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A Control Theory for Cartesian Flexible Robot Arms

Xuezhang Hou and Sze-kai Tsui*

Department of Mathematical Sciences, Oakland University, Rochester, Michigan 48309-4401

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In this article, we first use a fourth order partial differential equation with boundary conditions to model a flexible robot arm on a moving base with a payload at the tip end. Through the state-space formulation, we show that such a system is both controllable and observable in an infinite dimensional Hilbert space. We also show that the system is stabilizable via a feedback control.

Key Words: the beam equation; control systems; controllability; observability; stability; stabilization; Lyapunov functions; compact resolvents; self-adjoint operators.

1. INTRODUCTION

Structural flexibility in robotic systems has been emerging as an issue of increasing concern, for it is only realistic to include the vibration of such a system in the design of control to secure a certain degree of accuracy. The demands for high speed and low cost are driving the research for control of lightweight flexible robots. In this paper, we first formulate a mathematical model for a flexible robot arm on a moving base with a payload at the tip end. In general a Cartesian robot consists of components which are flexible robot arms with a payload at the tip end. There have been many investigations of the subject of flexible beam and its control. Among them we list a few, such as works of Cannon and Schmitz [1] in 1984, and works of Goong Chen *et al*. (see references in [2]), F. L. Huang [7], and Z. H. Luo *et al*. [9, 10] We take the fourth order well-known Euler–Bernoulli beam equation to model the transverse vibration of the flexible robot arm with four boundary conditions, which are similar to but not identical with that in [9]. This model is quite unique and different from all other models

* E-mail: tsui@oakland.edu.

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found in [2], [7], [9], and [10]. Our approach is more theoretical than that in [9, 10] and in general. Our methodology starts with a corresponding state-space control system in which the parameter matrix has as its entries differential operators. In this setting we are able to determine the spectrum of the parameter matrix (see Section 2) and, subsequently, to show that the robotic system with a sliding base is both approximately controllable and observable (see Section 3). In this infinite dimensional control analysis, one needs a heavy dose of functional analysis and operator theory in order to investigate approximation controllability and observability of the system. We also have investigated the stability in a specified feedback control, and we show that we may design a boundary point feedback control loop to stabilize the system (see Section 4). This work has laid down a foundation for the design of a real-time closed-loop feedback control for a flexible Cartesian robot. It is becoming more urgent that the traditional design of robot arms dependent on only the kinematics needs a makeover to include the dynamics of the system in the control. Our work fits nicely in this thrust of research which is becoming the focus of the research of dynamical robotics. Further work along this line is presently being pursued [6].

2. THE MODEL AND SPECTRAL STRUCTURE OF A FLEXIBLE ROBOT SYSTEM

2.1. *The Model and the Evolution Equation of the System*

Consider a Cartesian robot with a long tip arm, illustrated in Fig. 1. Since any motion in the x–y plane can be decomposed into its x and y components, the vibrations in the x-direction and y-direction can be considered independently. Figure 2 shows motion of an x–y robot in the x-direction. m represents the mass of a moving body driven by a control motor. The one end with a payload M of the flexible arm is attached to this moving body.

Let the amplitude of vibration of the flexible arm at time t and position r be $w(t,r)$. Then the dynamic model for vibration of this flexible arm in the x-direction can be written as follows [9]:

$$\begin{aligned} \rho\ddot{w}(t,r) + EIw''''(t,r) &= -\rho\ddot{x}(t), \qquad 0 < r < t,\ t > 0, \\ w(t,0) &= w'(t,0) = 0, \\ M[\ddot{w}(t,l) + \ddot{x}(t)] - EIw'''(t,l) &= 0, \\ J\ddot{w}'(t,l) + EIw''(t,l) &= 0, \\ w(0,r) &= w_0(r), \qquad \dot{w}(0,r) = w_1(r), \end{aligned} \tag{2.1}$$

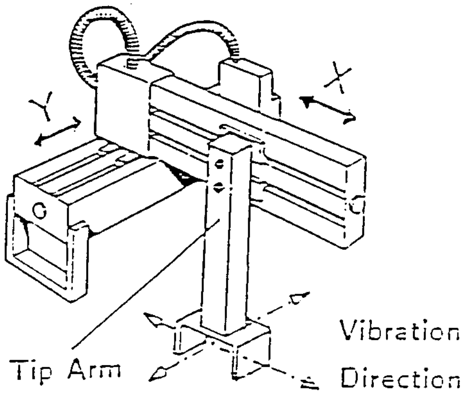

FIG. 1. A Cartesian robot with a long tip arm.

where $\dot{x}(t)$ denotes the acceleration of the moving body, "." denotes the time derivative, and "′" denotes the spatial derivative, ρ denotes the line density of mass for the arm, EI denotes the bending rigid degree of the flexible arm, l denotes the length of the arm, $w_0(r)$ and $w_1(r)$ denote the initial displacement and initial velocity of the arm, respectively, J denotes the turning inertia, and the first equation is the Euler−Bernoulli equation.

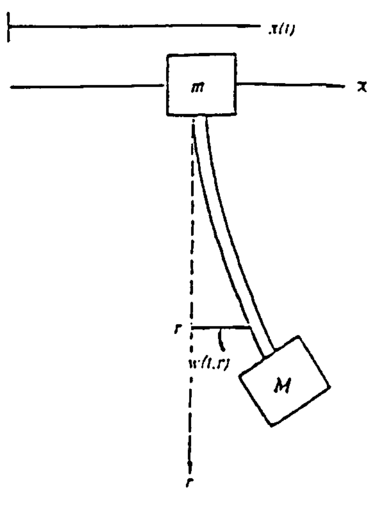

FIG. 2. Motion of an x−y robot in the x-direction.

For the motor system, we shall establish the following control equation:

$$m\dot{x}(t) = u(t) - EIw'''(t,0), \tag{2.2}$$

where the sliding friction was neglected and $u(t)$ is a control.

Let $y(t,r)$ be the total displacement in the x-direction of the flexible arm. Thus, we have

$$y(t,r) = w(t,r) + x(t). \tag{2.3}$$

Substituting (2.3) into (2.1) yields the following controlled closed-loop system equation about state $y(t,r)$:

$$\begin{aligned}
\rho\ddot{y}(t,r) + EIy''''(t,r) &= 0, \qquad 0 < r < l,\ t > 0,\\
y'(t,0) &= 0,\\
m\ddot{y}(t,0) + EIy'''(t,0) &= u(t),\\
M\ddot{y}(t,l) - EIy'''(t,l) &= 0,\\
J\ddot{y}'(t,l) + EIy''(t,l) &= 0.
\end{aligned} \tag{2.4}$$

In order to investigate the system (2.4) under the abstract frame, we now consider a real Hilbert space $H = R^3 \times L^2_\rho(0,l)$ equipped with the inner product as

$$(\Phi_1, \Phi_2)_H = m\xi_1\xi_2 + M\eta_1\eta_2 + J\zeta_1\zeta_2 + \langle\varphi_1, \varphi_2\rangle_\rho,$$

where $\Phi_i = [\xi_i, \eta_i, \zeta_i, \varphi_i]^\tau \in H$, $i = 1,2$, $\langle\varphi_1, \varphi_2\rangle_\rho = \int_0^l \rho\varphi_1(x)\overline{\varphi}_2(x)\,dx$, and τ means the transpose. We define a linear operator A with domain $D(A)$ in H as follows:

$$A\tilde{\varphi} = \begin{bmatrix} \dfrac{EI}{m}\varphi'''(0) \\ -\dfrac{EI}{M}\varphi'''(l) \\ \dfrac{EI}{J}\varphi''(l) \\ \dfrac{EI}{\rho}\varphi''''(\cdot) \end{bmatrix}, \qquad \text{for } \tilde{\varphi} = \begin{bmatrix} \varphi(0) \\ \varphi(l) \\ \varphi'(l) \\ \varphi(\cdot) \end{bmatrix} \in D(A),$$

where $D(A) = \{\tilde{\varphi} \in H\colon \varphi, \varphi', \varphi'', \varphi''', \varphi'''' \in L^2_\rho(0,l),\ \varphi'(0) = 0\}$.

