J. Math. Anal. Appl. 491 (2020) 124263

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Law of large numbers and central limit theorem for a class of pure jump Markov process

Mohsen Chebbi [a], Salwa Toumi [b,*]

[a] *Lamsin-Enit, Tunis El Manar University, Tunisia*
[b] *Lamsin-Enit, Tunis El Manar University, Insat, Carthage University, Tunisia*

ARTICLE INFO

Article history:
Received 9 September 2018
Available online 28 May 2020
Submitted by V. Pozdnyakov

Keywords:
Sacled processes
Pure jump Markov processes
LLN
TCL
Population dynamics
Interacting particles

ABSTRACT

This paper studies the law of large numbers (LLN) and central limit theorem (TCL) for a class of scaled pure jump Markov processes where the processes state variables are in $\mathbb{R}^d$ and where jump amplitudes depend on the state variables. Non-explosion property and semi-martingale decomposition are studied first for a class of stochastic processes, allowing to study in a second step these same properties for the scaled pure jump Markov processes we consider characterized by infinitely small jumps and rapid jumps rates. Then, asymptotic behavior is derived and convergence results are obtained.

1. Introduction

Many studies are interested in scaled pure jump Markov processes $(X_t^n)_{t\geq 0}$, in particular research works linked to applications with interacting particle systems like stochastic biochemical systems or population evolution. Specifically at a microscopic level, jump sizes ν^n are often considered to decrease proportionally to some scaling parameter n, and jump rates λ^n are considered to increase again proportionally to the same parameter. Different types of asymptotic behaviors are then studied and convergence results like the law of large numbers (LLN) and the central limit theorem (TCL) are derived for these scaled processes.

In this context, and to derive (LLN) or (TCL) results, the most of the considered models for scaled processes $(X_t^n)_{t\geq 0}$ dealing with real scenarios have fixed jump amplitudes - being independent of the process state itself - and such that the state variables are constrained to have values in a predefined grid [16–18,8, 15,1,2,13].

In this paper our aim is to prove these results for scaled jump Markov processes having values in $\mathbb{R}_+^d$, where the state variables are not constrained to be on a predefined grid since jump amplitudes $\nu^n(x)$ depend on the state variable x of the system and in a non-uniform way.

* Corresponding author.
E-mail addresses: mohsen.chebbi@enit.utm.tn (M. Chebbi), salwa.toumi@lamsin.rnu.tn (S. Toumi).

https://doi.org/10.1016/j.jmaa.2020.124263

More precisely, we allow to consider pure jump processes models where both frequency rates $\lambda^n(x)$ and jump amplitudes $\nu^n(x)$ depend on the state.

To situate our work, we remind first that when dealing with populations evolution, the state variable is usually taken to be the particles or the population density. Moreover, actually, in the usual mathematical models dealing with biochemical reactions, stoichiometric coefficients are considered to be constant and generally, for models dealing with population dynamics (logistic growth, epidemics), the jump amplitudes are assumed to be constant, independent of the state. However, the jump amplitudes may depend on the state for many natural systems. For example, in [7], the decline in twinning rate has been related to an increase in population density. In [23, Chapter3], it is also mentioned that effects of an increase in population density on reproduction are manifested through reductions in pregnancy, fecundity, twinning rate, number of offspring per female, depending on the species. So if propensity measures the probability of breeding event, the offspring may vary with the population density and then jump amplitude of the population size depends on the state.

Mathematically, the effects of state dependence in both frequency and amplitude have been examined together in terms of transition probability density functions (PDFs) as it was studied in [3]. They consider a model with stochastic Langevin equation resulting form the sum of a deterministic variation term and a pure jump process. Assuming state dependence of both the jump amplitude and frequency, they prove that the resulting jump process converges to a diffusion process under the limiting scenario of infinitely small jumps occurring infinitely often: the proof is achieved by considering the master equation which converges to Fokker-Planck equation.

As for our model, we study differently the asymptotic behavior of jump processes with state dependence for both propensity and amplitude, focusing on the convergence in law of the process itself and on the convergence speed. Such a model can allow researchers to think at infinitesimal scale first. In fact, to link microscopic behavior to macroscopic one, theorists often consider first the macroscopic deterministic equation (ODE), particularly the flow term $F(x)$, and then break it down into a product to deduce the propensity term and the jump amplitude, taking the later to be constant. Instead, when following our model, we can consider for example the case of alternative single/twining birth jumps as state/density-dependent jumps amplitudes $\nu^n(x)$ for a same reproduction event and then for a same propensity function $\lambda^n(x)$.

Another example which we can cite (and which is more detailed at the end of section 4 below) is the one studied in [27]. This example is dealing with biotechnologies Anaerobic model and the infinitesimal parameters are state dependent and verify:

$$\nu_j^n(x) \overset{\text{def}}{=} [x + \frac{1}{n}\tilde{\nu}_j]^+ - x, \qquad \lambda_j^n(x) \overset{\text{def}}{=} n\,\tilde{\lambda}_j(x) \tag{1}$$

for j being the index of the j-th reaction and $[x]^+$ denoting the orthogonal projection of x on $\mathbb{R}_+^d$.

In this paper, we derive (LLN) and (TCL) theorems using theoretical results as martingale functional limit theorems [8,28] and weak convergence theorems for stochastic integrals [19,20]. We mention that before dealing with asymptotic behavior, non-explosion and integrability results are given for a class of processes $(X_t)_{t\geq 0}$ we are considering in $\mathbb{R}_+^d$ and which will allow to study/define later the limit behavior of the considered scaled processes $(X_t^n)_{t\geq 0}$.

For this, in the first section, we consider a class of pure jump Markov processes $(X_t)_{t\geq 0}$ subject to J reactions (or events) and we establish non explosion property under adequate assumptions. For non-explosion property, being the feature to have finite number of jumps in any finite time interval, our setting is multi-dimensional in $\mathbb{R}_+^d$ when compared to the work of [14] where they consider jump-processes in the one-dimensional space $\mathbb{R}_+$.

For the second section, we study integrability and semi-martingale representation of the considered process $(X_t)_{t\geq 0}$ for any function with polynomial-growth Φ, when applied to the process. As for integrability property allowing the validity of what is known as Dynkin's formula, we remind that the result is essentially established in [8, p. 376] for bounded functions and in [10] for some general functions other than bounded ones, when the processes are taking values in $\mathbb{N}$ ($\mathbb{R}_+^d$ in our settings).

In the third section, we present our class of scaled process $(X_t^n)_{t\geq 0}$ with adequate assumptions on the asymptotic behavior of the sequence of the infinitesimal parameters of the jump sizes ν_j^n and jump rates λ_j^n for $j \in 1, ..., J$. Using results of section 2 and section 3, we deal then with LLN theorem where the convergence of the scaled process to some deterministic ODE is derived in L^2 and then in probability (the weak form LLN convergence). Finally, and in the last section, we deal with TCL theorem proof.

2. A general model

Consider a pure jump Markov process $(X_t)_{t\geq 0}$, taking values in $\mathbb{R}_+^d$, where X_t represents the size of a population of d species, or their concentrations, subject to J reactions: each reaction is represented by an intensity function $\lambda_j(x) \geq 0$ and a jump $\nu_j(x) \in \mathbb{R}^d$ ($j = 1, \ldots, J$). Suppose that the reactions are stochastically independent and that $x + \nu_j(x) \in \mathbb{R}_+^d$ for all $x \in \mathbb{R}_+^d$ and for all j such that $\lambda_j(x) > 0$.

The infinitesimal description of the process is as follows: conditionally to $X_t = x$:

$$X_{t+h} = \begin{cases} x + \nu_j(x)\,, & \text{with probability } \lambda_j(x)\,h + o(h) \text{ for } j = 1, \ldots, J\,, \\ x\,, & \text{with probability } 1 - \lambda(x)\,h + o(h)\,, \end{cases}$$

where

$$\lambda(x) \stackrel{\text{def}}{=} \sum_{j=1}^{J} \lambda_j(x)$$

or equivalently, $(X_t)_{t\geq 0}$ is a pure jump Markov process with the following infinitesimal generator:

$$\mathcal{L}\phi(x) \stackrel{\text{def}}{=} \sum_{j=1}^{J} \lambda_j(x)\,[\phi(x + \nu_j(x)) - \phi(x)] \tag{2}$$

for any bounded test function $\phi : \mathbb{R}_+^d \mapsto \mathbb{R}$. The infinitesimal generator can be rewritten:

$$\mathcal{L}\phi(x) = \lambda(x) \int_{\mathbb{R}_+^d} [\phi(y) - \phi(x)]\,\rho(x, \mathrm{d}y) \tag{3}$$

where the transition kernel $\rho(x, \mathrm{d}y)$ is:

$$\rho(x, \mathrm{d}y) \stackrel{\text{def}}{=} \sum_{j=1}^{J} p_j(x)\,\delta_{x+\nu_j(x)}(\mathrm{d}y)$$

with

$$p_j(x) \stackrel{\text{def}}{=} \lambda_j(x)/\lambda(x)\,.$$

2.1. Simulation and representation of the process

The simulation of trajectories $(X_t)_{t\geq 0}$ leads to a representation of the process. To simulate one trajectory of the process, let $T_0 = 0$ and $X_0 = Y_0 \sim \mu$ (the given initial distribution of X_0), and independently from the past and for all $k \geq 1$, simulate recursively:

(*i*) S_k according to an exponential distribution of parameter $\lambda(Y_{k-1})$ and let $T_k = S_k + T_{k-1}$,
(*ii*) Y_k according to $\rho(Y_{k-1}, \mathrm{d}y)$,

then let:

$$X_t = \begin{cases} Y_{k-1}\,, & \text{for } T_{k-1} < t < T_k\,, \\ Y_k\,, & \text{for } t = T_k\,. \end{cases}$$

$(Y_k)_{k\in\mathbb{N}}$ is the sequence of the states taken by the process $(X_t)_{t\geq 0}$, it is a Markov chain in itself with transition probability $\rho(x, \mathrm{d}y)$ and initial distribution μ, called the *embedded Markov chain* corresponding to the pure jump Markov process $(X_t)_{t\geq 0}$; $(T_k)_{k>0}$ is the sequence of the consecutive jump instants. Note that $Y_k = X_{T_k}$.

