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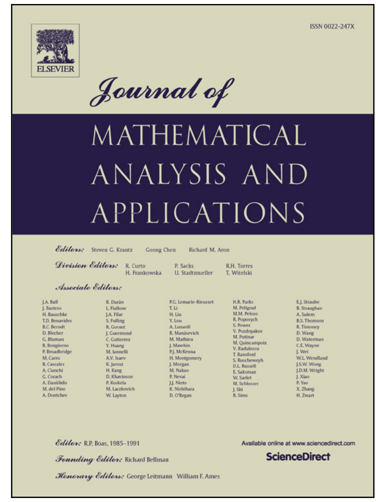

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On asymptotic properties of solutions to fractional differential equations

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Abstract

We present some distinct asymptotic properties of solutions to Caputo fractional differential equations (FDEs). First, we show that the non-trivial solutions to a FDE cannot converge to the fixed points faster than $t^{-\alpha}$, where α is the order of the FDE. Then, we introduce the notion of Mittag-Leffler stability which is suitable for systems of fractional-order. Next, we use this notion to describe the asymptotic behavior of solutions to FDEs by two approaches: Lyapunov's first method and Lyapunov's second method. Finally, we give a discussion on the relation between Lipschitz condition, stability and speed of decay, separation of trajectories to scalar FDEs.

Key words: *Fractional calculus, Fractional differential equation, Singular integral equations, Comparison principle, Lyapunov's first method, Lyapunov's second method, Asymptotic behavior, Asymptotic stability, Mittag-Leffler stability.*

2010 Mathematics Subject Classification: 26A33, 34A08, 34D05, 34D20.

1 Introduction

The theory of fractional calculus is an excellent instrument for describing memory and hereditary properties of various processes. This is the main advantage in comparison to classical integer-order models, in which such effects are often neglected [1]. Therefore, this theory has been applied to many

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fields of science, engineering, and mathematics as viscoelasticity and rheology, electrical engineering, electrochemistry, biology, biophysics and bioengineering, signal and image processing, mechanics, mechatronics, physics, and control theory. For more details, we refer the reader to the monographs [6, 1, 2, 3, 4, 5, 7] and the references therein. The mathematical modeling and simulation of systems and processes, based on the description of their properties in terms of fractional derivatives, naturally leads to differential equations of fractional order and it is necessary to solve such equations. However, most of the fractional differential equations used to describe practical problems can not be solved explicitly.

Many important problems of the qualitative theory of dynamical systems deal with stability properties of their solutions. In particular, the following questions are usually asked in studying fractional order systems: how do the trajectories of solutions change under small perturbations? will the solutions starting near to a given equilibrium point converge to that equilibrium point, and, if yes, with what rate of convergence? In his seminal 1892 thesis [8], Lyapunov proposed two main methods for investigating asymptotic properties of solution of ODEs as follows.

• Lyapunov's first method (reduction method): the key feature of this method is that one reduces the original problem to a much simpler one - linearization of the nonlinear equation near an equilibrium point. Then the stability of the resulting linearized equation can be solved and used for deducting the asymptotic properties of the original equation.

• Lyapunov's second method (direct method): this method uses the action of the system on a specific function (called Lyapunov function) to deduct the asymptotic properties of the system without the need to solve the system's fractional differential equations explicitly.

The two Lyapunov's methods have been powerful tools in the classical theory of ordinary differential equations. It is natural to expect that the Lyapunov's methods may work for FDEs as well since the fractional-order systems are generalizations of integer-order systems. However, one should take care of many distinct features of "purely" fractional-order systems, especially the nonlocal property and the long memory of the system.

The first work concerning with Lyapunov's first method for fractional-order systems was the paper [9]. Using linearization, the authors proposed a criterion to test the stability of a fractional-order predator-prey model and a fractional-order rabies model. They also examined these results by a numerical example. However, no rigorous mathematical proof was given in that paper. After that in [11, 10, 12], the authors formulated theorems on linearized stability. Unfortunately, as showed in [13, Remark 3.7], these papers contain some serious flaws in the proof of the linearization theorems. Using other tools, the authors of [13] improved the assertions presented in [11, 10, 12] and gave a powerful stability criterion. The global attractivity of solutions to some class of fractional-order systems in Riemann–Liouville sense was reported in [14]. However, so far, the convergence rate of solutions to an equilibrium point is still unavailable.

There are many papers on the Lyapunov's second method have been published. We only list here some typical contributions [15, 17, 16, 18, 20, 21, 19, 22, 23]. However, the development of this theory is still in its infancy and requires further investigation. One of the reasons for this might be that computation and estimation of fractional derivatives of Lyapunov candidate functions are very complicated due to the fact that the well-known Leibniz rule does not hold true for such derivatives. On the other hand, in contrast to classical derivatives, there is no acceptable geometrical nor physical interpretation of fractional derivatives. To the best of our knowledge, the common strategy in study the stability of FDEs by Lyapunov's second method is as follows. The authors combined effective fractional derivative inequalities [16, inequalities (6) and (16)], [19, inequality (24)], [17, inequality (10)] and the main results in [20, 21] to obtain the estimation of solutions to FDEs. However, in those papers there are some shortcomings of that approach and some flaws in the proofs, which were shown in [24]. Recently, using other tools, the authors of [24] were able to avoid the shortcomings and flaws mentioned above and proposed a rigorous method of fractional Lyapunov candidate functions to study the weakly asymptotical stability of FDEs.

In this paper, we develop a rigorous framework to study the asymptotic behavior of solution to FDEs by two directions: Lyapunov's first method and Lyapunov's second method. We improve existing results and provide new insightful results on the study of asymptotic properties of solutions to fractional differential equations that have not yet been available in the literature.

The rest of the paper is organized as follows. In Section 2, some important notions and elementary results concerning with fractional calculus and FDEs are recalled. In Section 3, we first show that every nontrivial solution to a FDE having Lipschitz continuous "vector field" does not converge to the equilibrium point of the FDE with a rate faster than $t^{-\alpha}$ where α is the order of the equation. Then, based on the role of Mittag-Leffler function, we introduce the notion of Mittag-Leffler stability to characterize the decay rate of solution to FDEs around the fixed points of the "vector field". The suitability and usefulness of this definition will be specified in the next sections. In Section 4, we develop a Lyapunov's first method for a FDE linearized around its equilibrium points. Our strategy is to combine a variation of constants formula, properties of Mittag-Leffler function, Lyapunov–Perron approach and a new weighted norm to obtain the Mittag-Leffler stability of fixed points. We also discuss on an application of this method in stabilizing some fractional-order chaotic systems. Using comparison principles, a characterization of functions having fractional derivative and an inequality concerning with fractional derivative of convex functions, in Section 5 we develop a Lyapunov's second method for FDEs. Some examples are also presented to illustrate the theoretical results. Finally, in Section 6, we discuss relations between Lipschitz condition, stability and speed of decay, separation of trajectories to FDEs. In particular, we give an example to show that Mittag-Leffler stability is strictly stronger than asymptotic stability, and another example showing that without Lipschitz condition we may encounter the non-uniqueness of solutions to FDE and that alone may lead to instability of the equilibrium point although almost all solutions tend to the equilibrium point with a power rate. At the end of this section, we prove a distinct property of solution to scalar FDEs in comparison to solutions of general higher dimensional FDEs: two trajectories starting from two different initial conditions do not intersect.

To conclude this part, we introduce notations which are used through the paper. Denote by $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{C}$ the set of real numbers, non-negative numbers and complex numbers, respectively. For some arbitrary positive constant integer d, let $\mathbb{R}^d$ and $\mathbb{C}^d$ be the d-dimensional Euclidean spaces with the scalar product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. For a Banach space X with the norm $\|\cdot\|$, $x \in X$ and $r > 0$, let $B_X(x,r)$ be the closed ball with the center at x and the radius $r > 0$. For some $T > 0$, denote by $C([0,T],X)$ the linear space of continuous functions $\varphi : [0,T] \to X$ and by $C_\infty([0,T],X)$ the normed space of $C([0,T],X)$ equipped with the norm

$$\|\varphi\|_\infty := \sup_{t\in[0,T]} \|\varphi(t)\| < \infty$$

for any $\varphi \in C_\infty([0,T],X)$. It is obvious that $(C_\infty([0,T],X), \|\cdot\|_\infty)$ is a Banach space. Finally, for $\alpha \in (0,1]$ we mean $\mathcal{H}^\alpha([0,T],\mathbb{R}^d)$ the standard Hölder space consisting of functions $v \in C([0,T],\mathbb{R}^d)$ such that

$$\|v\|_{\mathcal{H}^\alpha} := \max_{0\leq t\leq T} \|v(t)\| + \sup_{0\leq s<t\leq T} \frac{\|v(t)-v(s)\|}{(t-s)^\alpha} < \infty$$

and by $\mathcal{H}^\alpha_0([0,T],\mathbb{R}^d)$ the closed subspace of $\mathcal{H}^\alpha([0,T],\mathbb{R}^d)$ consisting of functions $v \in \mathcal{H}^{\alpha}([0,T],\mathbb{R}^d)$ such that

$$\sup_{0\le s<t\le T, t-s\le\varepsilon} \frac{\|v(t)-v(s)\|}{(t-s)^{\alpha}} \to 0 \quad \text{as } \varepsilon \to 0.$$

2 Preliminaries

We recall briefly important notions of fractional calculus and some fundamental results concerning with fractional differential equations.

