×10^{-6}	-6.38977×10^{-9}	7.65243×10^{-12}	-1.22097×10^{-14}
200	1.249969×10^{-7}	-6.24938×10^{-12}	4.68645×10^{-16}	-4.68563×10^{-20}
1000	1.00000×10^{-9}	-2.00000×10^{-15}	5.9999×10^{-21}	-2.39996×10^{-26}

Table 2
Simulations for $r_k(x)$, $1 \le k \le 4$, when $\lambda_1 = 2$, $\lambda_2 = 1$.

x	$r_1(x)$	$r_2(x)$	$r_3(x)$	$r_4(x)$
50	7.99680×10^{-6}	-6.38978×10^{-9}	7.65243×10^{-12}	-1.22097×10^{-14}
200	1.24997×10^{-7}	-6.24937×10^{-12}	4.68645×10^{-16}	-4.68563×10^{-20}
1000	9.99999×10^{-9}	-2.00000×10^{-15}	5.99995×10^{-21}	-2.39996×10^{-26}

When $\lambda_2 = 0$, we have

$$
\begin{aligned}
I_1^*(x) &= \frac{\bar{M}_1(x)}{\bar{m}_1(x)} = \frac{1}{x} + \frac{2\lambda_1}{x^2} + \frac{4\lambda_1^2}{x^3} + O\left(\frac{1}{x^4}\right), \\
I_2^*(x) &= \frac{\tilde{M}_2(x)}{\tilde{m}_2(x)} = \frac{1}{x} - \frac{1}{x^3} + \frac{1}{x^5} + O\left(\frac{1}{x^7}\right), \\
I_3^*(x) &= \frac{\bar{M}_3(x)}{\bar{m}_3(x)} = \frac{1}{x} + \frac{2\lambda_1}{x^2} + \frac{4\lambda_1^2 - 1}{x^3} + \frac{8\lambda_1^3 - 6\lambda_1}{x^4} + O\left(\frac{1}{x^5}\right), \\
I_4^*(x) &= \frac{\tilde{M}_4(x)}{\tilde{m}_4(x)} = \frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5} - \frac{15}{x^7} + \frac{81}{x^9} + O\left(\frac{1}{x^9}\right), \\
I_5^*(x) &= \frac{\bar{M}_5(x)}{\bar{m}_5(x)} = \frac{1}{x} + \frac{2\lambda_1}{x^2} + \frac{4\lambda_1^2 - 1}{x^3} + \frac{8\lambda_1^3 - 6\lambda_1}{x^4} + \frac{16\lambda_1^4 - 24\lambda_2 + 3}{x^5} + O\left(\frac{1}{x^6}\right), \\
I_6^*(x) &= \frac{\tilde{M}_6(x)}{\tilde{m}_6(x)} = \frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5} - \frac{15}{x^7} + \frac{105}{x^9} + O\left(\frac{1}{x^{11}}\right),
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
r_1^*(x) &= \frac{\bar{M}_1(x)}{\bar{m}_1(x)} - \frac{\tilde{M}_2(x)}{\tilde{m}_2(x)} = \frac{2\lambda_1}{x^2} + \frac{4\lambda_1^2 + 1}{x^3} + O\left(\frac{1}{x^4}\right), \\
r_2^*(x) &= \frac{\tilde{M}_2(x)}{\tilde{m}_2(x)} - \frac{\bar{M}_3(x)}{\bar{m}_3(x)} = -\frac{2\lambda_1}{x^2} - \frac{4\lambda_1^2}{x^3} - \frac{8\lambda_1^3 - 6\lambda_1}{x^4} + O\left(\frac{1}{x^5}\right), \\
r_3^*(x) &= \frac{\bar{M}_3(x)}{\bar{m}_3(x)} - \frac{\tilde{M}_4(x)}{\tilde{m}_4(x)} = \frac{2\lambda_1}{x^2} + \frac{4\lambda_1}{x^3} + \frac{8\lambda_1^3 - 6\lambda_1}{x^4} + O\left(\frac{1}{x^5}\right), \\
r_4^*(x) &= \frac{\tilde{M}_4(x)}{\tilde{m}_4(x)} - \frac{\bar{M}_5(x)}{\bar{m}_5(x)} = -\frac{2\lambda_1}{x^2} - \frac{4\lambda_1^2}{x^3} - \frac{8\lambda_1^3 - 6\lambda_1}{x^4} - \frac{16\lambda_4^3 - 24\lambda_1}{x^5} + O\left(\frac{1}{x^6}\right), \\
r_5^*(x) &= \frac{\bar{M}_5(x)}{\bar{m}_5(x)} - \frac{\tilde{M}_6(x)}{\tilde{m}_6(x)} = \frac{2\lambda_1}{x^2} + \frac{4\lambda_1}{x^3} + \frac{8\lambda_1^3 - 6\lambda_1}{x^4} + \frac{16\lambda_4^3 - 24\lambda_1}{x^5} + O\left(\frac{1}{x^6}\right).
\end{aligned}
\tag{3.7}
$$