In (*i*) if $\lambda(Y_{k-1}) = 0$, then $S_k = +\infty$ and $X_t = Y_{k-1}$ for all $t \geq T_{k-1}$ and Y_{k-1} is an absorbing state. This trajectory is thus simulated until the random time $T_\infty \stackrel{\text{def}}{=} \lim_{k\to\infty} T_k$, called explosion time. In the following paragraph, we will verify that $T_\infty = \infty$ a.s., namely that the process is *regular*.

Let us now make the following assumptions, for the infinitesimal parameters of the processes we are considering

Hypotheses 2.1.

(*i*) *For all* $x \in \mathbb{R}^d_+$, *for all* j *such that* $\lambda_j(x) > 0$, $x + \nu_j(x) \in \mathbb{R}^d_+$.
(*ii*) $\lambda(x) \geq c_\lambda > 0\,,\ \forall x \in \mathbb{R}^d_+$.
(*iii*) $\lambda(x) \leq C_\lambda\,(1 + |x|)\,,\ \forall x \in \mathbb{R}^d_+$.
(*iv*) $|\nu_j(x)| \leq C_\nu\,,\ \forall x \in \mathbb{R}^d_+\,,\ j = 1,\ldots,J$.
(*v*) $x \to \lambda_j(x)\nu_j(x)$ *is Lipschitz continuous.*
(*vi*) $x \to \lambda_j(x)\nu_j(x)\nu_j(x)^*$ *is continuous.*

2.2. Non-explosion

Note that:

$$X_t = X_0 + \sum_{0<s\leq t} \Delta X_s \quad \text{where } \Delta X_s \stackrel{\text{def}}{=} X_s - X_{s^-}\,. \tag{4}$$

The sum here is a discrete sum over all the jump times until t and we will verify below that we have finite number of jumps in every finite time interval (non explosion property).

We introduce the total variation process [6, Remark 1.19, p. 319]:

$$\xi_t \stackrel{\text{def}}{=} |X_0| + \sum_{0<s\leq t} |\Delta X_s| \tag{5}$$

it is a scalar, real and positive process with the same jump times T_k as X_t, it increases of $|\Delta X_{T_k}|$ at each jump time T_k. We have $|X_t| \leq \xi_t$ but as ξ_t is non-decreasing, we also have:

$$\sup_{s\leq t} |X_s| \leq \xi_t\,,\ \forall t \geq 0 \tag{6}$$

Lemma 2.2. *Under Hypotheses 2.1-(iii) and (iv), if* $X_0 < \infty$ *a.s. then the process* X_t *is non-explosive, i.e.* $T_k \to \infty$ *a.s.*

Proof. The classical necessary and sufficient condition for non-explosion is (see [22, Proposition 15.43, section 7, chapter 15])

$$\sum_{k=1}^{\infty} \frac{1}{\lambda(X_{T_k})} = \infty \ a.s. \tag{7}$$

According to Hypothesis 2.1-(*iii*):

$$\lambda(X_{T_k}) \le C_\lambda\,(1 + |X_{T_k}|)$$

and according to (5), (6), and Hypothesis 2.1-(*iv*):

$$|X_{T_k}| \le \xi_{T_k} \le |X_0| + C_\nu\, k$$

that implies (7). □

We introduce the counting process N_t, the number of jumps of the process during the interval $[0,t]$: it starts from 0 and increase by one at the same jumping times T_k:

$$N_t = \sum_{k\ge 1} \mathbf{1}_{\{T_k \le t\}}\,.$$

In particular for any k, $N_{T_k} = k$.

Corollary 2.3. *If* $\mathbb{E}\,[|X_0|^p] < \infty$ *then:*

$$\mathbb{E}\,[\sup_{s\le t} |X_s|^p] < \infty\,,\ \forall t \ge 0\,,\ p \ge 1\,.$$

Proof. According to the proof of Lemma 2.2, for any $p \ge 1$:

$$\begin{aligned} \xi_t^p &= \Big(|X_0| + \sum_{0<s\le t} |\Delta X_s|\Big)^p \\ &\le 2^{p-1} \Big(|X_0|^p + \big(\sum_{0<s\le t} |\Delta X_s|\big)^p\Big) \\ &\le 2^{p-1} \Big(|X_0|^p + C_\nu^p\, N_t^p\Big)\,. \end{aligned}$$

The counting process N_t starts from 0 and conditionally to $N_t = k$, it will increase by one at a rate upper-bounded by $C_\lambda\,(1 + C_\nu\, k)$. Next let ζ_t be a pure jump Markov process taking values in $\mathbb{N}$, starting from 0 and which, conditionally to $\zeta_t = k$ will increase by one at rate $C_\lambda\,(1 + C_\nu\, k)$. Clearly ζ_t is a simple immigration-birth process with immigration rate C_λ and birth rate $C_\lambda\, C_\nu$; it is well known that ζ_t admits a negative binomial distribution with finite mean and variance [24, p. 86]. Finally ζ_t is *stochastically larger* than N_t [25, Ch. 9], i.e. $\mathbb{P}(\zeta_t > z) \ge \mathbb{P}(N_t > z)$, so $\mathbb{E}\,[N_t^p] \le \mathbb{E}\,[\zeta_t^p]$, which proves the Corollary. □

3. Semi-martingale representation of the process

Another formulation of the previous simulation leads to the following representation of the process:

$$X_t = X_0 + \sum_{j=1}^{J} \int_0^t \int_0^\infty \mathbf{1}_{[0,\lambda_j(X_{s^-})]}(u)\, \nu_j(X_{s^-})\, \mathcal{N}_j(\mathrm{d}s, \mathrm{d}u) \tag{8}$$

where $\mathcal{N}_j(\mathrm{d}s, \mathrm{d}u)$ are independent Poisson random measures of intensity measure $\mathrm{d}s \times \mathrm{d}u$, the Lebesgue measure on $\mathbb{R}_+^2$.

Let us introduce the compensated Poisson measure:

$$\tilde{\mathcal{N}_j}(\mathrm{d}s, \mathrm{d}u) \stackrel{\text{def}}{=} \mathcal{N}_j(\mathrm{d}s, \mathrm{d}u) - \mathrm{d}s\, \mathrm{d}u$$

where $\mathrm{d}s \times \mathrm{d}u$ is the Lebesgue measure on $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_+)$. A possible decomposition of X can be:

$$X_t = X_0 + \int_0^t F(X_s)\, \mathrm{d}s + M_t \tag{9}$$

where

$$F(x) \stackrel{\text{def}}{=} \sum_{j=1}^{J} \lambda_j(x)\, \nu_j(x) \tag{10}$$

$$M_t \stackrel{\text{def}}{=} \sum_{j=1}^{J} \int_0^t \int_0^\infty G_j(X_{s^-}, u)\, \tilde{\mathcal{N}_j}(\mathrm{d}s, \mathrm{d}u) \tag{11}$$

with

$$G_j(x, u) \stackrel{\text{def}}{=} \mathbf{1}_{[0, \lambda_j(x)]}(u)\, \nu_j(x) \tag{12}$$

Let $(\mathcal{F}_t)_{t \geq 0}$ be the natural filtration of $(X_t)_{t \geq 0}$. We will justify later that this decomposition is in fact a semi-martingale one. The SDE (8) has a unique d-dimensional adapted cadlag process (see [12, theorem 9.1]) since the following two conditions are easily verified:

i) $F(x)^* F(x) + \int_0^\infty G_j(x, u)^* G_j(x, u)\, \mathrm{d}u \leq C(1 + |x|^2)$;

ii) $(F(x) - F(y))^*(F(x) - F(y)) + \int_0^\infty (G_j(x, u) - G_j(y, u))^* \left(G_j(x, u) - G_j(y, u)\right) \mathrm{d}u \leq C\, |x - y|^2$.

Remark 3.1. *Note that (8) is the natural generalization of a well-known expression obtained in the case where the jumps are constant (see [8, theorem 4.1, chapter 6, p. 327]), namely:*

$$X_t = X_0 + \sum_{j=1}^{J} \mathcal{P}_j\Big(\int_0^t \lambda_j(X_{s^-})\, \mathrm{d}s\Big)\, \nu_j \tag{13}$$

where $t \to \mathcal{P}_j(t)$ are independent Poisson processes with unit intensity.

