



Fractional integrals and their commutators on martingale Orlicz spaces



Ryutaro Arai^a, Eiichi Nakai^{a,*}, Gaku Sadasue^b

^a Department of Mathematics, Ibaraki University, Mito, Ibaraki 310-8512, Japan

^b Department of Mathematics, Osaka Kyoiku University, Kashiwara, Osaka 582-8582, Japan

ARTICLE INFO

Article history:

Received 3 December 2019

Available online 28 February 2020

Submitted by C. Gutierrez

Keywords:

Martingale transform

Fractional integral

Orlicz space

Campanato space

Commutator

ABSTRACT

On the dyadic martingales, Chao and Ombe (1985) proved the L_p - L_q boundedness of the fractional integrals I_α and their commutators $[b, I_\alpha]$, where b is in BMO or Lipschitz spaces. We extend these results to L_Φ - L_Ψ boundedness of generalized fractional integrals I_γ and their commutators $[b, I_\gamma]$, where L_Φ and L_Ψ are Orlicz spaces and b is in generalized martingale Campanato spaces on more general martingales. We also prove the L_Φ - $F_{L_\Psi}^\phi$ boundedness of $[b, I_\gamma]$, where $F_{L_\Psi}^\phi$ is the Triebel-Lizorkin-Orlicz space. Moreover, we characterize b such that the commutator is bounded or compact.

© 2020 Elsevier Inc. All rights reserved.

1. Introduction

It is well known as the Hardy-Littlewood-Sobolev theorem that the fractional integral operators I_α on the Euclidean space $\mathbb{R}^n$ is bounded from L_p to L_q for $1 < p < q < \infty$, $0 < \alpha < n$ and $-n/p + \alpha = -n/q$. For any BMO function b , Chanillo [2] proved the same boundedness of the commutator $[b, I_\alpha]$. This boundedness was extended to Morrey spaces by Di Fazio and Ragusa [5]. For the boundedness on Orlicz spaces, see [6,25]. On the other hand, Paluszynski [21] proved that, for any β -Lipschitz function b , $0 < \beta < 1$, the commutator $[b, I_\alpha]$ is bounded from L_p to L_q for $-n/p + \alpha + \beta = -n/q$ and from L_p to the Triebel-Lizorkin space $\dot{F}_{p,\infty}^\beta$.

In martingale theory, based on the result by Watari [26, Theorem 1.1], Chao and Ombe [3] proved the boundedness of the fractional integrals I_α for H_p , L_p , BMO and Lipschitz spaces of the dyadic martingales. They also proved the L_p - L_q boundedness of the commutator $[b, I_\alpha]$. These fractional integrals were extended to more general martingales in [23] and the boundedness on martingale Morrey spaces was established by [15–18]. See also Hao and Jiao [7] for fractional integrals on Hardy spaces with variable exponent.

* Corresponding author.

E-mail addresses: 18nd201t@vc.ibaraki.ac.jp, araryu314159@gmail.com (R. Arai), eiichi.nakai.math@vc.ibaraki.ac.jp (E. Nakai), sadasue@cc.osaka-kyoiku.ac.jp (G. Sadasue).

In this paper we consider generalized fractional integrals I_γ as martingale transforms. We prove the boundedness of I_γ and the commutator $[b, I_\gamma]$ on Orlicz spaces, where b is a function in generalized martingale Campanato spaces which contain martingale BMO and Lipschitz spaces as special cases. Moreover, we prove the compactness of the commutators.

Let $(\Omega, \mathcal{F}, P)$ be a non-atomic probability space and let $\{\mathcal{F}_n\}_{n \geq 0}$ be a nondecreasing sequence of sub- σ -algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$. We suppose that every σ -algebra $\mathcal{F}_n$ is generated by countable atoms, where $B \in \mathcal{F}_n$ is called an atom (more precisely a $(\mathcal{F}_n, P)$ -atom), if any $A \subset B$ with $A \in \mathcal{F}_n$ satisfies $P(A) = P(B)$ or $P(A) = 0$. Denote by $A(\mathcal{F}_n)$ the set of all atoms in $\mathcal{F}_n$. The expectation operator and the conditional expectation operators relative to $\mathcal{F}_n$ are denoted by E and E_n , respectively.

We say a sequence $(f_n)_{n \geq 0}$ in L_1 is a martingale relative to $\{\mathcal{F}_n\}_{n \geq 0}$ if it is adapted to $\{\mathcal{F}_n\}_{n \geq 0}$ and satisfies $E_n[f_m] = f_n$ for every $n \leq m$. It is known as the Doob theorem that, if $p \in (1, \infty)$, then any L_p -bounded martingale converges in L_p . Moreover, if $p \in [1, \infty)$, then, for any $f \in L_p$, its corresponding martingale $(f_n)_{n \geq 0}$ with $f_n = E_n f$ is an L_p -bounded martingale and converges to f in L_p (see for example [19]). For this reason a function $f \in L_1$ and the corresponding martingale $(f_n)_{n \geq 0}$ will be denoted by the same symbol f .

We first recall the definition of generalized fractional integrals of martingales.

Definition 1.1 ([18]). Let $(\gamma_n)_{n \geq 0}$ be a decreasing (i.e. non-increasing) sequence of non-negative bounded functions adapted to $\{\mathcal{F}_n\}_{n \geq 0}$ such that $\lim_{n \rightarrow \infty} \gamma_n = 0$. For a martingale $(f_n)_{n \geq 0}$, its generalized fractional integral $I_\gamma f = ((I_\gamma f)_n)_{n \geq 0}$ is defined as a martingale by

$$(I_\gamma f)_n = \sum_{k=0}^n \gamma_{k-1} (f_k - f_{k-1})$$

with convention $\gamma_{-1} = \gamma_0$ and $f_{-1} = 0$.

Our definition of I_γ is based on the notion of martingale transform in the sense of Burkholder [1]. For quasi-normed spaces M_1 and M_2 of functions, we denote by $B(M_1, M_2)$ the set of all bounded martingale transforms from M_1 to M_2 , that is, $I_\gamma \in B(M_1, M_2)$ means that

$$\sup_{n \geq 0} \|(I_\gamma f)_n\|_{M_2} \leq C \sup_{n \geq 0} \|f_n\|_{M_1},$$

for all M_1 -bounded martingales $f = (f_n)_{n \geq 0}$.

Let

$$\beta_n = \sum_{B \in A(\mathcal{F}_n)} P(B) \chi_B, \quad n = 0, 1, 2, \dots \quad (1.1)$$

For $\alpha > 0$, let $\gamma_n = \beta_n^\alpha$, $n \geq 0$. Then $I_\gamma f$ is the fractional integral of f introduced in [15] which is denoted by $I_\alpha f$. It is known that $I_\alpha \in B(L_p, L_q)$, if $1 < p < q < \infty$ and $-1/p + \alpha = -1/q$, see [15, Corollary 5.4].

In this paper we prove $I_\gamma \in B(L_\Phi, L_\Psi)$ for the Orlicz spaces L_Φ and L_Ψ under suitable conditions. Moreover, we consider the commutator $[b, I_\gamma]$ generated by a function b . For $f \in L_\infty$, which is regarded as an L_∞ -bounded martingale $f = (f_n)_{n \geq 0}$ with $f_n = E_n f$, $((I_\gamma f)_n)_{n \geq 0}$ is also an L_∞ -bounded martingale. Then, by the martingale convergence theorem, the sequence $((I_\gamma f)_n)_{n \geq 0}$ converges almost everywhere and in L_1 ([12, Theorem 1.3.2.9]). We denote by $I_\gamma f$ the limit function, that is, $I_\gamma f = ((I_\gamma f)_n)_{n \geq 0}$. In this case the commutator $[b, I_\gamma]f = bI_\gamma f - I_\gamma(bf)$ is well-defined for all $b \in L_\infty$. In this paper we prove that, for any function b in generalized Campanato spaces (see Definition 1.2 below), the commutator $[b, I_\gamma]$ is well-defined as a bounded operator from L_Φ to L_Ψ and from L_Φ to $F_{L_\Psi}^\phi$ under suitable conditions, where $F_{L_\Psi}^\phi$ is the

Triebel-Lizorkin-Orlicz space, see Definition 3.1 below. Conversely, we also prove that, if $[b, I_\gamma]$ is bounded from L_Φ to L_Ψ or from L_Φ to $F_{L_\Psi}^\phi$, then b is in a generalized Campanato space. Moreover, we characterize b so that $[b, I_\gamma]$ is compact.

The definition of the generalized Campanato space is the following:

Definition 1.2 ([13]). For $p \in [1, \infty)$ and $\phi : (0, 1] \rightarrow (0, \infty)$, let

$$\mathcal{L}_{p,\phi}^- = \{f \in L_p : \|f\|_{\mathcal{L}_{p,\phi}^-} < \infty\},$$

where

$$\|f\|_{\mathcal{L}_{p,\phi}^-} = \sup_{n \geq 0} \sup_{B \in \mathcal{A}(\mathcal{F}_n)} \frac{1}{\phi(P(B))} \left(\frac{1}{P(B)} \int_B |f - E_{n-1}f|^p dP \right)^{1/p},$$

with convention $E_{-1}f = 0$.

From [13, Theorem 2.9] it follows that, if ϕ is almost increasing, i.e., there exists a positive constant C such that $\phi(r) \leq C\phi(s)$ if $r < s$, then

$$\|f\|_{\mathcal{L}_{1,\phi}^-} \leq \|f\|_{\mathcal{L}_{p,\phi}^-} \leq C_p \|f\|_{\mathcal{L}_{1,\phi}^-}, \quad (1.2)$$

for each $p \in (1, \infty)$, see Theorem 5.1 below.

In the second section we recall the definitions and basic properties of Young functions and Orlicz spaces. Then we state main results in Section 3. More precisely, in Subsection 3.1 we state the boundedness of the fractional integrals I_γ as martingale transforms, and the boundedness of the fractional maximal operator M_γ from L_Φ to L_Ψ . In Subsection 3.2 we state the equivalence between $b \in \mathcal{L}_{1,\phi}^-$ and the boundedness of $[b, I_\gamma]$ from L_Φ to L_Ψ . Moreover, we define Triebel-Lizorkin-Orlicz spaces $F_{L_\Phi}^\phi$ by using the sharp maximal operators $M_\phi^\sharp$ and state the equivalence between $b \in \mathcal{L}_{1,\phi}^-$ and the boundedness of $[b, I_\gamma]$ from L_Φ to $F_{L_\Psi}^\phi$. In Subsection 3.3 we define a subset $\mathcal{C}_{1,\phi}^-$ in $\mathcal{L}_{1,\phi}^-$ and state the equivalence between $b \in \mathcal{C}_{1,\phi}^-$ and the compactness of $[b, I_\gamma]$ from L_Φ to L_Ψ or from L_Φ to $F_{L_\Psi}^\phi$. We prove the boundedness of I_γ and M_γ in Section 4 and the pointwise estimates of the commutators by sharp maximal functions in Section 5. Then, combining these boundedness and estimates, we prove the results on the boundedness and compactness of $[b, I_\gamma]$ in Sections 6 and 7, respectively.

At the end of this section, we make some conventions. Throughout this paper, we always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_p , is dependent on the subscripts. If $f \leq Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$.

2. Orlicz space

First we define a set $\bar{\Phi}$ of increasing functions $\Phi : [0, \infty] \rightarrow [0, \infty]$ and give some properties of functions in $\bar{\Phi}$.

For an increasing function $\Phi : [0, \infty] \rightarrow [0, \infty]$, let

$$a(\Phi) = \sup\{t \geq 0 : \Phi(t) = 0\}, \quad b(\Phi) = \inf\{t \geq 0 : \Phi(t) = \infty\},$$

with convention $\sup \emptyset = 0$ and $\inf \emptyset = \infty$. Then $0 \leq a(\Phi) \leq b(\Phi) \leq \infty$. Let $\bar{\Phi}$ be the set of all increasing functions $\Phi : [0, \infty] \rightarrow [0, \infty]$ such that

$$0 \leq a(\Phi) < \infty, \quad 0 < b(\Phi) \leq \infty, \quad (2.1)$$

$$\lim_{t \rightarrow +0} \Phi(t) = \Phi(0) = 0, \quad (2.2)$$

$$\Phi \text{ is left continuous on } [0, b(\Phi)), \quad (2.3)$$

$$\text{if } b(\Phi) = \infty, \text{ then } \lim_{t \rightarrow \infty} \Phi(t) = \Phi(\infty) = \infty, \quad (2.4)$$

$$\text{if } b(\Phi) < \infty, \text{ then } \lim_{t \rightarrow b(\Phi)-0} \Phi(t) = \Phi(b(\Phi)) (\leq \infty). \quad (2.5)$$

In what follows, if an increasing and left continuous function $\Phi : [0, \infty) \rightarrow [0, \infty)$ satisfies (2.2) and $\lim_{t \rightarrow \infty} \Phi(t) = \infty$, then we always regard that $\Phi(\infty) = \infty$ and that $\Phi \in \bar{\Phi}$.

Definition 2.1. A function $\Phi \in \bar{\Phi}$ is called a Young function (or sometimes also called an Orlicz function) if Φ is convex on $[0, b(\Phi))$.

By the convexity, any Young function Φ is continuous on $[0, b(\Phi))$ and strictly increasing on $[a(\Phi), b(\Phi)]$. Hence Φ is bijective from $[a(\Phi), b(\Phi)]$ to $[0, \Phi(b(\Phi))]$. Moreover, Φ is absolutely continuous on any closed subinterval in $[0, b(\Phi))$. That is, its derivative Φ' exists a.e. and

$$\Phi(t) = \int_0^t \Phi'(s) ds, \quad t \in [0, b(\Phi)). \quad (2.6)$$

For $\Phi, \Psi \in \bar{\Phi}$, we write $\Phi \approx \Psi$ if there exists a positive constant C such that

$$\Phi(C^{-1}t) \leq \Psi(t) \leq \Phi(Ct) \quad \text{for all } t \in [0, \infty].$$

Definition 2.2.

- (i) Let Φ_Y be the set of all Young functions.
- (ii) Let $\bar{\Phi}_Y$ be the set of all $\Phi \in \bar{\Phi}$ such that $\Phi \approx \Psi$ for some $\Psi \in \Phi_Y$.
- (iii) Let $\mathcal{Y}$ be the set of all $\Phi \in \bar{\Phi}_Y$ such that $a(\Phi) = 0$ and $b(\Phi) = \infty$.

For $\Phi \in \bar{\Phi}$, we recall the generalized inverse of Φ in the sense of O'Neil [20, Definition 1.2].

Definition 2.3. For $\Phi \in \bar{\Phi}$ and $u \in [0, \infty]$, let

$$\Phi^{-1}(u) = \begin{cases} \inf\{t \geq 0 : \Phi(t) > u\}, & u \in [0, \infty), \\ \infty, & u = \infty. \end{cases} \quad (2.7)$$

Let $\Phi \in \bar{\Phi}$. Then Φ^{-1} is finite, increasing and right continuous on $[0, \infty)$ and positive on $(0, \infty)$. If Φ is bijective from $[0, \infty]$ to itself, then Φ^{-1} is the usual inverse function of Φ . Moreover, we have the following proposition, which is a generalization of Property 1.3 in [20].

Proposition 2.1 ([25, Proposition 2.2]). Let $\Phi \in \bar{\Phi}$. Then

$$\Phi(\Phi^{-1}(u)) \leq u \leq \Phi^{-1}(\Phi(u)) \quad \text{for all } u \in [0, \infty]. \quad (2.8)$$

For functions $P, Q : [0, \infty] \rightarrow [0, \infty]$, we write $P \sim Q$ if there exists a positive constant C such that

$$C^{-1}P(t) \leq Q(t) \leq CP(t) \quad \text{for all } t \in [0, \infty].$$

Then, for $\Phi, \Psi \in \bar{\Phi}$,

$$\Phi \approx \Psi \quad \Leftrightarrow \quad \Phi^{-1} \sim \Psi^{-1}. \quad (2.9)$$

For a Young function Φ , its complementary function is defined by

$$\tilde{\Phi}(t) = \begin{cases} \sup\{tu - \Phi(u) : u \in [0, \infty)\}, & t \in [0, \infty), \\ \infty, & t = \infty. \end{cases}$$

Then $\tilde{\Phi}$ is also a Young function, and $(\Phi, \tilde{\Phi})$ is called a complementary pair. For example, if $\Phi(t) = t^p/p$, $p \in (1, \infty)$, then $\tilde{\Phi}(t) = t^q/q$ with $1/p + 1/q = 1$. If $\Phi(t) = t$, then

$$\tilde{\Phi}(t) = \begin{cases} 0, & t \in [0, 1], \\ \infty, & t \in (1, \infty]. \end{cases}$$

Definition 2.4. For a function $\Phi \in \bar{\Phi}_Y$, let

$$\begin{aligned} L_\Phi &= \{f \in L^0 : E[\Phi(\epsilon|f|)] < \infty \text{ for some } \epsilon > 0\}, \\ \|f\|_{L_\Phi} &= \inf \{\lambda > 0 : E[\Phi(|f|/\lambda)] \leq 1\}, \\ \text{w}L_\Phi &= \left\{ f \in L^0 : \sup_{t \in (0, \infty)} \Phi(t)P(\epsilon f, t) < \infty \text{ for some } \epsilon > 0 \right\}, \\ \|f\|_{\text{w}L_\Phi} &= \inf \left\{ \lambda > 0 : \sup_{t \in (0, \infty)} \Phi(t)P(f/\lambda, t) \leq 1 \right\}, \\ \text{where } P(f, t) &= P(\{\omega \in \Omega : |f(\omega)| > t\}). \end{aligned}$$

Then L_Φ and $\text{w}L_\Phi$ are complete quasi-normed spaces and $L_\Phi \subset \text{w}L_\Phi$. If $\Phi \in \bar{\Phi}_Y$, then L_Φ is a Banach space. Let $\Phi, \Psi \in \bar{\Phi}$. If $\Phi \approx \Psi$, then $L_\Phi = L_\Psi$ and $\text{w}L_\Phi = \text{w}L_\Psi$ with equivalent quasi-norms, respectively.