Using the operator A, (2.4) becomes the following second-order abstract evolution equation in H:

$$\frac{d^2\tilde{y}(t)}{dt^2} + A\tilde{y}(t) = bu(t), \tag{2.5}$$

where $\tilde{y}(t) = [y(t,0), y(t,l), y'(t,l), y(t,\cdot)]^{\tau}$, $b = [1/m, 0, 0, 0]^{\tau}$.

2.2. *The Spectral Structure of the Operator A*

In this section we shall investigate the spectrum of the operator A in (2.5).

THEOREM 2.1. *A: $D(A) \to H$ is a nonnegative self-adjoint operator.*

Proof. It is clear that $\overline{D(A)} = H$. For any $\tilde{\varphi}, \tilde{\psi} \in D(A)$, using integration by parts, we have

$$\left(A\tilde{\varphi}, \tilde{\psi}\right)_H = EI\int_0^l \varphi''(x)\psi''(x)\,dx = \left(\tilde{\varphi}, A\tilde{\psi}\right)_H,$$

and A is the symmetric operator; moreover,

$$(A\tilde{\varphi}, \tilde{\varphi})_H = EI\int_0^l |\varphi''(x)|^2\,dx \geq 0, \qquad \tilde{\varphi} \in D(A).$$

Thus, A is nonnegative. It can be checked that A is a closed linear operator. Consider the restriction of A on the orthogonal complement K of the kernel of A. $A|_K$ is densely defined and closed, and by the symmetry of $A|_K$ we know that $A(K)$ is dense in K. Hence by the open mapping theorem and the fact that $A(K)$ is of second category in K, we have that the range of $A|_K$ is open and the range of A is K [14]. Therefore, $A|_K$ and A are self-adjoint.

Let λ be an eigenvalue of A, and suppose that $\tilde{\varphi}$ is the eigenvector corresponding to λ. Then, $A\tilde{\varphi} = \lambda\tilde{\varphi}$, namely,

$$\begin{aligned} EI\varphi''''(x) &= \lambda\rho\varphi(x), \qquad 0 < x < l, \\ \varphi'(0) &= 0, \\ EI\varphi'''(0) - \lambda m\varphi(0) &= 0, \\ EI\varphi'''(l) + \lambda M\varphi(l) &= 0, \\ EI\varphi''(l) - \lambda J\varphi'(l) &= 0. \end{aligned} \tag{2.6}$$

It is obvious that $\lambda_0 = 0$ is an eigenvalue of $\mathcal{A}$ with its eigenvectors of the form $\beta[1,1,0,1]^\tau$ for some scalar β. We denote $[1,1,0,1]^\tau$ by $\tilde{\varphi}_0$. Let $\nu^4 = \lambda\rho/EI$ ($\lambda > 0$). Then it follows from (2.6) that

$$\begin{aligned} \varphi''''(x) &= \nu^4\varphi(x), \\ \varphi'(0) &= 0, \\ \varphi'''(0) - \nu^4\frac{m}{\rho}\varphi(0) &= 0, \\ \varphi'''(l) + \nu^4\frac{M}{\rho}\varphi(l) &= 0, \\ \varphi''(l) - \nu^4\frac{J}{\rho}\varphi'(l) &= 0. \end{aligned} \tag{2.7}$$

The general solution of (2.7) can be obtained as follows:

$$\begin{aligned} \varphi(x) = {} & \frac{\tilde{a}}{2}(\cosh\nu x + \cos\nu x) + \frac{\tilde{b}}{2\nu^2}(\cosh\nu x - \cos\nu x) \\ & + \frac{\tilde{c}}{2\nu^3}(\sinh\nu x - \sin\nu x), \end{aligned} \tag{2.8}$$

where $\tilde{a} = \varphi(0)$, $\tilde{b} = \varphi''(0)$, $\tilde{c} = \varphi'''(0)$. Let $y = \nu l$. Then it follows that

$$\begin{aligned} \varphi(l) = {} & \frac{\tilde{a}}{2}(\cosh z + \cos z) + \frac{\tilde{b}}{2\nu^2}(\cosh z - \cos z) \\ & + \frac{\tilde{c}}{2\nu^3}(\sinh z - \sin z), \\ \varphi'(l) = {} & \frac{\tilde{a}\nu}{2}(\sinh z - \sin z) + \frac{\tilde{b}}{2\nu}(\sinh z + \sin z) \\ & + \frac{\tilde{c}}{2\nu^2}(\cosh z - \cos z), \\ \varphi''(l) = {} & \frac{\tilde{a}\nu^2}{2}(\cosh z - \cos z) + \frac{\tilde{b}}{2}(\cosh z + \cos z) \\ & + \frac{\tilde{c}}{2\nu}(\sinh z + \sin z), \\ \varphi'''(l) = {} & \frac{\tilde{a}\nu^3}{2}(\sinh z + \sin z) + \frac{\tilde{b}\nu}{2}(\sinh z - \sin z) \\ & + \frac{\tilde{c}}{2}(\cosh z + \cos z). \end{aligned}$$

THEOREM 2.2. (i) *Let* $\tilde{\varphi} = [\varphi(0), \varphi(l), \varphi'(l), \varphi(\cdot)]^\tau$ *be any nonzero eigenvector corresponding to an eigenvalue* $\lambda > 0$ *of* A. *Then* $\varphi(0) \neq 0$ *and* $\varphi'(l) \neq 0$.

(ii) *Every eigenvalue of* A *is of multiplicity* 1; *that is, every eigenspace corresponding to an eigenvalue of* A *is one-dimensional.*

Proof. (i) If $\varphi(0) = 0$, then in view of the boundary conditions of (2.7) it follows that $\varphi'''(0) = 0$, and subsequently $\varphi(x) = \tilde{b}'(\cosh \nu x - \cos \nu x)$; here $\tilde{b}' = \tilde{b}/2\nu^2$, $\nu^4 = \lambda\rho/EI$. Hence, we can derive from the boundary condition of (1.7) that

$$\sinh z - \sin z + \alpha z(\cosh z - \cos z) = 0.$$

It is easy to see that the above equation has no positive root. This contradicts the hypothesis that $\lambda > 0$. Thus, $\varphi(0) \neq 0$.

If $\varphi'(l) = 0$, then it follows from (2.7) that $\varphi''(l) = 0$, that is,

$$\tilde{a}\nu^3(\sinh z - \sin z) + \tilde{b}\nu(\sinh z + \sin z) + \tilde{c}(\cosh z - \cos z) = 0,$$

$$\tilde{a}\nu^3(\cosh z - \cos z) + \tilde{b}\nu(\cosh z + \cos z) + \tilde{c}(\sinh z + \sin z) = 0.$$

Since $\tilde{c} = \nu^4(m/\rho)\tilde{a}$, by eliminating the $\tilde{b}$ term from the above two equations, we get

$$\tilde{a}\left[(\sinh z \cos z - \sin z \cosh z) - \frac{\nu m}{\rho}\sinh z \sin z\right] = 0.$$

For $z > 0$, it can easily be seen that $\cos z \sinh z - \sin z \cosh z - (\nu m/\rho)\sinh z \sin z \neq 0$, and so $\tilde{a} = 0$ and $\tilde{c} = 0$. Thus, $\varphi(x) = \tilde{b}(\cosh \nu x - \cos \nu x)$, but $\varphi'''(l) + \nu^4(m/\rho)\varphi(l) = 0$. Therefore,

$$\sinh z - \sin z + \alpha z(\cosh z - \cos z) = 0.$$

Note that the above equation has no positive root, a contradiction to the assumption, $\varphi'(l) \neq 0$. Thus, $\varphi'(l) \neq 0$.