In our setting, we used

$$\sum_{j=1}^{J} \int_0^t \mathcal{P}_j(ds \times\, \lambda_j(X_{s^-}))\nu_j(X_{s^-}) \stackrel{\mathcal{L}}{=} \sum_{j=1}^{J} \int_0^t \int_0^\infty \mathbf{1}_{[0, \lambda_j(X_{s^-})]}(u)\, \nu_j(X_{s^-})\, \mathcal{N}_j(\mathrm{d}s, \mathrm{d}u)$$

and for justification of the last equality in law, see [9,5].

An intuitive explanation allows us to consider the process $X(t)$ as resulting from the sum of independent successive jumps occurring in each infinitesimal time interval ds, with respective rate jumps equal to '"$ds \times \lambda_j(X_{s^-})$"'. Since $\mathcal{N}_j(\mathrm{d}s, \mathrm{d}u)$ is a Poisson random measure, $\mathcal{P}_j(ds \times \lambda_j(X_{s^-}))$ can be seen as equivalent in law to $\int_0^{\lambda_j(X_{s^-})} \mathcal{N}_j(\mathrm{d}s, \mathrm{d}u)$ due to the property of Poisson random measure on Borel sets

$\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_+^d)$ which allows to define random Poisson variables. Then, $\mathcal{P}_j(ds \times \lambda_j(X_{s^-}))$ is equivalent to $\int_0^\infty \mathbf{1}_{[0,\lambda_j(X_{s^-})]}(u)\, \mathcal{N}_j(\mathrm{d}s, \mathrm{d}u)$.

Hence, (8) is an equivalent version of the initially defined pure Markov process in (3): they have the same finite-dimensional distributions and then the same finite moments, so we can use Corollary 2.3 for what follows.

Henceforth, the counting process will be considered as

$$N_t \stackrel{\text{def}}{=} \sum_{j=1}^{J} \int_0^t \int_0^\infty \mathbf{1}_{[0,\lambda_j(X_{s^-})]}(u)\, \mathcal{N}_j(\mathrm{d}s, \mathrm{d}u)$$

and it is equivalent in distribution to the one defined previously.

3.1. Semi-martingale decomposition $\phi(x) = x$

Proposition 3.2. *If $\mathbb{E}\,[|X_0|] < \infty$, then M_t defined by (11) is a square integrable vector martingale with predictable quadratic variation matrix:*

$$\langle M \rangle_t = \int_0^t \Gamma(X_s)\, \mathrm{d}s \tag{14}$$

with

$$\Gamma(x) \stackrel{\text{def}}{=} \sum_{j=1}^{J} \lambda_j(x)\, \nu_j(x)\, \nu_j(x)^* \tag{15}$$

where $*$ denotes the transpose of any vector.

Proof. In (11), the integrand

$$G_j(X_{s^-}, u) = \mathbf{1}_{[0,\lambda_j(X_{s^-})]}(u)\, \nu_j(X_{s^-}) \tag{16}$$

is adapted to $(\mathcal{F}_t)_{t\geq 0}$ and left-continuous so it is predictable. From Hypotheses 2.1-(*iii*) and (*iv*):

$$\begin{aligned} \textstyle\sum_{j=1}^{J} \int_0^t \int_{\mathbb{R}_+} |G_j(X_{s^-}, u)|\, \mathrm{d}s\, \mathrm{d}u &= \textstyle\sum_{j=1}^{J} \int_0^t \lambda_j(X_s)\, |\nu_j(X_s)|\, \mathrm{d}s \\ &\leq C_\nu \int_0^t \lambda(X_s)\, \mathrm{d}s \\ &\leq C_\nu\, C_\lambda \int_0^t (1 + |X_s|)\, \mathrm{d}s \leq C_\nu\, C_\lambda t(1 + \sup_{s\leq t} |X_s|) \end{aligned}$$

and then, it follows from Corollary 2.3 that

$$\mathbb{E}\,[\sum_{j=1}^{J} \int_0^t \int_{\mathbb{R}_+} |G_j(X_{s^-}, u)|\, \mathrm{d}s\, \mathrm{d}u] < \infty \text{ for all } t > 0$$

likewise we show that

$$\mathbb{E}\,[\sum_{j=1}^{J}\int_0^t\int_{\mathbb{R}_+}|G_j(X_{s^-},u)|^2\,\mathrm{d}s\,\mathrm{d}u]<\infty \text{ for all } t>0$$

As a consequence, according to Ikeda and Watanabe [12, Theorem II-3.1 and Lemma II-3.1, p. 60-62] - each component $M_{t,\ell}$ of M_t

$$M_{t,\ell}\stackrel{\text{def}}{=}\textstyle\sum_{j=1}^{J}\int_0^t\int_0^\infty \mathbf{1}_{[0,\lambda_j(X_{s^-})]}(u)\,\nu_{j,\ell}(X_{s^-})\,\tilde{\mathcal{N}}_j(\mathrm{d}s,\mathrm{d}u)$$

is a square integrable $\mathcal{F}_t$-martingale for $1\le\ell\le d$ with predictable quadratic covariations

$$\langle M_{\cdot,\ell},M_{\cdot,\ell'}\rangle_t=\textstyle\sum_{j=1}^{J}\int_0^t\lambda_j(X_s)\,\nu_{j,\ell}(X_s)\,\nu_{j,\ell'}(X_s)\,\mathrm{d}s$$

for $1\le\ell,\ell'\le d$ which gives the (matrix) predictable quadratic variation (14). □

3.2. Semi-martingale decomposition for $\phi(x)$ with polynomial growth

Let us denote by $\mathcal{C}_p(\mathbb{R}_+^d)$ the space of continuous functions $\mathbb{R}_+^d\mapsto\mathbb{R}^{d'}$ with polynomial growth. We say that a function ϕ has polynomial growth if $\exists\,C$ such that $|\phi(x)|\le C\,(1+|x|^p)$ for some $p\ge 1$ (it includes trivially the space $\mathcal{C}_0(\mathbb{R}_+^d)$ of continuous vanishing functions).

For any test function $\phi:\mathbb{R}_+^d\mapsto\mathbb{R}^{d'}$ in $\mathcal{C}_p(\mathbb{R}_+^d)$ we have:

$$\phi(X_t)=\phi(X_0)+\sum_{j=1}^{J}\int_0^t\int_0^\infty \mathbf{1}_{[0,\lambda_j(X_{s^-})]}(u)\,\big[\phi(X_{s^-}+\nu_j(X_{s^-}))-\phi(X_{s^-})\big]\,\mathcal{N}_j(\mathrm{d}s,\mathrm{d}u)$$

Otherwise, we can write:

$$\phi(X_t)=\phi(X_0)+\int_0^t\mathcal{L}\phi(X_s)\,\mathrm{d}s+M_t^\phi \tag{17}$$

where

$$M_t^\phi\stackrel{\text{def}}{=}\sum_{j=1}^{J}\int_0^t\int_{\mathbb{R}_+}\mathbf{1}_{[0,\lambda_j(X_{s^-})]}(v)\,\left[\phi(X_{s^-}+\nu_j(X_{s^-}))-\phi(X_{s^-})\right]\tilde{\mathcal{N}}_j(\mathrm{d}s,\mathrm{d}u)$$

Proposition 3.3. *Let $\phi:\mathbb{R}_+^d\to\mathbb{R}^{d'}\in\mathcal{C}_p(\mathbb{R}_+^d)$ and let us assume $\mathbb{E}\,[|X_0|^{2p+1}]<\infty$, then: M_t^ϕ is a square integrable martingale, with predictable quadratic variation matrix*

$$\langle M^\phi\rangle_t=\sum_{j=1}^{J}\int_0^t\lambda_j(X_s)\,\big[\phi(X_s+\nu_j(X_s))-\phi(X_s)\big]\,\big[\phi(X_s+\nu_j(X_s))-\phi(X_s)\big]^*\,\mathrm{d}s \tag{18}$$

Proof. First suppose $d'=1$.

In (11), the integrand

$$\mathbf{1}_{[0,\lambda_j(X_{s^-})]}(v)\,\left[\phi(X_{s^-}+\nu_j(X_{s^-}))-\phi(X_{s^-})\right] \tag{19}$$

is adapted to $(\mathcal{F}_t)_{t\ge 0}$ and left-continuous so it is predictable.

Then, and using Corollary 2.3:

$$\begin{aligned}&\mathbb{E}\left[\sum_{j=1}^{J}\int_0^t \lambda_j(X_s)\ |\phi(X_s+\nu_j(X_s))-\phi(X_s)|\ \mathrm{d}s\right]\\&\qquad\le C_t\,\mathbb{E}\left[\int_0^t \left(1+|X_s|^{p+1}\right)\mathrm{d}s\right]\\&\qquad\le C_t\left(1+\mathbb{E}\left[\sup_{s\le t}|X_s|^{p+1}\right]\right)<\infty\end{aligned}$$

Moreover

$$\begin{aligned}&\mathbb{E}\left[\sum_{j=1}^{J}\int_0^t \lambda_j(X_s)\left[\phi(X_s+\nu_j(X_s))-\phi(X_s)\right]^2\right]\mathrm{d}s\\&\qquad\le C_p\,\mathbb{E}\left[\int_0^t \left(1+|X_s|^{2\,p+1}\right)\right]\mathrm{d}s\\&\qquad\le C_{p,t}\left(1+\mathbb{E}\left[\sup_{s\le t}|X_s|^{2\,p+1}\right]\right)<\infty\end{aligned}$$

Hence, according to Ikeda and Watanabe [12, Theorem II-3.1 and Lemma II-3.1, p. 60-62], M_t^{ϕ} is a square integrable martingale with predictable quadratic variation (18).