Let $\alpha \in (0,1)$, $[0,T] \subset \mathbb{R}$ and $x : [0,T] \to \mathbb{R}$ satisfy $\int_0^T |x(\tau)|\, d\tau < \infty$. Then, the *Riemann–Liouville integral of order* α is defined by

$$I_{0+}^{\alpha} x(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x(\tau)\, d\tau \quad \text{for } t \in (0,T],$$

where the Gamma function $\Gamma : (0,\infty) \to \mathbb{R}$ is defined as

$$\Gamma(\alpha) := \int_0^{\infty} \tau^{\alpha-1} \exp(-\tau)\, d\tau,$$

see e.g., Diethelm [3]. The *Riemann–Liouville derivative of fractional-order* α is given by

$${}^{R}D_{0+}^{\alpha} x(t) := (D I_{0+}^{1-\alpha} x)(t) \quad \forall t \in (0,T],$$

where $D = \frac{d}{dt}$ is the classical derivative. In the case the Riemann–Liouville derivative of $x(\cdot)$ exists, the *Caputo fractional derivative* ${}^{C}D_{0+}^{\alpha} x$ of this function is defined by

$${}^{C}D_{0+}^{\alpha} x(t) := {}^{R}D_{0+}^{\alpha}(x(t) - x(0)), \qquad \text{for } t \in (0,T],$$

see [3, Definition 3.2, pp. 50]. The Caputo fractional derivative of a d-dimensional vector function $x(t) = (x_1(t), \ldots, x_d(t))^T$ is defined componentwise as

$${}^{C}D_{0+}^{\alpha} x(t) = ({}^{C}D_{0+}^{\alpha} x_1(t), \ldots, {}^{C}D_{0+}^{\alpha} x_d(t))^T.$$

Denote by $I_{0+}^{\alpha} C([0,T],\mathbb{R}^d)$ the space of functions $\varphi : [0,T] \to \mathbb{R}^d$ such that there exists a function $\psi \in C([0,T],\mathbb{R}^d)$ satisfying $\varphi = I_{0+}^{\alpha}\psi$. Due to [25, Theorem 5.2, pp. 475], we have the following characterization of functions having Caputo fractional derivative.

Theorem 1. For $\alpha \in (0,1)$ and a function $v \in C([0,T],\mathbb{R}^d)$, the following conditions (i), (ii), (iii) are equivalent:

(i) the fractional derivative ${}^{C}D_{0+}^{\alpha} v \in C([0,T],\mathbb{R}^d)$ exists;

(ii) a finite limit $\lim_{t\to 0} \frac{v(t)-v(0)}{t^\alpha} := \gamma$ exists, and

$$\sup_{0<t\le T} \left\| \int_{\theta t}^{t} \frac{v(t)-v(\tau)}{(t-\tau)^{\alpha+1}}\, d\tau \right\| \to 0 \quad \text{as } \theta \to 1;$$

(iii) v has the structure $v - v(0) = t^\alpha \gamma + v_0$, where γ is a constant vector, $v_0 \in \mathcal{H}_0^\alpha([0,T],\mathbb{R}^d)$, and $\int_0^t (t-\tau)^{-\alpha-1}(v(t)-v(\tau))d\tau =: w(t)$ converges for every $t \in (0,T]$ defining a function $w \in C((0,T],\mathbb{R}^d)$ which has a finite limit $\lim_{t\to 0} w(t) =: w(0)$.

For $v \in C([0,T],\mathbb{R}^d)$ having fractional derivative ${}^C D_{0+}^\alpha v \in C([0,T],\mathbb{R}^d)$, it holds ${}^C D_{0+}^\alpha v(0) = \Gamma(\alpha+1)\gamma$, and

$$\begin{aligned} {}^C D_{0+}^\alpha v(t) &= \frac{v(t)-v(0)}{\Gamma(1-\alpha)t^\alpha} \\ &\quad + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{v(t)-v(\tau)}{(t-\tau)^{\alpha+1}} d\tau, \quad 0 < t \le T. \end{aligned}$$

Let $x_0 \in \mathbb{R}^d$, $K > 0$, $G := \{(t,x) : 0 \le t \le T,\ \|y - x_0\| \le K\}$ and $f : G \to \mathbb{R}^d$ is a continuous. Consider an initial value problem of order α in the form

$$ {}^C D_{0+}^\alpha x(t) = f(t, x(t)), \quad t > 0, \tag{1}$$

$$ x(0) = x_0 \tag{2}$$

Using Theorem 1 and the arguments as in [3, Lemma 6.2, p. 86] we obtain the following result.

Lemma 2. A function $y \in B_{C([0,T],\mathbb{R}^d)}(x_0, K)$ is a solution of the problem (1)-(2) if and only if it satisfies the Volterra integral equation

$$y(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau))\, d\tau, \quad t \in [0,T].$$

Using this lemma one can derive the following results on existence and uniqueness of solution to the problem (1)-(2).

Theorem 3 (Local existence)**.** There exists $T_b(x_0) \in (0,T)$ such that the problem (1)-(2) has a solution $\varphi(\cdot, x_0) \in C([0, T_b(x_0)], \mathbb{R}^d)$. Moreover, for any $0 \le t \le T_b(x_0)$ we have $(t, \varphi(t, x_0)) \in G$.

Proof. Using the same arguments as in the proof of [3, Theorem 6.1, p. 86]. □

Theorem 4 (Existence of unique solution on maximal interval of existence)**.** Assume additionally that the function $f(\cdot,\cdot)$ is uniformly Lipschitz continuous with respect to the second variable on G. Then there exists (a maximal time) $T_b(x_0) \in (0,T]$ such that the problem (1)-(2) has a unique solution $\varphi(\cdot,x_0) \in C([0,T_b(x_0)],\mathbb{R}^d)$. Moreover, for any $0 \le t \le T_b(x_0)$ we have

$$(t,\varphi(t,x_0)) \in G, \text{ and } (T_b(x_0),\varphi(T_b(x_0),x_0)) \in \partial G,$$

i.e. either $T_b(x_0) = T$, or $T_b(x_0) < T$ and $\|\varphi(T_b(x_0),x_0) - x_0\| = K$.

Proof. The proof is followed directly from [26, Proposition 4.6, p. 2892]. □

3 Asymptotic behavior of solutions to FDEs

In this section, we study asymptotic properties of solutions to fractional differential equations and show some distinct features compared to that of solutions to ordinary differential equations. We first show that a solution of fractional differential equations does not converge to an equilibrium point with exponential rate. Then, we present various notions of stability of solutions to FDEs, some of them are completely analogous to that of the ODEs, but one, namely the Mittag-Leffler stability is a new notion of stability which is suitable for systems of fractional-order.

3.1 Solution of FDEs cannot decay faster than power rate

Consider a nonlinear fractional system of order $\alpha \in (0,1)$ in the form

$$^{C}D^{\alpha}_{0+}x(t) = g(t,x(t)), \quad t > 0, \tag{3}$$

where $g : \mathbb{R}_{\ge 0} \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies the three conditions:

(g.1) $g(\cdot,\cdot)$ is continuous;

(g.2) $g(t,0) = 0$ for all $t \ge 0$;

(g.3) $g(\cdot,\cdot)$ is global Lipschitz continuous with respect to the second variable, i.e., there exists a constant $L > 0$ such that $\|g(t,x) - g(t,y)\| \le L\|x - y\|$ for all $t \ge 0$ and $x, y \in \mathbb{R}^d$.

It is well known that the initial value problem for the fraction differential equation (3) has unique solution defined on the whole $\mathbb{R}_{\ge 0}$, for any given initial value in $\mathbb{R}^d$ (see [38, Theorem 2]). We will prove that there is no nontrivial solution of (3) converging to the origin with exponential rate.

Lemma 5. Every nontrivial solution of (3) does not converge to the origin with exponential rate.

Proof. Due to the existence and uniqueness of solution to (3), for any $x_0 \neq 0$, the initial value problem (3) with the condition $x(0) = x_0$ has the unique solution $\Phi(\cdot, x_0)$ on the interval $[0, \infty)$. Assume that this solution converges to the origin with the exponential rate, then there exist positive constants λ and T_1 such that

$$\|\Phi(t, x_0)\| < \frac{1}{\exp(\lambda t)}, \quad \text{for all } t \geq T_1. \tag{4}$$

Take and fix a positive number $K > 0$ satisfying $K\|x_0\| > 1$. We recall here the notion of Mittag-Leffler functions; namely, the *Mittag-Leffler matrix function* $E_{\alpha,\beta}(A)$, for $\beta > 0$ and a matrix $A \in \mathbb{R}^{d\times d}$ is defined as

$$E_{\alpha,\beta}(A) := \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k + \beta)}, \qquad E_\alpha(A) := E_{\alpha,1}(A),$$

see, e.g., Diethelm [3]. In case $d = 1$ the above formula gives definition of Mittag-Lefler function of a real variable. From the asymptotic behavior of the exponential functions and Mittag-Leffler functions, there is a constant $T_2 > 0$ such that

$$\frac{1}{\exp(\lambda t)} < \frac{E_\alpha(-Lt^\alpha)}{K}, \quad \text{for all } t \geq T_2. \tag{5}$$