(ii) From the paragraph following (2.6) we know that the eigenspace of zero is one-dimensional. Suppose that there are two eigenvectors $\tilde{\varphi}$ and $\tilde{\psi}$ for the same eigenvalue $\lambda > 0$ of A. Then $\tilde{f} = \psi(0)\tilde{\varphi} - \varphi(0)\tilde{\psi}$ is also an eigenvector for λ of A, but $f(0) = 0$, and it follows from (i) that $\tilde{f} \equiv 0$. This implies that $\tilde{\varphi}$ and $\tilde{\psi}$ are linear dependent, and therefore the eigenspace corresponding to this eigenvalue of A is one-dimensional.

THEOREM 2.3. *The resolvent of* A *is a compact operator.*

Proof. Since A is the self-adjoint operator by Theorem 2.1, it follows from a theorem in [3, XIII.4] that the resolvent $R(\lambda, A)$ of A is compact.

THEOREM 2.4. *The spectrum of A consists of only nonnegative eigenvalues with single multiplicity.*

Proof. This is a direct consequence of a theorem in [5, XIII.4], and Theorems 2.1, 2.2, and 2.3.

3. THE WELL-POSEDNESS, CONTROLLABILITY, AND OBSERVABILITY OF THE SYSTEM

3.1. *The Existence and Uniqueness of the Solution to the System*

For any $r > 0$, let $A_r \triangleq A + rI$ (I denotes the identity operator on H). Denote $\mathscr{H} \triangleq H \times H$ equipped with the inner product

$$\langle \Phi, \Psi \rangle = (\xi_1, \eta_1)_H + (\xi_2, \eta_2)_H,$$
$$\text{for every } \Phi = [\xi_1, \xi_2]^\tau, \Psi = [\eta_1, \eta_2]^\tau \in \mathscr{H}.$$

It is easy to see that with the inner product defined above $\mathscr{H}$ is a Hilbert space.

We now define a linear operator $\mathscr{A}$ on $\mathscr{H}$ below. For r in $\rho(A)$, the resolvent set of A, we define

$$\mathscr{A} = \begin{bmatrix} 0 & A_r^{1/2} \\ -A_r^{1/2} + rA_r^{-1/2} & 0 \end{bmatrix}$$

with $D(\mathscr{A}) = D(A_r^{1/2}) \times D(A_r^{1/2})$. We also denote $[0, b]^\tau$ by $\mathscr{B}$, where b is defined in (2.5). Consider a subspace $\mathscr{S}$ of $\mathscr{H}$ consisting of $z = [z_1, z_2]^\tau$, where $z_1 = A_r^{1/2}\tilde{y}$ and $z_2 = \dot{\tilde{y}}$ and $\tilde{y}$ is defined in (2.5) In this notation, (2.5) with initial conditions

$$\begin{aligned} \tilde{y}(0) &= \tilde{y}_0 \\ \dot{\tilde{y}}(0) &= \tilde{y}_1 \end{aligned} \tag{3.1}$$

becomes a first-order evolution equation in $\mathscr{S}$ with initial conditions as follows:

$$\begin{aligned} \frac{dz}{dt} &= \mathscr{A}z + \mathscr{B}u, \\ z(0) &= \left[A_r^{1/2}\tilde{y}_0, \tilde{y}_1\right]^\tau. \end{aligned} \tag{3.2}$$

We denote the spectrum of $\mathscr{A}$ by $\sigma(\mathscr{A})$, a resolvent of A by $R(\lambda, A)$, and the resolvent set of A by $\rho(A)$.

LEMMA 3.1.

(i) $\mathscr{A}^* = \left[\begin{smallmatrix} 0 & -A_r^{1/2} + rA_r^{-1/2} \\ A_r^{1/2} & 0 \end{smallmatrix}\right]$.

(ii) $D(\mathscr{A}) = D(\mathscr{A}^*)$.

(iii) $\sigma(\mathscr{A}) = \sigma(\mathscr{A}^*)$.

(iv) *The resolvents of $\mathscr{A}$ and $\mathscr{A}^*$ are compact.*

(v) *$\mathscr{A}$ is densely defined closed linear operator and the spectrum of $\mathscr{A}$ denoted by $\sigma(\mathscr{A})$ is equal to $\{\mu_n\colon\ \mu_0 = 0,\ \mu_n = \pm i\sqrt{\lambda_n},\ n = 1, 2, \ldots\}$, where λ_n, $n = 1, 2, \ldots,$ are the eigenvalues of A.*

Proof. (i)–(iii) Parts (i), (ii), and (iii) follow from A being self-adjoint.

(iv) Let

$$\mathscr{A}_0 \begin{bmatrix} 0 & A_r^{1/2} \\ -A_r^{1/2} & 0 \end{bmatrix} \quad \text{and} \quad \mathscr{D} = \begin{bmatrix} 0 & 0 \\ rA_r^{-1/2} & 0 \end{bmatrix}.$$

Then it is clear that $D(\mathscr{A}_0) = D(A_r^{1/2}) \times D(A_r^{1/2})$, $d(\mathscr{D}) = \mathscr{H}$, and $\mathscr{D}$ is bounded on $\mathscr{H}$. It is also clear that $\mathscr{A} = \mathscr{A}_0 + \mathscr{D}$ and $(i\mathscr{A}_0)^* = i\mathscr{A}_0$. By Stone's theorem on infinitesimal generator of a one-parameter group of unitary operators. By the Hille–Yosida theorem [3] we know that $\|R(\lambda, \mathscr{A}_0)\| \leq 1/\lambda$ for $\lambda > 0$. Let $\lambda \in \rho(\mathscr{A}_0)$. We have

$$R(\lambda, \mathscr{A}_0) = \begin{bmatrix} \lambda\left(A_r + \lambda^2\right)^{-1} & A_r^{1/2}\left(A_r + \lambda^2\right)^{-1} \\ -A_r^{1/2}\left(A_r + \lambda^2\right)^{-1} & \lambda\left(A_r + \lambda^2\right)^{-1} \end{bmatrix}.$$

Since every entry in $R(\lambda, \mathscr{A}_0)$ is a compact operator, it follows that $R(\lambda, \mathscr{A}_0)$ is compact.

Since $\mathscr{D}$ is bounded, it follows that $\|\mathscr{D}\| < \lambda_0$ for some λ_0. Thus,

$$\begin{aligned} R(\lambda_0, \mathscr{A}) &= \left(\lambda_0 - (\mathscr{A}_0 + \mathscr{D})\right)^{-1} = \left((\lambda_0 - \mathscr{A}_0) - \mathscr{D}\right)^{-1} \\ &= R(\lambda_0, \mathscr{A}_0)\left[I - R(\lambda_0, \mathscr{A}_0)\mathscr{D}\right]^{-1} \\ &= R(\lambda_0, \mathscr{A}_0) \sum_{n=0}^{\infty} \left(R(\lambda_0, \mathscr{A}_0)\mathscr{D}\right)^n. \end{aligned}$$

The convergence of the infinite series in the above equation is due to $\|R(\lambda_0, \mathscr{A}_0)\mathscr{D}\| \leq \|R(\lambda_0, \mathscr{A}_0)\|\,\|\mathscr{D}\| < \|R(\lambda_0, \mathscr{A}_0)\|\lambda_0 < 1$. Thus $R(\lambda_0, \mathscr{A})$ is compact, and hence $R(\lambda, \mathscr{A})$ is compact for each λ in $\rho(\mathscr{A})$. Similarly, one can show that $R(\lambda, \mathscr{A}^*)$ is also compact.