From (3.4), (3.6), (3.5) and (3.7), we have

$$
I_2^*(x) < I_4^*(x) < I_6^*(x) < I_5^*(x) < I_3^*(x) < I_1^*(x) \tag{3.8}
$$

for x large enough.

Using (3.2) and (3.5), Table 1, Table 2, Table 3, Table 4 and Table 5 are obtained.

On the one hand, according to Table 1, Table 2, Table 3 and Table 4, we can see that $|r_k(x)|$ is always much smaller than $|r_{k-1}(x)|$ for $k > 1$. According to Table 5, $|r_2^*(x)|$ is smaller than $|r_1^*(x)|$, as well as $|r_3^*(x)|$ is smaller than $|r_2^*(x)|$, but $|r_k^*(x)|$ is gradually stable as k increases from 3.

Table 3
Simulations for $r_k(x)$, $1 \le k \le 4$, when $\lambda_1 = 2$, $\lambda_2 = 4$.

x	$r_1(x)$	$r_2(x)$	$r_3(x)$	$r_4(x)$
50	7.99680×10^{-6}	-6.38977×10^{-9}	7.65242×10^{-12}	-1.22097×10^{-14}
200	1.24997×10^{-7}	-6.24937×10^{-12}	4.68645×10^{-16}	-4.68563×10^{-20}
1000	1.00000×10^{-9}	-2.00000×10^{-15}	5.99995×10^{-21}	-2.39997×10^{-26}

Table 4
Simulations for $r_k(x)$, $1 \le k \le 4$, when $\lambda_1 = 5$, $\lambda_2 = 3$.

x	$r_1(x)$	$r_2(x)$	$r_3(x)$	$r_4(x)$
50	7.99680×10^{-6}	-6.38977×10^{-9}	7.65243×10^{-12}	-1.22097×10^{-14}
200	1.24997×10^{-7}	-6.24937×10^{-12}	4.68644×10^{-16}	-4.68563×10^{-20}
1000	1.00000×10^{-9}	-1.99999×10^{-15}	5.99995×10^{-21}	-2.39996×10^{-26}

Table 5
Simulations for $r_k^*(x)$, $1 \le k \le 4$, when $\lambda_1 = 5$, $\lambda_2 = 3$.

x	$r_1^*(x)$	$r_2^*(x)$	$r_3^*(x)$	$r_4^*(x)$	$r_5^*(x)$
50	8.003×10^{-6}	-1.253×10^{-8}	6.1513×10^{-9}	-6.1437×10^{-9}	6.1437×10^{-9}
200	1.250×10^{-7}	-1.226×10^{-11}	6.0136×10^{-12}	-6.0132×10^{-12}	6.0132×10^{-9}
1000	1.000×10^{-9}	-3.924×10^{-15}	1.9245×10^{-15}	-1.9245×10^{-15}	1.9245×10^{-9}

On the other hand, comparing Table 4 and Table 5, we can see that $I_{2k}(x)$ and $I_{2k+1}(x)$, as lower and upper bounds, have better estimate than $I^*_{2k}(x)$ and $I^*_{2k+1}(x)$ when $k \ge 2$. However, $r_2(x)$ and $r_2^*(x)$ are small enough, that is, $I_2(x)$ and $I_3(x)$ as well as $I_2^*(x)$ and $I_3^*(x)$, as lower and upper bounds for $I(x)$, are sharp enough.