For the case $d'>1$, each component of M_t^{ϕ} is an $\mathcal{F}_t^X$-martingale and we can compute the predictable quadratic covariation $\langle M^{\phi}_{\cdot,\ell}, M^{\phi}_{\cdot,\ell'}\rangle_t$ like in the case $d'=1$. □

Let us remark that

$$M_t^{\phi}=\phi(X_t)-\phi(X_0)-\int_0^t \mathcal{L}\phi(X_s)\,\mathrm{d}s \tag{20}$$

and then

$$\mathbb{E}\left[\phi(X_t)\right]=\mathbb{E}\left[\phi(X_0)\right]+\mathbb{E}\left[\int_0^t \mathcal{L}\phi(X_s)\,\mathrm{d}s\right] \tag{21}$$

$$=\mathbb{E}\left[\phi(X_0)\right]+\int_0^t \mathbb{E}\left[\mathcal{L}\phi(X_s)\right]\mathrm{d}s \tag{22}$$

for all $t\ge 0$, the last integral inversion is due to Fubini's theorem since we can verify easily that $\int_0^t \mathbb{E}\left[|\mathcal{L}\phi(X_s)\,\mathrm{d}s|\right]<\infty$ a.s. due to Corollary 2.3.

Remark 3.4. *The previous proposition is valid particularly for the set of continuous, vanishing at infinity functions and $(X_t)_{t\ge0}$ is the unique solution of the martingale problem associated with the generator $\mathcal{L}$, i.e. it is the only Markov process such that (20) is a martingale for all $\phi\in\mathcal{C}_0(\mathbb{R}_+^d)$ (definition in [8, Ch. 4]).*

4. Law of large numbers

We consider a sequence of Markov pure jump processes $(X_t^n)_{t\ge0}$ taking values in $\mathbb{R}_+^d$ with intensity functions $\lambda_j^n(x)$ and with jump functions $\nu_j^n(x)$ satisfying $x+\nu_j^n(x)\in\mathbb{R}_+^d$ for all $x\in\mathbb{R}_+^d$, for all j such that $\lambda_j(x)>0$ and for all n. Hence,

$$X_t^n=X_0^n+\int_0^t F^n(X_s^n)\,\mathrm{d}s+M_t^n \tag{23}$$

where

$$F^n(x) \stackrel{\text{def}}{=} \sum_j \lambda_j^n(x)\,\nu_j^n(x) \tag{24}$$

$$M_t^n \stackrel{\text{def}}{=} \sum_{j=1}^{J} \int_0^t \int_{\mathbb{R}_+} G_j^n(X_{s^-}^n, u)\,\tilde{N}_j(\mathrm{d}s, \mathrm{d}u)\,, \tag{25}$$

with

$$G_j^n(x,u) = \mathbf{1}_{[0,\lambda_j^n(x)]}(u)\,\nu_j^n(x), \text{ for } x \in \mathbb{R}^d\,.$$

We suppose $\mathbb{E}(|X_0^n|) < \infty$ so that M_t^n is square integrable martingale with predictable quadratic covariation matrix:

$$\langle M^n \rangle_t = \frac{1}{n} \int_0^t \Gamma^n(X_s^n)\,\mathrm{d}s \tag{26}$$

where

$$\Gamma^n(x) \stackrel{\text{def}}{=} n \sum_{j=1}^{J} \lambda_j^n(x)\,\nu_j^n(x)\,\nu_j^n(x)^* \tag{27}$$

For i.i.d. random variables $Y_i, i = 1, ..., n$ where $Y_i \stackrel{\mathcal{L}}{=} Y$ for some random variable Y, the law of large numbers $\sum_{i=1}^n \frac{Y_i}{n} \underset{n\to\infty}{\longrightarrow} E[Y]$ can be interpreted as the value reached by a random walk at time $t = 1$, where the jump size is divided by n and where simultaneously the jump rate is increased by the same factor n (the number of jumps is multiplied by n). Here, $t = 1$ is an arbitrary choice to insist that we focus on a fixed time interval $[0, t]$. The random walk position at time t=1 is then almost surely deterministic when n is big enough.

For a jump pure random processes, we should then intuitively guess that if the jump size and the jump rate are scaled in the same way (respectively reduced/multiplied by n), and if the jumps are with independent increments, then the position of the process at any time should be deterministic. We make then the following assumptions

Hypotheses 4.1.

(*i*) *There exists functions $\nu_j(x)$ and $\lambda_j(x)$ defined on $\mathbb{R}_+^d$ verifying Hypotheses 2.1 for some constants named again C_λ et C_ν and such that we have*

$$\forall x \in \mathbb{R}_+^d \qquad n\,\nu_j^n(x) \underset{n\to\infty}{\longrightarrow} \nu_j(x)\,, \qquad \frac{1}{n}\lambda_j^n(x) \underset{n\to\infty}{\longrightarrow} \lambda_j(x)\,;$$

(*ii*) *Asymptotically*

$$\forall n \in \mathbb{N}, \forall x \in \mathbb{R}_+^d \qquad n\,\nu_j^n(x) \le C_\nu\,, \qquad \frac{1}{n}\lambda_j^n(x) \le C_\lambda(1 + |x|)\,;$$

(iii) We have the **uniform convergence** *on* $\mathbb{R}_+^d$ *for the sequence of flow functions*

$$\|F^n - F\|_\infty \xrightarrow[n\to\infty]{} 0;$$

where

$$F(x) \stackrel{\text{def}}{=} \sum_j \lambda_j(x)\,\nu_j(x)$$

and where $\|.\|_\infty$ *denotes the standard infinity norm.*

Let us now consider the deterministic function $x(t)$ given by

$$x(t) = x_0 + \int_0^t F(x(s))\,\mathrm{d}s \tag{28}$$

The solution for (28) exists and is unique due to Cauchy Lipschitz Theorem since $x \to F(x)$ is overall Lipschitz on $\mathbb{R}_+^d$.

Proposition 4.2. *Suppose first that* $\mathbb{E}\,[|X_0^n - x_0|^2] \xrightarrow[n\to\infty]{} 0$. *Under Hypotheses 4.1;*

$$\mathbb{E}\,[\sup_{s\le t} |X_s^n - x(s)|^2] \xrightarrow[n\to\infty]{} 0$$

Proof. First, we remark that due to the quadratic convergence of X_0^n, $\mathbb{E}[X_0^n]$ is uniformly bounded in n by some constant X_0^{max}. Let

$$\delta_s^n \stackrel{\text{def}}{=} |X_s^n - x(s)|^2\,.$$

From:

$$\begin{aligned}
|X_t^n - x(t)|^2 &= \left|X_0^n + \textstyle\int_0^t F^n(X_s^n)\,\mathrm{d}s + M_t^n - x_0 - \int_0^t F(x(s))\,\mathrm{d}s\right|^2 \\
&\le 3\left(|X_0^n - x_0|^2 + \left|\textstyle\int_0^t \left(F^n(X_s^n) - F(x(s))\right)\mathrm{d}s\right|^2 + |M_t^n|^2\right) \\
&\le 3\left(|X_0^n - x_0|^2 + t\,\textstyle\int_0^t \left|F^n(X_s^n) - F(x(s))\right|^2 \mathrm{d}s + |M_t^n|^2\right)
\end{aligned}$$

Since F is C-Lipschitz for some $C > 0$ as a consequence of Hypothesis 2.1 (*vii*), then,

$$\begin{aligned}
|F^n(X_s^n) - F(x(s))|^2 &\le 2\,|F(X_s^n) - F(x(s))|^2 + 2\,|F^n(X_s^n) - F(X_s^n)|^2 \\
&\le 2\,\|F^n - F\|_\infty^2 + 2\,C^2\,|X_s^n - x(s)|^2 \\
&\le 2\,\|F^n - F\|_\infty^2 + 2\,C^2\,\delta_s^n
\end{aligned}$$

we deduce that

$$\delta_t^n \le 3\left(|X_0^n - x_0|^2 + 2\,C^2 t\,\textstyle\int_0^t \delta_s^n\,\mathrm{d}s + 2\,t^2\,\|F^n - F\|_\infty^2 + |M_t^n|^2\right),$$

and then, for all $s \le t$

$$\delta_s^n \leq 3\left(|X_0^n - x_0|^2 + 2\,C^2 t \int\limits_0^s \delta_r^n\,\mathrm{d}r + 2\,t^2\ \|F^n - F\|_\infty^2 + \sup_{s\leq t}|M_s^n|^2\right) \tag{29}$$

From Gronwall's inequality we get for all $s \leq t$

$$\delta_s^n \leq 3\left(|X_0^n - x_0|^2 + 2\,t^2\ \|F^n - F\|_\infty^2 + \sup_{s\leq t}|M_s^n|^2\right) e^{6C^2 t\times s} \tag{30}$$