(v) By Theorems 2.2 and 2.4, let $\{\tilde{\varphi}_n\}_{n=0}^{\infty}$ be an orthonormal set of eigenvectors of A corresponding to the eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ with $\lambda_0 = 0 < \lambda_1 < \lambda_2 < \cdots$. Let μ be an eigenvalue of $\mathscr{A}$. Thus,

$$\mathscr{A}\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix} = \mu\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}, \qquad \text{for } \begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix} \in \mathscr{H}\times\mathscr{H},$$

which is $A_r^{1/2}\varphi_2 = \mu\varphi_1$ and $(-A_r^{1/2} + rA_r^{-1/2})\varphi_1 = \mu\varphi_2$. From this we have $A^{-1/2}(-A_r + r)\varphi_1 = \mu(A_r^{-1/2}\mu\varphi_1)$ and thus $-A\varphi_1 = \mu^2\varphi_1$. Hence μ has to be $\pm i\sqrt{\lambda_n}$ for $n > 0$ (denoted by $\pm\mu_n$), and $[\tilde{\varphi}_n, (\pm\mu_n/\sqrt{\lambda_n + r})\tilde{\varphi}_n]^\tau$, denoted by $\Phi_{\pm n}$, are eigenvectors corresponding to $\pm\mu_n$, respectively. Zero is an eigenvalue of $\mathscr{A}$, for $\mathscr{A}[\varphi_1, \varphi_2]^\tau = [0,0]^\tau$ implies $[\varphi_1, \varphi_2] = \beta[\tilde{\varphi}_0, 0]$ for some scalar β, and we denote $[\tilde{\varphi}_0, 0]^\tau$ by Φ_{+0}. By a theorem in [5, XIII.4], that the spectrum of $\mathscr{A}$, $\sigma(\mathscr{A})$, is $\{\mu_n\colon \mu_0 = 0,\ \mu_n = \pm i\sqrt{\lambda_n},\ n \le 1\}$, where λ_n, $n = 1, 2, \ldots,$ are the eigenvalues of A.

LEMMA 3.2. *The operator $\mathscr{A}$ is the infinitesimal generator of a C_0 semigroup $T(t)$, $t \ge 0$, on $\mathscr{H}$ satisfying $\|T(t)\| \le e^{\|\mathscr{D}\|t}$, where $\mathscr{D}$ is defined in the proof of Lemma* 3.1.

Proof. Since $(i\mathscr{A}_0)^* = i\mathscr{A}_0$, we know from the celebrated Stone theorem [8] that $\mathscr{A}_0$ is the infinitesimal generator of a C_0 group of unitary operators on $\mathscr{H}$. Since $\mathscr{D}$ is a bounded linear operator of $\mathscr{H}$, it follows from the perturbation theory of the semigroups of linear operators [4] that $\mathscr{A} = \mathscr{A}_0 + \mathscr{D}$ generate a C_0 semigroup on $\mathscr{H}$, denoted by $T(t)$, $t \ge 0$, and $\|T(t)\| \le e^{\|\mathscr{D}\|t}$.

THEOREM 3.3. (i) *If $\tilde{z}_0 \in D(A^{1/2})$, $\tilde{z}_1 \in H$, $u \in L^2(0,T)$ $(0 < T \le +\infty)$, then the abstract Cauchy problem* (3.2) *has a unique mild solution*:

(ii) *if $\tilde{z}_0 \in D(A)$, $\tilde{z}_1 \in D(A^{1/2})$, $u \in C^1(0,T)$ $(0 < T \le +\infty)$, the Cauchy problem* (3.2) *has a unique classical solution*.

Proof. We first note that $D(A_r^{1/2}) = D(A^{1/2}) \supset D(A)$. It follows from the assumption in (i) that $z_0 = [(A_r^{1/2}\tilde{y}_0)^\tau, \tilde{y}_1^\tau]^\tau \in \mathscr{H}$, and then problem (3.2) has a unique mild solution. Since the hypothesis (ii) implies that $z_0 \in D(\mathscr{S})$, and hence problem (3.2) has a unique classical solution [12].

Remark. The existence and uniqueness of problem (3.2) are equivalent to the existence and uniqueness of original flexible robot control system.

3.2. *The Approximate Controllability and Observability of the System*

The following two theorems can be found in [3], and their proofs are not included here.

THEOREM 3.4. *Let $T(t)$ be the C_0 semigroup generated by $\mathscr{A}$. System* (3.2) *is approximately controllable if and only if one of the following is satisfied*:

(i) *If $\mathscr{B}^* T(t)^* \Psi = 0$, then $\Psi = 0$ for Ψ in $\mathscr{H}$.*

(ii) *If $\mathscr{B}^* R(\lambda, \mathscr{A})^* \Psi = 0$ for all λ in $\rho(\mathscr{A})$, then $\Psi = 0$ for Ψ in $\mathscr{H}$.*

THEOREM 3.5. *Let $W(t) = \langle C, z \rangle_{\mathscr{H}}$ be an observation equation. System* (3.2) *is observable if and only if one of the following is satisfied*:

(i) *If $C^* T(t)\Phi = 0$, then $\Phi = 0$ for Φ in $\mathscr{H}$.*

(ii) *If $C^* R(\lambda, \mathscr{A})\Phi = 0$ for all λ in $\rho(\mathscr{A})$, then $\Phi = 0$ for Φ in $\mathscr{H}$.*

It is clear from the proof of (v) of Lemma 3.1 that the positive eigenvalues of $\mathscr{A}$ have multiplicity 1. Now we show that the zero eigenvalue has multiplicity 2. $\mathscr{A}([\varphi_1, \varphi_2]^\tau) = \Phi_{+0}$ implies $[\varphi_1, \varphi_2] = [\beta\tilde{\varphi}_0, (1/\sqrt{r})\tilde{\varphi}_0]$ for some scalar β, and we denote $[0, \tilde{\varphi}_0]^\tau$ by Φ_{-0}. Since $\mathscr{A}([\varphi_1, \varphi_2]^\tau) = \Phi_{-0}$ has no solution for φ_1, φ_2, we see that the eigenspace of 0 is spanned by Φ_{+0} and Φ_{-0}. Similarly, we can get a family

$$\left\{ \Psi_{\pm n} = \left[\tilde{\varphi}_n, \pm \frac{\sqrt{\lambda_n + r}}{\mu_n} \tilde{\varphi}_n \right]^\tau \right\}_{n=1}^{\infty}$$

of eigenvectors of $\mathscr{A}^*$ corresponding to $\pm\mu_n$. The eigenspace of 0 for $\mathscr{A}^*$ is spanned by Φ_{+0} and Φ_{-0} with $\mathscr{A}^* \Phi_{-0} = 0$ and $\mathscr{A}^* \Phi_{+0} = \Phi_{-0}$. We normalize $\{\Phi_n\}, \{\Psi_n\}$, $n = \pm 1, \pm 2, \ldots,$ in the following way. Let $\tilde{\Phi}_n = \Phi_n / \|\Phi_n\|$ and $\langle \tilde{\Psi}_n, \tilde{\Phi}_n \rangle = 1$ for all $n = \pm 1, \pm 2, \ldots$. Then $\{\tilde{\Phi}_n\}$ and $\{\tilde{\Psi}_n\}$ form a biorthogonal basis for $\mathscr{H}$, i.e., $\langle \tilde{\Psi}_m, \tilde{\Phi}_n \rangle = \delta_{m,n}$, such that every Φ in $\mathscr{H}$ can be uniquely expressed as

$$\Phi = \sum_{|n|=1}^{\infty} \langle \Phi, \tilde{\Psi}_n \rangle \tilde{\Phi}_n + \langle \Phi, \Phi_{+0} \rangle \Phi_{+0} + \langle \Phi, \Phi_{-0} \rangle \Phi_{-0} \tag{3.3}$$

for $\langle \tilde{\Phi}_n, \tilde{\Psi}_k \rangle = \delta_{nk}$ and $\langle \tilde{\Psi}_n, \Phi_{+0} \rangle = \langle \tilde{\Psi}_n, \Phi_{-0} \rangle = \langle \Phi_{+0}, \Phi_{-0} \rangle = 0$. Now we are ready for the following theorem.