4. Proofs

In this section, we give the proofs of Proposition 2.5, Proposition 2.6, as well as Theorem 2.1 in Section 2.

4.1. Proof of Proposition 2.5

Let $F(t) = \frac{1}{1+t^2}$, $a_1 = \frac{\lambda_1^2}{2\sqrt{\lambda_2}}$, $a_2 = 3\sqrt{\lambda_2}$ and $t = \sqrt{\lambda_2}x$. Using variable substitution from (2.2), we have

$$g(x) \equiv -a_1 F^{(1)}(t) + a_2 t F(t) \triangleq \tilde{g}(t).$$

Thus,

$$\tilde{g}^{(n)}(t) = -a_1 F^{(n+1)}(t) + a_2 t F^{(n)}(t) + a_2 n F^{(n-1)}(t). \tag{4.1}$$

We know that

$$F^{(1)}(t) = \frac{-2t}{(1+t^2)^2},\ F^{(2)}(t) = \frac{6t^2-2}{(1+t^2)^3},$$
$$F^{(3)}(t) = \frac{-24t^3+24t}{(1+t^2)^4},\ F^{(4)}(t) = \frac{120t^4-240t^2+24}{(1+t^2)^5}.$$

Denote $F^{(n)}(t)$ as $\frac{f_n(t)}{(1+t^2)^{n+1}}$, and $A_{n,i}$ as the coefficient of t^i in $f_n(t)$, then

$$f_1(t) = -2t,\ f_2(t) = 6t^2 - 2,\ f_3(t) = -24t^3 + 24t,\ f_4(t) = 120t^4 - 240t^2 + 24,$$

and

$$\begin{cases} A_{1,1} = -2, \\ A_{1,0} = 0, \end{cases} \quad \begin{cases} A_{2,2} = 6, \\ A_{2,1} = 0, \\ A_{2,0} = -2, \end{cases} \quad \begin{cases} A_{3,3} = -24, \\ A_{3,2} = 0, \\ A_{3,1} = 24, \\ A_{3,0} = 0. \end{cases} \quad \begin{cases} A_{4,4} = 120, \\ A_{4,3} = 0, \\ A_{4,2} = -240, \\ A_{4,1} = 0. \\ A_{4,0} = 24. \end{cases}$$

Assume that $f_{2k}(t) = \sum_{i=0}^{i=k} A_{2k,2i} t^{2i}$, $k \geq 1$, then

$$F^{(2k)}(t) = \frac{\sum_{i=0}^{i=k} A_{2k,2i} t^{2i}}{(1+t^2)^{2k+1}},$$

and

$$\begin{aligned} F^{(2k+1)}(t) =& \frac{\sum_{i=1}^{i=k} 2i A_{2k,2i} t^{2i-1}}{(1+t^2)^{2k+1}} - \frac{\sum_{i=0}^{i=k} 2(2k+1) A_{2k,2i} t^{2i+1}}{(1+t^2)^{2k+2}} \\ =& \frac{(-2k-2) A_{2k,2k} t^{2k+1} + \sum_{i=1}^{i=k} [(-4k-4+2i) A_{2k,2i-2} + 2i A_{2k,2i}] t^{2i-1}}{(1+t^2)^{2k+2}}. \end{aligned} \tag{4.2}$$

(4.2) can also be induced to

$$F^{(2k+1)}(t) = \frac{\sum_{i=0}^{i=k} A_{2k+1,2i+1} t^{2i+1}}{(1+t^2)^{2k+2}},$$

where

$$\begin{cases} A_{2k+1,2k+1} = -(2k+2) A_{2k,2k}, \\ A_{2k+1,2i-1} = 2i A_{2k,2i} - (4k+4-2i) A_{2k,2i-2}, \quad i = 1, \dots, k. \end{cases} \tag{4.3}$$

Furthermore, we also get

$$F^{(2k+2)}(t) = \frac{\sum_{i=0}^{i=k+1} A_{2k+2,2i} t^{2i}}{(1+t^2)^{2k+3}},$$

where

$$\begin{cases} A_{2k+2,2k+2} = -(2k+3) A_{2k+1,2k+1}, \\ A_{2k+2,2i} = (2i+1) A_{2k+1,2i+1} - (4k+5-2i) A_{2k+1,2i-1}, \quad i = 1, \dots, k, \\ A_{2k+2,0} = A_{2k+1,1}. \end{cases} \tag{4.4}$$