Hence

$$\sup_{s\leq t}\delta_s^n \leq 3\left(|X_0^n - x_0|^2 + 2\,t^2\ \|F^n - F\|_\infty^2 + \sup_{s\leq t}|M_s^n|^2\right)\, e^{6C^2t^2}$$

and then

$$\mathbb{E}\,[\sup_{s\leq t}\delta_s^n] \leq 3\left(\mathbb{E}\,[|X_0^n - x_0|^2] + 2\,t^2\ \|F^n - F\|_\infty^2 + \mathbb{E}\,[\sup_{s\leq t}|M_s^n|^2]\right)\, e^{6C^2t^2}$$

Using Doob inequality

$$\begin{aligned}
\mathbb{E}\,[\sup_{s\leq t}\delta_s^n] &\leq 3\Big(\ \mathbb{E}\,[|X_0^n - x_0|^2] + 2\,t^2\ \|F^n - F\|_\infty^2 + 4\,\mathbb{E}\,\langle M^n\rangle_t\Big)\, e^{6C^2t^2} \\
&\leq 3\Big(\ \mathbb{E}\,[|X_0^n - x_0|^2] + 2\,t^2\ \|F^n - F\|_\infty^2 + \frac{4}{n}\,\mathbb{E}\int\limits_0^t \Gamma^n(X_s^n)\,\mathrm{d}s\Big)\, e^{6C^2t^2} \\
&\leq 3\Big(\ \mathbb{E}\,[|X_0^n - x_0|^2] + 2\,t^2\ \|F^n - F\|_\infty^2 + \frac{4\,c}{n}\,\mathbb{E}\int\limits_0^t (1+|X_s^n|)\,\mathrm{d}s\Big)\, e^{6C^2t^2} \\
&\leq 3\Big(\ \mathbb{E}\,[|X_0^n - x_0|^2] + 2\,t^2\ \|F^n - F\|_\infty^2 + \frac{4\,c\,t}{n}\,(1+\mathbb{E}\,[\sup_{s\leq t}|X_s^n|])\Big)\, e^{6C^2t^2}
\end{aligned}$$

It remains to show that $\mathbb{E}\,[\sup_{s\leq t}|X_s^n|]$ is uniformly bounded. In fact, let $C_{\lambda^n} \equiv n\ C_\lambda$ and $C_{\nu^n} \equiv \frac{1}{n}\ C_\nu$. Under Hypotheses 4.1, let ζ_t^n be a pure jump Markov process taking values in $\mathbb{N}$, starting from 0 and which, conditionally to $\zeta_t^n = k$ will increase by one at rate $C_{\lambda^n}\ (1+C_{\nu^n}\ k)$. Clearly ζ_t^n is a simple immigration-birth process with immigration rate C_{λ^n} and birth rate $C_{\lambda^n}\,C_{\nu^n}$, admits a negative binomial distribution and:

$$\mathbb{E}\,[\zeta_t^n] = \frac{C_{\lambda^n}}{C_{\lambda^n}C_{\nu^n}}(exp(C_{\lambda^n}\ C_{\nu^n}t) - 1) \leq \frac{1}{C_{\nu^n}}(exp(C_\lambda C_\nu t) - 1)\,.$$

Similarly to Corollary 2.3, we have

$$\begin{aligned}
\mathbb{E}\,[\sup_{s\leq t}|X_s^n|] &\leq \mathbb{E}\,[|X_0^n|] + C_{\nu^n}\,\mathbb{E}\,[\sup_{s\leq t}\ \zeta_s^n] \\
&\leq\ X_0^{max} + (exp(C_\lambda C_\nu t) - 1)
\end{aligned} \tag{31}$$

As a consequence

$$\mathbb{E}\,[\sup_{s\leq t}\delta_s^n] \leq \left(3\,\mathbb{E}\,[|X^n(0) - x_0|^2] + 6\,t^2\ \|F^n - F\|_\infty^2 + \tfrac{12\,c\,t}{n}\ (X_0^{max} + exp(C_\lambda C_\nu t))\right) e^{6C^2t^2} \tag{32}$$

From the last inequality and taking into account (30), we deduce that:

$$\mathbb{E}\,[\sup_{s\leq t}\delta_s^n] \underset{n\to\infty}{\to} 0\,. \quad \square$$

The quadratic convergence obtained in Proposition 4.2 implies convergence in probability (weak LLN result). For this, we mainly used the uniform boundedness on n of $E[\sup_{s\leq t,n} X_s^n]$ for fixed t. As for the almost sure convergence (strong LLN), it is not immediate (λ is not bounded). However, the process X_t^n can be uniformly bounded on n (we cite the example of conservative reaction systems with X_0^n being uniformly bounded on n, see [11]). In this case, we have this result:

Proposition 4.3. *Let us assume that* $|X_0^n - x(0)| \underset{n\to\infty}{\overset{a.s}{\to}} 0$. *Under Hypotheses 4.1 and if the process X_t^n defined in (23) is uniformly bounded on n, we have*

$$\sup_{s\leq t} |X_s^n - x(s)| \underset{n\to\infty}{\overset{a.s}{\to}} 0$$

Proof. If we use the following inequality

$$\begin{aligned}|X_t^n - x(t)| &\leq |X_0^n - x_0| + \Big|\int_0^t \big(F^n(X_s^n) - F(x(s))\big)\,\mathrm{d}s\Big| + |M_t^n| \\ &\leq |X_0^n - x_0| + \Big|\int_0^t \big(F^n(X_s^n) - F((X_s^n)\big)\,\mathrm{d}s\Big| + C\Big|\int_0^t |X_s^n - x(s)|\,\mathrm{d}s\Big| + |M_t^n|\end{aligned}$$

and then Gronwall lemma, we conclude that $|M_t^n| \to 0$ *a.s.* In fact,

$$\begin{aligned}|M_t^n| &= |\sum_{j=1}^{J}\int_0^t\int_0^\infty \nu_j^n(X_{s-}^n)\mathbf{1}_{[0,\lambda_j^n(X_{s-}^n)]}(u)\tilde{\mathcal{N}}_j(\mathrm{d}s,\mathrm{d}u)| \\ &\leq 2C_\nu \sum_{j=1}^{J}\int_0^t\int_{\mathbb{R}_+} \mathbf{1}_{[0,\lambda_j^n(X_{s-}^n)]}(u)\frac{\tilde{\mathcal{N}}_j(\mathrm{d}s,\mathrm{d}u)}{n} \\ &\leq 2C_\nu \sum_{j=1}^{J}\int_0^t\int_{\mathbb{R}_+} \mathbf{1}_{[0,\frac{1}{n}\lambda_j^n(X_{s-}^n)]}(u)\frac{\tilde{\mathcal{N}}_j(\mathrm{d}s,n\mathrm{d}u)}{n}\end{aligned}$$

Since $\frac{1}{n}\lambda_j^n(X_s^n) \leq 2C_\lambda(1+|X_s^n|)$ and having X_s^n uniformly bounded on n, we can deduce the convergence of $|M_t^n|$ to the null measure. This deduction is due to LLN for the compensated measure [26]: $\frac{\tilde{\mathcal{N}}_j(\mathrm{d}s,n\mathrm{d}u)}{n} \Rightarrow 0$, where 0 denotes here the null measure: $(\forall A \in \mathcal{B}(\mathbb{R}_+),\ \int_A \frac{\tilde{\mathcal{N}}_j(\mathrm{d}s,n\mathrm{d}u)}{n} \overset{a.s}{\to} 0\,)$. $\square$

Example of application

As an application of the LLN Proposition 4.2, we can cite the anaerobic model AM2b [27] since the infinitesimal parameters of the model they consider depend on the variable states. In fact, AM2b model describes the dynamics of biological and anaerobic wastewater treatment, where in summarized words S_1 and S_2 denote substrates degraded by biomasses B_1 and B_2. The state of the AM2b model is described by $x = (s_1, b_1, s_2, b_2, s)$ where s_1, b_1, s_2, b_2, s are the concentrations in $\mathrm{S}_1, \mathrm{B}_1, \mathrm{S}_2, \mathrm{B}_2, \mathrm{S}$.

The reaction scheme of AM2b model correspond to $J = 15$ reactions classified in three sets: The first set correspond to Biological reactions ($j = 1 : 5$) and the second and third set of reactions ($j = 6 : 15$) are not biochemical reactions since they just describe the inflows and outflows in the AM2b process. Here, we will detail only the first reaction ($j = 1$) because it is an example of biological one:

$$k_1\,\mathrm{S}_1 + \mathrm{B}_1 \xrightarrow{\lambda_1} 2\,\mathrm{B}_1 + c_{12}\,\mathrm{S}_2 + c_{10}\,\mathrm{S} + \kappa_1\,\mathrm{CO}_2\,,$$

with k_1, c_{12}, c_{10} and κ_1 being fixed parameters.