THEOREM 3.6. *Let $b \in H$, $\mathscr{B} = [0, b]^\tau$. Then system* (3.2) *is approximately controllable if and only if*

$$(b, \tilde{\varphi}_n)_H \neq 0, \qquad n \geq 0. \tag{3.4}$$

Proof. For $\lambda \in \rho(\mathscr{A})$ and $\Phi \in \mathscr{H}$, we have by (3.3)

$$R(\lambda, \mathscr{A})\Phi = \sum_{|n|=1} \frac{\langle \Phi, \tilde{\Psi}_n \rangle}{\lambda - \mu_n} \tilde{\Phi}_n + \frac{1}{\lambda} \langle \Phi, \Phi_{+0} \rangle \Phi_{+0}$$
$$+ \frac{1}{\lambda} \langle \Phi, \Phi_{-0} \rangle \Phi_{-0} + \frac{\sqrt{r}}{\lambda^2} \langle \Phi, \Phi_{-0} \rangle \Phi_{+0}.$$

For every $u \in L^2[0, T]$ $(0 < T \leq \infty)$, we have

$$\begin{aligned}
&\langle \mathscr{B}^* R(\lambda, \mathscr{A})^* \Psi, u \rangle \\
&\quad = \langle \Psi, R(\lambda, \mathscr{A}) \mathscr{B} u \rangle \\
&\quad = \sum_{|n|=1}^{\infty} \frac{\langle \Psi, \tilde{\Phi}_n \rangle}{\lambda - \mu_n} \langle \mathscr{B} u, \tilde{\Psi}_n \rangle + \frac{1}{\lambda} \langle Bu, \Phi_{+0} \rangle \langle \Psi, \Phi_{+0} \rangle \\
&\qquad + \frac{1}{\lambda} \langle \mathscr{B} u, \Phi_{-0} \rangle \langle \Psi, \Phi_{-0} \rangle + \frac{\sqrt{r}}{\lambda^2} \langle \mathscr{B} u, \Psi_{-0} \rangle \langle \Psi, \Phi_{+0} \rangle \\
&\quad = \sum_{|n|=1}^{\infty} \beta_n \frac{\langle \Psi, \tilde{\Phi}_n \rangle}{\lambda - \mu_n} \frac{\sqrt{\lambda_n + \gamma}}{2\mu_n} (bu, \tilde{\varphi}_n)_H + \frac{1}{\lambda} (bu, \tilde{\varphi}_0)_n \langle \Psi, \Phi_{-0} \rangle \\
&\qquad + \frac{\sqrt{r}}{\lambda^2} (bu, \tilde{\varphi}_0)_H \langle \Psi, \Phi_{+0} \rangle,
\end{aligned} \tag{3.5}$$

where $\tilde{\Psi}_n = \beta_n \Psi_n$. It is clear that $\{1/(\lambda - \mu_n), 1/\lambda, 1/\lambda^2\}$ are linearly independent functions in λ. Thus, the above equation equals zero for all μ in $L^2[0, T]$ if and only if

$$\begin{aligned}
\langle \Psi, \tilde{\Phi}_n \rangle (bu, \tilde{\varphi}_n)_H &= 0, \\
\langle \Psi, \Phi_{-0} \rangle (bu, \tilde{\varphi}_0)_H &= 0, \\
\langle \Psi, \Phi_{+0} \rangle (bu, \tilde{\varphi}_0)_H &= 0,
\end{aligned} \tag{3.6}$$

for $|n| = 1, 2, \dots$ and all u in $L^2[0, T]$. Now (3.4) implies that there exists μ_n in $L^2[0, T]$ such that $(bu_n, \tilde{\varphi}_n)_H \neq 0$ for $n = 0, 1, 2, \dots$. Letting $u = u_n$ in (3.6) we see that (3.3) implies $\langle \Psi, \tilde{\Phi}_n \rangle = \langle \Psi, \Phi_{+0} \rangle = \langle \Psi, \Phi_{-0} \rangle = 0$ for $|n| = 1, 2, \dots,$ in (3.5), and hence $\Psi = 0$. By this, we have shown that condition (3.3) is sufficient via Theorem 3.4. Conversely suppose that $(b, \tilde{\varphi}_n)_H = 0$ for all $n = 0, 1, 2, \dots$. It follows from (3.4) that $\langle \mathscr{B}^* \mathscr{R}(\lambda, \mathscr{A})^* \Psi, \mu \rangle_0 = 0$ for all Ψ, and (ii) in Theorem 3.4 does not hold. Hence, Theorem 3.6 is proved.

THEOREM 3.7. *The flexible robot system* (3.2) *is approximately controllable.*

Proof. We first note that, in system (3.2), $b = [1/m, 0, 0, 0]^{\tau}$, and then $(b, \tilde{\varphi}_n)_H = \varphi_n(0)$ $(n \geq 0)$. However, in view of Theorem 2.4 we see that $\varphi_n(0) \neq 0$ and thus we know from Theorem 3.6 that flexible robot system (3.2) is approximately controllable.

Next, we shall discuss the observability of the flexible robot system. Let

$$b_1 = b, \qquad b_2 = [0, 1/M, 0, 0]^{\tau}, \qquad b_3 = [0, 0, 1/J, 0]^{\tau}. \tag{3.7}$$

Now we consider the following six major point observations for system (3.5):

$$W_{i1}(t) = (b_i, \tilde{y}(t))_H, \qquad W_{i2}(t) = \left(b_i, \dot{\tilde{y}}(t)\right)_H, \qquad i = 1, 2, 3, \tag{3.8}$$

where $W_{i1}(t), W_{i2}(t)$, $i = 1, 2, 3$, denote the displacement $x(t)$, the line velocity $\dot{x}(t)$ of the moving body on the end of the arm, the total displacement $y(t, l)$, its velocity $\dot{y}(t, l)$ of the end of the arm, the turning angle $w'(t, l)$, and the angular velocity $\dot{w}'(t, l)$ of the end of the arm, respectively.

Similarly, we consider the following observation equations for system (3.2):

$$W_i(t) = \langle C_i, z(t) \rangle, \qquad i = 1, 2, \ldots, 6. \tag{3.9}$$

Here

$$C_1 = \begin{bmatrix} A_\gamma^{-1/2} b_1 \\ 0 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 0 \\ b_1 \end{bmatrix}, \qquad C_3 = \begin{bmatrix} A_\gamma^{-1/2} b_2 \\ 0 \end{bmatrix},$$
$$C_4 = \begin{bmatrix} 0 \\ b_2 \end{bmatrix}, \qquad C_5 = \begin{bmatrix} A_\gamma^{-1/2} b_3 \\ 0 \end{bmatrix}, \qquad C_6 = \begin{bmatrix} 0 \\ b_3 \end{bmatrix}.$$

Explicitly, $W_{2i-1}(t) = W_{i1}(t)$, $W_{2i}(t) = W_{i2}(t)$, $i = 1, 2, 3$.

THEOREM 3.8. *The system* (3.2) *is observable for the measured data* $W_1(t)$ *and* $W_5(t)$, *while it is not observable for the measured data* $W_2(t)$, $W_3(t), W_4(t), W_6(t)$. *In other words, the flexible robot system* (2.4) *is observable for the displacement* $x(t)$ *of the moving body on the end of the arm and the angular velocity* $W'(t, l)$ *of the end of the arm, but for other measured data the flexible robot system is not observable.*

Proof. By Theorem 3.5, (3.2) with $W_i(t), \mathscr{S}$, $i = 1, 2, \dots, 6$, is observable if and only if

$$
\begin{aligned}
& C_i^* R(\lambda, \mathscr{A})\Phi = 0, \\
& \forall \lambda \in \rho(\mathscr{A}) \Rightarrow \Phi = 0, \qquad \forall \Phi \in \mathscr{H} \\
& \qquad \Leftrightarrow \sum_{|n|=1}^{\infty} \frac{\langle \Phi, \tilde{\Psi}_n \rangle}{\lambda - \mu_n} \langle C_i, \tilde{\Phi}_n \rangle + \frac{1}{\lambda} \langle \Phi, \Psi_{+0} \rangle \langle C_i, \Phi_{+0} \rangle \\
& \qquad\qquad + \frac{1}{\lambda} \langle \Phi, \Psi_{-0} \rangle \langle C_i, \Phi_{-0} \rangle \\
& \qquad\qquad + \frac{\sqrt{r}}{\lambda^2} \langle C_i, \Phi_{+0} \rangle \langle \Phi, \Psi_{-0} \rangle = 0 \quad \Rightarrow \quad \Phi = 0, \quad \forall \Phi \in \mathscr{H} \\
& \qquad \Leftrightarrow \left\{ \begin{array}{ll} \langle C_i, \tilde{\Phi}_n \rangle \langle \Phi, \tilde{\Psi}_n \rangle = 0, & n = \pm 1, \pm 2, \dots \\ \langle C_i, \Phi_{+0} \rangle \langle \Phi, \Psi_{+0} \rangle = 0 & \\ \langle C_i, \Phi_{-0} \rangle + \dfrac{\sqrt{r}}{\lambda} \langle C_i, \Phi_{+0} \rangle \langle \phi, \psi_{-0} \rangle = 0 & \end{array} \right\} \\
& \qquad \Rightarrow \quad \Phi = 0, \qquad \forall \Phi \in \mathscr{H}.
\end{aligned}
$$