Through mathematical induction, we can verify that

$$F^{(2k)}(t) = \frac{\sum_{i=0}^{i=k} A_{2k,2i} t^{2i}}{(1+t^2)^{2k+1}}, \; F^{(2k+1)}(t) = \frac{\sum_{i=0}^{i=k} A_{2k+1,2i+1} t^{2i+1}}{(1+t^2)^{2k+2}} \tag{4.5}$$

hold for any integer $k > 0$. Meanwhile, combining (4.3) and (4.4), it's easy to have

$$A_{2k,2k} = (2k+1)! > 0, \; A_{2k+1.2k+1} = -(2k+2)! < 0.$$

Plugging (4.1) into (4.5) yields that

$$\begin{aligned}\tilde{g}^{(2k)}(t) =& -a_1F^{(2k+1)}(t)+a_2tF^{(2k)}(t)+a_2 2kF^{(2k-1)}(t)\\ =&\frac{1}{(1+t^2)^{2k+2}}[a_1(2k+2)!t^{2k+1}-a_1\sum_{i=0}^{k-1}A_{2k+1,2i+1}t^{2i+1}\\ &+a_2t(1+t^2)(2k+1)!t^{2k}+a_2t(1+t^2)\sum_{i=0}^{k-1}A_{2k,2i}t^{2i}\\ &-a_2 2k(1+t^2)^2(2k)!t^{2k-1}+a_2 2k(1+t^2)^2\sum_{i=0}^{k-2}A_{2k-1,2i+1}t^{2i+1}]\\ =&\frac{1}{(1+t^2)^{2k+2}}[a_2(2k)!t^{2k+3}+\widetilde{f}_{2k+1}(t)],\end{aligned}$$

where $\widetilde{f}_{2k+1}(t)$ is a polynomial of t with $2k+1$ degree. Therefore, there exists a positive constant t_{2k} such that

$$\tilde{g}^{(2k)}(t)>0 \text{ for } t>t_{2k}. \tag{4.6}$$

Analogously,

$$\begin{aligned}\tilde{g}^{(2k+1)}(t) =& -a_1F^{(2k+2)}(t)+a_2tF^{(2k+1)}(t)+a_2(2k+1)F^{(2k)}(t)\\ =&\frac{1}{(1+t^2)^{2k+3}}[-a_1(2k+3)!t^{2k+2}-a_1\sum_{i=0}^{k}A_{2k+2,2i}t^{2i}\\ &-a_2t(1+t^2)(2k+2)!t^{2k+1}+a_2t(1+t^2)\sum_{i=0}^{k-1}A_{2k+1,2i+1}t^{2i+1}\\ &+a_2(2k+1)(2k+1)!(1+t^2)^2t^{2k}+a_2(2k+1)(1+t^2)^2\sum_{i=0}^{k-1}A_{2k,2i}t^{2i}]\\ =&\frac{1}{(1+t^2)^{2k+3}}[-a_2(2k+1)!t^{2k+4}+\widetilde{f}_{2k+2}(t)],\end{aligned}$$

where $\widetilde{f}_{2k+2}(t)$ is a polynomial of t with $2k+2$ degree. Therefore, there exists a positive constant t_{2k+1} such that

$$\tilde{g}^{(2k+1)}(t)<0 \text{ for } t>t_{2k+1}. \tag{4.7}$$

For any fixed integer $n>1$, combining (4.6) and (4.7), there exists a positive constant t_n^*, such that when $t>t_n^*$ and $1\le k\le n/2$,

$$\tilde{g}^{(2k-1)}(t)<0 \text{ and } \tilde{g}^{(2k)}(t)>0$$

both hold.

Let $x_n=t_n^*/\sqrt{\lambda_2}$, then $g(x)$ is (n)-cmf on (x_n,∞).