Remark 2.1. Let $\Phi \in \bar{\Phi}_Y$. It is known that

$$\sup_{t \in (0, \infty)} \Phi(t)P(f, t) = \sup_{t \in (0, \infty)} t P(\Phi(|f|), t), \quad (2.10)$$

see [8, Proposition 4.2] for example.

Let $(\Phi, \tilde{\Phi})$ be a complementary pair of functions in $\bar{\Phi}_Y$. Then it is known that

$$t \leq \Phi^{-1}(t)\tilde{\Phi}^{-1}(t) \leq 2t, \quad t \in [0, \infty]. \quad (2.11)$$

It is also known that

$$E[|fg|] \leq 2\|f\|_{L_\Phi}\|g\|_{L_{\tilde{\Phi}}}. \quad (2.12)$$

By the definitions of $\|\cdot\|_{L_\Phi}$ and $\|\cdot\|_{\text{w}L_\Phi}$ we have the following lemma:

Lemma 2.2. Let $\Phi \in \Phi_Y$. Then, for all $A \in \mathcal{F}$, its characteristic function χ_A is in $L_\Phi \subset \mathbf{w}L_\Phi$ and

$$\|\chi_A\|_{L_\Phi} = \|\chi_A\|_{\mathbf{w}L_\Phi} = \frac{1}{\Phi^{-1}(1/P(A))}. \quad (2.13)$$

Definition 2.5.

- (i) A function $\Phi \in \bar{\Phi}$ is said to satisfy the Δ_2 -condition, denote $\Phi \in \bar{\Delta}_2$, if there exists a constant $C > 0$ such that

$$\Phi(2t) \leq C\Phi(t) \quad \text{for all } t > 0. \quad (2.14)$$

- (ii) A function $\Phi \in \bar{\Phi}$ is said to satisfy the ∇_2 -condition, denote $\Phi \in \bar{\nabla}_2$, if there exists a constant $k > 1$ such that

$$\Phi(t) \leq \frac{1}{2k}\Phi(kt) \quad \text{for all } t > 0. \quad (2.15)$$

- (iii) Let $\Delta_2 = \Phi_Y \cap \bar{\Delta}_2$ and $\nabla_2 = \Phi_Y \cap \bar{\nabla}_2$.

Remark 2.2.

- (i) $\Delta_2 \subset \mathcal{Y}$ and $\bar{\nabla}_2 \subset \bar{\Phi}_Y$ ([11, Lemma 1.2.3]).
(ii) Let $\Phi \in \bar{\Phi}_Y$. Then $\Phi \in \bar{\Delta}_2$ if and only if $\Phi \approx \Psi$ for some $\Psi \in \Delta_2$, and, $\Phi \in \bar{\nabla}_2$ if and only if $\Phi \approx \Psi$ for some $\Psi \in \nabla_2$.
(iii) Let $\Phi \in \Phi_Y$. Then $\Phi \in \Delta_2$ if and only if the set of simple functions is dense in L_Φ .
(iv) Let $\Phi \in \Phi_Y$. Then Φ^{-1} satisfies the condition

$$\Phi^{-1}(u) \leq \Phi^{-1}(2u) \leq 2\Phi^{-1}(u) \quad \text{for all } u \in [0, \infty],$$

by its concavity.

- (v) If $\Phi \in \bar{\nabla}_2$, then there exists $\theta \in (0, 1)$ such that $\Phi((\cdot)^\theta) \in \bar{\nabla}_2$ ([25, Lemma 4.5]).

3. Main results

To state the main results we first recall some notion on positive real-valued functions. We say that a function $\theta : (0, \infty) \rightarrow (0, \infty)$ satisfies the doubling condition if there exists a positive constant C such that, for all $r, s \in (0, \infty)$,

$$\frac{1}{C} \leq \frac{\theta(r)}{\theta(s)} \leq C, \quad \text{if } \frac{1}{2} \leq \frac{r}{s} \leq 2. \quad (3.1)$$

We say that a function $\theta : (0, \infty) \rightarrow (0, \infty)$ is almost increasing (resp. almost decreasing) if there exists a positive constant C such that, for all $r, s \in (0, \infty)$,

$$\theta(r) \leq C\theta(s) \quad (\text{resp. } \theta(s) \leq C\theta(r)), \quad \text{if } r < s. \quad (3.2)$$

3.1. Fractional integrals and fractional maximal functions

For a martingale $f = (f_n)_{n \geq 0}$ relative to $\{\mathcal{F}_n\}_{n \geq 0}$, the maximal function is defined by

$$Mf = \sup_{n \geq 0} |f_n|.$$

We denote by $\mathcal{M}_{L_\Phi}$ the set of all L_Φ bounded martingales $f = (f_n)_{n \geq 0}$. It is known that, for all $f \in \mathcal{M}_{L_\Phi}$,

$$\begin{aligned} \|Mf\|_{\mathbf{w}L_\Phi} &\leq C_\Phi \sup_n \|f_n\|_{L_\Phi}, \quad \text{if } \Phi \in \bar{\Phi}_Y, \\ \|Mf\|_{L_\Phi} &\leq C_\Phi \sup_n \|f_n\|_{L_\Phi}, \quad \text{if } \Phi \in \bar{\nabla}_2, \end{aligned}$$

see Section 4 for their references. Our first result is the following:

Theorem 3.1. *Let $\Phi, \Psi \in \bar{\Phi}_Y$. Assume that $u \mapsto \Psi^{-1}(u)/\Phi^{-1}(u)$ is almost decreasing and that there exists a positive constant C such that, for all $n \geq 0$,*

$$\sum_{k=0}^n (\gamma_{k-1} - \gamma_k) \Phi^{-1} \left(\frac{1}{\beta_k} \right) + \gamma_n \Phi^{-1} \left(\frac{1}{\beta_n} \right) \leq C \Psi^{-1} \left(\frac{1}{\beta_n} \right). \quad (3.3)$$

Then, for any positive constant C_Φ , there exists a positive constant C'_Φ such that, for all $f \in \mathcal{M}_{L_\Phi}$ with $f \not\equiv 0$,

$$\Psi \left(\frac{M(I_\gamma f)}{C'_\Phi \sup_{n \geq 0} \|f_n\|_{L_\Phi}} \right) \leq \Phi \left(\frac{Mf}{C_\Phi \sup_{n \geq 0} \|f_n\|_{L_\Phi}} \right). \quad (3.4)$$

Consequently, $I_\gamma \in B(L_\Phi, \mathbf{w}L_\Psi)$. Moreover, if $\Phi \in \bar{\nabla}_2$, then $I_\gamma \in B(L_\Phi, L_\Psi)$.

Let β_n be as in (1.1). For a function $\rho : (0, 1] \rightarrow (0, \infty)$ satisfying the condition

$$\int_0^1 \frac{\rho(t)}{t} dt < \infty, \quad (3.5)$$

let

$$\gamma_n = \int_0^{\beta_n} \frac{\rho(t)}{t} dt, \quad n = 0, 1, 2, \dots. \quad (3.6)$$

In this case we denote I_γ by I_ρ . Namely, for a martingale $f = (f_n)_{n \geq 0}$, let

$$I_\rho f = ((I_\rho f)_n)_{n \geq 0}, \quad (I_\rho f)_n = \sum_{k=0}^n \left(\int_0^{\beta_{k-1}} \frac{\rho(t)}{t} dt \right) (f_k - f_{k-1}). \quad (3.7)$$

If $\rho(t) = \alpha t^\alpha$ and $\alpha > 0$, then $\int_0^{\beta_{k-1}} \frac{\rho(t)}{t} dt = (\beta_{k-1})^\alpha$ and I_ρ is the fractional integrals I_α introduced by [15].

The stochastic basis $\{\mathcal{F}_n\}_{n \geq 0}$ is said to be regular, if there exists a constant $R \geq 2$ such that

$$f_n \leq R f_{n-1} \quad (3.8)$$

holds for all $n \geq 1$ and all nonnegative martingales $(f_n)_{n \geq 0}$. If $\{\mathcal{F}_n\}_{n \geq 0}$ is regular, then the inequality $\beta_n \leq \beta_{n-1} \leq R \beta_n$ holds, see [15, Lemma 3.1]. Hence,

$$\begin{aligned}
\sum_{k=0}^n (\gamma_{k-1} - \gamma_k) \Phi^{-1}(1/\beta_k) &= \sum_{k=1}^n \Phi^{-1}(1/\beta_k) \int_{\beta_k}^{\beta_{k-1}} \frac{\rho(t)}{t} dt \\
&\sim \sum_{k=1}^n \int_{\beta_k}^{\beta_{k-1}} \frac{\Phi^{-1}(1/t) \rho(t)}{t} dt \\
&= \int_{\beta_n}^{\beta_0} \frac{\Phi^{-1}(1/t) \rho(t)}{t} dt.
\end{aligned}$$

That is, (3.3) is equivalent to

$$\int_0^{\beta_n} \frac{\rho(t)}{t} dt \Phi^{-1}(1/\beta_n) + \int_{\beta_n}^{\beta_0} \frac{\rho(t) \Phi^{-1}(1/t)}{t} dt \leq C \Psi^{-1}(1/\beta_n) \quad \text{for all } n \geq 0. \quad (3.9)$$

For I_ρ we have the following corollary.

Corollary 3.2. *Let $\{\mathcal{F}_n\}_{n \geq 0}$ be regular, and let $\Phi, \Psi \in \bar{\Phi}_Y$. Assume that $u \mapsto \Psi^{-1}(u)/\Phi^{-1}(u)$ is almost decreasing and that there exists a positive constant A such that, for all $r \in (0, 1]$,*

$$\int_0^r \frac{\rho(t)}{t} dt \Phi^{-1}(1/r) + \int_r^1 \frac{\rho(t) \Phi^{-1}(1/t)}{t} dt \leq A \Psi^{-1}(1/r). \quad (3.10)$$

Then, for any positive constant C_Φ , there exists a positive constant C'_Φ such that, for all $f \in \mathcal{M}_{L_\Phi}$ with $f \not\equiv 0$,

$$\Psi \left(\frac{M(I_\rho f)}{C'_\Phi \sup_{n \geq 0} \|f_n\|_{L_\Phi}} \right) \leq \Phi \left(\frac{Mf}{C_\Phi \sup_{n \geq 0} \|f_n\|_{L_\Phi}} \right). \quad (3.11)$$

Consequently, $I_\rho \in B(L_\Phi, wL_\Psi)$. Moreover, if $\Phi \in \bar{\nabla}_2$, then $I_\rho \in B(L_\Phi, L_\Psi)$.

The above result can be compared with the boundedness of I_ρ on Orlicz spaces defined on the Euclidean space $\mathbb{R}^n$ by [4,14].

Example 3.1. Let $\alpha \in (0, \infty)$, and let $\rho : (0, 1] \rightarrow (0, \infty)$ satisfy the condition (3.5) and

$$\rho(r) = (\log(1/r))^{-\alpha-1} \quad \text{for small } r. \quad (3.12)$$

Then

$$\int_0^r \frac{\rho(t)}{t} dt \sim (\log(1/r))^{-\alpha} \quad \text{for small } r.$$

We denote by $L_p(\log L)_{p_1}$ and $\exp L_p$ the Orlicz spaces L_Φ generated by $\Phi \in \bar{\Phi}$ such that

$$\Phi(t) = t^p (\log t)^{p_1} \quad \text{for large } t, \quad \begin{cases} p = 1, & p_1 \in [0, \infty), \\ p \in (1, \infty), & p_1 \in (-\infty, \infty), \end{cases} \quad \text{or}$$

and

$$\Phi(t) = \exp(t^p) \quad \text{for large } t, \quad p \in (0, \infty),$$

respectively. Then

$$\Phi^{-1}(u) \sim u^{1/p} (\log u)^{-p_1/p} \quad \text{for large } u,$$

and

$$\Phi^{-1}(u) \sim (\log u)^{1/p} \quad \text{for large } u,$$

respectively. In this case I_ρ is in

$$\begin{aligned} & B(L_1(\log L)_{p_1}, \text{w}L_1(\log L)_{p_1+\alpha}), \\ & B(L_p(\log L)_{p_1}, L_p(\log L)_{p_1+\alpha p}), \quad \text{if } p > 1, \\ & B(\exp(L_p), \exp(L_q)), \quad \text{if } p < q < \infty \text{ and } -1/p + \alpha = -1/q. \end{aligned}$$

We also consider the cases $\rho(r) = r^\alpha (\log(1/r))^{-\alpha_1} (\log \log(1/r))^{-\alpha_2}$, etc.

Example 3.2. Let $\rho : (0, 1] \rightarrow (0, \infty)$ satisfy the condition (3.5), and let $\rho^*(r) = \int_0^r \frac{\rho(t)}{t} dt$. For a Young function Φ , if ρ satisfies the doubling condition and $r \mapsto r^\epsilon \rho^*(r) \Phi^{-1}(1/r)$ is almost decreasing for some $\epsilon > 0$, then there exists a Young function Ψ such that I_ρ is bounded from L_Φ to L_Ψ . To show this we first prove that

$$\int_r^1 \frac{\rho(t) \Phi^{-1}(1/t)}{t} dt \lesssim \int_0^r \frac{\rho(t)}{t} dt \Phi^{-1}(1/r). \quad (3.13)$$

Since $\rho(r) \sim \int_{r/2}^r \frac{\rho(t)}{t} dt \leq \rho^*(r)$, if $1/2 < r \leq 1$, then

$$\int_r^1 \frac{\rho(t) \Phi^{-1}(1/t)}{t} dt \sim \rho(r) \Phi^{-1}(1/r) \lesssim \rho^*(r) \Phi^{-1}(1/r).$$

If $0 < r < 1/2$, then

$$\begin{aligned} \int_r^1 \frac{\rho(t) \Phi^{-1}(1/t)}{t} dt & \lesssim \int_r^1 \frac{t^\epsilon \rho^*(t) \Phi^{-1}(1/t)}{t^{1+\epsilon}} dt \lesssim r^\epsilon \rho^*(r) \Phi^{-1}(1/r) \int_r^1 \frac{1}{t^{1+\epsilon}} dt \\ & \sim \int_0^r \frac{\rho(t)}{t} dt \Phi^{-1}(1/r). \end{aligned}$$

Hence we have (3.13). Next we set $Q(u) = \rho^*(1/u) \Phi^{-1}(u)$ and show that Q is pseudo-concave, which is equivalent to

$$Q(su) \lesssim \max(1, s) Q(u) \quad \text{for all } s, u \in (0, \infty), \quad (3.14)$$

see [22, Lemma]. By the assumption we see that $u \mapsto u^{-\epsilon}Q(u)$ is almost increasing and then Q is also. Hence, if $0 < s < 1$, then (3.14) is true. If $s > 1$, (3.14) is also true, since ρ^* is increasing and Φ^{-1} is concave. Therefore we can find a concave function $\tilde{Q}$ such that $Q \sim \tilde{Q}$. By the assumption we also see that $Q(u) \rightarrow \infty$ as $u \rightarrow \infty$. Then we may assume that $\tilde{Q}$ is continuous, increasing and bijective from $[1, \infty)$ to $[\tilde{Q}(1), \infty)$. Let Ψ be a Young function such that $\Psi^{-1}(u) = \tilde{Q}(u)$ for $u \geq 1$. Then

$$\int_0^r \frac{\rho(t)}{t} dt \Phi^{-1}(1/r) + \int_r^1 \frac{\rho(t) \Phi^{-1}(1/t)}{t} dt \sim \Psi^{-1}(1/r), \quad r \in (0, 1].$$

By Corollary 3.2 we have the boundedness of I_ρ from L_Φ to L_Ψ .

Next, for a sequence $\gamma = (\gamma_n)_{n \geq 0}$ of positive measurable functions, let

$$M_\gamma f = \sup_{n \geq 0} \gamma_n |E_n f|, \quad f \in L_1. \quad (3.15)$$

If $\gamma_n \equiv 1$, then $M_\gamma f$ is denoted by Mf , which is the maximal function of f .