Put $T_0 = \max\{T_1, T_2\}$. Using the equivalent integral form of (3), by virtue of (4) and (5), we have

$$\begin{aligned}
\frac{\Gamma(\alpha)\|x_0\|}{L} &\le \limsup_{t\to\infty} \int_0^{T_0} (t-s)^{\alpha-1} \|\Phi(s,x_0)\|\, ds \\
&+ \limsup_{t\to\infty} \int_{T_0}^{t} (t-s)^{\alpha-1} \|\Phi(s,x_0)\|\, ds \\
&\le \sup_{s\in[0,T_0]} \|\Phi(s,x_0\| \limsup_{t\to\infty} \int_0^{T_0} (t-s)^{\alpha-1}\, ds \\
&+ \limsup_{t\to\infty} \int_{T_0}^{t} (t-s)^{\alpha-1} \frac{1}{\exp(\lambda s)}\, ds \\
&\le \sup_{s\in[0,T_0]} \|\Phi(s,x_0\| \limsup_{t\to\infty} \frac{t^\alpha-(t-T_0)^\alpha}{\alpha} \\
&+ \limsup_{t\to\infty} \frac{1}{K} \int_0^t (t-s)^{\alpha-1} E_\alpha(-Ls^\alpha)\, ds \\
&= \limsup_{t\to\infty} \frac{1}{K} \int_0^t (t-s)^{\alpha-1} E_\alpha(-Ls^\alpha)\, ds \qquad (6)
\end{aligned}$$

It is worth mentioning that $E_\alpha(-Lt^\alpha)$ is the solution of the initial value problem

$$\begin{aligned}
{}^{C}D^{\alpha}_{0+}x(t) &= -Lx(t), \quad t>0, \\
x(0) &= 1,
\end{aligned}$$

see, e.g., [4, Example 4.9, pp. 231]. Hence,

$$E_\alpha(-Lt^\alpha) = 1 - \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E_\alpha(-Ls^\alpha)\, ds, \quad \forall t \ge 1,$$

and

$$\lim_{t\to\infty} \int_0^t (t-s)^{\alpha-1} E_\alpha(-Ls^\alpha)\, ds = \frac{\Gamma(\alpha)}{L},$$

a contradiction with (6). Therefore, there do not exist any nontrivial solution of (3) converging to the origin with the exponential rate. The proof is complete. □

A closer look at the proof of Lemma 5 allows us to have an even stronger statement on the decaying rate of solutions to fractional differential equations.

Theorem 6 (Power rate decay of solution of FDEs)**.** Any nontrivial solution of the FDE (3) cannot decay to 0 faster than $t^{-\alpha}$. More precisely, let $\Phi(\cdot, x_0)$ be an arbitrary solution of the FDE (3) with initial value $\Phi(0, x_0) = x_0 \neq 0$ and $\beta > 0$ be an arbitrary positive number satisfying $\beta > \alpha$, then

$$\limsup_{t\to+\infty} t^\beta \|\Phi(t, x_0)\| = +\infty.$$

Proof. Assume, in contrary, that there exists an $\beta > \alpha$ such that

$$\limsup_{t\to+\infty} t^\beta \|\Phi(t, x_0)\| = M < \infty.$$

It suffices to use the arguments of the proof of Lemma 5, modifying the relations (4) and (5) by changing $\exp(\lambda t)$ there to $t^\beta/(M + 1)$, to derive a contradiction. □

Remark 7. Lemma 5 remains true if we replace the strong condition of global Lipschitz property (g.3) by a weaker condition of local Lipschitz property of g at the origin:

(g.3') There are positive constants $a > 0, L > 0$ such that $\|g(t, x)-g(t, y)\| \leq L\|x - y\|$ for all $t \geq 0$ and $x, y \in \mathbb{R}^d$, $\|x\| \leq a$, $\|y\| \leq a$.

Similarly, nonuniform Lipschitz property (g.3') of g suffices for Theorem 6.

3.2 Notions of stability for FDE systems

Consider the nonlinear fractional differential equation (3)

$${}^C D^\alpha_{0+} x(t) = g(t, x(t)), \quad t > 0,$$

where $g : \mathbb{R}_{\geq 0} \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous and satisfies the condition (g.1)-(g.2)-(g.3'). Since g is local Lipschitz continuous, Theorem 3 and Theorem 4 imply unique existence of solution to the initial value problem (3), $x(0) = x_0$ for $x_0 \in \mathbb{R}^d, \|x\| \leq a$. Let $\Phi : I \times \mathbb{R}^d \to \mathbb{R}^d$ denote the solution of (8), $x(0) = x_0$, on its maximal interval of existence $I = [0, T_b(x_0))$ with $0 < T_b(x_0) \leq \infty$. We recall notions of stability and asymptotic stability of the trivial solution of (3) which is a direct application of the stability notions from classical ordinary differential equations theory to the FDE case, cf. [3, Definition 7.2, p. 157].

Definition 8. (i) The trivial solution of the nonlinear fractional differential equation (3) is called *stable* if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $\|x_0\| < \delta$ we have $T_b(x_0) = \infty$ and $\|\Phi(t, x_0)\| < \varepsilon$ for all $t \geq 0$.

(ii) The trivial solution is called *asymptotically stable* if it is stable and there exists some $\tilde{\delta} > 0$ such that $\lim_{t\to\infty} \|\Phi(t, x_0)\| = 0$ whenever $\|x_0\| < \tilde{\delta}$.

It is well known that there is a notion of exponential stability of solution of ordinary differential equations which related to the exponential rate of convergence to solutions. However, the results of Section 3 show that the non-trivial solution to FDEs cannot decay with exponential rate but at most power rate. Therefore, it make sense to investigate the power rate of decay of solution to FDEs.

In the equation (3) if $g(t, x) = Ax$ for all $t \geq 0$, $x \in \mathbb{R}^d$ and $A \in \mathbb{R}^{d\times d}$, then for any $x_0 \in \mathbb{R}^d$, this system with the initial condition $x(0) = x_0$ has the unique solution $E_\alpha(t^\alpha A)x_0$ on the interval $[0, \infty)$. This suggests us to use the Mittag-Leffler function in establishing a suitable stability definition for systems of fractional-order.

Motivated by Lemma 5, we now propose a new definition to characterize the convergent rate to the equilibrium points of solutions to FDEs. This is similar to that introduced by several authors (see Li *et al.* [20, 21] and Stamova [27]).

Definition 9. The equilibrium point $x^* = 0$ of (3) is called *Mittag-Leffler stable* if there exist positive constants β, m and δ such that

$$\sup_{t\geq 0} t^\beta \|\Phi(t, x_0)\| \leq m \tag{7}$$

for all $\|x_0\| \leq \delta$.

Remark 10. (i) Our definition of Mittag-Leffler stability is formulated in the form similar to the notion of exponential stability in the classical theory of ordinary differential equations. It reveals the power rate of decay of solutions to Mittag-Leffler stable systems.

(ii) Due to the asymptotic behavior of the Mittag-Leffler function our definition is equivalent to the definition of Mittag-Leffler stability by several other authors (see Li *et al.* [20, 21], and Stamova [27]).

(iii) In light of Theorem 6 the parameter β in the Definition 9 must satisfy $\beta \leq \alpha$.

4 Linearized Mittag-Leffler stability of fractional systems

In this section, we propose a Lyapunov's first method to study the asymptotic behavior of solutions to FDEs. Based on a variation of constants formula, properties of Mittag-Leffler functions, Lyapunov-Perron approach and a new weighted norm which first appears in the literature, we obtain the Mittag-Leffler stability of fixed points to a class of nonlinear FDEs linearized about its equilibrium points.

4.1 Formulation of the result

Consider a nonlinear fractional differential equation in the form

$$ {}^{C}D^{\alpha}_{0+}x(t) = Ax(t) + f(x(t)), \tag{8} $$

where $A \in \mathbb{R}^{d\times d}$ and $f : \mathbb{R}^d \to \mathbb{R}^d$ is continuous on $\mathbb{R}^d$ and Lipschitz continuous in a neighborhood of the origin satisfying

$$ f(0) = 0 \quad \text{and} \quad \lim_{r\to 0} \ell_f(r) = 0, \tag{9} $$

in which

$$ \ell_f(r) := \sup_{x,y\in B_{\mathbb{R}^d}(0,r)} \frac{\|f(x) - f(y)\|}{\|x - y\|}. $$

Furthermore, let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of A. Suppose that

$$ \lambda_i \in \Lambda^s_\alpha := \left\{\lambda \in \mathbb{C} \setminus \{0\} : |\arg(\lambda)| > \frac{\alpha\pi}{2}\right\},\ i = 1, \ldots, n. \tag{10} $$

Our task is to study the asymptotic behavior of solutions to (8) around the origin. In [13], the authors give a linearized stability theorem for the trivial solution of (8) as follows.

Theorem 11 (see [13, Theorem 3.1])**.** Assume that A satisfies the condition (10) and $f(\cdot)$ satisfies the condition (9). Then the trivial solution of the system (8) is asymptotically stable.

After the work [13, Theorem 3.1], a natural question now arises: what is the rate of convergence to the origin for solutions of the nonlinear FDE (8)? As shown above (see Theorem 6) the trivial solution of fractional-order systems cannot be exponentially stable. Hence, the best rate of convergence one may expect is the polynomial rate, and one of our main contributions is the following result on Mittag-Leffler stability of FDEs.

Theorem 12 (Lyapunov's first method for Mittag-Leffler stability)**.** Assume that A satisfies (10) and $f(\cdot)$ satisfies the condition (9). Then the trivial solution of the system (8) is Mittag-Leffler stable.