The last equivalence relation is due to the fact that

$$\left\{ \frac{1}{\lambda - \mu_n}, \frac{1}{\lambda}, \frac{1}{\lambda^2} \right\}_{|n|=1}^{\infty}$$

is linearly independent. Clearly, one necessary condition for the above implication to be valid is that $\langle C_i, \Phi_{+0} \rangle \neq 0$, for $\langle C_2, \Phi_{+0} \rangle = \langle C_4, \Phi_{+0} \rangle = \langle C_6, \Phi_{+0} \rangle = 0$. Thus W_2, W_4, and W_6 are not observable.

Since, for any $n \geq 0$,

$$\|\Phi_n\| \langle C_1, \tilde{\Phi}_{\pm n} \rangle = \frac{1}{\sqrt{\lambda_n + \gamma}} \varphi_n(0),$$

$$\|\Phi_n\| \langle C_3, \tilde{\Phi}_{\pm n} \rangle = \frac{1}{\sqrt{\lambda_n + \gamma}} \varphi_n(l),$$

$$\|\Phi_n\| \langle C_5, \tilde{\Phi}_{\pm n} \rangle = \frac{1}{\sqrt{\lambda_n + \gamma}} \varphi_n'(l),$$

and $\varphi_n(0) \neq 0$, $\varphi_n'(l) \neq 0$ (see Theorem 2.4) ($n \geq 0$), it follows that

$$\langle \Phi, \Psi_{\pm n} \rangle = 0 \quad \text{and} \quad \Phi = 0, \qquad \forall \Phi \in \mathscr{H}.$$

This implies that the measured data W_1, W_5 are observable.

Because we cannot assure that $\varphi_n(l) \neq 0$ for $n \geq 0$, we cannot obtain from $\varphi_n(l)\langle \Phi, \Psi_n \rangle = 0$ ($n = 0, \pm 1, \pm 2 \ldots$) that $\Phi = 0$. Therefore, the system with measured data W_3 is not observable.

4. THE FEEDBACK CONTROL FOR THE BOUNDARY POINTS OF THE SYSTEM

In this section we shall consider the feedback control of flexible arms under the general control law (2.2). In practical applications, it is necessary to control not only the vibrations but also positions of the moving body. It should be pointed out that in general the control for the robot is realized by the sensors and the actuators which were collected at the two ends of the arm of the robot. Therefore, it is imperative to design the feedback control for the boundary points of the robot system.

4.1. *A Feedback Control for the Output of the Boundary Points*

In system (2.1), if we add the controllers $v_1(t), v_2(t)$ to the root end of the arm and assume $\ddot{x}(t) = 0$, then system (2.1) becomes as follows:

$$\begin{aligned} \rho \ddot{w}(t,r) + EIw''''(t,r) &= 0, \\ w'(t,0) = w(t,0) &= 0, \\ EIw'''(t,l) - M\ddot{w}(t,l) + v_1(t) &= 0, \\ EIw''(t,l) + J\ddot{w}'(t,l) - v_2(t) &= 0, \end{aligned} \tag{4.1}$$

where the controllers are defined as

$$\begin{aligned} v_1(t) &= M\ddot{w}(t,l) - \beta\dot{w}(t,l) - EI, \\ v_2(t) &= -J\ddot{w}'(t,l) - \gamma\dot{w}(t,l). \end{aligned} \tag{4.2}$$

Then the closed loop system (4.1) becomes as follows:

$$\begin{aligned} \rho \ddot{w}(t,r) + EIw''''(t,r) &= 0, \\ w'(t,0) = w(t,0) &= 0, \\ EIw'''(t,l) &= \beta\dot{w}(t,l), \\ EIw''(t,l) &= \gamma\dot{w}'(t,l). \end{aligned} \tag{4.3}$$

It is appropriate to point out that system (4.1) follows control law (2.2). So this system is both approximately controllable and observable. Now we shall stabilize it with additional controllers $v_1(t), v_2(t)$.

Now, we introduce the space

$$\tilde{\mathscr{H}} = \left\{ W = (w, \dot{w})^{\tau} = h \in H_0^2(0, l), \dot{w} \in L^2(0, l) \right\},$$

where $H_0^2(0, l)$ is the Sobolev space

$$H_0^2(0, l) = \left\{ \varphi \in L^2(0, l) = \varphi,\ \varphi',\ \varphi'' \in L^2(0, l),\ \varphi(0) = 0,\ \varphi'(0) = 0 \right\}.$$

The inner product in $\tilde{\mathscr{H}}$ is defined as follows:

$$\langle W, V \rangle_E = EI \int_0^l w'' \bar{v}''\, dr + \rho \int_0^l \dot{w} \bar{\dot{v}}\, dr,$$

where $W = (w, \dot{w})^{\tau}$, $V = (v, \dot{v})^{\tau} \in \tilde{H}$, $i = 1, 2$. We can verify that the space $\tilde{H}$ with the inner product defined above is a Hilbert space. We shall also define an energy norm as follows:

$$\tilde{E}(t) = \tfrac{1}{2} EI \int_0^l |w''|^2\, dr + \tfrac{1}{2} \rho \int_0^l |\dot{w}|^2\, dr, \qquad W \in \tilde{\mathscr{H}}. \tag{4.4}$$

We can describe (4.1) as a dynamical system

$$\begin{aligned} \dot{W}(t) &= \tilde{\mathscr{A}} W(t), \\ W(0) &= W_0, \end{aligned} \tag{4.5}$$

where

$$\tilde{\mathscr{A}} = \begin{bmatrix} 0 & 1 \\ -\dfrac{EI}{\rho} \dfrac{\partial^4}{\partial r^4} & 0 \end{bmatrix}$$

and $D(\tilde{\mathscr{A}}) = \{(w, \dot{w})^{\tau}\colon\ w \in H^4(0, l),\ \dot{w} \in H_0^2(0, l),\ EIw'''(l) = \beta \dot{w}(t, l), EIw''(l) = -\gamma \dot{w}'(l)\}$ while $H^4(0, l) = \{\varphi\colon\ \varphi, \varphi', \varphi'', \varphi''', \varphi'''' \in L^2(0, l)\}$. In this notation we show the following lemmas.

LEMMA 4.1. *For all* $t \geq 0$, *we have* $\dot{\tilde{E}}(t) \leq 0$.