Theorem 3.3. *Let $\Phi, \Psi \in \bar{\Phi}_Y$. Assume that $u \mapsto \Psi^{-1}(u)/\Phi^{-1}(u)$ is almost decreasing and that there exists a positive constant A such that, for all $n \geq 0$,*

$$\gamma_n \Phi^{-1}(1/\beta_n) \leq A \Psi^{-1}(1/\beta_n). \quad (3.16)$$

Then, for any positive constant C_Φ , there exists a positive constant C'_Φ such that, for all $f \in L_\Phi$ with $f \neq 0$,

$$\Psi \left(\frac{M_\gamma f}{C'_\Phi \|f\|_{L_\Phi}} \right) \leq \Phi \left(\frac{Mf}{C_\Phi \|f\|_{L_\Phi}} \right). \quad (3.17)$$

Consequently, M_γ is bounded from L_Φ to wL_Ψ . Moreover, if $\Phi \in \bar{\nabla}_2$, then M_γ is bounded from L_Φ to L_Ψ .

3.2. Boundedness of commutators

To state the results for the commutator $[b, I_\gamma]f = bI_\gamma f - I_\gamma(bf)$, we first define a function space $\mathcal{L}_{1,\phi}^-(\mathcal{F}_0)$ as the set of all $b \in L_1$ such that

$$\|b\|_{\mathcal{L}_{1,\phi}^-(\mathcal{F}_0)} = \sup_{B \in A(\mathcal{F}_0)} \frac{1}{\phi(P(B))P(B)} \int_B |b| dP < \infty, \quad (3.18)$$

where $A(\mathcal{F}_0)$ is the set of all $\mathcal{F}_0$ -atoms. Then $\mathcal{L}_{1,\phi}^- \subset \mathcal{L}_{1,\phi}^-(\mathcal{F}_0) \subset L_1$ and

$$\frac{1}{\phi(1)} \|b\|_{L_1} \leq \|b\|_{\mathcal{L}_{1,\phi}^-(\mathcal{F}_0)} \leq \|b\|_{\mathcal{L}_{1,\phi}^-}$$

if ϕ is increasing. If the number of the elements in $A(\mathcal{F}_0)$ is finite, then $\mathcal{L}_{1,\phi}^-(\mathcal{F}_0) = L_1$ and

$$\frac{1}{\phi(1)} \|b\|_{L_1} \leq \|b\|_{\mathcal{L}_{1,\phi}^-(\mathcal{F}_0)} \leq \frac{1}{\min_{B \in A(\mathcal{F}_0)} \phi(P(B))P(B)} \|b\|_{L_1}.$$

For the commutator $[b, I_\gamma]$ we have the following theorems.

Theorem 3.4. Let $\phi : (0, 1] \rightarrow (0, \infty)$, and let $\Phi, \Psi \in \bar{\Phi}_Y$.

- (i) Assume that $\Phi, \Psi \in \bar{\Delta}_2 \cap \bar{\nabla}_2$. Assume also that ϕ is almost increasing and that there exists a positive constant A and a function $\Theta \in \bar{\nabla}_2$ such that $u \mapsto \Theta^{-1}(u)/\Phi^{-1}(u)$ is almost decreasing and that, for all $n \geq 0$,

$$\sum_{k=0}^n (\gamma_{k-1} - \gamma_k) \Phi^{-1} \left(\frac{1}{\beta_k} \right) + \gamma_n \Phi^{-1} \left(\frac{1}{\beta_n} \right) \leq A \Theta^{-1} \left(\frac{1}{\beta_n} \right), \quad (3.19)$$

$$\phi(\beta_n) \Theta^{-1} \left(\frac{1}{\beta_n} \right) \leq A \Psi^{-1} \left(\frac{1}{\beta_n} \right). \quad (3.20)$$

Then there exists a positive constant C such that, for all $b \in \mathcal{L}_{1,\phi}^-$ and all $f \in L_\Phi$,

$$\|[b, I_\gamma]f\|_{L_\Psi} \leq C \|b\|_{\mathcal{L}_{1,\phi}^-} \|f\|_{L_\Phi}. \quad (3.21)$$

- (ii) Conversely, let $\{\mathcal{F}_n\}_{n \geq 0}$ be regular and $\alpha > 0$. Assume that ϕ satisfies the doubling condition and that there exists a positive constant A such that, for all $n \geq 0$,

$$\Psi^{-1} \left(\frac{1}{\beta_n} \right) \leq A \beta_n^\alpha \phi(\beta_n) \Phi^{-1} \left(\frac{1}{\beta_n} \right). \quad (3.22)$$

If $b \in \mathcal{L}_{1,\phi}^-(\mathcal{F}_0)$ and $[b, I_\alpha]$ is bounded from L_Φ to L_Ψ with operator norm $\|[b, I_\alpha]\|_{L_\Phi \rightarrow L_\Psi}$, then b is in $\mathcal{L}_{1,\phi}^-$ and there exists a positive constant C , independently of b , such that

$$\|b\|_{\mathcal{L}_{1,\phi}^-} \leq C \left(\|[b, I_\alpha]\|_{L_\Phi \rightarrow L_\Psi} + \|b\|_{\mathcal{L}_{1,\phi}^-(\mathcal{F}_0)} \right).$$

For $\phi : (0, 1] \rightarrow (0, \infty)$, we define the sharp maximal function $M_\phi^\sharp$ by

$$M_\phi^\sharp f = \sup_{n \geq 0} \phi(\beta_n)^{-1} E_n |f - E_{n-1} f|, \quad f \in L_1, \quad (3.23)$$

with the convention $E_{-1} f = 0$ and $\beta_{-1} = \beta_0$. If $\phi \equiv 1$, we denote $M_\phi^\sharp$ by $M^\sharp$, that is,

$$M^\sharp f = \sup_{n \geq 0} E_n |f - E_{n-1} f|. \quad (3.24)$$

Then we define the Triebel-Lizorkin-Orlicz space as follows.

Definition 3.1. For $\Phi \in \bar{\Phi}$ and $\phi : (0, 1] \rightarrow (0, \infty)$, let

$$F_{L_\Phi}^\phi = \{f \in L_1 : \|f\|_{F_{L_\Phi}^\phi} < \infty\},$$

where

$$\|f\|_{F_{L_\Phi}^\phi} = \|M_\phi^\sharp f\|_{L_\Phi}.$$

If $\Phi(t) = t^p$ and $\phi(r) = r^s$, then $F_{L_\Phi}^\phi$ coincides with the Triebel-Lizorkin space $F_{p,\infty}^s$ introduced by Sadasue [24].

Let

$$\phi^-(\beta_n) = \phi(\beta_{n-1}) \quad (3.25)$$

with convention $\beta_{-1} = \beta_0$. We define $F_{L_\Phi}^{\phi^-}$ by $\|f\|_{F_{L_\Phi}^{\phi^-}} = \|M_{\phi^-}^\# f\|_{L_\Phi}$, where $M_{\phi^-}^\# f$ is defined by (3.23) with ϕ^- instead of ϕ . Then we have the following theorem.

Theorem 3.5. *Let $\phi : (0, 1] \rightarrow (0, \infty)$, and let $\Phi, \Psi \in \bar{\Phi}_Y$.*

- (i) *Assume that $\Phi, \Psi \in \bar{\Delta}_2 \cap \bar{\nabla}_2$ and that $u \mapsto \Psi^{-1}(u)/\Phi^{-1}(u)$ is almost decreasing. Assume also that ϕ is almost increasing and that there exists a positive constant A such that, for all $n \geq 0$,*

$$\sum_{k=0}^n (\gamma_{k-1} - \gamma_k) \Phi^{-1} \left(\frac{1}{\beta_k} \right) + \gamma_n \Phi^{-1} \left(\frac{1}{\beta_n} \right) \leq A \Psi^{-1} \left(\frac{1}{\beta_n} \right). \quad (3.26)$$

Then there exists a positive constant C such that, for all $b \in \mathcal{L}_{1,\phi}^-$ and all $f \in L_\Phi$,

$$\|[b, I_\gamma]f\|_{F_{L_\Psi}^{\phi^-}} \leq C \|b\|_{\mathcal{L}_{1,\phi}^-} \|f\|_{L_\Phi}. \quad (3.27)$$

- (ii) *Conversely, let $\{\mathcal{F}_n\}_{n \geq 0}$ be regular and let $\alpha > 0$. Assume that ϕ satisfies the doubling condition and that there exists a positive constant A such that, for all $n \geq 0$,*

$$\Psi^{-1} \left(\frac{1}{\beta_n} \right) \leq A \beta_n^\alpha \Phi^{-1} \left(\frac{1}{\beta_n} \right). \quad (3.28)$$

If $b \in \mathcal{L}_{1,\phi}^-(\mathcal{F}_0)$ and $[b, I_\alpha]$ is bounded from L_Φ to $F_{L_\Psi}^{\phi^-}$ with operator norm $\|[b, I_\alpha]\|_{L_\Phi \rightarrow F_{L_\Psi}^{\phi^-}}$, then b is in $\mathcal{L}_{1,\phi}^-$ and there exists a positive constant C , independently b , such that

$$\|b\|_{\mathcal{L}_{1,\phi}^-} \leq C \left(\|[b, I_\alpha]\|_{L_\Phi \rightarrow F_{L_\Psi}^{\phi^-}} + \|b\|_{\mathcal{L}_{1,\phi}^-(\mathcal{F}_0)} \right).$$

Corollary 3.6. *Let $\{\mathcal{F}_n\}_{n \geq 0}$ be regular and $\alpha > 0$. Assume that $\Phi, \Psi \in \bar{\Delta}_2 \cap \bar{\nabla}_2$. Assume also that ϕ is almost increasing and satisfies the doubling condition. If $b \in \mathcal{L}_{1,\phi}^-(\mathcal{F}_0)$, then the following conditions are equivalent:*

- (i) $b \in \mathcal{L}_{1,\phi}^-$.
- (ii) $[b, I_\alpha]$ is bounded from L_Φ to L_Ψ , if $\beta_n^\alpha \phi(\beta_n) \Phi^{-1}(1/\beta_n) \sim \phi(\beta_n) \Theta^{-1}(1/\beta_n) \sim \Psi^{-1}(1/\beta_n)$ for some $\Theta \in \bar{\nabla}_2$.
- (iii) $[b, I_\alpha]$ is bounded from L_Φ to $F_{L_\Psi}^{\phi^-}$, if $\beta_n^\alpha \Phi^{-1}(1/\beta_n) \sim \Psi^{-1}(1/\beta_n)$.

3.3. Compactness of commutators

Finally, we state the compactness of the commutators. Let L_0 be the set of all $\mathcal{F}$ -measurable functions, and let

$$L = \{f \in L_0 : f \text{ is } \mathcal{F}_n\text{-measurable for some } n \geq 0\}.$$

Define

$$\mathcal{W}_{1,\phi}^- = \overline{\mathcal{L}_{1,\phi}^- \cap L},$$

where $\overline{\mathcal{L}_{1,\phi}^- \cap L}$ stands for the closure of $\mathcal{L}_{1,\phi}^- \cap L$ in $\mathcal{L}_{1,\phi}^-$. Further, let $A(\mathcal{F}_0) = \{A_k\}_{k=1}^\infty$ and let $D_n = \bigcup_{k=1}^n A_k$. Let

$$L_C = \{f \in L_0 : f = \chi_{D_n} f \text{ for some } n \geq 0\},$$

and define

$$\mathcal{C}_{1,\phi}^- = \overline{\mathcal{L}_{1,\phi}^- \cap L \cap L_C},$$

where $\overline{\mathcal{L}_{1,\phi}^- \cap L \cap L_C}$ stands for the closure of $\mathcal{L}_{1,\phi}^- \cap L \cap L_C$ in $\mathcal{L}_{1,\phi}^-$. If the number of the elements in $A(\mathcal{F}_0)$ is finite, then $L_C = L_0$ and $\mathcal{W}_{1,\phi}^- = \mathcal{C}_{1,\phi}^-$.

For the compactness of $[b, I_\gamma]$, we have the following theorems.

Theorem 3.7. *Let $\{\mathcal{F}_n\}_{n \geq 0}$ be regular. Let $\phi : (0, 1] \rightarrow (0, \infty)$ be almost increasing and $\Phi, \Psi \in \bar{\Phi}_Y$. Let $b \in \mathcal{L}_{1,\phi}^-$.*

- (i) *Assume that $\Phi, \Psi \in \bar{\Delta}_2 \cap \bar{\nabla}_2$. Assume also that there exists a positive constant A and a function $\Theta \in \bar{\nabla}_2$ such that $u \mapsto \Theta^{-1}(u)/\Phi^{-1}(u)$ is almost decreasing and that, for all $n \geq 0$, (3.19) and (3.20) hold. If $b \in \mathcal{C}_{1,\phi}^-$, then $[b, I_\gamma]$ is compact from L_Φ to L_Ψ .*
- (ii) *Conversely, let $\alpha > 0$. Assume that $u \mapsto u^{-\alpha-\epsilon}\Phi^{-1}(u)$ is almost increasing for some $\epsilon > 0$, that ϕ satisfies the doubling condition and that there exists a positive constant A such that, for all $n \geq 0$, (3.22) holds. Assume also that*

$$\lim_{n \rightarrow \infty} \|b - b\chi_{D_n}\|_{\mathcal{L}_{1,\phi}^-(\mathcal{F}_0)} = 0. \quad (3.29)$$

If $[b, I_\alpha]$ is compact from L_Φ to L_Ψ , then b is in $\mathcal{C}_{1,\phi}^-$.

Remark 3.1. (i) If $b \in \mathcal{C}_{1,\phi}^-$, then b satisfies (3.29).

(ii) If the number of the elements in $A(\mathcal{F}_0)$ is finite, then every $b \in L_1$ satisfies (3.29).

Recall that ϕ^- is defined by (3.25).

Theorem 3.8. *Let $\{\mathcal{F}_n\}_{n \geq 0}$ be regular. Let $\phi : (0, 1] \rightarrow (0, \infty)$ be almost increasing and let $\Phi, \Psi \in \bar{\Phi}_Y$. Let $b \in \mathcal{L}_{1,\phi}^-$.*

- (i) *Assume that $\Phi, \Psi \in \bar{\Delta}_2 \cap \bar{\nabla}_2$ and that $u \mapsto \Psi^{-1}(u)/\Phi^{-1}(u)$ is almost decreasing. Assume also that there exists a positive constant A such that, for all $n \geq 0$, (3.26) holds. If $b \in \mathcal{C}_{1,\phi}^-$, then $[b, I_\gamma]$ is compact from L_Φ to $F_{L_\Psi}^{\phi^-}$.*
- (ii) *Conversely, let $\alpha > 0$. Assume that $u \mapsto u^{-\alpha-\epsilon}\Phi^{-1}(u)$ is almost increasing for some $\epsilon > 0$, that ϕ satisfies the doubling condition and that there exists a positive constant A such that, for all $n \geq 0$, (3.28) holds. Assume also that b satisfies (3.29). If $[b, I_\alpha]$ is compact from L_Φ to $F_{L_\Psi}^{\phi^-}$, then b is in $\mathcal{C}_{1,\phi}^-$.*

Corollary 3.9. *Let $\{\mathcal{F}_n\}_{n \geq 0}$ be regular and $\alpha > 0$. Assume that $\Phi, \Psi \in \bar{\Delta}_2 \cap \bar{\nabla}_2$. Assume also that $u \mapsto u^{-\alpha-\epsilon}\Phi^{-1}(u)$ is almost increasing for some $\epsilon > 0$, and that ϕ is almost increasing and satisfies the doubling condition. If b is in $\mathcal{L}_{1,\phi}^-$ and satisfies (3.29), then the following conditions are equivalent:*

- (i) $b \in \mathcal{C}_{1,\phi}^-$.
- (ii) $[b, I_\alpha]$ is compact from L_Φ to L_Ψ , if $\beta_n^{-\alpha} \phi(\beta_n) \Phi^{-1}(1/\beta_n) \sim \phi(\beta_n) \Theta^{-1}(1/\beta_n) \sim \Psi^{-1}(1/\beta_n)$ for some $\Theta \in \bar{\nabla}_2$.
- (iii) $[b, I_\alpha]$ is compact from L_Φ to $F_{L_\Psi}^{\phi^-}$, if $\beta_n^{-\alpha} \Phi^{-1}(1/\beta_n) \sim \Psi^{-1}(1/\beta_n)$.