To prove Theorem 12 we need the lemmas below.

Lemma 13. (i) For any $\lambda \in \Lambda_\alpha^s$, there exists a constant $C_1 > 0$ such that

$$|E_\alpha(\lambda t^\alpha)| \leq C_1 E_\alpha(-t^\alpha), \quad \forall t \geq 0.$$

(ii) There is a constant $C_2 > 0$ such that

$$t^\alpha \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha) s^{-\alpha}\, ds \leq C_2, \quad \forall t \geq 0.$$

Proof. (i) The proof of this statement is obvious.

(ii) The proof is deduced by using [1, Formula (1.100)] and the asymptotic behavior of Mittag-Leffler function $E_\alpha(-t^\alpha)$. □

Lemma 14. Let $\lambda \in \Lambda_\alpha^s$. Then, there exists a positive constant C_3 satisfying

$$\int_0^\infty \tau^{\alpha-1} |E_{\alpha,\alpha}(\lambda \tau^\alpha)|\, d\tau < C_3.$$

Proof. See [28, Theorem 3(ii)]. □

4.2 Proof of Theorem 12

We follow the approach in our preceding paper [13] to complete the proof of Theorem 12. This proof contain two main steps:

- Transformation of the linear part: we transform the matrix A in (8) to a Jordan normal form which is "very close" to a diagonal matrix. This step helps us to reduce the difficulty in the estimation of the matrix valued Mittag-Leffler function in the next step.

- Construction of an appropriate Lyapunov-Perron operator: in this step, we establish a family of operators with the property that any solution of the nonlinear system (8) can be interpreted as a fixed point of these operators. On the other hand, these operators are contractive in a suitable space and hence their fixed points can be estimated.

Transformation of the linear part

By virtue [29, Theorem 6.37, pp. 146], we can find a nonsingular matrix $T \in \mathbb{C}^{d\times d}$ such that

$$T^{-1}AT = \operatorname{diag}(A_1, \dots, A_n),$$

where for $i = 1, \dots, n$ the block A_i has the form

$$A_i = \lambda_i \operatorname{id}_{d_i\times d_i} + \delta_i N_{d_i\times d_i},$$

with λ_i is an eigenvalue, $\delta_i \in \{0, 1\}$ and the nilpotent matrix $N_{d_i\times d_i}$ is given by

$$N_{d_i\times d_i} := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{d_i\times d_i}.$$

Let η be an arbitrary but fixed positive number. Applying the transformation $P_i := \operatorname{diag}(1, \eta, \dots, \eta^{d_i-1})$ leads to

$$P_i^{-1}A_iP_i = \lambda_i \operatorname{id}_{d_i\times d_i} + \eta_i N_{d_i\times d_i},$$

$\eta_i \in \{0, \eta\}$. Hence, under the transformation $y := (TP)^{-1}x$ system (8) becomes

$$^{C}D^{\alpha}_{0+}y(t) = \operatorname{diag}(J_1, \dots, J_n)y(t) + h(y(t)), \tag{11}$$

where $J_i := \lambda_i\operatorname{id}_{d_i\times d_i}$ for $i = 1, \dots, n$ and the function h is given by

$$h(y) := \operatorname{diag}(\eta_1 N_{d_1\times d_1}, \dots, \eta_n N_{d_n\times d_n})y + (TP)^{-1}f(TPy).$$

Remark 15 (see [13, Remark 3.2]). The map

$$x \mapsto \operatorname{diag}(\eta_1 N_{d_1\times d_1}, \dots, \eta_n N_{d_n\times d_n})x$$

is a Lipschitz continuous function with Lipschitz constant η. Thus, by (9) we have

$$h(0) = 0, \qquad \lim_{r\to 0} \ell_h(r) = \begin{cases} \eta & \text{if there exists } \eta_i = \eta, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 16. The type of stability of the trivial solution to equations (8) and (11) is the same, i.e., they are both stable (asymptotic/Mittag-Leffler stable) or not stable (asymptotic/Mittag-Leffler stable).

Construction of an appropriate Lyapunov-Perron operator

We now concentrate only on the equation (11) and introduce a Lyapunov-Perron operator associated with this equation.

For any $x = (x^1, \dots, x^n) \in \mathbb{C}^d = \mathbb{C}^{d_1} \times \cdots \times \mathbb{C}^{d_n}$, the operator

$$\mathcal{T}_x : C([0,\infty), \mathbb{C}^d) \to C([0,\infty), \mathbb{C}^d)$$

is defined by

$$(\mathcal{T}_x\xi)(t) = ((\mathcal{T}_x\xi)^1(t), \dots, (\mathcal{T}_x\xi)^n(t)) \qquad \text{for } t \in \mathbb{R}_{\geq 0},$$

where for $i = 1, \dots, n$

$$\begin{aligned}(\mathcal{T}_x\xi)^i(t) &= E_\alpha(t^\alpha J_i)x^i + \\ &\quad \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha J_i) h^i(\xi(\tau))\, d\tau,\end{aligned}$$

is called the *Lyapunov-Perron operator associated with* (11). The relationship between a fixed point of the operator $\mathcal{T}_x(\cdot)$ and a solution to the equation (11) is described in the lemma below.

Lemma 17. Consider (11) and assume that the function $h(\cdot)$ is global Lipschitz continuous. Let $x \in \mathbb{C}^d$ be arbitrary and $\xi : \mathbb{R}_{\geq 0} \to \mathbb{C}^d$ be a continuous function with $\xi(0) = x$. Then, the following two statements are equivalent:

(i) ξ is a solution of (11) with the initial condition $x(0) = x$.

(ii) ξ is a fixed point of the operator $\mathcal{T}_x$.

Proof. The proof is obtained by using the theorem on existence and uniqueness of solutions and the variation of constants formula for fractional differential equations, see e.g., [30]. □

Our novel contribution in the present work is to combine the approach in [13] and a new weighted norm as follows. In $C([0,\infty), \mathbb{C}^d)$ we define a function $\|\cdot\|_w$ by

$$\|x\|_w = \max\{\sup_{t\in[0,1]} \|x(t)\|, \sup_{t\geq 1} t^\alpha \|x(t)\|\}.$$

Then $C_w := \{x \in C([0,\infty), \mathbb{C}^d) : \|x\|_w < \infty\}$ is also a Banach space with the norm $\|\cdot\|_w$.

Next, we give some estimates concerning the operator $\mathcal{T}_x$ in the space C_w.

358 **Proposition 18.** Consider system (11) and suppose that

$$\lambda_i \in \Lambda_\alpha^s, \qquad i = 1, \ldots, n,$$

where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A. Then, we can find a constant $C(\alpha, A)$ depending on α and $\lambda := (\lambda_1, \ldots, \lambda_n)$ such that

$$\begin{aligned}
&\|\mathcal{T}_x\xi - \mathcal{T}_{\widehat{x}}\widehat{\xi}\|_w \\
&\leq \max_{1\leq i\leq n}\{\sup_{t\in[0,1]}|E_\alpha(\lambda_i t^\alpha)| + \sup_{t\geq 1} t^\alpha |E_\alpha(\lambda_i t^\alpha)|\}\|x - \widehat{x}\| \\
&\qquad + C(\alpha, A)\, \ell_h(\max\{\|\xi\|_\infty, \|\widehat{\xi}\|_\infty\})\|\xi - \widehat{\xi}\|_w
\end{aligned} \tag{12}$$

for all $x, \widehat{x} \in \mathbb{C}^d$ and $\xi, \widehat{\xi} \in C_w$. Consequently, $\mathcal{T}_x$ considered as an operator on the Banach space C_w endowed with the norm $\|\cdot\|_w$ is well-defined and

$$\|\mathcal{T}_x\xi - \mathcal{T}_x\widehat{\xi}\|_w \leq C(\alpha, A)\, \ell_h(\max\{\|\xi\|_\infty, \|\widehat{\xi}\|_\infty)\}\|\xi - \widehat{\xi}\|_w.$$

Proof. For $i = 1, \ldots, n$, we get

$$\begin{aligned}
&\|(\mathcal{T}_x\xi)^i(t) - (\mathcal{T}_{\widehat{x}}\widehat{\xi})^i(t)\| \\
&\quad\leq \|x - \widehat{x}\||E_\alpha(\lambda_i t^\alpha)| + \ell_h(\max\{\|\xi\|_\infty, \|\widehat{\xi}\|_\infty\})\times \\
&\int_0^t (t-\tau)^{\alpha-1}|E_{\alpha,\alpha}(\lambda_i(t-\tau)^\alpha)|\|(\xi - \widehat{\xi})(\tau)\|\, d\tau.
\end{aligned}$$

In the case $t \in [0, 1]$, we have

$$\begin{aligned}
&\sup_{t\in[0,1]}\|(\mathcal{T}_x\xi - \mathcal{T}_x\widehat{\xi})^i(t)\| \leq \sup_{t\in[0,1]}|E_\alpha(\lambda_i t^\alpha)|\|x - \widehat{x}\| \\
&+ \ell_h(\max\{\|\xi\|_\infty, \|\widehat{\xi}\|_\infty\})\int_0^\infty u^{\alpha-1}|E_{\alpha,\alpha}(-\lambda_i u^\alpha)|\, du\, \|\xi - \widehat{\xi}\|_w.
\end{aligned} \tag{13}$$