The rate functions λ_1 is density-dependent and then can be expressed as function of the mass-density if we substitute molar density by mass density. Then, for m being the 'mean' molecular mass, and for $n \stackrel{\text{def}}{=} \frac{vol}{m}$, we have $X^n(t) \equiv \frac{X(t)}{\frac{vol}{m}}$, describing the state of the scaled process and for $X^n(t) = x$, the infinitesimal parameters are such that:

$$\lambda_1^n(x) \stackrel{\text{def}}{=} n\,\lambda_1(x)\,, \qquad \nu_1^n(x) \stackrel{\text{def}}{=} [x + \tfrac{1}{n}\nu_1]^+ - x$$

where we remind that $[x]^+$ is the orthogonal projection of x onto $\mathbb{R}_+^5$ and where:

$$\begin{aligned}
&\lambda_1(x) = \mu_1(s_1)\,b_1,\\
&\nu_1(x) = (-k_1, 1, c_{12}, 0, c_{10})^* \text{ if } x_1 > 0 \qquad \nu_1(x) = (0, 1, c_{12}, 0, c_{10})^* \text{ if } x_1 = 0
\end{aligned}$$

for μ_1 being the (Monod) growth function

$$\mu_1(s_1) = \mu_1^{max}\,\frac{s_1}{K_1 + s_1}$$

with K_1 being the half saturation constant associated with S_1.

We remark that for this biological reaction, $\lambda_1^n(x)$ does not cancel for $s_1 \le \frac{k_1}{n}$ and then we need to impose the non-negativity of $x + \nu_1^n(x)$.

We precise that applying the result of our proposition to this model is possible since Hypotheses 2.1 are verified for the considered model, particularly $\nu_1(x)\lambda_1(x)$ is continuous and we verify moreover the uniform convergence of $F^n(x)$ to $F(x)$ on $\mathbb{R}_+^d$ and all Hypotheses 4.1 are verified.

5. Central limit theorem

Let us denote $D \equiv D([0,\infty), \mathbb{R})$ the usual space of cadlag functions defined on $[0,\infty)$ endowed with the Skorohod d^0 topology for which D is separable and complete (see [4,8]). In consequence $\mathcal{P}(D) \equiv \mathcal{P}(D([0,\infty), \mathbb{R}))$ is separable and complete. Let us consider $D^d \equiv D([0,\infty), \mathbb{R})^d$. D^d is made complete and separable in the standard way.

Let us consider $V_t^n \equiv \sqrt{n}\,(X_t^n - x(t))$ where X^n satisfy (23) and x satisfy (28):

$$\begin{aligned}
V_t^n &= \sqrt{n}\,(X_0^n - x_0) + \int_0^t \sqrt{n}\,\big(F^n(X_s^n) - F(x(s))\big)\,\mathrm{d}s + \sqrt{n}\,M_t^n\\
&= V_0^n + \int_0^t \sqrt{n}\,\big(F^n(X_s^n) - F(x(s))\big)\,\mathrm{d}s + \sqrt{n}\,M_t^n
\end{aligned}$$

Hypotheses 5.1.

i) $\sqrt{n}\,\|F^n - F\|_\infty \underset{n\to\infty}{\longrightarrow} 0$;

ii) $\exists\, V_0$ *with bounded variance such that* $V_0^n \Rightarrow V_0$;

iii) F *is differentiable and* $DF \equiv (\partial_j F_i)_{1\le i,j\le d}$ *is a continuous application on* $\mathbb{R}^d$;

iv) $\|\Gamma^n - \Gamma\|_\infty \underset{n\to\infty}{\longrightarrow} 0$;

Theorem 5.2. *Under Hypotheses 5.1, $V^n \Rightarrow V$ where:*

$$V_t = V_0 + \int_0^t DF(x(s)).V_s \mathrm{d}s + \sum_{j=1}^{J} \int_0^t \int_{\mathbb{R}_+} G_j(x(s), u)\, W_j(\mathrm{d}s, \mathrm{d}u) \tag{33}$$

with W_j being independent Gaussian white noises determined each one by

$$\mathbb{E}[W_j(t, A) W_j(s, B)] = l(A \cap B)\, \min(t, s)$$

where l is the usual Lebesgue measure on $\mathbb{R}$.

Proof. This proof is done in three steps:

Step 1

At this step we prove that V^n is tight (it induces tight measures in D^d). We should mention that V^n is tight if and only if each one of its components is tight in D. For this, we apply Lemma A.2:

First We verify that V^n is stochastically bounded (see Definition A.1), which is equivalent to have stochastic boundedness for each $V^n_{t,\ell}$: the ℓth component of $V^n_t, 1 \leq \ell \leq d$). In fact, due to (32):

$$\begin{aligned}
\mathbb{E}\,[|V^n_t|^2] &\leq \mathbb{E}\,[\sup_{s\leq t}|V^n_s|^2] \equiv n \sup_{s\leq t}\mathbb{E}\,[\delta^n_s] \\
&\leq \Big(3\,\mathbb{E}\,[|V^n_0|^2] + 6\,t^2\, n\, \|F^n - F\|^2_\infty + 12\, c\, t\, (X_0^{max} + exp(C_\lambda C_\nu t))\Big)\, e^{6C^2t^2} \\
&\leq \bar{C}_t
\end{aligned}$$

where $\bar{C}_t$ is a constant independent of n such that the last inequality is true for $n \geq n_0$ for some integer n_0 (taking into account Hypotheses 5.1 i) and ii)).

From Tchebychev inequality, $\forall n \geq n_0$:

$$\mathbb{P}(|V^n_t| > \alpha) \leq \frac{1}{\alpha^2}\mathbb{E}\,[|V^n_t|^2] \leq \frac{1}{\alpha^2}\mathbb{E}\,[\sup_{s\leq t}|V^n_s|^2] \leq \frac{1}{\alpha^2}\bar{C}_t$$

Then, $\forall \varepsilon > 0, \exists\, K_{t,\varepsilon} \equiv [-\alpha_\varepsilon, \alpha_\varepsilon]$ such that $\sup_{n\geq n_0} P(|V^n_t|) \notin K_{t,\varepsilon})| < \varepsilon$ where α_ε is chosen such that $\alpha_\varepsilon > \sqrt{\frac{\bar{C}_t}{\varepsilon}}$.

Second We verify ii) of Lemma A.2: for all $t \leq T$:

$$\begin{aligned}
|V^n_{t+u} - V^n_t|^2 &\leq 2\Big(2\,C^2 u\, \int_t^{t+u} n\, \delta^n_s\, \mathrm{d}s + 2\, u^2\, n\, \|F^n - F\|^2_\infty + n|M^n_{t+u} - M^n_t|^2\Big) \\
&\leq 2\Big(2\,C^2\, n\, u^2 \sup_{t\leq s\leq t+u} \delta^n_s + 2\, u^2\, n\, \|F^n - F\|^2_\infty + n|M^n_{t+u} - M^n_t|^2\Big)
\end{aligned}$$

Then

$$\begin{aligned}
&\mathbb{E}_t\,[|V^n_{t+u} - V^n_t|^2] \\
&\leq 2\Big(2\,C^2\, n\, u^2\, \mathbb{E}_t\,[\sup_{t\leq s\leq t+u} \delta^n_s] + 2\, u^2\, n\, \|F^n - F\|^2_\infty + n\mathbb{E}_t\,[|M^n_{t+u} - M^n_t|^2]\Big) \\
&\leq 2\Big(2\,C^2\, n\, u^2\, \mathbb{E}_t\,[\sup_{t\leq s\leq t+u} \delta^n_s] + 2\, u^2\, n\, \|F^n - F\|^2_\infty + \mathbb{E}_t\,[\sum_{j=1}^{J}\int_t^{t+u} n\, \lambda^n_j(X^n_{s-})\, \nu^n_j(X^n_{s-})\, \nu^n_j(X^n_{s-})^* ds]\Big)
\end{aligned}$$

$$\leq 2\Big(2\,C^2\,n\,u^2\,\mathbb{E}_t\,[\underset{t\leq s\leq t+u}{sup}\;\delta_s^n]+2\,u^2\,n\;\|F^n-F\|_\infty^2+c\,u\,(1+\mathbb{E}_t\,[\sup_{t\leq s\leq t+u}|X_s^n|])\Big)$$

$$\leq 2\Big(2\,C^2\,n\,u^2\,\mathbb{E}_t\,[\underset{s\leq t+u}{sup}\;\delta_s^n]+2\,u^2\,n\;\|F^n-F\|_\infty^2+c\,u\,(1+\mathbb{E}_t\,[\sup_{s\leq t+u}|X_s^n|])\Big)$$

Let

$$Z_n(\eta,T)=\;2\Big(2\,C^2\,n\,\eta^2\;\underset{s\leq T+\eta}{sup}\;\delta_s^n\;+2\,\eta^2\,n\;\|F^n-F\|_\infty^2+c\,\eta\,(1+\sup_{s\leq T+\eta}|X_s^n|)\Big) \tag{34}$$

Then we have $\forall 1\leq \ell\leq d$, for $0\leq t\leq T$ and $0\leq u\leq \eta$:

$$\begin{aligned}\mathbb{E}_t\,[|V^n_{t+u,\ell}-V^n_{t+u,\ell}|^2]&\leq \mathbb{E}_t\,[|V^n_{t+u}-V^n_t|^2]\\&\leq \mathbb{E}_t\,[Z_n(\eta,T)]\;\;w.p.1\,.\end{aligned}$$

Given (32), we have:

$$\mathbb{E}\,[\underset{s\leq T+\eta}{sup}\;\delta_s^n]\leq\Big(3\,\mathbb{E}\,[|X_0^n-x(0)|^2]+6\,(T+\eta)^2\;\|F^n-F\|_\infty^2+\frac{12\,c\,(T+\eta)}{n}\;(X_0^{max}+exp(C_\lambda C_\nu(T+\eta)))\Big)\,e^{6C^2(T+\eta)^2}$$

and

$$\mathbb{E}(1+\sup_{s\leq T+\eta}|X_s^n|)\leq (X_0^{max}+exp(C_\lambda C_\nu(T+\eta)))$$

Form the last two inequalities combined with (34), we deduce that:

$$\underset{\eta\to 0}{lim}\;\underset{n\to\infty}{limsup}\;\mathbb{E}\,[Z_n(\eta,T)]=0$$

In consequence, we have $\forall 1\leq\ell\leq d$, $V^n_{t,\ell}$ is tight in D and then V^n_t is tight in D^d. Hence, there exists a subsequence of V^n, which will be still denoted V^n such that $V^n\Rightarrow V$ where V is a process with paths in D^d.