Proof. All functions considered here are real-valued. By the first equation in (4.1) we have

$$\frac{d\tilde{E}(t)}{dt} = EI \int_0^l w'' \dot{w}''\, dr + \rho \int_0^l \dot{w} \ddot{w}\, dr = EI \int_0^l w'' \dot{w}''\, dr - EI \int_0^l \dot{w} w''''\, dr.$$

By integration by parts, with $\dot{w} \in H_0^2$, the above equation becomes

$$\begin{aligned}\frac{d\tilde{E}(t)}{dt} &= EI\int_0^l w''\dot{w}''\,dr - EI\left\{(\dot{w}w''')|_0^l - \int_0^l \dot{w}'w'''\,dr\right\}\\ &= EI\int_0^l w''\dot{w}''\,dr - EI\left\{(\dot{w}w''')|_0^l\right\} + EI\left\{(\dot{w}'w'')|_0^l - \int_0^l \dot{w}''w''\,dr\right\}\\ &= EI\left\{(\dot{w}w''')|_0^l\right\} + EI\left\{(\dot{w}'w'')|_0^l\right\}. \qquad (4.6)\end{aligned}$$

It is easily seen that $\dot{w}'(t,0) = (d/dt)(w'(t,0)) = 0$, $\dot{w}(t,0) = 0$ from (4.1). It also follows from the last two equations in (4.3) that $-EI\{\dot{w}(t,l)w'''(t,l) - \dot{w}(t,0)w'''(t,0)\} = -\beta\dot{w}^2(t,l)$ and $EI\{\dot{w}'(t,l)w''(t,l) - \dot{w}'(t,0)w''(t,0)\} = -\gamma[\dot{w}'(t,l)]^2$. Thus, (4.6) becomes $dE(t)/dt = -\beta\dot{w}^2(t,l) - \gamma[\dot{w}'(t,l)]^2 < 0$.

To investigate the stability of the robot control system of (4.5), we shall define a Lyapunov functional as follows:

$$V(t) = 2(1-\epsilon)t\tilde{E}(t) + 2\rho\int_0^l rw'\dot{w}\,dr.$$

LEMMA 4.2. *For all $t \geq 0$ we have*

$$[2(1-\epsilon)t - \Omega]\tilde{E}(t) \leq V(t) \leq [2(1-\epsilon)t + \Omega]\tilde{E}(t).$$

Proof. It is easy to see that $2\int_0^l rw'\dot{w}\,dr \leq \int_0^l r(w'^2 + \dot{w}^2)\,dr \leq l\int_0^l w'^2 + \dot{w}^2\,dr$. Now

$$\begin{aligned}w'(t,r)^2 = \left(\int_0^r w'(t,r)\,dr\right)^2 &\leq \int_0^r dr \cdot \int_0^r w''(t,r)^2\,dr\\ &\leq r\cdot\int_0^l w''^2\,dr \leq l\int_0^l w''^2\,dr.\end{aligned}$$

Applying Cauchy's inequality we get

$$\int_0^l w'^2\,dr \leq l\int_0^l w''^2\,dr \cdot \int_0^l dr = l^2\int_0^l w''^2\,dr.$$

Thus,

$$\begin{aligned}\rho 2\int_0^l rw'\dot{w}\,dr &\leq \rho l\left\{l^2\int_0^l w''^2\,dr + \int_0^l \dot{w}^2\,dr\right\}\\ &= \rho l^3\int_0^l w''^2\,dr + l\rho\int_0^l \dot{w}^2\,dr\\ &\leq \Omega\tilde{E}(t)\end{aligned}$$

with $\Omega = 2\rho l^3/EI + 2l$.

Then the result in Lemma 4.2 follows easily.

LEMMA 4.3. *For any real number t large enough, we have*

$$\dot{V}(t) \le 0.$$

Proof. Applying integration by parts, we can obtain that

$$2\rho\int_0^l r\dot{w}'\dot{w}\,dr = \rho l\dot{w}^2(t,l) - \rho\int_0^l \dot{w}^2\,dr.$$

By the first equation in (4.2) we have

$$\begin{aligned}
2\rho\int_0^l rw\ddot{w}\,dr &= -2EI\int_0^l rw'w'''' \,dr \\
&= -2EI(rw'w''')|_0^l + 2EI\int_0^l w'''[w' + rw'']\,dr \\
&= 2EIlw'(t,l)w'''(t,l) \\
&\quad + 2EI\left\{\left(\frac{r}{2}w''^2\right)\Big|_0^l - \int_0^l \frac{1}{2}w''^2\,dr + \int_0^l w'\,dw''\right\} \\
&= 2EIlw'(t,l)w''(t,l) \\
&\quad + 2EI\left\{\frac{1}{2}w''^2(t,l) - \int_0^l \frac{1}{2}w''^2\,dr + (w'w'')|_0^l - \int_0^l w''^2\,dr\right\} \\
&= -3EI\int_0^l (w'')^2\,dr + EIl[w''(l)]^2 \\
&\quad - 2EIw'(t,l)[lw'''(t,l) - w''(t,l)] \\
&\le -3EI\int_0^l (w'')^2\,dr + \left(EIl + \frac{EI}{2\epsilon_2}\gamma^2\right)(\dot{w}'(t,l))^2 \\
&\quad + 2EI(\epsilon_1 + \epsilon_2)l^3\int_0^l (w'')^2\,dr + \frac{EI}{\epsilon_1}l^2\beta^2[\dot{w}(t,l)]^2,
\end{aligned}$$

whenever taking ϵ_1 and ϵ_2 small enough.

Combining the above equality and inequality, we have

$$\begin{aligned}\dot{V}(t) &= 2(1-\epsilon)t\dot{\tilde{E}}(t) + 2(1-\epsilon)\tilde{E}(t) + 2\rho\int_0^l r\dot{w}'\ddot{w}\,dr + 2\rho\int_0^l r\dot{w}'\dot{w}\,dr \\ &\le \left[-2(1-\epsilon)t\beta + \frac{\beta^2 l^2}{2\epsilon_1}EI\right](\dot{w}(t,l))^2 \\ &\quad + \left[-2(1-\epsilon)tr + \left(\frac{r^2}{2\epsilon_2} + l\right)EI\right](\dot{w}(t,l))^2 \\ &\quad + \left[-2-\epsilon + 2(\epsilon_1+\epsilon_2)l^3\right]EI\int_0^l (w'')^2\,dr - \epsilon\rho\int_0^l \dot{w}^2\,dr,\end{aligned}$$

and if $t \ge T_1$ (T_1 is large enough), ϵ_1 and ϵ_2 are small enough, it is clear that $\dot{V}(t) \le 0$.

THEOREM 4.4. *For system* (4.5), *we have the following results*:
$\tilde{\mathscr{A}}$ *is the infinitesimal generator of a* C_0 *semigroup* $S(t)$, $t \ge 0$, *on* $\tilde{\mathscr{H}}$ *that decays exponentially, and, for any* $w(0) \in D(\tilde{\mathscr{A}})$, *there is a unique classical solution* $W(t)$ *to the system* (4.5) *satisfying*

$$\|W(t)\| \le M_0 e^{-\mu_0 t},$$

where M_0 *and* μ_0 *are positive constant numbers.*
Thus, the robot feedback control system (4.3) *is asymptotically stable with exponential decay.*

Proof. We decompose $\tilde{\mathscr{A}} = \tilde{\mathscr{A}}_0 + \tilde{\mathscr{D}}$, where

$$\tilde{\mathscr{A}}_0 = \begin{bmatrix} 0 & 0 \\ -\dfrac{EI}{\rho}\dfrac{\partial^4}{\partial r^4} & 0 \end{bmatrix} \quad \text{and} \quad \tilde{\mathscr{D}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

It is clear that $\tilde{\mathscr{A}}_0$ is the infinitesimal generator of a c_0 semigroup, and thus $\tilde{\mathscr{A}}$, the perturbation of $\tilde{\mathscr{A}}_0$ by a bounded operator $\tilde{\mathscr{D}}$, is also an infinitesimal generator of a c_0 semigroup $S(t)$, $t \ge 0$. By Lemma 4.2, we see that if there is T_1 such that $2(1-\epsilon)T_1 - \Omega > 0$, and

$$\tilde{E}(t) \le V(T_2)/(2(1-\epsilon)t - \Omega) \qquad (t \ge T_2),$$

where $T_2 = \max\{T_1, \Omega/(1-\epsilon)\}$. It is known from Lemma 4.3 that $V(t)$ is uniformly bounded, and therefore $\tilde{E}(t) = O(1/t)$ for n large enough, thus

$$\int_0^\infty (\tilde{E}(t))^2\,dt < +\infty.$$

It follows from a theorem in [8] that for a solution to decay exponentially there are constants M_1 and $\mu_0 > 0$ such that

$$\tilde{E}(t) \le \tilde{E}(0) M_1 \epsilon^{-\mu_0 t}.$$

Since $\tilde{E}(t) = \frac{1}{2}\|W(t)\|^2 = \frac{1}{2}\|S(t)W(0)\|^2$ and $\int_0^\infty E^2(t)\,dr < \infty$, we have $\int_0^\infty \|S(t)\|^4 < \infty$. Thus, we have, for some constant M_2,

$$\|S(t)\| \le M_2 e^{-\mu_0 t}.$$

Since the solution of (4.5) is $W(t) = S(t)W(0)$ [8], it should satisfy

$$\|W(t)\| \le \|W(0)\|\,\|S(t)\| \le M_0 e^{-\mu_0 t},$$

where $M_0 = \|W(0)\|M_2$. This implies that the solution of system (4.5) is exponentially stable.