Furthermore,

$$\begin{aligned}
&\sup_{t\geq 1} t^\alpha\|(\mathcal{T}_x\xi - \mathcal{T}_x\widehat{\xi})^i(t)\| \\
&\leq \sup_{t\geq 1} t^\alpha|E_\alpha(\lambda_i t^\alpha)|\, \|x - \widehat{x}\| + C_{\lambda_i}\ell_h(\max\{\|\xi\|_\infty, \|\widehat{\xi}\|_\infty\})\times \\
&\quad \sup_{t\geq 1} t^\alpha\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-(t-\tau)^\alpha)\tau^{-\alpha}\, d\tau\|\xi - \widehat{\xi}\|_w,
\end{aligned} \tag{14}$$

where C_{λ_i} is a constant chosen as in Lemma 13 (i). Now by combining Lemma 13, (13) and (14), we have

$$\begin{aligned}
&\|\mathcal{T}_x\xi) - \mathcal{T}_{\widehat{x}}\widehat{\xi}\|_w \\
&\quad \le \max_{1\le i\le n}\Big\{\sup_{t\in[0,1]}|E_\alpha(\lambda_i t^\alpha)| + \sup_{t\ge 1} t^\alpha |E_\alpha(\lambda_i t^\alpha)|\Big\}\|x-\widehat{x}\| \\
&\qquad + C(\alpha,A)\,\ell_h(\max\{\|\xi\|_\infty,\|\widehat{\xi}\|_\infty\})\|\xi-\widehat{\xi}\|_w,
\end{aligned}$$

where

$$\begin{aligned}
C(\alpha,A) := &\max_{1\le i\le n}\int_0^\infty u^{\alpha-1}|E_{\alpha,\alpha}(\lambda_i u^\alpha)|\,du \\
&+ C_\lambda \sup_{t\ge 1} t^\alpha \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-(t-\tau)^\alpha)\tau^{-\alpha}\,d\tau
\end{aligned}$$

with $C_\lambda := \max\{C_{\lambda_1},\dots,C_{\lambda_n}\}$. The proof is complete. □

We have showed that $\mathcal{T}_x(\cdot)$ is well-defined and that it is Lipschitz continuous with the constant $C(\alpha,A)$. Moreover, $C(\alpha,A)$ is independent of the constant η. From now, we choose $\eta = \frac{1}{2C(\alpha,A)}$.

Lemma 19. Let

$$\lambda_i \in \Lambda_\alpha^s, \qquad i=1,\dots,n,$$

where $\lambda_1,\dots,\lambda_n$ are eigenvalues of A and $C(\alpha,A)$ be the constant defined in Proposition 18. We have the following assertions.

(i) There is a $r>0$ such that

$$q := C(\alpha,A)\,\ell_h(r) < 1. \tag{15}$$

(ii) Choose $r>0$ satisfying (15) and let

$$\gamma := \max_{1\le i\le n}\Big\{\sup_{t\in[0,1]}|E_\alpha(\lambda_i t^\alpha)| + \sup_{t\ge 1} t^\alpha|E_\alpha(\lambda_i t^\alpha)|\Big\}$$

and

$$r^* := \frac{r(1-q)}{\gamma}. \tag{16}$$

Define $B_{C_w}(0,r) := \{\xi \in C_\infty([0,\infty),\mathbb{C}^d) : ||\xi||_w \le r\}$. Then, for any $x\in B_{\mathbb{C}^d}(0,r^*)$, we have $\mathcal{T}_x(B_{C_w}(0,r)) \subset B_{C_w}(0,r)$ and

$$\|\mathcal{T}_x\xi - \mathcal{T}_x\widehat{\xi}\|_w \le q\|\xi-\widehat{\xi}\|_w \quad \text{for all } \xi,\widehat{\xi}\in B_{C_w}(0,r).$$

Proof. (i) Due to Remark 15, $\lim_{r\to 0}\ell_h(r) \le \eta$. Hence $\eta C(\alpha, A) = \frac{1}{2}$ and the proof of (i) is complete.

(ii) Let $x \in B_{\mathbb{C}^d}(0, r^*)$ and $\xi \in B_{C_w}(0, r)$. According to (12) in Proposition 18, we obtain that

$$\begin{aligned}
&\|\mathcal{T}_x\xi\|_w \\
&\quad\le \max_{1\le i\le n}\{\sup_{t\in[0,1]}|E_\alpha(\lambda_i t^\alpha)| + \sup_{t\ge 1} t^\alpha |E_\alpha(\lambda_i t^\alpha)|\}\|x\| \\
&\qquad\quad + C(\alpha, A)\,\ell_h(r)\|\xi\|_w \\
&\quad\le (1-q)r + qr,
\end{aligned}$$

which proves that $\mathcal{T}_x(B_{C_w}(0,r)) \subset B_{C_w}(0,r)$. Moreover, from Proposition 18 and part (i), we have

$$\begin{aligned}
\|\mathcal{T}_x\xi - \mathcal{T}_x\widehat{\xi}\|_w &\le C(\alpha, A)\ell_h(r)\,\|\xi - \widehat{\xi}\|_w \\
&\le q\|\xi - \widehat{\xi}\|_w
\end{aligned}$$

for all $x \in B_{\mathbb{C}^d}(0, r^*)$ and $\xi, \widehat{\xi} \in B_{C_w}(0, r)$. This ends the proof. □

Proof of Theorem 12. Due to Remark 16, it is sufficient to prove the Mittag-Leffler stability for the trivial solution of (11). To do this, taking r^* is a constant defined as in (16). For any $x \in B_{\mathbb{C}^d}(0, r^*)$, by Lemma 19 and the Contraction Mapping Principle, there is a unique fixed point $\xi \in B_{C_w}(0, r)$ of $\mathcal{T}_x$. This fixed point is also the unique solution of (11) satisfying $\xi(0) = x$ (see Lemma 17). Together existence and uniqueness of solutions for initial value problems for the equation (11) in a neighborhood of the origin, this shows that the trivial solution is stable in the Lyapunov's sense. Furthermore,

$$\sup_{t\ge 0} t^\alpha \|\xi(t)\| \le r,$$

which shows that the solution 0 of (11) is Mittag-Leffler stable. The proof is complete. □

Remark 20. In [13, Theorem 3.1], we proved that the trivial solution to (8) is asymptotically stable. However, we did not know the decay rate of non-trivial solutions to this equation. Now by Theorem 12, this question is answered fully. Namely, in the proof of Theorem 12 we showed the convergence rate of solutions around the equilibrium as $t^{-\alpha}$.

Remark 21. In the case the linear part A of the equation (8) is hyperbolic, that is the spectrum $\sigma(A)$ satisfies

$$\sigma(A) \cap \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg(\lambda) = \frac{\alpha\pi}{2}\} = \emptyset, \quad \sigma(A) \cap \Lambda_\alpha^u \neq \emptyset,$$

where $\Lambda_\alpha^u := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg(\lambda) < \frac{\alpha\pi}{2}\}$, Cong *et al.* [31] showed the existence of a stable manifold around the origin. Now using the weighted norm $\|\cdot\|_w$ and the approach as in the proof of Theorem 12, we may prove that solutions of (8) starting from its stable manifold converge to the origin with the rate $t^{-\alpha}$.

Remark 22. In the case $A = 0$, we can not use the linearization method around an equilibrium point to analyze the Mittag-Leffler stability of (8). To overcome this obstacle, in Section 5 we will develop the Lyapunov's second method for fractional differential equations.

4.3 Application of Theorem 12 in the stabilization of fractional-order chaotic systems

In this subsection, we discuss on an application in the stabilization of some fractional-order chaotic systems.

Example 23 (Fractional Lorenz system). Consider the fractional-order Lorenz system of the order $\alpha \in (0,1)$ at the origin as follows.

$$\begin{aligned} {}^C D_{0+}^\alpha x_1(t) &= -\sigma x_1(t) + \sigma x_2(t), \\ {}^C D_{0+}^\alpha x_2(t) &= \rho x_1(t) - x_2(t) - x_1(t)x_3(t), \\ {}^C D_{0+}^\alpha x_3(t) &= -\beta x_3(t) + x_1(t)x_2(t), \end{aligned} \tag{17}$$

where σ is called the Prandtl number and ρ is called the Rayleigh number. For $\alpha = 0.995$ and $(\sigma, \rho, \beta) = (10, 28, 8/3)$, this system is chaotic [32, pp. 134–137]. It is obvious that 0 is a solution of (17). On the other hand, its linear part has three eigenvalues as $\lambda_1 = -\frac{8}{3}$, $\lambda_2 = \frac{1}{2}(\sqrt{1201} - 11)$, $\lambda_3 = \frac{1}{2}(-\sqrt{1201} - 11)$. Hence, from [33, Theorem 5], this solution is unstable.

Now consider a controlled system of (17) with a linear feedback control input:

$$ {}^C D_{0+}^\alpha x(t) = A_1 x(t) + f_1(x(t)) + B_1 u(t), \tag{18}$$

$$ u(t) = K_1 x(t), \tag{19}$$

where

$$A_1 = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{pmatrix},$$

$$f_1(x(t)) = \begin{pmatrix} 0 \\ -x_1(t)x_3(t) \\ x_1(t)x_2(t) \end{pmatrix},$$

the state-space matrix $B_1 \in \mathbb{R}^{3\times 1}$ and a feedback gain $K_1 \in \mathbb{R}^{1\times 3}$ needs to be determined. For simplicity, let $B_1 = (1,1,1)^T$. By virtue Theorem 12, for $K_1 = (0,-10,0)$, then the trivial solution of (18)–(19) is Mittag-Leffler stable for any $\alpha \in (0,1)$.