Moreover, applying Theorem A.3, the limit process V is of continuous paths. In fact, for the jumps ν_j^n of X^n, we have $n\|\nu_j^n\|_\infty\leq 2C_\nu$ for $n\geq n_0$ and then $\nu_j^n=O(\frac{1}{n})$. In consequence, due to scaling, the jumps in $\sqrt{n}(X_s^n-x(s))$ are at most of size $\frac{1}{\sqrt{n}}$ and then, $j(V^n)\equiv\sup_{0\leq t\leq T}|V_t^n-V_{t^-}^n|\Rightarrow 0$. We conclude that V is continuous.

Step 2

Here we prove that:

$$\sqrt{n}\,M_t^n\Rightarrow\sum_{j=1}^{J}\int_0^t\int_{\mathbb{R}_+}G_j(x(s),u)\;W_j(\mathrm{d}s,\mathrm{d}u) \tag{35}$$

From (26), we remind that:

$$\langle\sqrt{n}M^n\rangle_t=\int_0^t\Gamma^n(X_s^n)\,\mathrm{d}s$$

Applying the martingale central limit Theorem A.4, we can verify that:

$$\langle \sqrt{n} M^n \rangle_t \Rightarrow \int_0^t \Gamma(x(s))\,\mathrm{d}s$$

We do this in two sub-steps:

First, we verify the two conditions in i) of Theorem A.4:

$$\lim_{n\to\infty} \mathbb{E}\,[\sup_{0<t\leq T} n(\langle M_i^n, M_j^n\rangle_t - \langle M_i^n, M_j^n\rangle_{t-})] = 0$$

This is well verified since $\langle M_i^n, M_j^n\rangle_t$ is continuous: the discontinuity may occur in t which is of null measure according to the Lebesgue measure on $[0,\infty)$. Moreover:

$$\lim_{n\to\infty} \mathbb{E}\,[n(\sup_{0<t\leq T} |M_t^n - M_{t^-}^n|)^2] = 0$$

In fact, as explained in step 1, and due to scaling, the jumps in $\sqrt{n} M^n$ are at most of size $\frac{1}{\sqrt{n}}$.

Second, we verify condition ii):

$$\langle \sqrt{n} M^n \rangle_t = \int_0^t [\Gamma^n(X_s^n) - \Gamma(X_s^n)]\,\mathrm{d}s + \int_0^t \Gamma(X_s^n)\,\mathrm{d}s$$
$$\Rightarrow \int_0^t \Gamma(x(s))\,\mathrm{d}s$$

In fact, due to the uniform convergence of Γ^n to Γ (Hypotheses 4.1), the first term on the right of the equality $\xrightarrow[n\to\infty]{P} 0$.

For the second term on the right of the equality, since Γ is continuous, and since the mapping $z \mapsto \int_0^t \Gamma(z_s)\,ds$ is continuous on D^d (see Lemma A.6), we can then apply the mapping Theorem A.5.

Since $(X_s^n)_{s\geq 0} \xrightarrow[n\to\infty]{P} (x(s))_{s\geq 0}$, then we have the convergence in probability (and then particularly convergence in law):

$$\int_0^t \Gamma^n(X_s^n)\,\mathrm{d}s \xrightarrow[n\to\infty]{P} \int_0^t \Gamma(x(s))\,\mathrm{d}s$$

Finally, we use Slutsky theorem (Theorem A.7 i)) for $f(x,y) \equiv x + y$:

$$\sqrt{n} M_t^n \Rightarrow \sum_{j=1}^{J} \int_0^t \nu_j(x(s)) \sqrt{\lambda_j(x(s))}\,\mathrm{d}W_j(s) \tag{36}$$

where $W_j(t)$ here designates standard Brownian motion.

Finally (36) is in accordance with (35) since:

$$\int_0^t \int_{\mathbb{R}_+} \nu_j(x(s))\,\mathbf{1}_{[0,\lambda_j(x(s))]}(u)\,W_j(\mathrm{d}s,\mathrm{d}u) = \int_0^t \nu_j(x(s)) W_j(\mathrm{d}s, \lambda_j(x(s)))$$
$$= \int_0^t \nu_j(x(s)) \sqrt{\lambda_j(x(s))}\,\mathrm{d}W_j(s)$$

Step 3

At this step we decompose V_t^n in the following way

$$V_t^n = \sqrt{n}\,(X_0^n - x_0) + \sqrt{n}\,M_t^n + \int_0^t \sqrt{n}\,\big(F^n(X_s^n) - F(X_s^n(s))\big)\,\mathrm{d}s + \int_0^t \sqrt{n}\,\big(F(X_s^n) - F(x(s))\big)\,\mathrm{d}s$$

Equivalently

$$V_t^n = \sqrt{n}\,(X_0^n - x_0) + \sqrt{n}\,M_t^n + \int_0^t \sqrt{n}\,\big(F^n(X_s^n) - F(X_s^n(s))\big)\,\mathrm{d}s + \int_0^t DF(x(s)).V_s^n\,\mathrm{d}s + \int_0^t |V_s^n|\theta\big(X_s^n - x(s)\big)\,\mathrm{d}s$$

where θ is the error $\mathbb{R}^d$-valued function derived when differentiating F. We should mention that θ is continuous since DF is assumed to be continuous. Now, for the convergence of V_t^n we have

The first term of $V_0^n \Rightarrow 0$ by Hypotheses 5.1.

The second term $\Rightarrow \sum_{j=1}^{J}\int_0^t\int_{\mathbb{R}_+} G_j(x(s),u)\,W_j(\mathrm{d}s,\mathrm{d}u)$ (step 2).

The third term $\to 0$ with probability 1, due to uniform convergence of F^n.

The fourth term $\Rightarrow \int_0^t DF(x(s)).V_s\,\mathrm{d}s$ using mapping theorem since the mapping $z \mapsto \int_0^t DF(x(s)) \cdot z(s)\,\mathrm{d}s$ for $z \in D$ is continuous on D, see Lemma A.6.

The fifth term $\Rightarrow 0$: $\theta\big(X_s^n - x(s)\big) \xrightarrow[n\to\infty]{P} 0$ due again to mapping theorem (θ being continuous). Moreover, $|V_t^n| \Rightarrow |V(t)|$ (using the fact that the mapping $|\,.\,|$ is continuous. Using Slutsky theorem, we have then $|V_s^n|\;\theta\big(X_s^n - x(s)\big) \Rightarrow 0$. Due to the continuity of $z \mapsto \int_0^t z(s)ds$ in D, we conclude that the fifth term $\Rightarrow 0$

In consequence, $V^n \Rightarrow V$ which verifies necessarily (33).

Recapitulation Since in step 1, we proved that V^n is tight, and there exists a subsequence of V^n such that $V^n \Rightarrow V$ where V is a process in $D^d([0,\infty))$, and since in step 3, we proved that necessarily, V verifies a unique stochastic SDE in the weak sense, we prove in consequence that the whole sequence $V^n \Rightarrow V$. □

6. Conclusion

In this paper we studied LLN and TCL results in a specific case of jump Markov processes, where the jump sizes are reduced and the jump rates are multiplied by some scaling factor n. Other questions deserve interest like the speed convergence of X^n to the diffusion approximation:

$$\tilde{V}_t = \tilde{X}_0 + \int_0^t F(\tilde{V}_t(s))\mathrm{d}s + \sum_{j=1}^{J}\int_0^t\int_{\mathbb{R}_+} G_j(\tilde{V}(s),u)\,W_j(\mathrm{d}s,\mathrm{d}u) \tag{37}$$

Appendix A

Definition A.1. *(Stochastic boundedness) [28, Definition 3.4] A sequence $X_n, n \geq 1$ is stochastically bounded in D^d if $\{|X_n|_T, n \geq 1\}$ is stochastically bounded in $\mathbb{R}$ for each $T > 0$, where $|z|_T \equiv \sup_{0\leq t\leq T} |z(t)|$ for $z \in D^d$:*

$$\forall\ T>0, \forall \varepsilon>0, \exists K_{T,\varepsilon} \text{ compact in } \mathbb{R} \text{ such that } \underset{n\geq 0}{P}(|X^n|_T \notin K_{T,\varepsilon}) < \varepsilon$$

The following theorem is a step in [8, proof of theorem 7.1] and an equivalent version of the theorem is given by [21, Theorem 2.1 §9].