4. Proofs of Theorems 3.1 and 3.3

Recall that, for a martingale $f = (f_n)_{n \geq 0}$ relative to $\{\mathcal{F}_n\}_{n \geq 0}$, the maximal functions are defined by

$$Mf = \sup_{n \geq 0} |f_n|, \quad M_n f = \sup_{0 \leq m \leq n} |f_m|.$$

Recall also that a function $f \in L_1$ and its corresponding martingale $(f_n)_{n \geq 0}$ with $f_n = E_n f$ will be denoted by the same symbol f . In this case, for $f \in L_1$,

$$Mf = \sup_{n \geq 0} |E_n f|, \quad M_n f = \sup_{0 \leq m \leq n} |E_m f|.$$

It is known as Doob's inequality that (see for example [27, pages 20–21])

$$\|Mf\|_{L_p} \leq \frac{p}{p-1} \|f\|_{L_p}, \quad f \in L_p \ (p > 1), \quad (4.1)$$

$$\|Mf\|_{wL_1} \leq \|f\|_{L_1}, \quad f \in L_1. \quad (4.2)$$

Doob's inequality (4.1) were extended to Orlicz spaces in [12, Theorem 3.3.3]. Moreover, (4.2) were extended to a class of Banach function spaces including Orlicz spaces by [9, Theorem 1] and [10, Theorems 1 and 3]. That is, for any Young function Φ , there exists a positive constant C_Φ such that, for all $f \in L_\Phi$,

$$\|Mf\|_{wL_\Phi} \leq C_\Phi \|f\|_{L_\Phi}.$$

Moreover, if $\Phi \in \nabla_2$, then

$$\|Mf\|_{L_\Phi} \leq C_\Phi \|f\|_{L_\Phi}.$$

In other words, for all $f \in \mathcal{M}_{L_\Phi}$,

$$\|Mf\|_{wL_\Phi} \leq C_\Phi \sup_n \|f_n\|_{L_\Phi}, \quad (4.3)$$

$$\|Mf\|_{L_\Phi} \leq C_\Phi \sup_n \|f_n\|_{L_\Phi}, \quad \text{if } \Phi \in \nabla_2. \quad (4.4)$$

To prove Theorem 3.1 we state two lemmas.

Lemma 4.1. *Let $\Phi \in \Phi_Y$. Then, for any $f \in L_\Phi$ with $\sup_{n \geq 0} \|f_n\|_{L_\Phi} = 1$ and for any $n \geq 0$,*

$$|(I_\gamma f)_n| \leq 2 \left(\sum_{k=0}^n (\gamma_{k-1} - \gamma_k) \Phi^{-1}(1/\beta_k) + \gamma_n \Phi^{-1}(1/\beta_n) \right). \quad (4.5)$$

Proof. First note that

$$|f_n| \leq 2\Phi^{-1}(1/\beta_n), \quad n \geq 0.$$

Actually, using (2.12), (2.13), (2.11) and $\|f_n\|_{L_\Phi} \leq 1$, we have that, for any $B \in A(\mathcal{F}_n)$,

$$\begin{aligned} |f_n(\omega)| &= \frac{1}{P(B)} \left| \int_B f_n dP \right| \leq \frac{2}{P(B)} \|f_n\|_{L_\Phi} \|\chi_B\|_{L_{\tilde{\Phi}}} \\ &\leq 2\Phi^{-1}(1/P(B)) = 2\Phi^{-1}(1/\beta_n(\omega)), \quad \text{a.s. } \omega \in B. \end{aligned}$$

Next, we use the equality

$$\sum_{k=0}^n \gamma_{k-1} (f_k - f_{k-1}) = \sum_{k=0}^n (\gamma_{k-1} - \gamma_k) f_k + \gamma_n f_n, \quad n \geq 0, \quad (4.6)$$

with convention $\gamma_{-1} = \gamma_0$ and $f_{-1} = 0$. Then

$$\begin{aligned} |(I_\gamma f)_n| &\leq \sum_{k=0}^n (\gamma_{k-1} - \gamma_k) |f_k| + \gamma_n |f_n| \\ &\leq 2 \left(\sum_{k=0}^n (\gamma_{k-1} - \gamma_k) \Phi^{-1}(1/\beta_k) + \gamma_n \Phi^{-1}(1/\beta_n) \right). \end{aligned}$$

This is the conclusion. $\square$

Lemma 4.2 ([16, Lemma 2]). *Let $(\gamma_n)_{n \geq 0}$ be a decreasing sequence of non-negative bounded functions adapted to $\{\mathcal{F}_n\}_{n \geq 0}$. Let $f = (f_n)_{n \geq 0}$ be a martingale. Then,*

$$|(I_\gamma f)_m - (I_\gamma f)_{n-1}| \leq 2\gamma_{n-1} M f \quad (4.7)$$

for all $0 \leq n \leq m$ with convention $(I_\gamma f)_{-1} = f_{-1} = 0$ and $\gamma_{-1} = \gamma_0$.

Proof of Theorem 3.1. We may assume that $\Phi, \Psi \in \Phi_Y$. We may also assume that $\sup_{n \geq 0} \|f_n\|_{L_\Phi} = 1$. Combining (3.3) and Lemma 4.1, we have

$$|(I_\gamma f)_n| \lesssim \Psi^{-1} \left(\frac{1}{\beta_n} \right) \quad \text{for } n \geq 0. \quad (4.8)$$

Since (3.3) implies $\gamma_{n-1} \leq C\Psi^{-1} \left(\frac{1}{\beta_n} \right) / \Phi^{-1} \left(\frac{1}{\beta_n} \right)$, using Lemma 4.2, we have

$$|(I_\gamma f)_m - (I_\gamma f)_{n-1}| \lesssim \frac{\Psi^{-1} \left(\frac{1}{\beta_n} \right)}{\Phi^{-1} \left(\frac{1}{\beta_n} \right)} M f \quad \text{for } 0 \leq n \leq m, \quad (4.9)$$

with convention $(I_\gamma f)_{-1} = 0$. Let

$$N = \sum_{n=0}^{\infty} \chi_{\left\{ \frac{1}{\beta_n} \leq \Phi \left(\frac{Mf}{C\Phi} \right) \right\}},$$

and define measurable subsets Ω_1, Ω_2 and Ω_3 of Ω by

$$\Omega_1 = \{N = \infty\}, \quad \Omega_2 = \{N = 0\}, \quad \Omega_3 = \{0 < N < \infty\}.$$

Case 1. Let $\omega \in \Omega_1$. Since $\frac{1}{\beta_n(\omega)}$ is increasing with respect to n , we have $\frac{1}{\beta_n(\omega)} \leq \Phi\left(\frac{Mf(\omega)}{C_\Phi}\right)$ for all $n \geq 0$. Hence, by (4.8) we have

$$|(I_\gamma f)_n(\omega)| \lesssim \Psi^{-1}\left(\frac{1}{\beta_n(\omega)}\right) \leq \Psi^{-1}\left(\Phi\left(\frac{Mf(\omega)}{C_\Phi}\right)\right) \quad \text{for all } n \geq 0.$$

Then (3.4) holds on Ω_1 .

Case 2. Let $\omega \in \Omega_2$. Then $\frac{1}{\beta_0(\omega)} > \Phi\left(\frac{Mf(\omega)}{C_\Phi}\right)$. If $Mf(\omega) = 0$, then $(I_\gamma f)_m(\omega) = 0$ for all $m \geq 0$. Let $Mf(\omega) > 0$. Using (4.9) with $n = 0$, the almost decreasingness of $u \mapsto \Psi^{-1}(u)/\Phi^{-1}(u)$ and (2.8), we have

$$\begin{aligned} |(I_\gamma f)_m(\omega)| &\lesssim \frac{\Psi^{-1}\left(\frac{1}{\beta_0(\omega)}\right)}{\Phi^{-1}\left(\frac{1}{\beta_0(\omega)}\right)} Mf(\omega) \lesssim \frac{\Psi^{-1}\left(\Phi\left(\frac{Mf(\omega)}{C_\Phi}\right)\right)}{\Phi^{-1}\left(\Phi\left(\frac{Mf(\omega)}{C_\Phi}\right)\right)} Mf(\omega) \\ &\leq \frac{\Psi^{-1}\left(\Phi\left(\frac{Mf(\omega)}{C_\Phi}\right)\right)}{\frac{Mf(\omega)}{C_\Phi}} Mf(\omega) \\ &= C_\Phi \Psi^{-1}\left(\Phi\left(\frac{Mf(\omega)}{C_\Phi}\right)\right) \quad \text{for all } m \geq 0. \end{aligned}$$

Then (3.4) holds on Ω_2 .

Case 3. Let $\omega \in \Omega_3$. Then, we can take an integer $n \geq 1$ such that

$$\frac{1}{\beta_{n-1}(\omega)} \leq \Phi\left(\frac{Mf(\omega)}{C_\Phi}\right) \quad \text{and} \quad \frac{1}{\beta_n(\omega)} > \Phi\left(\frac{Mf(\omega)}{C_\Phi}\right).$$

If $m \leq n - 1$, then, using (4.8), we have

$$|(I_\gamma f)_m(\omega)| \lesssim \Psi^{-1}\left(\frac{1}{\beta_m(\omega)}\right) \leq \Psi^{-1}\left(\frac{1}{\beta_{n-1}(\omega)}\right) \leq \Psi^{-1}\left(\Phi\left(\frac{Mf(\omega)}{C_\Phi}\right)\right).$$

If $m \geq n$, then, using (4.8), (4.9), the almost decreasingness of $u \mapsto \Psi^{-1}(u)/\Phi^{-1}(u)$ and (2.8), we have

$$\begin{aligned} |(I_\gamma f)_m(\omega)| &\leq |(I_\gamma f)_{n-1}(\omega)| + |(I_\gamma f)_m(\omega) - (I_\gamma f)_{n-1}(\omega)| \\ &\lesssim \Psi^{-1}\left(\frac{1}{\beta_{n-1}(\omega)}\right) + \frac{\Psi^{-1}\left(\frac{1}{\beta_n(\omega)}\right)}{\Phi^{-1}\left(\frac{1}{\beta_n(\omega)}\right)} Mf(\omega) \\ &\lesssim \Psi^{-1}\left(\Phi\left(\frac{Mf(\omega)}{C_\Phi}\right)\right) + \frac{\Psi^{-1}\left(\Phi\left(\frac{Mf(\omega)}{C_\Phi}\right)\right)}{\frac{Mf(\omega)}{C_\Phi}} Mf(\omega) \\ &\sim \Psi^{-1}\left(\Phi\left(\frac{Mf(\omega)}{C_\Phi}\right)\right). \end{aligned}$$

That is,

$$|(I_\gamma f)_m| \lesssim \Psi^{-1}\left(\Phi\left(\frac{Mf}{C_\Phi}\right)\right) \quad \text{for all } m \geq 0.$$

Then (3.4) holds on Ω_3 .

Consequently, for the constant C_Φ in (4.3), we have

$$\sup_{t \in (0, \infty)} t P \left(\Psi \left(\frac{M(I_\gamma f)}{C'_\Phi} \right), t \right) \leq \sup_{t \in (0, \infty)} t P \left(\Phi \left(\frac{Mf}{C_\Phi} \right), t \right) \leq 1,$$

which and (2.10) show

$$\sup_n \|(I_\gamma f)_n\|_{\text{w}L_\Psi} \leq \|M(I_\gamma f)\|_{\text{w}L_\Psi} \leq C'_\Phi.$$

Moreover, if $\Phi \in \nabla_2$, for the constant C_Φ in (4.4), we have

$$E \left[\Psi \left(\frac{M(I_\gamma f)}{C'_\Phi} \right) \right] \leq E \left[\Phi \left(\frac{Mf}{C_\Phi} \right) \right] \leq 1,$$

which shows

$$\sup_n \|(I_\gamma f)_n\|_{L_\Psi} \leq \|M(I_\gamma f)\|_{L_\Psi} \leq C'_\Phi.$$

Therefore, we have the conclusion. $\square$

Proof of Theorem 3.3. We may assume that $\Phi, \Psi \in \Phi_Y$. We may also assume that $\|f\|_{L_\Phi} = 1$. Let $B \in A(\mathcal{F}_n)$, $n \geq 0$, and $\omega \in B$. If $Mf(\omega) = 0$, then $M_\gamma f(\omega) = 0$. Let $Mf(\omega) > 0$. If

$$\Phi \left(\frac{Mf(\omega)}{C_\Phi} \right) \geq \frac{1}{\beta_n},$$

then, using (2.12), (2.13), (2.11) and $\|f\|_{L_\Phi} = 1$,

$$\begin{aligned} \gamma_n \left| \frac{1}{P(B)} \int_B f dP \right| &\leq 2 \frac{\gamma_n}{P(B)} \|f\|_{L_\Phi} \|\chi_B\|_{L_{\tilde{\Phi}}} \\ &\leq 2\gamma_n \Phi^{-1} \left(\frac{1}{P(B)} \right) \\ &\leq 2A \Psi^{-1} \left(\frac{1}{\beta_n} \right) \\ &\leq 2A \Psi^{-1} \left(\Phi \left(\frac{Mf(\omega)}{C_\Phi} \right) \right). \end{aligned}$$

Conversely, if

$$\Phi \left(\frac{Mf(\omega)}{C_\Phi} \right) \leq \frac{1}{\beta_n},$$

then, by (3.16), the almost decreasingness of $u \mapsto \Psi^{-1}(u)/\Phi^{-1}(u)$ and (2.8) we have

$$\gamma_n \leq A \frac{\Psi^{-1} \left(\frac{1}{\beta_n} \right)}{\Phi^{-1} \left(\frac{1}{\beta_n} \right)} \leq AC \frac{\Psi^{-1} \left(\Phi \left(\frac{Mf(\omega)}{C_\Phi} \right) \right)}{\Phi^{-1} \left(\Phi \left(\frac{Mf(\omega)}{C_\Phi} \right) \right)} \leq ACC_\Phi \frac{\Psi^{-1} \left(\Phi \left(\frac{Mf(\omega)}{C_\Phi} \right) \right)}{Mf(\omega)},$$

for some positive constant C , which implies

$$\gamma_n \left| \frac{1}{P(B)} \int_B f dP \right| \leq ACC_\Phi \Psi^{-1} \left(\Phi \left(\frac{Mf(\omega)}{C_\Phi} \right) \right).$$

Hence, letting $C'_\Phi = 2A + ACC_\Phi$, we have

$$M_\gamma f(\omega) \leq C'_\Phi \Psi^{-1} \left(\Phi \left(\frac{Mf(\omega)}{C_\Phi} \right) \right),$$

which shows (3.17) by (2.8). $\square$

5. Pointwise estimate by the sharp maximal function

We first state a John-Nirenberg type inequality without the regularity of $\{\mathcal{F}_n\}_{n \geq 0}$.

Theorem 5.1. *Assume that $\phi : (0, 1] \rightarrow (0, \infty)$ is almost increasing. Then, for every $p \in [1, \infty)$, there exists a positive constant C_p such that*

$$\|f\|_{\mathcal{L}_{p,\phi}^-} \leq C_p \|f\|_{\mathcal{L}_{1,\phi}^-}. \quad (5.1)$$

Actually, we already proved that, under the same assumption,

$$\|f\|_{\mathcal{L}_{p,\phi}} \leq C'_p \|f\|_{\mathcal{L}_{1,\phi}^-} \quad (5.2)$$

in [13, Theorem 2.9], where

$$\|f\|_{\mathcal{L}_{p,\phi}} = \sup_{n \geq 0} \sup_{B \in A(\mathcal{F}_n)} \frac{1}{\phi(P(B))} \left(\frac{1}{P(B)} \int_B |f - E_n f|^p dP \right)^{1/p}. \quad (5.3)$$

Then, for $B \in A(\mathcal{F}_n)$.

$$\begin{aligned} & \left(\frac{1}{P(B)} \int_B |f - E_{n-1} f|^p dP \right)^{1/p} \\ & \leq \left(\frac{1}{P(B)} \int_B |f - E_n f|^p dP \right)^{1/p} + \left(\frac{1}{P(B)} \int_B |E_n f - E_{n-1} f|^p dP \right)^{1/p} \\ & = \left(\frac{1}{P(B)} \int_B |f - E_n f|^p dP \right)^{1/p} + \left(\frac{1}{P(B)} \int_B \left| \frac{1}{P(B)} \int_B (f - E_{n-1} f) dP \right|^p dP \right)^{1/p} \\ & \leq \left(\frac{1}{P(B)} \int_B |f - E_n f|^p dP \right)^{1/p} + \frac{1}{P(B)} \int_B |f - E_{n-1} f| dP \\ & \leq (C'_p + 1) \phi(P(B)) \|f\|_{\mathcal{L}_{1,\phi}^-}. \end{aligned}$$

This shows (5.1).

In this section, we prove the following two propositions. We always assume that $(\gamma_n)_{n \geq 0}$ is a decreasing sequence of non-negative bounded functions adapted to $\{\mathcal{F}_n\}_{n \geq 0}$.