Example 24 (Fractional-order Liu system). Consider the fractional Liu system of the order $\alpha \in (0,1)$:

$$^{C}D^{\alpha}_{0+}x(t) = A_2x(t) + f_2(x(t)), \tag{20}$$

where

$$A_2 = \begin{pmatrix} -a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -c \end{pmatrix},$$

$$f_2(x(t)) = \begin{pmatrix} -ex_2^2(t) \\ -kx_1(t)x_3(t) \\ mx_1(t)x_2(t) \end{pmatrix},$$

with a,b,c,e,k,m are positive constants. In this example we let $(a,b,c,e,k,m) = (1,2.5,5,1,4,4)$. As known in [32, pp. 142–145], the system (20) is chaotic for $\alpha = 0.95$. Note that in this case, its trivial solution is unstable. Using the same approach as above, we can stabilize (20) as below.
Consider a controlled system of (20) with a linear feedback control input is described by

$$^{C}D^{\alpha}_{0+}x(t) = A_2x(t) + f_2(x(t)) + B_2u(t), \tag{21}$$

$$u(t) = K_2x(t). \tag{22}$$

Let $B_2 = (1,1,1)^T$ and choose $K_2 = (0,-3.5,0)$, then the trivial solution of the controlled system (21)–(22) is Mittag-Leffler stable.

5 Lyapunov's second method and Mittag-Leffler stability

This section is devoted to develop a Lyapunov's second method for systems of fractional-order equations. Our approach is based on a comparison principle for FDE and an inequality concerning with fractional derivatives of a convex function. For this purpose, we introduce the following preparation results.

Lemma 25. Let $m : [0, T] \to \mathbb{R}$ be continuous and Caputo derivative ${}^C D^\alpha_{0+} m$ exists on the interval $(0, T]$. If there is a $t_0 \in (0, T]$ such that

$$m(t) \le 0 \quad \forall t \in [0, t_0) \quad \text{and} \quad m(t_0) = 0,$$

then ${}^C D^\alpha_{0+} m(t_0) \ge 0$.

Proof. The proof of this lemma is obtained by using arguments as in the proof of [34, Lemma 2.1]. □

Based on arguments as in [34, Theorem 2.3], the following comparison proposition holds.

Proposition 26. Let $L : \mathbb{R} \to \mathbb{R}$ be continuous and non-increasing (it means that for $x_1 \le x_2$ then $L(x_1) \ge L(x_2)$, $m_1 : [0, T] \to \mathbb{R}$, $m_2 : [0, T] \to \mathbb{R}$ be continuous. Assume that ${}^C D^\alpha_{0+} m_1$, ${}^C D^\alpha_{0+} m_1$ exist on $(0, T]$. If

$$ {}^C D^\alpha_{0+} m_1(t) \ge L(m_1(t)), \quad t \in (0, T], \quad m_1(0) \ge m_0, \tag{23}$$
$$ {}^C D^\alpha_{0+} m_2(t) \le L(m_2(t)), \quad t \in (0, T], \quad m_2(0) \le m_0, \tag{24}$$

then $m_1(t) \ge m_2(t)$ for all $t \in (0, T]$.

Proof. We first assume that one of the inequalities in (23) and (24) is strict, say ${}^C D^\alpha_{0+} m_2(t) < L(m_2(t))$ and $m_2(0) < m_0 \le m_1(0)$. Then, for all $t \in [0, T]$ the following inequality holds

$$m_2(t) < m_1(t).$$

Indeed, suppose that there is a $t_0 \in (0, T]$ such that $m_2(t_0) = m_1(t_0)$ and $m_2(t) < m_1(t)$ on the interval $[0, t_0)$. Set $m(t) = m_2(t) - m_1(t)$ it follows that $m(t_0) = 0$ and $m(t) < 0$ for $t \in [0, t_0)$. By virtue of Lemma 25, it implies that ${}^C D^\alpha_{0+} m(t_0) \ge 0$. However, since $m_2(t_0) = m_1(t_0)$, we get

$$\begin{aligned} L(m_2(t_0)) &> {}^C D^\alpha_{0+} m_2(t_0) \\ &\ge {}^C D^\alpha_{0+} m_1(t_0) \\ &\ge L(m_1(t_0)) \\ &= L(m_2(t_0)), \end{aligned}$$

a contradiction. Hence, $m_2(t) < m_1(t)$ on $[0, T]$. Now assume that the inequalities in (23) are non-strict. We will show that $m_2(t) \le m_1(t)$ for all

$t \in [0,T]$. Set $m_1^\varepsilon(t) = m_1(t) + \varepsilon\lambda(t)$ where $\varepsilon > 0$ and $\lambda(t) = E_\alpha(t^\alpha)$. Noting that $\lambda(\cdot)$ is positive and $L(\cdot)$ is non-increasing, we have

$$\begin{aligned} {}^C D^\alpha_{0+} m_1^\varepsilon(t) &= {}^C D^\alpha_{0+} m_1(t) + \varepsilon\lambda(t) \\ &\geq L(m_1(t)) + \varepsilon\lambda(t) \\ &= L(m_1^\varepsilon(t)) + L(m_1(t)) - L(m_1^\varepsilon(t)) + \varepsilon\lambda(t) \\ &\geq L(m_1^\varepsilon(t)) + \varepsilon\lambda(t) \\ &> L(m_1^\varepsilon(t)), \quad \forall t \in (0,T]. \end{aligned} \tag{25}$$

Due to (25) and the result above for strict inequalities, we get that $m_2(t) < m_1^\varepsilon(t)$ for all $t \in [0,T]$. Consequently, letting $\varepsilon \to 0$ leads to $m_2(t) \leq m_1(t)$, $\forall t \in [0,T]$. The proof is complete. □

Remark 27. Proposition 26 improved [34, Theorem 2.3] in the way that we do not need to require continuous differentiability of $m_1(\cdot), m_2(\cdot)$, and Lipschitz property of $L(\cdot)$. This improvement is very useful for our purpose in the next steps.

5.1 Lyapunov's second method for fractional differential equations

Let D be an open set in $\mathbb{R}^d$ and $0 \in D$. Consider a fractional order equation with the order $\alpha \in (0,1)$ in the form

$${}^C D^\alpha_{0+} x(t) = f(x(t)), \quad \text{for all } t \in (0,\infty), \tag{26}$$

where $f : D \to \mathbb{R}^d$ satisfies the two conditions:

(f.1) $f(0) = 0$;

(f.2) $f(\cdot)$ is Lipchitz continuous in a neighborhood of the origin.

The main result in this section is the following theorem.

Theorem 28 (Mittag-Leffler stability by Lyapunov's second method)**.** Consider the equation (26). Let $V : \mathbb{R}^d \to \mathbb{R}_+$ be a function satisfying three conditions:

(V.1) $V(\cdot)$ is convex and differentiable on $\mathbb{R}^d$;

(V.2) there are constants $a, b, C_1, C_2, r > 0$ such that

$$C_1\|x\|^a \leq V(x) \leq C_2\|x\|^b$$

for all $x \in B_{\mathbb{R}^d}(0,r)$;

(V.3) there exists constants $C_3, c \geq 0$ such that

$$\langle \nabla V(x), f(x) \rangle \leq -C_3 \|x\|^c$$

for all $x \in B_{\mathbb{R}^d}(0, r)$.

Then, the following statements hold

(a) if $C_3 = 0$, then the trivial solution of (26) is stable;

(b) if $C_3 > 0$, then the trivial solution of (26) is Mittag-Leffler stable .

Proof. (a) See the proof of [24, Theorem 3(a)].

(b) Due to the fact that the trivial solution to (26) is stable, for any $\varepsilon > 0$, there is a $\delta > 0$ such that the solution $\varphi(t, x_0)$ to (26) with $\|x_0\| < \delta$ satisfies $\|\varphi(t, x_0)\| < \varepsilon$ for all $t \geq 0$. Moreover, from [24, Theorem 2] and the hypotheses (V.2) and (V.3), we have

$$\begin{aligned}
{}^C D^\alpha_{0+} V(\varphi(t, x_0)) &\leq \langle \nabla V(\varphi(t, x_0)), {}^C D^\alpha_{0+} \varphi(t, x_0) \rangle \\
&\leq -C_3 \|\varphi(t, x_0)\|^c \\
&\leq -\frac{C_3}{C_2^{c/b}} (V(\varphi(t, x_0)))^{c/b}, \quad \forall t \geq 0.
\end{aligned}$$

Put $A := -\frac{C_3}{C_2^{c/b}}$, $p := \frac{c}{b}$ and consider the following initial value problem

$$\begin{cases} {}^C D^\alpha_{0+} y(t) = A y^p(t), & t > 0, \\ y(0) = V(x_0) > 0. \end{cases} \tag{27}$$

Then $V(\varphi(\cdot, x_0))$ is a sub-solution of (27) (for the definition of sub-solution see [35]). Furthermore, from the construction of a super-solution to (27) (see [35, p. 333]), we can find a super-solution w of (27) on $[0, \infty)$ defined by

$$w(t) = \begin{cases} V(x_0), & t \in [0, t_1], \\ C t^{-\frac{\alpha}{p}}, & t \geq t_1, \end{cases}$$

where $C = V(x_0) t_1^{\frac{\alpha}{p}}$ and

$$t_1^\alpha = \frac{V(x_0)^{1-p}}{-A} \left(\frac{2^\alpha}{\Gamma(1-\alpha)} + \frac{\alpha}{p} \frac{2^{\alpha + \frac{\alpha}{p}}}{\Gamma(2-\alpha)} \right).$$

Now using the comparison proposition 26, we obtain

$$V(\varphi(t, x_0)) \leq w(t), \quad \forall t \geq 0.$$

This implies that for any $x_0 \in B_{\mathbb{R}^d}(0,\delta) \setminus \{0\}$, there exists a constant $d > 0$ such that

$$\|\varphi(t,x_0)\| \leq \left(\frac{1}{C_1}V(\varphi(t,x_0))\right)^{1/a} \leq \left(\frac{d}{C_1(1+t^{\alpha/p})}\right)^{1/a}$$

for all $t \geq 0$. Note that from the existence and uniqueness of the solution to (26), if $x_0 = 0$ then $\varphi(\cdot,0) = 0$. So, the trivial solution to the original system (26) is Mittag-Leffler stable. The proof is complete. □

Remark 29. (i) Theorem 28 is still true if we replace the condition of global convex and differentiable property (V.1) by a condition of local convex and differentiable property in a neighborhood of the origin.