4.2. Proof of Proposition 2.6

Let

$$H(x)=\ln Q_u(x)=s(x)-l(x),$$

where

$$s(x) = \ln \Phi(\theta_{u+x}),\ l(x) = \ln \Phi(\theta_x),$$

and $u > 0$ be a constant. We respectively denote s_k, l_k as the kth order derivative of $s(x)$, $l(x)$. It's easy to know $s_1 < l_1$.

For the fixed integer $n > 1$, we want to find the expressions of s_n and l_n with support on (x_n, ∞).

$$\begin{aligned} s_1 =& s^{(1)}(x) = \frac{\phi(\theta_{u+x})}{\Phi(\theta_{u+x})}\theta^{(1)}_{u+x} > 0, \\ s_2 =& - s_1^2 - s_1 h(x) \\ =& - s_1^2 - s_1 \left(\theta_{x+u}\theta^{(1)}_{x+u} - \frac{\theta^{(2)}_{x+u}}{\theta^{(1)}_{x+u}}\right) < 0, \end{aligned}$$

where

$$h(x) = g(x+u) = \frac{\lambda_1^2(x+u)}{[1+\lambda_2(x+u)^2]^2} + \frac{3\lambda_2(x+u)}{1+\lambda_2(x+u)^2}.$$

And from Proposition 2.5, we know that $h(x)$ is (n)-cmf when $x + u > x_n$, that is, for $0 \le k \le n$,

$$(-1)^k h^{(k)}(x) > 0 \tag{4.8}$$

holds. From (4.8), we have

$$\begin{aligned} s_3 =& - 2s_1 s_2 - s_2 h(x) - s_1 h^{(1)}(x) \\ =& 2s_1^3 + 3h(x)s_1^2 + [h^2(x) - h^{(1)}(x)]s_1 > 0. \end{aligned}$$

Without loss of generality, let $s_j = A_j s_1^j + \sum_{i=1}^{j-1} A_{j,i}(x) s_1^i$, $j = 1, 2, \cdots$, and $A_{j,0}(x) = 0$. Then we have

$$\begin{array}{lll} A_1 = 1, & & \\ A_2 = -1, & A_{2,1}(x) = -h(x) < 0, & \\ A_3 = 2, & A_{3,2}(x) = 3h(x) > 0, & A_{3,1}(x) = h^2(x) - h^{(1)}(x) > 0. \end{array}$$

From Proposition 2.1 and Proposition 2.2, it is easy to judge that $A_{2,1}(x)$ is (n)-bnf, $A_{3,1}(x)$ is $(n$-$1)$-cmf, and $A_{3,2}(x) = 3h(x)$ is (n)-cmf respectively. For any $1 < k < n$, we have

$$\begin{aligned} s_{k+1} =& kA_k s_1^{k-1} s_2 + \sum_{i=1}^{k-1} \left(iA_{k,i}(x)s_1^{i-1}s_2 + A^{(1)}_{k,i}(x)s_1^i\right) \\ \triangleq& A_{k+1}s_1^{k+1} + \sum_{i=1}^{k} A_{k,i}(x)s_1^i, \end{aligned}$$

where

$$\begin{cases} A_{k+1} = -kA_k, \\ A_{k+1,k}(x) = -kA_k h(x) - (k-1)A_{k,k-1}(x), \\ A_{k+1,i}(x) = A^{(1)}_{k,i}(x) - iA_{k,i}(x)h(x) - (i-1)A_{k,i-1}(x),\ 1 \le i \le k-1. \end{cases} \tag{4.9}$$

From (4.9) we have $A_k = (-1)^{k-1}(k-1)!$. Assume that

$$A_{k,i}(x) = \begin{cases} \text{(n-k)-cmf,} & \text{if k is an odd number,} \\ \text{(n-k)-bnf,} & \text{if k is an even number.} \end{cases}$$

Combining formula (4.9) and Proposition 2.1-Proposition 2.4, it is obtained that

$$A_{k+1,i}(x) = \begin{cases} \text{(n-k-1)-bnf,} & \text{if k is an odd number,} \\ \text{(n-k-1)-cmf,} & \text{if k is an even number.} \end{cases}$$

Based on mathematical induction, it yields that

$$(-1)^k s_k < 0$$

holds for $1 \le k \le n$ on (x_n, ∞). In addition, (4.9) are also the recursion formulas for A_k and $A_{k,i}(x)$ in the expression of s_k.