Proposition 5.2. Assume that $\phi : (0, 1] \rightarrow (0, \infty)$ is almost increasing. Let

$$\gamma_n^- = \gamma_{n-1}, \quad n \geq 0. \quad (5.4)$$

Then, for any $\eta \in (1, \infty)$, there exists a positive constant C such that, for all $b \in \mathcal{L}_{1,\phi}^- \cap L_\infty$ and all $f \in L_\infty$,

$$M^\#([b, I_\gamma]f) \leq C \|b\|_{\mathcal{L}_{1,\phi}^-} \left(M_\phi(I_\gamma f) + (M_{(\phi\gamma^-)^\eta}(|f|^\eta))^{1/\eta} \right), \quad (5.5)$$

where

$$M_\phi f = \sup_{n \geq 0} \phi(\beta_n) |E_n f|, \quad M_{(\phi\gamma^-)^\eta} f = \sup_{n \geq 0} (\phi(\beta_n) \gamma_n^-)^\eta |E_n f|.$$

Proposition 5.3. Assume that $\phi : (0, 1] \rightarrow (0, \infty)$ is almost increasing. Then, for any $\eta \in (1, \infty)$, there exists a positive constant C such that, for all $b \in \mathcal{L}_{1,\phi}^- \cap L_\infty$ and $f \in L_\infty$,

$$M_{\phi^-}^\#([b, I_\gamma]f) \leq C \|b\|_{\mathcal{L}_{1,\phi}^-} \left(M(I_\gamma f) + (M_{(\gamma^-)^\eta}(|f|^\eta))^{1/\eta} \right), \quad (5.6)$$

where

$$M_{(\gamma^-)^\eta} f = \sup_{n \geq 0} (\gamma_n^-)^\eta |E_n f|, \quad M_{\phi^-}^\# f = \sup_{n \geq 0} \phi^-(\beta_n)^{-1} |E_n f - E_{n-1} f|.$$

To prove Propositions 5.2 and 5.3 we prepare several lemmas. For $n \geq 0$ and $f \in L_1$, let

$$I_\gamma^{(n)} f = I_\gamma f - (I_\gamma f)_n.$$

Then

$$I_\gamma^{(n)}(fg) = (I_\gamma^{(n)} f)g \quad \text{for all } \mathcal{F}_n\text{-measurable functions } g. \quad (5.7)$$

Let

$$I_\gamma^{[n]} f = I_\gamma^{(n)} f + \gamma_n E_n f.$$

Then

$$I_\gamma^{[n]} f = \sum_{k=n+1}^{\infty} \gamma_{k-1} (E_k f - E_{k-1} f) + \gamma_n E_n f = \sum_{k=n+1}^{\infty} (\gamma_{k-1} - \gamma_k) E_k f. \quad (5.8)$$

Note that $I_\gamma^{[0]} = I_\gamma$. Let $I_\gamma^{[-1]} = I_\gamma^{[0]}$. Then (5.8) holds for the case $n = -1$ also with convention $\gamma_{-1} = \gamma_0$ and $E_{-1} f = 0$.

In the following, let $\eta, \eta' \in (1, \infty)$ and $1/\eta + 1/\eta' = 1$. Moreover, let $C_{\eta'}$ be the constant in (5.1) for $p = \eta'$. For $b \in L_\infty$, let $b_n = E_n b$. Assume also that ϕ is almost increasing with the constant C_ϕ , that is,

$$\phi(r) \leq C_\phi \phi(s), \quad \text{if } r \leq s. \quad (5.9)$$

Then we have the following sequence of lemmas, whose proofs will be given later.

Lemma 5.4. Let f be in L_∞ . Let $\eta \in [1, \infty)$. Then, for all $n \geq 0$,

$$E_n(|I_\gamma^{[n-1]}f|^\eta)^{1/\eta} \leq \gamma_{n-1}E_n(|f|^\eta)^{1/\eta} \quad (5.10)$$

with convention $I_\gamma^{[-1]} = I_\gamma^{[0]}$ and $\gamma_{-1} = \gamma_0$.

Lemma 5.5. Let $b \in \mathcal{L}_{1,\phi}^- \cap L_\infty$ and $f \in L_\infty$. Then

$$E_0|[b, I_\gamma]f| \leq 2C_{\eta'}\|b\|_{\mathcal{L}_{1,\phi}^-} \phi(\beta_0)\gamma_0(E_0[|f|^\eta])^{1/\eta}. \quad (5.11)$$

Lemma 5.6. Let f and b be in L_∞ . Then, for all $n \geq 1$,

$$\begin{aligned} & E_n|[b, I_\gamma]f - E_{n-1}([b, I_\gamma]f)| \\ & \leq E_n|(b - b_{n-1})I_\gamma^{[n-1]}f| + E_{n-1}|(b - b_{n-1})I_\gamma^{[n-1]}f| \\ & \quad + E_n|I_\gamma^{[n-1]}((b - b_{n-1})f)| + \gamma_{n-1}E_{n-1}|(b - b_{n-1})f| \\ & \quad + |(I_\gamma f)_{n-1}|E_n|b - b_{n-1}|. \end{aligned}$$

Lemma 5.7. Let $b \in \mathcal{L}_{1,\phi}^- \cap L_\infty$ and $f \in L_\infty$. Then, for all $n \geq 1$,

$$E_n|(b - b_{n-1})I_\gamma^{[n-1]}f| \leq C_{\eta'}\|b\|_{\mathcal{L}_{1,\phi}^-} \phi(\beta_n)\gamma_{n-1}(E_n[|f|^\eta])^{1/\eta}, \quad (5.12)$$

$$E_n|I_\gamma^{[n-1]}((b - b_{n-1})f)| \leq C_{\eta'}\|b\|_{\mathcal{L}_{1,\phi}^-} \phi(\beta_n)\gamma_{n-1}(E_n[|f|^\eta])^{1/\eta}, \quad (5.13)$$

$$E_{n-1}|(b - b_{n-1})I_\gamma^{[n-1]}f| \leq C_\phi C_{\eta'}\|b\|_{\mathcal{L}_{1,\phi}^-} \phi(\beta_{n-1})\gamma_{n-1}(E_{n-1}[|f|^\eta])^{1/\eta}, \quad (5.14)$$

$$\gamma_{n-1}E_{n-1}|(b - b_{n-1})f| \leq C_\phi C_{\eta'}\|b\|_{\mathcal{L}_{1,\phi}^-} \phi(\beta_{n-1})\gamma_{n-1}(E_{n-1}[|f|^\eta])^{1/\eta}. \quad (5.15)$$

Now we prove Propositions 5.2 and 5.3.

Proof of Proposition 5.2. For $n = 0$, Lemma 5.5 implies

$$E_0|[b, I_\gamma]f| \leq 2C_{\eta'}\|b\|_{\mathcal{L}_{1,\phi}^-} (M_{(\phi\gamma^-)^\eta}(|f|^\eta))^{1/\eta},$$

since $\gamma_0 = \gamma_0^-$. For $n \geq 1$, Lemma 5.7 implies that

$$\begin{aligned} & E_n|(b - b_{n-1})I_\gamma^{[n-1]}f| + E_{n-1}|(b - b_{n-1})I_\gamma^{[n-1]}f| \\ & \leq (1 + C_\phi)C_{\eta'}\|b\|_{\mathcal{L}_{1,\phi}^-} (M_{(\phi\gamma^-)^\eta}(|f|^\eta))^{1/\eta}, \\ & E_n|I_\gamma^{[n-1]}((b - b_{n-1})f)| + \gamma_{n-1}E_{n-1}|(b - b_{n-1})f| \\ & \leq (1 + C_\phi)C_{\eta'}\|b\|_{\mathcal{L}_{1,\phi}^-} (M_{(\phi\gamma^-)^\eta}(|f|^\eta))^{1/\eta}, \end{aligned}$$

since $\gamma_{n-1} \leq \gamma_{n-1}^-$. From the almost increasingness of ϕ it follows that

$$\begin{aligned} |(I_\gamma f)_{n-1}|E_n|b - b_{n-1}| & \leq |(I_\gamma f)_{n-1}|\phi(\beta_n)\|b\|_{\mathcal{L}_{1,\phi}^-} \\ & \leq C_\phi|(I_\gamma f)_{n-1}|\phi(\beta_{n-1})\|b\|_{\mathcal{L}_{1,\phi}^-} \\ & \leq C_\phi\|b\|_{\mathcal{L}_{1,\phi}^-} M_\phi(I_\gamma f). \end{aligned}$$

Combining these and Lemma 5.6, we have the conclusion. $\square$

Proof of Proposition 5.3. For $n = 0$, Lemma 5.5 implies

$$E_0|[b, I_\gamma]f| \leq C_{\eta'}\phi(\beta_0)\|b\|_{\mathcal{L}_{1,\phi}^-} (M_{(\gamma^-)^\eta}(|f|^\eta))^{1/\eta}.$$

Next, let $n \geq 1$. Combining the almost increasingness of ϕ and Lemma 5.7, we have

$$\begin{aligned} E_n|(b - b_{n-1})I_\gamma^{[n-1]}f| + E_{n-1}|(b - b_{n-1})I_\gamma^{[n-1]}f| \\ \leq 2C_{\eta'}C_\phi\phi(\beta_{n-1})\|b\|_{\mathcal{L}_{1,\phi}^-} (M_{(\gamma^-)^\eta}(|f|^\eta))^{1/\eta}, \\ E_n|I_\gamma^{[n-1]}((b - b_{n-1})f)| + \gamma_{n-1}E_{n-1}|(b - b_{n-1})f| \\ \leq 2C_{\eta'}C_\phi\phi(\beta_{n-1})\|b\|_{\mathcal{L}_{1,\phi}^-} (M_{(\gamma^-)^\eta}(|f|^\eta))^{1/\eta}. \end{aligned}$$

Moreover,

$$\begin{aligned} |(I_\gamma f)_{n-1}|E_n|b - b_{n-1}| &\leq |(I_\gamma f)_{n-1}|\phi(\beta_n)\|b\|_{\mathcal{L}_{1,\phi}^-} \\ &\leq C_\phi\phi(\beta_{n-1})\|b\|_{\mathcal{L}_{1,\phi}^-} M(I_\gamma f). \end{aligned}$$

By Lemma 5.6 we have the conclusion. $\square$

In the left of this section we show the lemmas.

Proof of Lemma 5.4. Let $n \geq 0$. By (5.8) we have

$$|I_\gamma^{[n-1]}f| \leq \sum_{k=n}^{\infty} (\gamma_{k-1} - \gamma_k)E_k|f|. \quad (5.16)$$

Then, for $\eta = 1$,

$$E_n|I_\gamma^{[n-1]}f| \leq \sum_{k=n}^{\infty} E_n[(\gamma_{k-1} - \gamma_k)|f|] \leq \gamma_{n-1}E_n|f|.$$

For $\eta > 1$, applying Hölder's inequality to (5.16), we have

$$\begin{aligned} |I_\gamma^{[n-1]}f| &\leq \left(\sum_{k=n}^{\infty} (\gamma_{k-1} - \gamma_k)(E_k|f|^\eta) \right)^{1/\eta} \left(\sum_{k=n}^{\infty} (\gamma_{k-1} - \gamma_k) \right)^{1/\eta'} \\ &\leq \gamma_{n-1}^{1/\eta'} \left(\sum_{k=n}^{\infty} (\gamma_{k-1} - \gamma_k)E_k(|f|^\eta) \right)^{1/\eta}. \end{aligned}$$

Then

$$\begin{aligned} E_n[|I_\gamma^{[n-1]}f|^\eta]^{1/\eta} &\leq \gamma_{n-1}^{1/\eta'} \left(\sum_{k=n}^{\infty} E_n[(\gamma_{k-1} - \gamma_k)|f|^\eta] \right)^{1/\eta} \\ &\leq \gamma_{n-1}E_n(|f|^\eta)^{1/\eta}, \end{aligned}$$

which shows the conclusion. $\square$

Proof of Lemma 5.5. By Hölder's inequality we have

$$E_0 |b I_\gamma f| \leq \left(E_0 \left[|b|^{\eta'} \right] \right)^{1/\eta'} (E_0 [|I_\gamma f|^\eta])^{1/\eta}.$$

Using Theorem 5.1, we have

$$\left(E_0 \left[|b|^{\eta'} \right] \right)^{1/\eta'} \leq C_{\eta'} \phi(\beta_0) \|b\|_{\mathcal{L}_{1,\phi}^-}.$$

Applying Lemma 5.4 to $I_\gamma f = I_\gamma^{[-1]} f$, we have

$$(E_0 [|I_\gamma f|^\eta])^{1/\eta} \leq \gamma_0 (E_0 [|f|^\eta])^{1/\eta}.$$

Then

$$E_0 |b I_\gamma f| \leq C_{\eta'} \|b\|_{\mathcal{L}_{1,\phi}^-} \phi(\beta_0) \gamma_0 (E_0 [|f|^\eta])^{1/\eta}.$$

Next, using Lemma 5.4, we have

$$E_0 |I_\gamma(bf)| \leq \gamma_0 E_0 |bf|.$$

By Hölder's inequality and Theorem 5.1 we have

$$E_0 |I_\gamma(bf)| \leq \gamma_0 \left(E_0 \left[|b|^{\eta'} \right] \right)^{1/\eta'} (E_0 [|f|^\eta])^{1/\eta} \leq C_{\eta'} \|b\|_{\mathcal{L}_{1,\phi}^-} \gamma_0 \phi(\beta_0) (E_0 [|f|^\eta])^{1/\eta}.$$

The proof is complete. $\square$

Proof of Lemma 5.6. Let $n \geq 1$. Noting that

$$b_{n-1} I_\gamma^{(n-1)} f = I_\gamma^{(n-1)}(b_{n-1} f) \quad \text{and} \quad [b_{n-1}, I_\gamma^{(n-1)}] f = 0,$$

we have

$$\begin{aligned} [b, I_\gamma] f &= [b, I_\gamma^{(n-1)}] f + [b, E_{n-1} I_\gamma] f \\ &= [b - b_{n-1}, I_\gamma^{(n-1)}] f + b(I_\gamma f)_{n-1} - (I_\gamma(bf))_{n-1}. \end{aligned}$$

Noting that $E_{n-1}[I_\gamma^{(n-1)}(b - b_{n-1})f] = 0$, we obtain

$$\begin{aligned} & |[b, I_\gamma] f - E_{n-1}([b, I_\gamma] f)| \\ & \leq \left| [b - b_{n-1}, I_\gamma^{(n-1)}] f - E_{n-1}([b - b_{n-1}, I_\gamma^{(n-1)}] f) \right| + |(b - b_{n-1})(I_\gamma f)_{n-1}| \\ & \leq \left| (b - b_{n-1}) I_\gamma^{(n-1)} f - E_{n-1}[(b - b_{n-1}) I_\gamma^{(n-1)} f] \right| \\ & \quad + |I_\gamma^{(n-1)}(b - b_{n-1}) f| \\ & \quad + |(I_\gamma f)_{n-1}(b - b_{n-1})| \\ & \leq \left| (b - b_{n-1}) I_\gamma^{[n-1]} f - E_{n-1}[(b - b_{n-1}) I_\gamma^{[n-1]} f] \right| \\ & \quad + |I_\gamma^{[n-1]}(b - b_{n-1}) f| + \gamma_{n-1} |E_{n-1}[(b - b_{n-1}) f]| \\ & \quad + |(I_\gamma f)_{n-1}(b - b_{n-1})|, \end{aligned}$$

which implies the conclusion. $\square$

Proof of Lemma 5.7. Let $n \geq 1$. Then, by Hölder's inequality, Theorem 5.1 and Lemma 5.4, we have

$$\begin{aligned} E_n |(b - b_{n-1}) I_\gamma^{[n-1]} f| &\leq (E_n [| (b - b_{n-1}) |^{\eta'}])^{1/\eta'} (E_n [| I_\gamma^{[n-1]} f |^\eta])^{1/\eta} \\ &\leq C_{\eta'} \|b\|_{\mathcal{L}_{1,\phi}^-} \phi(\beta_n) \gamma_{n-1} (E_n [|f|^\eta])^{1/\eta}, \end{aligned}$$

which is (5.12). By (5.12) and the almost increasingness of ϕ , we also have

$$E_n |(b - b_{n-1}) I_\gamma^{[n-1]} f| \leq C_\phi C_{\eta'} \|b\|_{\mathcal{L}_{1,\phi}^-} \phi(\beta_{n-1}) \gamma_{n-1} (E_n [|f|^\eta])^{1/\eta}.$$

Taking the conditional expectation E_{n-1} of the both side, we have (5.14).

We now show (5.13). By Lemma 5.4, we have

$$E_n |I_\gamma^{[n-1]} ((b - b_{n-1}) f)| \leq \gamma_{n-1} E_n |(b - b_{n-1}) f|. \quad (5.17)$$

Then, by Hölder's inequality and Theorem 5.1, we have

$$\begin{aligned} \gamma_{n-1} E_n |(b - b_{n-1}) f| &\leq \gamma_{n-1} (E_n [| (b - b_{n-1}) |^{\eta'}])^{1/\eta'} (E_n [|f|^\eta])^{1/\eta} \\ &\leq C_{\eta'} \|b\|_{\mathcal{L}_{1,\phi}^-} \phi(\beta_n) \gamma_{n-1} (E_n [|f|^\eta])^{1/\eta}. \end{aligned} \quad (5.18)$$

Combining (5.17) and (5.18), we have (5.13).

Using the almost increasingness of ϕ and taking the conditional expectation E_{n-1} in (5.18), we have (5.15). $\square$

6. Proofs of Theorems 3.4 and 3.5

We first note that, for $\theta \in (0, \infty)$,

$$\| |g|^\theta \|_{L_\Phi} = (\|g\|_{L_{\Phi((\cdot)^\theta)}})^\theta. \quad (6.1)$$

We use the following theorem to prove Theorem 3.4 (i) and Theorem 3.5 (i).

Theorem 6.1 ([12, Theorem 3.5.7]). *Let $\Phi \in \Delta_2$. If $Mf \in L_\Phi$, then*

$$\|Mf\|_{L_\Phi} \leq C \|M^\sharp f\|_{L_\Phi}, \quad (6.2)$$

where C is a positive constant which is independent of f .