(ii) Theorem 28 is a new contribution in the theory of Lyapunov's second method for fractional differential equations. It improves and strengthens a recent result by Tuan and Trinh [24, Theorem 3]. In particular, we removed the condition $c > b$ in the statement of [24, Theorem 3(c)]. Moreover, we proved the Mittag-Leffler stability of the trivial solution instead of the weakly asymptotic stability.

5.2 Illustrative examples

Example 30 (Simple nonlinear one-dimensional FDE). Consider the nonlinear one-dimensional FDE of order $0 < \alpha < 1$ which is nonlinear of order $\beta \geq 1$:

$$^{C}D^{\alpha}_{0+}x(t) = f(x(t)), \quad x(0) = x_0, \tag{28}$$

where

$$f(x) := \begin{cases} -x^{\beta}, & \text{if} \quad x \geq 0, \\ |x|^{\beta}, & \text{if} \quad x < 0. \end{cases} \tag{29}$$

It is easy to see that $f(\cdot)$ is local Lipschitz continuous at the origin. Choosing $V(x) = x^2$, $x \in \mathbb{R}$. This function satisfies the conditions (V.1), (V.2) (with $C_1 = C_2 = 1$ and $a = b = 2$), and (V.3) (with $C_3 = 2$ and $c = 1+\beta$) in Theorem 28. Thus the trivial solution to (28) is Mittag-Leffler stable. More precisely, from the proof of Theorem 28, the non-trivial solutions of (28) converge to the origin with the rate at least $t^{-\alpha/(1+\beta)}$ as $t \to \infty$. A special case of (28) when $\beta = 3$ was studied by Li *et al.* [20, Example 14], Shen *et al.* [36, Remark 11], Zhou *et al.* [23], where they tried to prove asymptotic stability of (28). However, their proof is not correct, see Tuan and Trinh [24, Remark 3] for details. Our method now solves this problem completely: we showed that the trivial solution of (28) is Mittag-Leffler stable, hence asymptotically stable.

Example 31 (A more complicated nonlinear one-dimensional FDE). Consider an equation in form

$$^{C}D^{\alpha}_{0+}x(t) = -x^3 + g(x(t)), \quad t > 0, \tag{30}$$

where $g : \mathbb{R} \to \mathbb{R}$ is differentiable at the origin and satisfies

$$g(0) = 0, \quad \lim_{x\to 0} \frac{g(x)}{x^3} = 0.$$

Choosing the Lyapunov candidate function $V(x) = x^2$ for $x \in \mathbb{R}$ and $r > 0$ such that

$$2x(-x^3 + g(x)) \le -x^4, \quad \forall x \in B_{\mathbb{R}}(0, r).$$

Then the conditions of Theorem 28 are satisfied for $C_1 = C_2 = 1$, $a = b = 2$, $C_3 = 1$ and $c = 4$. Thus, the trivial solution of (30) is Mittag-Leffler stable.

Example 32 (Higher dimensional nonlinear FDE). Consider a two dimensional fractional-order nonlinear system

$$^{C}D^{\alpha}_{0+}x(t) = f(x(t)), \quad t > 0, \tag{31}$$

where $f(x) = (-x_1^3 + x_2^4, -x_2^3 - x_2 x_1^2)^{\mathrm{T}}$ for any $x = (x_1, x_2) \in \mathbb{R}^2$. In this case, we choose the Lyapunov candidate function $V(x) = \|x\|^2 = x_1^2 + x_2^2$ for $x = (x_1, x_2) \in \mathbb{R}^2$ and $r > 0$ such that

$$\langle (2x_1, 2x_2), (-x_1^3 + x_2^4, -x_2^3 - x_1^2 x_2)\rangle \le -x_1^4 - x_2^4$$

for all $x = (x_1, x_2) \in B_{\mathbb{R}^2}(0, r)$. The function $V(\cdot)$ now satisfies the conditions (V.1), (V.2) and (V.3) in Theorem 28 for $a = b = 2$, $c = 4$, $C_1 = C_2 = 1$ and $C_3 = 1$. Hence, the trivial solution of (31) is Mittag-Leffler stable.

6 Relation between Lipschitz condition, stability and speed of decay, separation of trajectories to Caputo FDEs

We first present here several examples of Caputo FDEs of various kinds of stability to illustrate the stability notions given in Section 3. It is obvious that Mittag-Leffler stability is stronger than asymptotic stability.

Example 33 (Linear autonomous FDE). Let us consider a linear autonomous FDE of order $\alpha \in (0,1)$:

$$^{C}D^{\alpha}_{0+}x(t) = Ax(t), \tag{32}$$

where

$$A = \mathrm{diag}(a_1, \ldots, a_d), \quad a_i < 0, i = 1, \ldots, d.$$

This FDE is solve explicitly and its solutions are of the form

$$\mathrm{diag}(E_\alpha(a_1 t^\alpha), \ldots, E_\alpha(a_d t^\alpha))x_0, \ x_0 \in \mathbb{R}^d,$$

see Diethelm [3, Theorem 7.2]. It is easy to see that the trivial solution of (32) is Mittag-Leffler stable and all non-trivial solutions have a decay rate $t^{-\alpha}$.

Unlike the linear autonomous case, solution to nonlinear FDEs may have decay rate smaller or bigger than the order of the equations. The FDE (28) treated in Example 30 is a nonlinear FDE with solutions decaying to 0 with rate slower than $t^{-\alpha}$. Actually we show in Example 30, using Theorem 28 that the decay rate of nontrivial solutions to the FDE (28) is at least $t^{-\alpha/(1+\beta)}$ as $t \to \infty$. An application of the result of Vergara and Zacher [35, Theorem 7.1, p. 334] shows that decay rate of nontrivial solutions to the FDE (28) is $\alpha/\beta < \alpha$ for $\beta > 1$.

Example 34 (One-dimensional FDE with non Lipschitz right-hand side). Consider the nonlinear one-dimensional FDE of order $0 < \alpha < 1$ which is nonlinear of order $\beta \in (0,1)$:

$$^{C}D^{\alpha}_{0+}x(t) = f(x(t)), \quad x(0) = x_0, \tag{33}$$

where

$$f(x) := \begin{cases} -x^\beta, & \text{if} \quad x \geq 0, \\ |x|^\beta, & \text{if} \quad x < 0. \end{cases} \tag{34}$$

It is worth mentioning that the function $f(\cdot)$ in right-hand side of the above FDE is continuous but non Lipschitzian in a neighborhood of the origin.

Let $x_0 > 0$, consider the FDE (33) in the area $x \in (0,\infty)$. From Theorem 4, the equation (33) has a unique solution, denoted by $\varphi(\cdot, x_0)$, on the maximal interval of existence $[0, T_b)$. If $T_b(x_0) < \infty$, then $\liminf_{t\to T_b(x_0)-} \varphi(t, x_0) = 0$ or $\limsup_{t\to T_b(x_0)-} \varphi(t, x_0) = \infty$ (see [37, Proposition 1]). However, using Proposition 26 and construction of a super-solution and a sub-solution to (33) (see [35, pp. 232–234]), we have

$$\limsup_{t\to T_b-} \varphi(t, x_0) \leq \frac{c_1}{1 + T_b^{\alpha/\beta}}$$

and

$$\liminf_{t\to T_b-} \varphi(t,x_0) \geq \frac{c_2}{1+T_b^{\alpha/\beta}}$$

for some $c_1, c_2 > 0$, a contradiction. Hence, $T_b = \infty$ and

$$\frac{c_2}{1+t^{\alpha/\beta}} \leq \varphi(t,x_0) \leq \frac{c_1}{1+t^{\alpha/\beta}}, \quad \forall t \geq 0.$$

On the other hand, due to the specific form of f in (34), if we multiply the solutions of (33) with negative initial values by -1 then we get solutions of (33) with the positive initial values, and vice versa. Therefore, the solution of (33) starting from $x_0 \neq 0$ has decay rate as $t^{-\gamma}$ with $\gamma = \alpha/\beta > \alpha$. This is different from the Lipschitz case (see Theorem 6).