Analogously, we can obtain that

$$l_k = B_k l_1^k + \sum_{i=1}^{k-1} B_{k,i}(x) l_1^i,$$

and the recursion formulas for B_k and $B_{k,i}(x)$, $i = 1, ..., k-1$, are as follows,

$$\begin{cases} B_{k+1} = -kB_k, \\ B_{k+1,k}(x) = -kB_k g(x) - (k-1)B_{k,k-1}(x), \\ B_{k+1,i}(x) = B_{k,i}^{(1)}(x) - iB_{k,i}(x)g(x) - (i-1)B_{k,i-1}(x),\ 1 \le i \le k-1. \end{cases} \tag{4.10}$$

We also have $B_k = (-1)^{k-1}(k-1)!$, and $B_{k,i}(x)$ is $(n\text{-}k)$-cmf when k is an odd number, or negative and $(n\text{-}k)$-bnf when k is an even number on (x_k, ∞). In addition, it yields that

$$(-1)^k l_k < 0.$$

On the one hand, it's easy to see that for $1 \le k \le n$ and $x \in (x_n, \infty)$, the signs of s_k and l_k alternate when k increases, namely

$$s_k s_{k-1} < 0,\ l_k l_{k-1} < 0.$$

Besides, s_k and l_k always have the same sign, that is,

$$s_k l_k > 0.$$

On the other hand, with $h(x)$ and $g(x)$ being (n)-cmf on (x_n, ∞), we have

$$|h^{(k)}(x)| < |g^{(k)}(x)| \tag{4.11}$$

for $0 \le k \le n$. Plugging (4.11) into (4.9) and (4.10), it yields that

$$|A_{k,i}(x)| < |B_{k,i}(x)|.$$

Returning to $H^{(k)}(x)$, we have

$$H^{(k)}(x) = s_k - l_k = A_k s_1^k + \sum_{i=1}^{k-1} A_{k,i}(x) s_1^i - B_k l_1^k - \sum_{i=1}^{k-1} B_{k,i}(x) l_1^i$$
$$= (-1)^{k-1}(k-1)!(s_1^k - l_1^k) + \sum_{i=1}^{k-1} [A_{k,i}(x) s_1^i - B_{k,i}(x) l_1^i].$$

Let

$$T_1(x) = (-1)^{k-1}(k-1)!(s_1^k - l_1^k),\ T_2(x) = \sum_{i=1}^{k-1} [A_{k,i}(x) s_1^i - B_{k,i}(x) l_1^i].$$

Now that $|A_{k,i}(x)| < |B_{k,i}(x)|$ and $0 < s_1 < l_1$,

$$(-1)^k \left(A_{k,i}(x) s_1^i - B_{k,i}(x) l_1^i\right) > 0$$

holds for $1 \le i \le k-1$. Hence, we have

$$(-1)^k T_1(x) > 0,\ (-1)^k T_2(x) > 0,$$

thus

$$(-1)^k H^{(k)}(x) = (-1)^k (T_1 + T_2) > 0$$

holds, that is, $Q_u(x)$ is (n)-logarithmically completely monotonic function, also (n)-cmf.

4.3. Proof of Proposition 2.7

The Proof is similar to Proposition 2.6, but here $g(x) = \lambda_1^2 x$, which is different from Proposition 2.6. Let

$$H(x) = \ln Q_u(x) = s(x) - l(x),$$

where

$$s(x) = \ln \Phi(\theta_{u+x}),\ l(x) = \ln \Phi(\theta_x),$$

and

$$\theta_{u+x} = \lambda_1(u+x),\ \theta_x = \lambda_1 x,$$

and let $u > 0$ be a constant.

We respectively denote s_k, l_k as the kth order derivative of $s(x)$, $l(x)$. It's easy to know $s_1 < l_1$.

For the fixed integer $n > 1$, we want to find the expressions of s_n and l_n with support on (x_n, ∞).