Proof of Theorem 3.4 (i). We may assume that $\Phi, \Psi \in \Delta_2 \cap \nabla_2$ and $\Theta \in \nabla_2$. We may also assume that b is real valued, since the commutator $[b, I_\gamma]f$ is linear with respect to b and $\|\Re(b)\|_{\mathcal{L}_{1,\phi}^-}, \|\Im(b)\|_{\mathcal{L}_{1,\phi}^-} \leq \|b\|_{\mathcal{L}_{1,\phi}^-}$. Let

$$b_k(x) = \begin{cases} k, & \text{if } b(x) > k, \\ b(x), & \text{if } -k \leq b(x) \leq k, \\ -k, & \text{if } b(x) < -k. \end{cases}$$

Then $b_k \in L_\infty$ and $\|b_k\|_{\mathcal{L}_{1,\phi}^-} \leq (9/4) \|b\|_{\mathcal{L}_{1,\phi}^-}$. For $f \in L_\infty$, $b_k f$ lies in L_∞ , thus $I_\gamma(b_k f)$ lies in L_Ψ since $L_\infty \subset L_\Psi$. Likewise, $b_k I_\gamma f$ also lies in L_Ψ . Then $M[b, I_\gamma]f$ is also in L_Ψ . From this fact and Proposition 5.2 and Theorem 6.1 it follows that

$$\begin{aligned} \| [b_k, I_\gamma] f \|_{L_\Psi} &\leq \| M([b_k, I_\gamma] f) \|_{L_\Psi} \lesssim \| M^\sharp([b_k, I_\gamma] f) \|_{L_\Psi} \\ &\lesssim \| b \|_{\mathcal{L}_{1,\phi}^-} \left(\| M_\phi(|I_\gamma f|) \|_{L_\Psi} + \left\| (M_{(\phi\gamma^-)^\eta}(|f|^\eta))^{1/\eta} \right\|_{L_\Psi} \right), \end{aligned}$$

here, we can choose $\eta \in (1, \infty)$ such that $\Phi((\cdot)^{1/\eta})$ and $\Psi((\cdot)^{1/\eta})$ are in $\bar{\nabla}_2$ by Remark 2.2 (v).

We now show that

$$\| M_\phi(|I_\gamma f|) \|_{L_\Psi} + \left\| (M_{(\phi\gamma^-)^\eta}(|f|^\eta))^{1/\eta} \right\|_{L_\Psi} \lesssim \| f \|_{L_\Phi}.$$

By Theorems 3.1 and 3.3 we see that I_γ is bounded from L_Φ to L_Θ and M_ϕ is bounded from L_Θ to L_Ψ , respectively. Then we have

$$\| M_\phi(|I_\gamma f|) \|_{L_\Psi} \lesssim \| I_\gamma f \|_{L_\Theta} \lesssim \| f \|_{L_\Phi}.$$

From (3.19) and (3.20) it follows that

$$(\phi(\beta_n)\gamma_n^-)^\eta (\Phi^{-1}(1/\beta_n))^\eta \leq A^{2\eta} (\Psi^{-1}(1/\beta_n))^\eta.$$

By using Theorem 3.3, we have the boundedness of $M_{(\phi\gamma^-)^\eta}$ from $L_{\Phi((\cdot)^{1/\eta})}$ to $L_{\Psi((\cdot)^{1/\eta})}$. That is,

$$\begin{aligned} \left\| (M_{(\phi\gamma^-)^\eta}(|f|^\eta))^{1/\eta} \right\|_{L_\Psi} &= (\| M_{(\phi\gamma^-)^\eta}(|f|^\eta) \|_{L_{\Psi((\cdot)^{1/\eta})}})^{1/\eta} \\ &\lesssim (\| |f|^\eta \|_{L_{\Phi((\cdot)^{1/\eta})}})^{1/\eta} = \| f \|_{L_\Phi}. \end{aligned}$$

Therefore, we obtain

$$\| [b_k, I_\gamma] f \|_{L_\Psi} \lesssim \| b \|_{\mathcal{L}_{1,\phi}^-} \| f \|_{L_\Phi} \quad \text{for all } f \in L_\infty.$$

By the standard argument we deduce that, for some subsequence of integers k_j , $[b_{k_j}, I_\gamma] f \rightarrow [b, I_\gamma] f$ a.e. Letting $j \rightarrow \infty$ and using Fatou's lemma, we have

$$\| [b, I_\gamma] f \|_{L_\Psi} \lesssim \| b \|_{\mathcal{L}_{1,\phi}^-} \| f \|_{L_\Phi} \quad \text{for all } f \in L_\infty.$$

Since L_∞ is dense in L_Φ (see Remark 2.2 (iii)), it follows that the commutator admits a bounded extension on L_Φ that satisfies (3.21). $\square$

Proof of Theorem 3.5 (i). We may assume that $\Phi, \Psi \in \Delta_2 \cap \nabla_2$. We may also assume that b is real valued. Define b_k in the same way as the proof of Theorem 3.4 (i). Then, from Proposition 5.3, it follows that

$$\begin{aligned} \| [b_k, I_\gamma] f \|_{F_{L_\Psi}^{\phi^-}} &= \| M_{\phi^-}^\sharp([b_k, I_\gamma] f) \|_{L_\Psi} \\ &\lesssim \| b \|_{\mathcal{L}_{1,\phi}^-} \left(\| M(I_\gamma f) \|_{L_\Psi} + \left\| (M_{(\gamma^-)^\eta}(|f|^\eta))^{1/\eta} \right\|_{L_\Psi} \right), \end{aligned}$$

here, we can choose $\eta \in (1, \infty)$ such that $\Phi((\cdot)^{1/\eta})$ and $\Psi((\cdot)^{1/\eta})$ are in $\bar{\nabla}_2$ by Remark 2.2 (v).

We show that

$$\| M(I_\gamma f) \|_{L_\Psi} + \left\| (M_{(\gamma^-)^\eta}(|f|^\eta))^{1/\eta} \right\|_{L_\Psi} \lesssim \| f \|_{L_\Phi}.$$

Since M is bounded on L_Ψ by (4.4) and I_γ is bounded from L_Φ to L_Ψ by Theorem 3.1, we see that

$$\|M(I_\gamma f)\|_{L_\Psi} \lesssim \|f\|_{L_\Phi}.$$

By Theorem 3.3 we have the boundedness of $M_{(\gamma^-)^\eta}$ from $L_{\Phi((\cdot)^{1/\eta})}$ to $L_{\Psi((\cdot)^{1/\eta})}$. That is,

$$\begin{aligned} \left\| \left(M_{(\gamma^-)^\eta}(|f|^\eta) \right)^{1/\eta} \right\|_{L_\Psi} &= \left(\|M_{(\gamma^-)^\eta}(|f|^\eta)\|_{L_{\Psi((\cdot)^{1/\eta})}} \right)^{1/\eta} \\ &\lesssim \left(\| |f|^\eta \|_{L_{\Phi((\cdot)^{1/\eta})}} \right)^{1/\eta} = \|f\|_{L_\Phi}. \end{aligned}$$

Therefore, we obtain

$$\|[b_k, I_\gamma]f\|_{F_{L_\Psi}^{\phi-}} \lesssim \|b\|_{\mathcal{L}_{1,\phi}^-} \|f\|_{L_\Phi} \quad \text{for all } f \in L_\infty.$$

By the density argument used in the proof of Theorem 3.4, we obtain the conclusion. $\square$

Next, to prove Theorem 3.4 (ii) and Theorem 3.5 (ii), we show the following lemmas.

Lemma 6.2. *Let $b \in L_1$. Let $B \in A(\mathcal{F}_n)$, $n \geq 0$, and let $f \in L_\infty$ with $f\chi_B = 0$. Then,*

$$\chi_B E_n |[b, I_\alpha]f - E_n [[b, I_\alpha]f]| = \chi_B |I_\alpha f| E_n |b - E_n b|. \quad (6.3)$$

Proof. From the assumption, it follows that

$$I_\alpha f = E_n [I_\alpha f], \quad I_\alpha (fb) = E_n [I_\alpha (fb)] \quad \text{on } B.$$

Hence, we have

$$[b, I_\alpha]f = bE_n [I_\alpha f] - E_n [I_\alpha (bf)] \quad \text{on } B.$$

Therefore,

$$[b, I_\alpha]f - E_n ([b, I_\alpha]f) = (b - E_n b)E_n [I_\alpha f] = (b - E_n b)I_\alpha f \quad \text{on } B,$$

and

$$E_n |[b, I_\alpha]f - E_n [[b, I_\alpha]f]| = |I_\alpha f| E_n |b - E_n b| \quad \text{on } B,$$

which shows the conclusion. $\square$

Lemma 6.3 ([15], Lemma 3.3). *Let $\{\mathcal{F}_n\}_{n \geq 0}$ be regular. Then every sequence*

$$B_0 \supset B_1 \supset \cdots \supset B_n \supset \cdots, \quad B_n \in A(\mathcal{F}_n),$$

has the following property: For each $n \geq 1$,

$$B_n = B_{n-1} \text{ or } (1 + 1/R)P(B_n) \leq P(B_{n-1}) \leq RP(B_n),$$

where R is the constant in (3.8).

Lemma 6.4. Assume that $\{\mathcal{F}_n\}_{n \geq 0}$ is regular. Let $B \in A(\mathcal{F}_n)$, $n \geq 2$, and let

$$B = B_n \subset B_{n-1} \subset \cdots \subset B_0, \quad B_k \in A(\mathcal{F}_k).$$

Assume that $0 \leq \ell < m < n$, where

$$m = \max\{k : P(B_n) < P(B_k)\}, \quad \ell = \max\{k : P(B_m) < P(B_k)\}.$$

Let $h = \chi_{B_m \setminus B}$ and let $b \in L_1$. Then

$$C_R \chi_B P(B)^\alpha E_n |b - E_n b| \leq \chi_B E_n |[b, I_\alpha]h - E_n([b, I_\alpha]h)|, \quad (6.4)$$

where C_R is a positive constant depending only on the constant R in (3.8).

Proof. Since $h\chi_B = 0$, we have $I_\alpha h = E_n[I_\alpha h]$ on B . Then, observing Lemma 6.3, we have

$$\begin{aligned} \chi_B I_\alpha h &= \chi_B \sum_{k=0}^n (\beta_{k-1})^\alpha (E_k h - E_{k-1} h) \\ &= \chi_B \sum_{k=1}^{n-1} ((\beta_{k-1})^\alpha - (\beta_k)^\alpha) E_k h + \chi_B (\beta_{n-1})^\alpha E_n h \\ &\geq \chi_B ((\beta_\ell)^\alpha - (\beta_m)^\alpha) E_m h \\ &= \chi_B (P(B_\ell)^\alpha - P(B_m)^\alpha) \left(1 - \frac{P(B)}{P(B_m)}\right) \\ &\geq C_R \chi_B P(B)^\alpha, \end{aligned} \quad (6.5)$$

where C_R is a positive constant depending only on R . Hence, by Lemma 6.2, we have

$$C_R \chi_B P(B)^\alpha E_n |b - E_n b| \leq \chi_B I_\alpha h E_n |b - E_n b| = \chi_B E_n |[b, I_\alpha]h - E_n([b, I_\alpha]h)|.$$

This is the conclusion. $\square$

We state one more definition:

Definition 6.1. Let $n \geq 0$. For $B \in \mathcal{F}_n$, take $B_k \in A(\mathcal{F}_k)$ ($0 \leq k \leq n$) such that

$$B = B_n \subset B_{n-1} \subset \cdots \subset B_0,$$

and let N_B be the number of elements in the set $\{k : 1 \leq k \leq n, B_k \neq B_{k-1}\}$.

Proof of Theorem 3.4 (ii). We show that

$$\sup_{n \geq 0} \sup_{B' \in A(\mathcal{F}_{n+1})} \frac{1}{\phi(P(B'))P(B')} \int_{B'} |b - E_n b| dP \lesssim \|b\|_{\mathcal{L}_{1,\phi}^-(\mathcal{F}_0)} + \|[b, I_\alpha]\|_{L_\Phi \rightarrow L_\Psi}. \quad (6.6)$$

Then we have

$$\begin{aligned} & \sup_{n \geq 0} \sup_{B \in A(\mathcal{F}_n)} \frac{1}{\phi(P(B))P(B)} \int_B |b - E_{n-1}b| dP \\ & \leq \|b\|_{\mathcal{L}_{1,\phi}^-(\mathcal{F}_0)} + \sup_{n \geq 0} \sup_{B' \in A(\mathcal{F}_{n+1})} \frac{1}{\phi(P(B'))P(B')} \int_{B'} |b - E_nb| dP \\ & \lesssim \|b\|_{\mathcal{L}_{1,\phi}^-(\mathcal{F}_0)} + \|[b, I_\alpha]\|_{L_\Phi \rightarrow L_\Psi}. \end{aligned}$$

This is the conclusion.

Now, let $n \geq 0$ and $B' \in A(\mathcal{F}_{n+1})$. Take $B \in A(\mathcal{F}_n)$ such that $B' \subset B$. Let $B_k \in A(\mathcal{F}_k)$ ($0 \leq k \leq n$) such that

$$B' = B_{n+1} \subset B = B_n \subset B_{n-1} \subset \cdots \subset B_0. \quad (6.7)$$

Let N_B be the number as in Definition 6.1.

Case 1: $N_B = 0$. In this case, $B = B_0 \in A(\mathcal{F}_0)$. Then $P(B') \leq P(B_0) \leq RP(B')$ by Lemma 6.3. Using the regularity condition on $\{\mathcal{F}_n\}_{n \geq 0}$ and the doubling condition on ϕ , we have

$$\begin{aligned} \chi_B E_{n+1} |b - E_nb| & \leq R\chi_B E_n |b - E_nb| = R\chi_B E_0 |b - E_0b| \\ & \leq 2R\chi_B E_0 |b| \leq 2R\phi(P(B_0)) \|b\|_{\mathcal{L}_{1,\phi}^-(\mathcal{F}_0)} \\ & \sim \phi(P(B')) \|b\|_{\mathcal{L}_{1,\phi}^-(\mathcal{F}_0)}. \end{aligned}$$

Case 2: $N_B = 1$. In this case, there exists an integer k such that $1 \leq k \leq n$ and that $B = B_k \subsetneq B_{k-1} = B_0$. Then $P(B') \leq P(B_0) \leq RP(B) \leq R^2P(B')$ by Lemma 6.3, and we have

$$\begin{aligned} \chi_B E_{n+1} |b - E_nb| & \leq R\chi_B E_k |b - E_kb| \leq 2R\chi_B E_k |b| \leq 2R^2\chi_B E_{k-1} |b| \\ & = 2R^2\chi_B E_0 |b| \leq 2R^2\phi(P(B_0)) \|b\|_{\mathcal{L}_{1,\phi}^-(\mathcal{F}_0)} \\ & \sim \phi(P(B')) \|b\|_{\mathcal{L}_{1,\phi}^-(\mathcal{F}_0)}. \end{aligned}$$

Case 3: $N_B \geq 2$. In this case, we can apply Lemma 6.4. Let h be as in Lemma 6.4. Then, using the boundedness of $[b, I_\alpha]$ from L_Φ to L_Ψ and $\|h\|_{L_\Phi} \lesssim \frac{1}{\Phi^{-1}\left(\frac{1}{P(B)}\right)}$, we have

$$\begin{aligned} \chi_B E_n |[b, I_\alpha]h - E_n([b, I_\alpha]h)| & \leq 2\chi_B E_n |[b, I_\alpha]h| \\ & \leq 4\chi_B \frac{1}{P(B)} \|\chi_B\|_{L_{\tilde{\Psi}}} \|\chi_B [b, I_\alpha]h\|_{L_\Psi} \\ & \leq 4\chi_B \Psi^{-1}\left(\frac{1}{P(B)}\right) \|h\|_{L_\Phi} \|[b, I_\alpha]\|_{L_\Phi \rightarrow L_\Psi} \\ & \lesssim \Psi^{-1}\left(\frac{1}{P(B)}\right) \frac{1}{\Phi^{-1}\left(\frac{1}{P(B)}\right)} \|[b, I_\alpha]\|_{L_\Phi \rightarrow L_\Psi} \\ & \lesssim \phi(P(B))P(B)^\alpha \|[b, I_\alpha]\|_{L_\Phi \rightarrow L_\Psi}. \end{aligned}$$

Combining this and Lemma 6.4, we have

$$\chi_B E_{n+1} |b - E_nb| \leq R\chi_B E_n |b - E_nb| \lesssim \phi(P(B')) \|[b, I_\alpha]\|_{L_\Phi \rightarrow L_\Psi}.$$

Thus, we have (6.6) and the conclusion. $\square$

Proof of Theorem 3.5 (ii). We show that

$$\sup_{n \geq 0} \sup_{B' \in A(\mathcal{F}_{n+1})} \frac{1}{\phi(P(B'))P(B')} \int_{B'} |b - E_n b| dP \lesssim \|b\|_{\mathcal{L}_{1,\phi}^-(\mathcal{F}_0)} + \|[b, I_\alpha]\|_{L_\Phi \rightarrow F_{L_\Psi}^{\phi-}}. \quad (6.8)$$