On the other hand, by a direct computation, we obtain a global solution of the initial value problem

$$\begin{cases} {}^C D^\alpha_{0+} x(t) & = (x(t))^\beta, \quad t > 0. \\ x(0) & = 0, \end{cases}$$

as $\varphi(t,0) = \left(\frac{\Gamma(1-\alpha)}{\frac{\alpha}{1-\beta}B(1-\alpha,\frac{\alpha}{1-\beta})}\right)^{1/(1-\beta)} t^{\alpha/(1-\beta)}$, where $\Gamma(\cdot)$ is Gamma function and $B(\cdot,\cdot)$ is Beta function. This implies that the trivial solution to (33) is unstable.

A consequence of the non-Lipschitz property at the origin of $f(\cdot)$ in this example is non-uniqueness of the solution: we have at least two solutions starting from the origin. This circumstance alone makes the system unstable although any solution starting from a point close to the origin but distinct from the origin tends to the origin with decay rate of $t^{-\gamma}$.

Now we show that the Mittag-Leffler stability is strictly stronger than asymptotic stability. For this, we give below an example of an asymptotically stable FDE which is not Mittag-Leffler stable.

Example 35 (Asymptotically stable nonlinear one-dimensional FDE which is not Mittag-Leffler stable). Consider a nonlinear one-dimensional FDE of order $0 < \alpha < 1$:

$$ {}^C D^\alpha_{0+} x(t) = f(x(t)), \quad x(0) = x_0, \tag{35}$$

where

$$f(x) := \begin{cases} -e^{-1/x}x, & \text{if} \quad x > 0, \\ 0, & \text{if} \quad x = 0, \\ -e^{1/x}x, & \text{if} \quad x < 0. \end{cases} \tag{36}$$

Clearly $f(\cdot) \in C^2(-\infty,\infty)$. Therefore, by [38, Theorem 2] the equation (35) has a unique solution $x(\cdot)$ which exists globally on $\mathbb{R}_{\geq 0}$.

Fix some $x_0 > 0$. By [30, Theorem 3.5], the solution of the FDE (35) cannot intersect the trivial solution, hence $x(t) > 0$ for all $t \in \mathbb{R}_{\geq 0}$.

Now let $n \geq 2$ be an arbitrary integer. Put $g(x) := -(n-1)!x^n$ on a neighborhood of 0 and extend it suitably to get $g(x) \leq f(x)$ on $(0,\infty)$. By Proposition 26, the solution $x(\cdot)$ of (35) is bounded by the solution of the FDE

$$^{C}D^{\alpha}_{0+}y(t) = g(y(t)), \quad y(0) = x_0. \tag{37}$$

Using construction of a sub-solution by Vergara and Zacher [35, pp. 332–334], we see that the solution $y(\cdot)$ of the FDE (37) has decay rate of $t^{-\alpha/n}$, hence the function $x(\cdot)$, which is bigger or equal to $y(\cdot)$, cannot converge faster than $t^{-\alpha/n}$. Since n is arbitrary, $x(\cdot)$ cannot decay with power-rate. Thus, the trivial solution of (35) is not Mittag-Leffler stable.

On the other hand, due to the fact that $f_{|(0,\infty)} \in C^2(0,\infty)$, using [37, Theorem 3.3], we see that the solution $x(\cdot)$ of (35) is strictly decreasing on the interval $[0,\infty)$. Now we assume that there exist $\delta \in (0,1)$ such that $x(t) \geq \delta$ for all $t \geq 0$. Then,

$$^{C}D^{\alpha}_{0+}x(t) \leq -e^{-1/\delta}x(t), \quad t > 0.$$

Using Proposition 26, we obtain

$$x(t) \leq x_0 E_\alpha(-e^{-1/\delta}t) \to 0 \quad as \quad t \to \infty,$$

and we arrive at a contradiction. Consequently, $x(\cdot)$ converges to 0 as t tends to ∞. It is easily seen that this assertion is also true for the solution of (35) starting from any $x_0 < 0$.

Finally, since $f(\cdot) \in C^2(-\infty,\infty)$ the equation (35) with the initial condition $x_0 = 0$ has the unique solution $x(t) \equiv 0$. Hence, the trivial solution of (35) is asymptotically stable.

To complete this section, we study the separation of trajectories of solutions to an one-dimensional FDE with local Lipschitz right-hand side defining on an interval $x \in (a,b) \subset \mathbb{R}$. We extend our previous result [30, Theorem 3.5] on separation of solution of one-dimensional FDE to this case. Let $-\infty \leq a < b \leq \infty$ and $f : [0,\infty) \times (a,b) \to \mathbb{R}$ be a continuous function and locally Lipschitz continuous with respect the second variable, that is, for any $T > 0$ and any compact interval $K \subset (a,b)$ there exists a positive constant $L_{K,T}$ such that

$$|f(t,x) - f(t,y)| \leq L_{K,T}|x-y|, \quad \forall x,y \in K,\ t \in [0,T]. \tag{38}$$

Consider the equation

$$^{C}D^{\alpha}_{0+}x(t) = f(t, x(t)), \quad t > 0. \tag{39}$$

Then, using the approach of [30] we obtain the following result.

Theorem 36. Assume that the function $f(\cdot,\cdot)$ satisfies the condition (38). Then for any pair of distinct points $x_1, x_2 \in (a, b)$, the solutions of the FDE (39) starting from x_1 and x_2, respectively, do not meet.

Proof. By virtue Theorem 4, for $x_i \in (a, b)$ the initial value problem (39), $x(0) = x_i$ $(i = 1, 2)$, has the unique solution denoted by $\varphi(\cdot, x_i)$ on the maximal interval of existence $[0, T_b(x_i))$. Without loss of generality we let $x_1 < x_2$. Assume that $\varphi(\cdot, x_1)$ and $\varphi(\cdot, x_2)$ meet at some $t \in (0, T_b(x_1)) \cap (0, T_b(x_2))$. Let $t_1 := \inf\{t \in (0, T_b(x_1)) \cap (0, T_b(x_2)) : \varphi(t, x_1) = \varphi(t, x_2)\}$. It is obvious that $0 < t_1 < \min\{T_b(x_1), T_b(x_1)\}$ and

$$\varphi(t_1, x_1) = \varphi(t_1, x_2), \quad \varphi(t, x_1) < \varphi(t, x_2), \quad \forall t \in [0, t_1).$$

Take $r_1, r_2 > 0$ such that $[x_1 - r_1, x_2 + r_2] \subset (a, b)$ and $\varphi(t, x_1), \varphi(t, x_2) \in [x_1 - r_1, x_2 + r_2]$ for all $t \in [0, t_1]$. Then following the assumption on the locally Lipschitz continuity of $f(\cdot,\cdot)$ (see the condition (38)), the function

$$f_1 := f_{|[0,t_1]\times[x_1-r_1,x_2+r_2]}$$

is continuous and Lipschitz continuous with respect to the second variable on the set $[0, t_1] \times [x_1 - r_1, x_2 + r_2]$.

Now we construct a extension of $f_1(\cdot,\cdot)$ as follows:

$$f_2(t, x) := \begin{cases} f_1(t, x), & \text{if} \quad (t, x) \in [0, t_1] \times [x_1 - r_1, x_2 + r_2], \\ f_1(t, x_2 + r_2), & \text{if} \quad t \in [0, t_1],\ x > x_2 + r_2, \\ f_1(t, x_1 - r_1), & \text{if} \quad t \in [0, t_1],\ x < x_1 - r_1. \end{cases}$$

This function is continuous and global Lipschitz continuous with respect to the second variable on the domain $[0, t_1] \times \mathbb{R}$. Therefore, by [38, Theorem 2] the FDE

$$^{C}D^{\alpha}_{0+}x(t) = f_2(t, x(t)), \quad t > 0, \quad x(0) = x_i,\ i = 1, 2, \tag{40}$$

has unique solutions $\tilde{\varphi}(\cdot, x_i)$, $i = 1, 2$, on $\mathbb{R}_{\geq 0}$. On the other hand, using [30, Theorem 3.5], we have

$$\tilde{\varphi}(t, x_1) < \tilde{\varphi}(t, x_2), \quad \forall t \in \mathbb{R}_{\geq 0}.$$

However, due to the fact $\varphi(t, x_1), \varphi(t, x_2) \in [x_1 - r_1, x_2 + r_2]$ for all $t \in [0, t_1]$, we also have

$$\begin{aligned} {}^C D^{\alpha}_{0+}\varphi(t, x_i) &= f(t, \varphi(t, x_i)) \\ &= f_2(t, \varphi(t, x_i)), \quad t \in (0, t_1],\ i = 1, 2. \end{aligned}$$

This implies that

$$\varphi(t, x_1) = \tilde{\varphi}(t, x_1) < \tilde{\varphi}(t, x_2) = \varphi(t, x_2)$$

for all $t \in [0, t_1]$, a contradiction. Thus two solutions $\varphi(\cdot, x_1)$ and $\varphi(\cdot, x_2)$ do not meet and the proof is complete. □

Remark 37. (i) Theorem 36 improves our preceding result [30, Theorem 3.5]. Here, we only used the assumption on the locally Lipschitz continuity of "vector field" $f(\cdot, \cdot)$ instead of the global Lipschitz continuity of this function.

(ii) This theorem also improved a recent result by Y. Feng *et al.* [37, proposition 2]. More precisely, we removed the condition on monotonity of the function $f(\cdot)$ in [37, Proposition 2] (see also [37, Remark 6]).

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