We have

$$s_1 = s^{(1)}(x) = \lambda_1 \frac{\phi(\theta_{u+x})}{\Phi(\theta_{u+x})} > 0,$$
$$s_2 = -s_1^2 - s_1 h(x) < 0,$$

where

$$h(x) = g(x+u) = \lambda_1^2(x+u).$$

Then

$$s_3 = 2s_1^3 + 3h(x)s_1^2 + [h^2(x) - h^{(1)}(x)]s_1 > 0,$$

when $x > 1/\lambda_1$.

$$s_4 = -6s_1^4 - 12h(x)s_1^3 + [8h^2(x) - 4\lambda_1^2]s_1^2 - [h^3(x) - 3\lambda_1^2 h(x)]s_1 < 0,$$

when $x > \sqrt{3}/\lambda_1$.

Through the proof of Proposition 2.6, $s_{k+1} = A_{k+1}s_1^{k+1} + \sum_{i=1}^{k} A_{k+1,i}(x)s_1^i$, $k = 1, 2, \cdots$, and $A_{k+1,0}(x) = 0$, where

$$\begin{cases} A_{k+1} = -kA_k, \\ A_{k+1,k}(x) = -kA_k h(x) - (k-1)A_{k,k-1}(x), \\ A_{k+1,i}(x) = A_{k,i}^{(1)}(x) - iA_{k,i}(x)h(x) - (i-1)A_{k,i-1}(x),\ 1 \le i \le k-1. \end{cases} \tag{4.12}$$

Without loss of generality, let

$$A_{k,i}(x) = \sum_{j=0}^{j=[\frac{k-i}{2}]} C_{k,i,j} h^{k-i-2j}(x), \tag{4.13}$$

which is a polynomial of $h(x)$ with $k-i$ degree, where $C_{k,i,j}$ is a constant, and $C_{k,i,0}$ is the coefficient of h^{k-i} in $A_{k,i}(x)$.

We have

$$\begin{aligned} &A_{2,1}(x) = -h(x) < 0, \\ &A_{3,1}(x) = 3h(x) > 0, \quad A_{3,2}(x) = h^2(x) - h^{(1)}(x) > 0, \end{aligned}$$

when $x > 1/\lambda_1$.

$$A_{4,1}(x) = -12h(x) < 0,\ A_{4,2}(x) = -8h^2(x) - 4\lambda_1^2 < 0,\ A_{4,3}(x) = h^3(x) - 3\lambda_1^3 < 0,$$

when $x > \sqrt{3}/\lambda_1$.

Assume that there exists a constant x_k, such that when $x > x_k$, $A_k \cdot A_{k,k-1}(x) > 0$, and $A_{k,i}(x) \cdot A_{k,i-1}(x) > 0$ both hold.

According to (4.12), it yields that

$$\begin{cases} A_{k+1} \cdot A_{k+1,k}(x) > 0, \\ A_{k+1,k}(x) \cdot A_{k,k-1}(x) < 0, \\ C_{k+1,i,0}(x) = -iC_{k,i,0}(x). \end{cases} \tag{4.14}$$

Combining (4.14) and (4.13), there exists a constant x_{k+1}, such that $A_{k+1,i}(x) \cdot A_{k+1} > 0$ and $A_{k+1,i}(x) \cdot A_k < 0$ when $x > x_{k+1}$.

Based on mathematical induction, it yields that, for any fixed integer k, there exists a constant x_n such that

$$(-1)^k s_k < 0$$

holds when $x > x_n$.

Analogously, we can obtain that

$$l_k = B_k l_1^k + \sum_{i=1}^{k-1} B_{k,i}(x) l_1^i,$$

and the recursion formulas for B_k and $B_{k,i}(x)$, $i = 1, ..., k-1$, are as follows,

$$\begin{cases} B_{k+1} = -kB_k, \\ B_{k+1,k}(x) = -kB_k g(x) - (k-1)B_{k,k-1}(x), \\ B_{k+1,i}(x) = B_{k,i}^{(1)}(x) - iB_{k,i}(x)g(x) - (i-1)B_{k,i-1}(x), \ 1 \le i \le k-1. \end{cases} \tag{4.15}$$

We also have $B_k = (-1)^{k-1}(k-1)!$, and $B_{k,i}(x)$ is $(n$-$k)$-cmf when k is an odd number, or negative and $(n$-$k)$-bnf when k is an even number on (x_k, ∞). In addition, it yields that

$$(-1)^k l_k < 0.$$

On the one hand, it's easy to see that for $1 \le k \le n$ and $x \in (x_n, \infty)$, the signs of s_k and l_k alternate when k increases, namely

$$s_k s_{k-1} < 0, \ l_k l_{k-1} < 0.$$

Besides, s_k and l_k always have the same sign, that is,

$$s_k l_k > 0.$$

The rest of the proof is the similar with Proposition 2.6, and we delete the rest proof.