Let $n \geq 0$ and $B' = B_{n+1} \in A(\mathcal{F}_{n+1})$. We use (6.7). If $N_B = 0$ or $N_B = 1$, then in the same way as the proof of Theorem 3.4 (ii), we have

$$\chi_B E_{n+1} |b - E_n b| \lesssim \phi(P(B')) \|b\|_{\mathcal{L}_{1,\phi}^-(\mathcal{F}_0)}.$$

For the case $N_B \geq 2$, let h be as in Lemma 6.4. Then, using the doubling condition on ϕ , the boundedness of $[b, I_\alpha]$ from L_Φ to $F_{L_\Psi}^{\phi-}$ and $\|h\|_{L_\Phi} \lesssim \frac{1}{\Phi^{-1}(\frac{1}{P(B)})}$, we have

$$\begin{aligned} \chi_B E_n |[b, I_\alpha]h - E_n([b, I_\alpha]h)| &\leq 2\chi_B \frac{\phi(P(B_{n-1}))}{\phi(\beta_{n-1})} E_n |[b, I_\alpha]h - E_{n-1}([b, I_\alpha]h)| \\ &\lesssim \phi(P(B)) \operatorname{ess\,inf}_{\omega \in B} M_{\phi-}^\#([b, I_\alpha]h) \\ &\leq \frac{\phi(P(B))}{P(B)} \int_B M_{\phi-}^\#([b, I_\alpha]h) dP \\ &\lesssim \frac{\phi(P(B))}{P(B)} \|\chi_B\|_{L_\Psi} \|M_{\phi-}^\#([b, I_\alpha]h)\|_{L_\Psi} \\ &\lesssim \phi(P(B)) \Psi^{-1}\left(\frac{1}{P(B)}\right) \|h\|_{L_\Phi} \|[b, I_\alpha]\|_{L_\Phi \rightarrow F_{L_\Psi}^{\phi-}} \\ &\lesssim \phi(P(B)) P(B)^\alpha \|[b, I_\alpha]\|_{L_\Phi \rightarrow F_{L_\Psi}^{\phi-}}. \end{aligned}$$

Combining this and Lemma 6.4, we have

$$\chi_B E_{n+1} |b - E_n b| \leq R \chi_B E_n |b - E_n b| \lesssim \phi(P(B')) \|[b, I_\alpha]\|_{L_\Phi \rightarrow F_{L_\Psi}^{\phi-}}.$$

Thus, we have (6.8) and the conclusion. $\square$

7. Proofs of Theorems 3.7 and 3.8

First we show a lemma to prove Theorem 3.7 (i) and Theorem 3.8 (i).

Lemma 7.1. *Let $f \in \mathcal{L}_{1,\phi}^-$. Then, f belongs to $\mathcal{W}_{1,\phi}^-$ if and only if*

$$\lim_{n \rightarrow \infty} \|f - E_n f\|_{\mathcal{L}_{1,\phi}^-} = 0. \quad (7.1)$$

Proof. For $B \in A(\mathcal{F}_m)$, if $m - 1 \geq n$, then

$$\frac{1}{P(B)} \int_B |E_n f - E_{m-1}(E_n f)| dP = 0$$

since $E_{m-1} E_n f = E_n f$. If $m \leq n$, then

$$\begin{aligned} \frac{1}{P(B)} \int_B |E_n f - E_{m-1}(E_n f)| dP &= \frac{1}{P(B)} \int_B |E_n f - E_n(E_{m-1} f)| dP \\ &\leq \frac{1}{P(B)} \int_B E_n |f - E_{m-1} f| dP \\ &= \frac{1}{P(B)} \int_B |f - E_{m-1} f| dP \\ &\leq \phi(P(B)) \|f\|_{\mathcal{L}_{1,\phi}^-}. \end{aligned}$$

Then,

$$\|E_n f\|_{\mathcal{L}_{1,\phi}^-} \leq \|f\|_{\mathcal{L}_{1,\phi}^-} \quad (7.2)$$

for all $n \geq 0$. Hence, we have $E_n f \in \mathcal{L}_{1,\phi}^- \cap L$. Therefore, (7.1) implies $f \in \mathcal{W}_{1,\phi}^-$.

For the converse, let $f \in \mathcal{W}_{1,\phi}^-$. Then, for any $\epsilon > 0$, there exists $g \in \mathcal{L}_{1,\phi}^-$ such that g is $\mathcal{F}_{n_0}$ -measurable for some $n_0 \geq 0$ and that $\|f - g\|_{\mathcal{L}_{1,\phi}^-} < \epsilon$. In this case, if $n \geq n_0$, then

$$\|f - E_n f\|_{\mathcal{L}_{1,\phi}^-} \leq \|f - g\|_{\mathcal{L}_{1,\phi}^-} + \|E_n(g - f)\|_{\mathcal{L}_{1,\phi}^-} \leq 2\|f - g\|_{\mathcal{L}_{1,\phi}^-} \leq 2\epsilon.$$

This shows the conclusion. $\square$

Since $\mathcal{C}_{1,\phi}^- = \overline{\mathcal{L}_{1,\phi}^- \cap L \cap L_C} = \overline{\mathcal{W}_{1,\phi}^- \cap L_C}$, we have the following corollary:

Corollary 7.2. *Let $f \in \mathcal{L}_{1,\phi}^-$. Then, f belongs to $\mathcal{C}_{1,\phi}^-$ if and only if*

$$\lim_{n \rightarrow \infty} \|f - \chi_{D_n} E_n f\|_{\mathcal{L}_{1,\phi}^-} = 0. \quad (7.3)$$

We now prove Theorem 3.7 (i) and Theorem 3.8 (i).

Proof of Theorem 3.7 (i). Let b be in $\mathcal{L}_{1,\phi}^-$. We show that, if $b \in \mathcal{C}_{1,\phi}^-$, then $[b, I_\gamma]$ is compact from L_Φ to L_Ψ . Note that $b \in \mathcal{C}_{1,\phi}^-$ implies that b satisfies (3.29). From

$$[\chi_{D_n} E_n b, I_\gamma] f = [\chi_{D_n} E_n b, I_\gamma^{(n)} + E_n I_\gamma] f = \chi_{D_n} E_n ([E_n b, I_\gamma] f),$$

we obtain that the range of the commutator $[\chi_{D_n} E_n b, I_\gamma]$ is finite dimensional, since the number of the elements in $\{B \in A(\mathcal{F}_n) : B \subset D_n\}$ is finite for each $n \geq 0$ by the regularity of $\{\mathcal{F}_n\}_{n \geq 0}$. Besides, by Theorem 3.4,

$$\|[b, I_\gamma] - [\chi_{D_n} E_n b, I_\gamma]\|_{L_\Phi \rightarrow L_\Psi} \leq C \|b - \chi_{D_n} E_n b\|_{\mathcal{L}_{1,\phi}^-}.$$

Combining this with Corollary 7.2, we obtain

$$\lim_{n \rightarrow \infty} \|[b, I_\gamma] - [\chi_{D_n} E_n b, I_\gamma]\|_{L_\Phi \rightarrow L_\Psi} = 0.$$

Thus, we have the compactness of $[b, I_\gamma]$ from L_Φ to L_Ψ . $\square$

Proof of Theorem 3.8 (i). Let b be in $\mathcal{L}_{1,\phi}^-$. We show that, if $b \in \mathcal{C}_{1,\phi}^-$, then $[b, I_\gamma]$ is compact from L_Φ to $F_{L_\Psi}^\phi$. In the same way as the proof of Theorem 3.7 (i), we obtain that the range of the commutator $[\chi_{D_n} E_n b, I_\gamma]$ is finite dimensional. Besides, by Theorem 3.5,

$$\|[b, I_\gamma] - [\chi_{D_n} E_n b, I_\gamma]\|_{L_\Phi \rightarrow F_{L_\Psi}^{\phi-}} \leq C \|b - \chi_{D_n} E_n b\|_{\mathcal{L}_{1,\phi}^-}.$$

Combining this with Corollary 7.2, we obtain

$$\lim_{n \rightarrow \infty} \|[b, I_\gamma] - [\chi_{D_n} E_n b, I_\gamma]\|_{L_\Phi \rightarrow F_{L_\Psi}^{\phi-}} = 0.$$

Thus, we have the compactness of $[b, I_\gamma]$ from L_Φ to $F_{L_\Psi}^{\phi-}$. $\square$

To prove Theorem 3.7 (ii) and Theorem 3.8 (ii) we show more three lemmas.

Lemma 7.3. Assume that $\{\mathcal{F}_n\}_{n \geq 0}$ is regular and that ϕ satisfies the doubling condition. Let $f \in \mathcal{L}_{1,\phi}^-(\mathcal{F}_0)$. Then

$$\|E_n f\|_{\mathcal{L}_{1,\phi}^-} \leq 2(RC_{\phi,R})^n \|f\|_{\mathcal{L}_{1,\phi}^-(\mathcal{F}_0)}, \quad (7.4)$$

where $C_{\phi,R}$ is a positive constant which satisfies $\phi(Rt) \leq C_{\phi,R} \phi(t)$ for all $0 < t \leq 1$.

Proof. Let $g = E_n f$. Then $g - E_{k-1}g = 0$ for $k \geq n+1$. For each $B \in A(\mathcal{F}_k)$, $1 \leq k \leq n$,

$$\chi_B E_k |g - E_{k-1}g| \leq R \chi_B E_{k-1} |g - E_{k-1}g| \leq 2R \chi_B E_{k-1} |g| \leq 2R^k \chi_B E_0 |g|.$$

Take $B_0 \in A(\mathcal{F}_0)$ such that $B \subset B_0$. Then $P(B_0) \leq R^k P(B)$ by Lemma 6.3, and we have

$$\chi_B E_k |g - E_{k-1}g| \leq 2R^k \phi(P(B_0)) \|g\|_{\mathcal{L}_{1,\phi}^-(\mathcal{F}_0)} \leq 2(RC_{\phi,R})^k \phi(P(B)) \|g\|_{\mathcal{L}_{1,\phi}^-(\mathcal{F}_0)},$$

which shows

$$\begin{aligned} \|g\|_{\mathcal{L}_{1,\phi}^-} &= \sup_{0 \leq k \leq n} \sup_{B \in A(\mathcal{F}_k)} \frac{1}{\phi(P(B))P(B)} \int_B |g - E_{k-1}g| dP \\ &\leq 2(RC_{\phi,R})^n \|f\|_{\mathcal{L}_{1,\phi}^-(\mathcal{F}_0)}. \end{aligned}$$

This is the conclusion. $\square$

Lemma 7.4. Let $f \in \mathcal{L}_{1,\phi}^-$. Assume that $\{\mathcal{F}_n\}_{n \geq 0}$ is regular, that ϕ satisfies the doubling condition, and that f satisfies (3.29). Then, f belongs to $\mathcal{C}_{1,\phi}^-$ if and only if f belongs to $\mathcal{W}_{1,\phi}^-$.

Proof. If $f \in \mathcal{C}_{1,\phi}^-$, then $f \in \mathcal{W}_{1,\phi}^-$ by the definition. Conversely, let $f \in \mathcal{W}_{1,\phi}^-$. Then, by Lemma 7.1, for any $\epsilon > 0$, there exists $n \geq 0$ such that $\|f - E_n f\|_{\mathcal{L}_{1,\phi}^-(\mathcal{F}_0)} < \epsilon$. By the assumption (3.29), there exists $k_0 \geq 0$ such that, if $k \geq k_0$, then $\|f - \chi_{D_k} f\|_{\mathcal{L}_{1,\phi}^-(\mathcal{F}_0)} < \epsilon / (2(RC_{\phi,R})^n)$. Hence, $\|E_n(f - \chi_{D_k} f)\|_{\mathcal{L}_{1,\phi}^-} < \epsilon$ by Lemma 7.3. In this case,

$$\begin{aligned} \|f - \chi_{D_k} f\|_{\mathcal{L}_{1,\phi}^-} &\leq \|f - E_n f\|_{\mathcal{L}_{1,\phi}^-} + \|E_n f - \chi_{D_k} E_n f\|_{\mathcal{L}_{1,\phi}^-} + \|\chi_{D_k}(E_n f - f)\|_{\mathcal{L}_{1,\phi}^-} \\ &\leq \|f - E_n f\|_{\mathcal{L}_{1,\phi}^-} + \|E_n(f - \chi_{D_k} f)\|_{\mathcal{L}_{1,\phi}^-} + \|f - E_n f\|_{\mathcal{L}_{1,\phi}^-} < 3\epsilon. \end{aligned}$$

This show that $f \in \mathcal{C}_{1,\phi}^-$. Therefore, we have the desired conclusion. $\square$

Lemma 7.5. Assume that $\{\mathcal{F}_n\}_{n \geq 0}$ is regular and that ϕ satisfies the doubling condition. Let $C_{\phi,R}$ be the same constant in Lemma 7.3. Let b be in $\mathcal{L}_{1,\phi}^-$ and satisfy (3.29) with $\|b\|_{\mathcal{L}_{1,\phi}^-} = 1$. For a positive constant c , choose an integer J such that

$$\|b - b\chi_{D_J}\|_{\mathcal{L}_{1,\phi}^-(\mathcal{F}_0)} \leq \frac{c}{8R^2C_{\phi,R}}.$$

If $B \in A(\mathcal{F}_n)$ satisfies

$$P(B) \leq R^{-2} \min\{P(A_j) : A_j \in A(\mathcal{F}_0), 1 \leq j \leq J\} \quad (7.5)$$

and

$$\int_B |b - E_nb| dP \geq cP(B)\phi(P(B))/2, \quad (7.6)$$

then $N_B \geq 2$, where N_B is the number as in Definition 6.1.

Proof. If $B \subset D_J$, then we see that $N_B \geq 2$ by (7.5) and the regularity on $\{\mathcal{F}_n\}_{n \geq 0}$. If $B \subset \Omega \setminus D_J$, we also conclude that $N_B \geq 2$. Indeed, if $N_B \leq 1$, using a similar way to the proof of Theorem 3.4 (ii), we have

$$\begin{aligned} \int_B |b - E_nb| dP &= \int_B \chi_{\Omega \setminus D_J} |b - E_nb| dP \\ &\leq 2R^2C_{\phi,R}P(B)\phi(P(B))\|b - b\chi_{D_J}\|_{\mathcal{L}_{1,\phi}^-(\mathcal{F}_0)} \leq cP(B)\phi(P(B))/4, \end{aligned}$$

which contradict (7.6). $\square$

Further, let $\mathcal{L}_{p,\phi}$ be the generalized Campanato space equipped with $\|f\|_{\mathcal{L}_{p,\phi}}$ defined by (5.3). We recall the boundedness of I_ρ :

Theorem 7.6 ([18, Theorem 11 and Corollary 17]). Assume that $\{\mathcal{F}_n\}_{n \geq 0}$ is regular. Let $p \in (1, \infty)$ and $\rho, \phi, \psi : (0, 1] \mapsto (0, \infty)$. Suppose that

$$\sup_{0 < t \leq 1} \frac{\phi(t)}{\psi(t)} \int_0^t \frac{\rho(s)}{s} ds < \infty. \quad (7.7)$$

Then I_ρ is bounded from $\mathcal{L}_{p,\phi}$ to $\mathcal{L}_{p,\psi}$.

Remark 7.1. (i) Let $p \in (1, \infty)$. By Hölder's inequality we have $\|I_\rho f\|_{\mathcal{L}_{1,\psi}} \leq \|I_\rho f\|_{\mathcal{L}_{p,\psi}}$. Assume that $\{\mathcal{F}_n\}_{n \geq 0}$ is regular. If ϕ is almost increasing and satisfies the doubling condition, then $\|f\|_{\mathcal{L}_{p,\phi}} \leq C_p\|f\|_{\mathcal{L}_{1,\phi}}$. Moreover, from $E_n|f - E_nf| \leq 2E_n|f - E_{n-1}f|$ it follows that $\|f\|_{\mathcal{L}_{1,\phi}} \leq 2\|f\|_{\mathcal{L}_{1,\phi}^-}$. Then, under the assumption (7.7), I_ρ is bounded from $\mathcal{L}_{1,\phi}^-$ to $\mathcal{L}_{1,\psi}$.

(ii) Assume that $\{\mathcal{F}_n\}_{n \geq 0}$ is regular and ϕ satisfies the doubling condition. If $E_0f = 0$, then $\|f\|_{\mathcal{L}_{p,\phi}^-} \leq C\|f\|_{\mathcal{L}_{p,\phi}}$, where C is independent of f , see [13, Lemma 5.1].

We now prove Theorem 3.7 (ii) and Theorem 3.8 (ii).

Proof of Theorem 3.7 (ii). Let $\|b\|_{\mathcal{L}_{1,\phi}^-} = 1$. We may assume that ϕ is increasing. We show that, if $b \notin \mathcal{L}_{1,\phi}^-$, then $[b, I_\alpha]$ is not compact from L_Φ to L_Ψ , which contradicts the assumption.