Lemma A.2. *(Tightness in D) [28, Lemma 3.11] A sequence of stochastic processes X_n in D is tight if:*

(i) The sequence $\{X_n\}$ is stochastically bounded.
(ii) For each $n\geq 1$, the stochastic process is adapted to a filtration $\mathcal{F}^n \equiv \{\mathcal{F}_{n,t}, t\geq 0\}$. In addition, for each $T>0$, there exists a family of nonnegative random variables $Z_n(\eta, T): n\geq 1, \delta>0$ such that:

$$\mathbb{E}_t\left[(X_n(t+u)-X_n(t))^2\right] \leq \mathbb{E}_t\left[Z_n(\eta,T)\right] w.p.1.\ for\ 0\leq t\leq T,\ 0\leq u\leq \eta$$

and

$$\lim_{\eta\to 0}\limsup_{n\to\infty}\mathbb{E}\left[Z_n(\eta,T)\right]=0$$

where $\mathbb{E}_t$ is the expectation with respect to $\mathcal{F}_{n,t}$.

Theorem A.3. *(Continuity Limit Theorem) [4] Let z a function in D and let $j_T(z) \equiv \sup_{0\leq t\leq T}|z(t)-z(t^-)|$. Let X_n and X be random elements in D and suppose $X_n \Rightarrow X$, then:*

$$j_T(X_n) \Rightarrow 0\ if\ and\ only\ if\ P[X\in\mathcal{C}]=1$$

where $\mathcal{C}$ is the set of continuous trajectories.

For the following FTCL theorem, an equivalent version is originally given in [8, theorem 7.1, p. 339]):

Theorem A.4. *(Multidimensional martingale FCLT) [28, Theorem 2.1] For $n\geq 1$, let $M_n \equiv \{M_{n,1},...,M_{n,d}\}$ be a local martingale in D^d with respect to a filtration $\mathcal{F}^n \equiv \{\mathcal{F}_{n,t}, t\geq 0\}$ satisfying $M_n(0)=\{0,...,0\}$. Let $C\equiv(c_{i,j})$ be a $d\times d$ covariance matrix (nonnegative definite symmetric matrix of real numbers). If*

i) The expected value of the maximum jumps in $\langle M_{n,i}, M_{n,j}\rangle$ and maximum squared jumps in M_n is asymptotically negligible; i.e., for each $T>0$

$$lim_{n\to\infty}\mathbb{E}\,[\sup_{0<t\leq T}(\langle M_{n,i},M_{n,j}\rangle_t - \langle M_{n,i},M_{n,j}\rangle_{t^-})]=0$$
$$lim_{n\to\infty}\mathbb{E}\,[(\sup_{0<t\leq T}|M_n(t)-M_n(t^-)|)^2]=0$$

ii) For each pair (i,j) with $1\leq i\leq d$ and $1\leq j\leq d$, and each $t>0$,

$$\langle M_{n,i}, M_{n,j}\rangle \Rightarrow c_{i,j}\ in\ \mathbb{R}\ as\ n\to\infty$$

Then,

$$M_n \Rightarrow M\ in\ D^d\ as\ n\to\infty$$

where M is a d-dimensional (0,C) Brownian motion:

$$\mathbb{E}[M(t)]=\{0,...,0\}\ and\ \mathbb{E}\left[M(t)M(t)^*\right]=Ct,\ t\geq 0$$

Theorem A.5. *(Mapping Theorem) [8, Corollary 1.9, p. 103] Let E be a metric space, and let $\{P_n, n < \infty\}$, and P be probability measures on E satisfying $P_n \Rightarrow P$. Let f be a real valued measurable function on S and define D_f to be the measurable set of points at which f is not continuous. Let X_n and X be random variables which induce the measures P_n and P on E, respectively. Then, $f(X_n) \Rightarrow f(X)$ whenever $P\{X \in D_f\} = 0$.*

Lemma A.6. *[19, lemma 5.2 and examples 5.3] For h continuous on $[0, \infty)$ and g continuous on $\mathbb{R}^d \times [0, \infty)$, the following mappings are continuous on $D^d([0, \infty))$ in the Skorohod topology*
i) $z \longrightarrow \int_0^t h(t-s)g(z(s), s)ds$
ii) $z \longrightarrow \sup_{s \le t} h(t-s)\, g(x(s) - x(s-), s)$

Theorem A.7. *(Slutsky Theorem) For any continuous function f on $\mathbb{R}^k \times \mathbb{R}^l$*

(i) *If $X^n \Rightarrow X \in \mathbb{R}^k$ and $Y^n \xrightarrow{P} c \in \mathbb{R}^l$, then:*

$$f(X^n, Y^n) \Rightarrow f(x, c)$$

(ii) *If X^n and Y^n are independent and if $X^n \Rightarrow X \in \mathbb{R}^k$ and $Y^n \Rightarrow c \in \mathbb{R}^l$, then:*

$$f(X^n, Y^n) \Rightarrow f(x, c)$$

References

[1] David F. Anderson, T.G. Kurtz, Continuous time Markov chain models for chemical reaction networks, in: Design and Analysis of Biomolecular Circuits, 2010.
[2] David F. Anderson, Thomas G. Kurtz, Stochastic Analysis of Biochemical Systems, Springer, 2015.
[3] Mark S. Bartlett, Amilcare Porporato, State-dependent jump processes: Itô-Stratonovich interpretations, potential, and transient solutions, Phys. Rev. E 98 (5) (2018) 052132.
[4] Patrick Billingsley, Convergence of Probability Measures, John Wiley & Sons, New York, 1968.
[5] Pierre Brémaud, Sampling cluster point processes: a review, preprint, arXiv:1604.06643, 2016.
[6] Erhan Çinlar, Probability and Stochastics, Springer, 2011.
[7] J.D.H. Doherty, P.A.L. Lancaster, The secular trend of twinning in Australia, 1853–1982, Acta Genet. Med. Gemellol.: Twin Res. 35 (1–2) (1986) 61–76.
[8] Stewart N. Ethier, Thomas G. Kurtz, Markov Processes – Characterization and Convergence, John Wiley & Sons, New York, 1986.
[9] B. Grigelionis, On the representation of integer-valued random measures by means of stochastic integrals with respect to the Poisson measure, Lith. Math. J. 11 (1) (1971) 93–108.
[10] Kais Hamza, Fima C. Klebaner, Conditions for integrability of Markov chains, J. Appl. Probab. 32 (2) (1995) 541–547.
[11] Fritz Horn, Roy Jackson, General mass action kinetics, Arch. Ration. Mech. Anal. 47 (2) (1972) 81–116.
[12] Nobuyuki Ikeda, Shinzo Watanabe, Stochastic Differential Equations and Diffusion Processes, second edition, North–Holland/Kodansha, 1989.
[13] Hye-Won Kang, Thomas G. Kurtz, Lea Popovic, Central limit theorems and diffusion approximations for multiscale Markov chain models, Ann. Appl. Probab. 24 (2) (2014) 721–759.
[14] G. Kersting, F.C. Klebaner, Sharp conditions for nonexplosions and explosions in Markov jump processes, Ann. Probab. 23 (1) (1995) 268–272.
[15] Fima C. Klebaner, Introduction to Stochastic Calculus with Applications, second edition, Imperial College Press, 2005.
[16] Thomas G. Kurtz, Solutions of ordinary differential equations as limits of pure jump Markov processes, J. Appl. Probab. 7 (1) (1970) 49–58.
[17] T.G. Kurtz, Limit theorems for sequences of jump Markov processes, J. Appl. Probab. 8 (2) (1971) 344–356.
[18] Thomas G. Kurtz, et al., Strong approximation theorems for density dependent Markov chains, Stoch. Process. Appl. 6 (3) (1978) 223–240.
[19] Thomas G. Kurtz, Philip Protter, Weak limit theorems for stochastic integrals and stochastic differential equations, Ann. Probab. 19 (3) (1991) 1035–1070.
[20] Thomas G. Kurtz, Philip E. Protter, Weak convergence of stochastic integrals and differential equations I, in: Probabilistic Models for Nonlinear Partial Differential Equations, Springer, 1996, pp. 1–41.
[21] Harold Kushner, Paul G. Dupuis, Numerical Methods for Stochastic Control Problems in Continuous Time, vol. 24, Springer Science & Business Media, 2013.
[22] Breiman Leo, Probability, Addison-Wesley, Mass., 1968.
[23] National Research Council, et al., Using Science to Improve the BLM Wild Horse and Burro Program: A Way Forward, National Academies Press, 2013.

[24] Eric Renshaw, Stochastic Population Processes: Analysis, Approximations, Simulations, Oxford University Press, 2015.
[25] M. Ross Sheldon, Stochastic Processes, John Wiley, 1996.
[26] Winfried Stute, On a generalization of the Glivenko-Cantelli theorem, Z. Wahrscheinlichkeitstheor. Verw. Geb. 35 (2) (1976) 167–175.
[27] Salwa Toumi, Mohsen Chebbi, Fabien Campillo, Stochastic modeling for biotechnologies anaerobic model am2b, ARIMA Rev. Afr. Rech. Inform. Math. Appl. 28 (2019).
[28] Ward Whitt, et al., Proofs of the martingale fclt, Probab. Surv. 4 (2007) 268–302.