4.4. Proof of Theorem 2.1

Through Proposition 2.6 and Proposition 2.7, it yields that for any integer $n \ge 1$, $Q_u(x)$ is (n)-cmf on (x_n, ∞). It is easy to see that $P_u(x) = e^{-ux}$ is also (n)-cmf on (x_n, ∞). Combining Proposition 2.2 and formula (2.1), we obtain that $I(x)$ is (n)-cmf on (x_n, ∞), that is,

$$(-1)^k I^{(k)}(x) \ge 0 \tag{4.16}$$

holds for any integer $0 \le k \le n$ on (x_n, ∞).

Meanwhile, from (1.1),

$$I(x) = \frac{e^{x^2/2}}{\Phi(\theta_x)} \int_x^\infty \Phi(\theta_t) e^{-t^2/2} dt$$

is obtained, which yields

$$\begin{aligned} I^{(1)}(x) =& \frac{x\Phi(\theta_x) - \phi(\theta_x)\theta_x^{(1)}}{\Phi(\theta_x)} I(x) - 1 \\ \triangleq& m_1(x) I(x) - M_1(x), \end{aligned}$$

where

$$\begin{cases} m_1(x) = \; x - \frac{\phi(\theta_x)}{\Phi(\theta_x)}\theta_x^{(1)}, \\ M_1(x) = 1. \end{cases}$$

Let

$$I^{(k)}(x) = m_k(x)I(x) - M_k(x), \tag{4.17}$$

then

$$\begin{aligned} I^{(k+1)}(x) =& m_k^{(1)}(x)I(x) + m_k(x)I^{(1)}(x) - M_k^{(1)}(x) \\ =& m_{k+1}(x)I(x) - M_{k+1}(x), \end{aligned}$$

where

$$\begin{cases} m_{k+1}(x) = m_k^{(1)}(x) + m_k(x)m_1(x), \\ M_{k+1}(x) = M_k^{(1)}(x) + m_k(x)M_1(x). \end{cases}$$

Combining (4.16) and (4.17), (2.3) holds. The proof is complete.

5. Conjecture

In view of (3.2), (3.3), Table 1, Table 2, Table 3, Table 4 and Table 5, we conjecture that

$$\begin{aligned} \frac{M_2(x)}{m_2(x)} < \frac{M_4(x)}{m_4(x)} < \cdots < \frac{M_{2n}(x)}{m_{2n}(x)} < \cdots < I(x) \\ < \cdots < \frac{M_{2n-1}(x)}{m_{2n-1}(x)} < \cdots < \frac{M_3(x)}{m_3(x)} < \frac{M_1(x)}{m_1(x)}, \end{aligned}$$

and

$$\lim_{n\to\infty} \frac{M_n(x)}{m_n(x)} = I(x)$$

for x large enough.

In addition, by power series expansion, $I(x)$ can be written as

$$I(x) = \frac{1}{x} + \frac{-1}{x^3} + \frac{d_1^*}{x^5} + \frac{-d_2^*}{x^7} + \frac{d_3^*}{x^9} + \frac{-d_4^*}{x^{11}} + \cdots,$$

for x large enough, where d_1^*, d_2^*, d_3^*, d_4^*, $\ldots$, are positive constants relying on the parameters λ_1 and λ_2.

$I(x)$ can also be written in the form of continued fraction as

$$I(x) = \cfrac{1}{x + \cfrac{1}{x + \cfrac{e_1}{x + \cfrac{e_2}{x + \cfrac{e_3}{\ddots}}}}},$$

where $e_1, e_2, e_3, \cdots$ are constants relying on λ_1 and λ_2.

These are interesting and challenging problems. Of course, this is also a direction of our future work.

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