We first note that we can find a Young function Θ such that I_α is bounded from L_Φ to L_Θ as in Example 3.2. In this case $\Theta^{-1}(1/r) \sim r^\alpha \Phi^{-1}(1/r)$ and $\Theta^{-1}(u) \rightarrow \infty$ as $u \rightarrow \infty$. We also note that I_α is bounded from $\mathcal{L}_{1,\phi}^-$ to $\mathcal{L}_{1,\psi}$ with $\psi(r) = r^\alpha \phi(r)$ by Remark 7.1 (i). We denote their operator norms by $\|I_\alpha\|_{L_\Phi \rightarrow L_\Theta}$ and $\|I_\alpha\|_{\mathcal{L}_{1,\phi}^- \rightarrow \mathcal{L}_{1,\psi}}$, respectively.

Now, by Lemma 7.4, $b \notin \mathcal{C}_{1,\phi}^-$ implies $b \notin \mathcal{W}_{1,\phi}^-$. Then $\limsup_{k \rightarrow \infty} \|b - E_k b\|_{\mathcal{L}_{1,\phi}^-} > 0$ by Lemma 7.1. Observing Remark 7.1 (ii), we see that $c := \limsup_{k \rightarrow \infty} \|b - E_k b\|_{\mathcal{L}_{1,\phi}} > 0$. Hence, the set

$$N(c) = \left\{ n \geq 0 : \frac{1}{P(B)\phi(P(B))} \int_B |b - E_n b| dP \geq \frac{c}{2} \text{ for some } B \in A(\mathcal{F}_n) \right\}$$

is infinite. By the fact $\lim_{n \rightarrow \infty} \max\{P(B) : B \in A(\mathcal{F}_n)\} = 0$ and Lemma 7.5, we can choose an increasing sequence $\{n_j\}_{j \geq 1}$ in $N(c)$ and a sequence $\{B_{n_j}\}_{j \geq 1}$ of $\mathcal{F}_{n_j}$ -atoms, inductively in the following manner. Let $n_1 = \min N(c)$ and let $B_{n_1} \in A(\mathcal{F}_{n_1})$ satisfy

$$\int_{B_{n_1}} |b - E_{n_1} b| dP \geq cP(B_{n_1})\phi(P(B_{n_1}))/2.$$

Next, we choose $n_2 < n_3 < \dots$ and $B_{n_j} \in A(\mathcal{F}_{n_j})$, $j = 2, 3, \dots$, satisfying

$$\int_{B_{n_j}} |b - E_{n_j} b| dP \geq cP(B_{n_j})\phi(P(B_{n_j}))/2. \quad (7.8)$$

$$P(B_{n_j}) < P(B_{n_{j-1}}), \quad (7.9)$$

$$\frac{C_R}{4R} P(B_{n_j})^\alpha \Phi^{-1}\left(\frac{1}{P(B_{n_j})}\right) > \Theta^{-1}\left(\frac{1}{P(B_{n_{j-1}})}\right) \|I_\alpha\|_{L_\Phi \rightarrow L_\Theta}, \quad (7.10)$$

$$\frac{cC_R}{8} \Phi^{-1}\left(\frac{1}{P(B_{n_j})}\right) > \Phi^{-1}\left(\frac{1}{P(B_{n_{j-1}})}\right) \|I_\alpha\|_{\mathcal{L}_{1,\phi}^- \rightarrow \mathcal{L}_{1,\psi}}, \quad (7.11)$$

$$N_{B_{n_j}} \geq 2, \quad (7.12)$$

where C_R is the constant in Lemma 6.4, $N_{B_{n_j}}$ is the constant as in Definition 6.1. Note that we do not need to choose as $B_{n_j} \subset B_{n_{j-1}}$.

For each n_j above, we define an integer m_j by

$$m_j = \max\{m : m < n_j \text{ and } \exists B \in A(\mathcal{F}_m) \text{ such that } B_{n_j} \subset B, P(B_{n_j}) \neq P(B)\},$$

and let

$$h_j = \Phi^{-1}\left(\frac{1}{P(B_{n_j})}\right) (\chi_{B_{m_j}} - \chi_{B_{n_j}}), \quad B_{m_j} \in A(\mathcal{F}_{m_j}), B_{n_j} \subset B_{m_j}.$$

By the regularity of $\{\mathcal{F}_n\}_{n \geq 0}$, it is easy to see that

$$\|h_j\|_{L_\Phi} \leq R \quad \text{for all } j \geq 1. \quad (7.13)$$

Therefore, to prove that $[b, I_\alpha]$ is not compact, we only have to show

$$\inf_{j \neq k} \|[b, I_\alpha](h_j - h_k)\|_{L_\Psi} > 0. \quad (7.14)$$

In the following we show (7.14). Let $k < j$. Then $h_j - h_k$ and $I_\alpha(h_j - h_k)$ are $\mathcal{F}_{n_j}$ -measurable. Hence

$$\begin{aligned} & \chi_{B_{n_j}} E_{n_j} |[b, I_\alpha](h_j - h_k) - E_{n_j} [[b, I_\alpha](h_j - h_k)]| \\ &= \chi_{B_{n_j}} E_{n_j} [|b I_\alpha(h_j - h_k) - E_{n_j}[b] I_\alpha(h_j - h_k) \\ &\quad - \{I_\alpha[b(h_j - h_k)] - E_{n_j}(I_\alpha[b(h_j - h_k)])\}|] \\ &\geq \chi_{B_{n_j}} |I_\alpha(h_j - h_k)| E_{n_j} |b - E_{n_j} b| - \chi_{B_{n_j}} |h_j - h_k| E_{n_j} |I_\alpha(b) - E_{n_j}(I_\alpha(b))| \\ &= J_1 - J_2, \quad \text{say.} \end{aligned} \quad (7.15)$$

In the above we use the property of (5.7). In the same way as (6.5) in the proof of Lemma 6.4, we have

$$\chi_{B_{n_j}} |I_\alpha h_j| \geq \chi_{B_{n_j}} C_R P(B_{n_j})^\alpha \Phi^{-1} \left(\frac{1}{P(B_{n_j})} \right). \quad (7.16)$$

Further, by (7.13) and (7.10)

$$\begin{aligned} \chi_{B_{n_j}} |I_\alpha h_k| &= \chi_{B_{n_j}} \frac{1}{P(B_{n_k})} \int_{B_{n_k}} |I_\alpha h_k| dP \\ &\leq 2 \chi_{B_{n_j}} \frac{1}{P(B_{n_k})} \|\chi_{B_{n_k}}\|_{L_{\Theta}} \|I_\alpha h_k\|_{L_{\Theta}} \\ &\leq 2R \chi_{B_{n_j}} \Theta^{-1} \left(\frac{1}{P(B_{n_k})} \right) \|I_\alpha\|_{L_{\Phi} \rightarrow L_{\Theta}} \\ &\leq \frac{C_R}{2} \chi_{B_{n_j}} P(B_{n_j})^\alpha \Phi^{-1} \left(\frac{1}{P(B_{n_j})} \right). \end{aligned} \quad (7.17)$$

Then (7.16) and (7.17) imply

$$\chi_{B_{n_j}} |I_\alpha(h_j - h_k)| \geq \frac{C_R}{2} \chi_{B_{n_j}} P(B_{n_j})^\alpha \Phi^{-1} \left(\frac{1}{P(B_{n_j})} \right).$$

Combining this with (7.8), we have

$$J_1 \geq \chi_{B_{n_j}} \frac{cC_R}{4} P(B_{n_j})^\alpha \phi(P(B_{n_j})) \Phi^{-1} \left(\frac{1}{P(B_{n_j})} \right). \quad (7.18)$$

Besides, by (7.11) we have

$$\begin{aligned} J_2 &= \chi_{B_{n_j}} |h_k| E_{n_j} |I_\alpha(b) - E_{n_j}(I_\alpha(b))| \\ &\leq \chi_{B_{n_j}} \Phi^{-1} \left(\frac{1}{P(B_{n_k})} \right) \psi(P(B_{n_j})) \|I_\alpha(b)\|_{\mathcal{L}_{1,\psi}} \\ &\leq \chi_{B_{n_j}} \Phi^{-1} \left(\frac{1}{P(B_{n_k})} \right) \psi(P(B_{n_j})) \|I_\alpha\|_{\mathcal{L}_{1,\phi}^- \rightarrow \mathcal{L}_{1,\psi}} \\ &\leq \chi_{B_{n_j}} \frac{cC_R}{8} \Phi^{-1} \left(\frac{1}{P(B_{n_j})} \right) P(B_{n_j})^\alpha \phi(P(B_{n_j})). \end{aligned} \quad (7.19)$$

Thus, combining (7.15), (7.18) and (7.19), we have

$$\begin{aligned} & \chi_{B_{n_j}} E_{n_j} |[b, I_\alpha](h_j - h_k) - E_{n_j} [[b, I_\alpha](h_j - h_k)]| \\ & \geq J_1 - J_2 \geq \frac{cC_R}{8} \chi_{B_{n_j}} P(B_{n_j})^\alpha \phi(P(B_{n_j})) \Phi^{-1} \left(\frac{1}{P(B_{n_j})} \right). \end{aligned} \quad (7.20)$$

By the assumption (3.22), we see that

$$\chi_{B_{n_j}} E_{n_j} |[b, I_\alpha](h_j - h_k) - E_{n_j} [[b, I_\alpha](h_j - h_k)]| \geq \frac{cC_R}{8A} \Psi^{-1} \left(\frac{1}{P(B_{n_j})} \right) \quad \text{on } B_{n_j},$$

which shows

$$E_{n_j} |[b, I_\alpha](h_k - h_j)| \geq \frac{cC_R}{16A} \Psi^{-1} \left(\frac{1}{P(B_{n_j})} \right) \quad \text{on } B_{n_j}.$$

By Jensen's inequality we have

$$E_{n_j} [\Psi(|[b, I_\alpha](h_k - h_j)|/C)] \geq \Psi(E_{n_j} |[b, I_\alpha](h_k - h_j)|/C) \geq \frac{1}{P(B_{n_j})} \quad \text{on } B_{n_j},$$

where $C = cC_R/(16A)$, that is

$$\|[b, I_\alpha](h_j - h_k)\|_{L_\Psi} \geq C.$$

Therefore, we have (7.14). The proof is complete. $\square$

Proof of Theorem 3.8 (ii). Let $\|b\|_{\mathcal{L}_{1,\phi}^-} = 1$. We show that, if $b \notin C_{1,\phi}^-$, then $[b, I_\alpha]$ is not compact from L_Φ to $F_{L_\Psi}^{\phi^-}$, which contradicts the assumption.

We take the same sequence $\{h_j\}_{j \geq 1}$ in the proof of Theorem 3.7 (ii). Then, using (7.20) and the assumption (3.28), we obtain that, for $k < j$,

$$\begin{aligned} & \chi_{B_{n_j}} E_{n_j} |[b, I_\alpha](h_j - h_k) - E_{n_j} [[b, I_\alpha](h_j - h_k)]| \\ & \geq \frac{cC_R}{8A} \chi_{B_{n_j}} \Psi^{-1} \left(\frac{1}{P(B_{n_j})} \right) \phi(P(B_{n_j})), \end{aligned}$$

which shows

$$\chi_{B_{n_j}} M_\phi^\sharp([b, I_\alpha](h_j - h_k)) \geq \frac{cC_R}{8A} \chi_{B_{n_j}} \Psi^{-1} \left(\frac{1}{P(B_{n_j})} \right).$$

Since $\phi(\beta_n) \sim \phi(\beta_{n-1})$ by the regularity of $\{\mathcal{F}_n\}_{n \geq 0}$ and by the doubling condition on ϕ , we see that

$$\|[b, I_\alpha](h_j - h_k)\|_{F_{L_\Psi}^{\phi^-}} \geq C.$$

This shows the conclusion. $\square$

Acknowledgment

The authors would like to thank the referee for her/his careful reading and useful comments. The second author was supported by Grant-in-Aid for Scientific Research (B), No. 15H03621, Japan Society for the Promotion of Science. The third author was supported by Grant-in-Aid for Scientific Research (C), No. 19K03543, Japan Society for the Promotion of Science. This work was also supported by the Research Institute for Mathematical Sciences, Kyoto University.

References

- [1] D.L. Burkholder, Martingale transforms, *Ann. Math. Stat.* 37 (1966) 1494–1504.
- [2] S. Chanillo, A note on commutators, *Indiana Univ. Math. J.* 31 (1) (1982) 7–16.
- [3] J.-A. Chao, H. Ombe, Commutators on dyadic martingales, *Proc. Jpn. Acad., Ser. A* 61 (1985) 35–38.
- [4] F. Deringoz, V.S. Guliyev, E. Nakai, Y. Sawano, M. Shi, Generalized fractional maximal and integral operators on Orlicz and generalized Orlicz-Morrey spaces of the third kind, *Positivity* 23 (3) (2019) 727–757, <https://doi.org/10.1007/s11117-018-0635-9>.
- [5] G. Di Fazio, M.A. Ragusa, Commutators and Morrey spaces, *Boll. Unione Mat. Ital., A* (7) 5 (3) (1991) 323–332.
- [6] X. Fu, D. Yang, W. Yuan, Generalized fractional integrals and their commutators over non-homogeneous metric measure spaces, *Taiwan. J. Math.* 18 (2) (2014) 509–557.
- [7] Z. Hao, Y. Jiao, Fractional integral on martingale Hardy spaces with variable exponents, *Fract. Calc. Appl. Anal.* 18 (5) (2015) 1128–1145.
- [8] R. Kawasumi, E. Nakai, Pointwise multipliers on weak Orlicz spaces, to appear in *Hiroshima Math. J.*, arXiv:1811.02858v1.
- [9] M. Kikuchi, On weighted weak type maximal inequalities for martingales, *Math. Inequal. Appl.* 6 (1) (2003) 163–175.
- [10] M. Kikuchi, Uniform boundedness of conditional expectation operators on a Banach function space, *Math. Inequal. Appl.* 16 (2) (2013) 483–499.
- [11] V. Kokilashvili, M. Krbeć, *Weighted Inequalities in Lorentz and Orlicz Spaces*, World Scientific Publishing Co., Inc., River Edge, NJ, 1991.
- [12] R.L. Long, *Martingale Spaces and Inequalities*, Peking University Press, Beijing, 1993.
- [13] T. Miyamoto, E. Nakai, G. Sadasue, Martingale Orlicz-Hardy spaces, *Math. Nachr.* 285 (5–6) (2012) 670–686, <https://doi.org/10.1002/mana.201000109>.
- [14] E. Nakai, On generalized fractional integrals, *Taiwan. J. Math.* 5 (3) (2001) 587–602.
- [15] E. Nakai, G. Sadasue, Martingale Morrey-Campanato spaces and fractional integrals, *J. Funct. Spaces Appl.* 2012 (2012) 673929, <https://doi.org/10.1155/2012/673929>.
- [16] E. Nakai, G. Sadasue, Characterizations of boundedness for generalized fractional integrals on martingale Morrey spaces, *Math. Inequal. Appl.* 20 (4) (2017) 929–947, <https://doi.org/10.7153/mia-2017-20-58>.
- [17] E. Nakai, G. Sadasue, Commutators of fractional integrals on martingale Morrey spaces, *Math. Inequal. Appl.* 22 (2) (2019) 631–655, <https://doi.org/10.7153/mia-2019-22-44>.
- [18] E. Nakai, G. Sadasue, Y. Sawano, Martingale Morrey-Hardy and Campanato-Hardy spaces, *J. Funct. Spaces Appl.* 2013 (2013) 690258, <https://doi.org/10.1155/2013/690258>.
- [19] J. Neveu, *Discrete-Parameter Martingales*, North-Holland, Amsterdam, 1975.
- [20] R. O’Neil, Fractional integration in Orlicz spaces. I, *Trans. Am. Math. Soc.* 115 (1965) 300–328.
- [21] M. Paluszyński, Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss, *Indiana Univ. Math. J.* 44 (1) (1995) 1–17.
- [22] J. Peetre, On interpolation functions, II, *Acta Sci. Math. (Szeged)* 29 (1968) 91–92.
- [23] G. Sadasue, Fractional integrals on martingale Hardy spaces for $0 < p \leq 1$, *Mem. Osaka Kyoiku Univ. Ser. III Nat. Sci. Appl. Sci.* 60 (1) (2011) 1–7.
- [24] G. Sadasue, Martingale Besov spaces and martingale Triebel-Lizorkin spaces, *Sci. Math. Jpn.* 82 (1) (2019) 57–82.
- [25] M. Shi, R. Arai, E. Nakai, Generalized fractional integral operators and their commutators with functions in generalized Campanato spaces on Orlicz spaces, *Taiwan. J. Math.* 23 (6) (2019) 1339–1364, <https://doi.org/10.11650/tjm/181211>.
- [26] C. Watari, Multipliers for Walsh Fourier series, *Tohoku Math. J.* 16 (1964) 239–251.
- [27] F. Weisz, *Martingale Hardy Spaces and Their Applications in Fourier Analysis*, Lecture Notes in Mathematics, vol. 1568, Springer-Verlag, Berlin, 1994.