4.2. *Dynamical Feedback Control*

Consider the following feedback control:

$$\begin{aligned} v_1(t) &= M\ddot{w}(t,l) - f_1(t), \\ v_2(t) &= -J\ddot{w}'(t,l) + f_2(t), \end{aligned} \tag{4.7}$$

where $f_i(t) = \tilde{c}_i^\tau z_i + \tilde{d}_i w_i(t)$ and the z_i's are solutions of $\dot{z}_i = A_i z_i + \tilde{b}_i w_i(t)$ with A_i an $n_i \times n_i$ matrix, $\tilde{b}_i, \tilde{c}^i \in R^{n_i}$ the constant column vectors, $\tilde{d}_i$ a real number, $w_1(t) = \dot{w}(t,l)$, $w_2(t) = \dot{w}'(t,l)$, and $z_i(0) = 0$, $i = 1,2$.

For controller (4.7), we assume that

(I) All the eigenvalues of the matrixes A_i $(i = 1,2)$ are located on the left half of the plane;

(II) $(A_i, \tilde{b}_i, \tilde{c}_i)$ is controllable and observable;

(III) $\tilde{d}_1 > 0$, $\tilde{d}_2 > 0$; moreover, there are $\gamma_1 > 0$, $\gamma_2 > 0$ such that $\tilde{d}_1 > \gamma_1$, $\tilde{d}_2 > \gamma_2$, and

$$Re\{g_i(x)\} > \gamma_i, \qquad i = 1,2,\ x \in R,$$

where $g_i(x) = \tilde{d}_i + \tilde{c}_i^\tau (xI - A_i)^{-1}\tilde{b}_i$ $(i = 1,2)$.

Since (I)–(III) are valid, we obtain from the Kalman–Yacubovitch lemma [15] that, for the given positive numbers ϵ_i, $i = 1,2$, and any positive defined symmetric matrixes $Q_i \in M_{n_i}$, the space of all $n_i \times n_i$ matrices, $i = 1,2$, there exist positive defined symmetric matrixes $P_i \in M_{n_i}$, $i = 1,2$, and columns vectors $q_i \in R^{n_i}$, $i = 1,2$, satisfying

$$\begin{aligned} A_i^\tau P_i + P_i A_i &= -q_i q_i^\tau - \epsilon_i Q_i, \\ P_i \tilde{b}_i - \tfrac{1}{2}\tilde{c}_i &= \sqrt{\tilde{d}_i - \gamma_i} \cdot q_i. \end{aligned}$$

Now, we introduce the following space of functions:

$$\mathscr{H}_1 = H_0^2(0,l) \times L^2(0,l) \times R^{n_1} \times R^{n_2}$$

equipped with inner product

$$\begin{aligned}&\left\langle (\varphi_1(\cdot), \psi_1(\cdot), b_1, c_1)^\tau, (\varphi_2(\cdot), \psi_2(\cdot), b_2, c_2)^\tau \right\rangle_{\mathscr{H}_1} \\ &\qquad = \int_0^l (EI\varphi_1''\varphi_2'' + \rho\psi_1\psi_2)\, dr + b_2^\tau P_1 b_1 + c_2^\tau P_2 c_1, \end{aligned} \tag{4.8}$$

and define the linear operator $\mathscr{A}\colon \mathscr{H}_1 \to \mathscr{H}_1$ by

$$\begin{aligned}&\mathscr{A}_1(\varphi(\cdot), \psi(\cdot), z_1, z_2)^\tau \\ &\qquad = \left\{ \psi(\cdot), \frac{-EI}{\rho}\varphi''''(\cdot), A_1 z_1 + \tilde{b}_1\psi(l), A_2 z_2 + \tilde{b}_2\psi'(l) \right\}^\tau\end{aligned}$$

with domain

$$\begin{aligned}D(\mathscr{A}_1) = \big\{ &(\varphi(\cdot), \psi(\cdot), z_1, z_2)\colon \varphi \in H^4(0,l), \\ &\psi \in H^2(0,l),\ \varphi(0) = \varphi'(0) = \psi'(0) = 0, \\ &w_1 \in R^n, w_2 \in R^{n_2},\ EI\varphi'''(l) + \tilde{c}_1^\tau w_1 + \tilde{d}_1\psi(l) = 0, \\ &-EI\varphi''(l)\tilde{c}_1^\tau w_2 + \tilde{d}_2\psi'(l) = 0\big\}.\end{aligned}$$

In terms of the above notation we can describe the system (4.1) with (4.7) as the following linear evolution equation in $\mathscr{H}_1$:

$$\begin{aligned}\frac{d\tilde{W}(t)}{dt} + \mathscr{A}_1\tilde{W}(t) &= 0, \\ \tilde{W}(0) &= \tilde{W}_0,\end{aligned} \tag{4.9}$$

where $\tilde{W}(t) = (w(t,\cdot), \dot{w}(t,\cdot), z_1, z_2)^\tau$.

It is easy to verify that the space $\mathscr{H}_1$ with inner product (4.8) is a Hilbert space, and so the energy of the system can be derived from (4.8) as follows:

$$E_1(t) = \tfrac{1}{2}\|\tilde{W}(t)\|^2 = \tfrac{1}{2}\int_0^l \left[\rho\dot{w}^2 + EI(w'')^2\right] dr + z_1^\tau P_1 z_1 + z_2^\tau P_2 z_2. \tag{4.10}$$

LEMMA 4.5. *Suppose system* (4.9) *satisfies conditions* (I)–(III). *Then the energy of the system given by* (4.10) *is a decreasing function of time t along with the classical solution to* (4.9).

Proof. Using control (4.7), we have

$$\begin{aligned}
\dot{E}_1(t) &= \int_0^l \left[\rho \dot{w}\ddot{w} + EIw''\dot{w}'' \right] dr + \dot{z}_1^\tau P_1 z_1 + z_1^\tau P_1 \dot{z}_1 + \dot{z}_2^\tau P_2 z_2 + z_2^\tau P_2 \dot{z}_2 \\
&= -EI\dot{w}(t,l)w'''(t,l) + EIw''(t,l)\dot{w}(t,l) \\
&\quad + \left(A_1^\tau z_1^\tau + \tilde{b}_1' \dot{w}(t,l) \right) P_1 z_1 + z_1^\tau P_1 \left(A_1 z_1 + \tilde{b}_1^\tau \dot{w}(t,l) \right) \\
&\quad + \left[A_2^\tau z_2^\tau + b_2^\tau \dot{w}'(t,l) \right] P_2 z_2 + z_2^\tau P_2 \left[A_2 z_2 + b_2 \dot{w}'(t,l) \right] \\
&= -\dot{w}(t,l)\left[\tilde{c}_1^\tau z_1 + \tilde{d}_1 \dot{w}(t,l) \right] - EI\dot{w}'(t,l)\left[\tilde{c}_2^\tau z_2 + \tilde{d}_2 \dot{w}'(t,l) \right] \\
&\quad + z_1^\tau (A_1^\tau P_1 + P_1 A_1) z_1 + 2\tilde{b}_1^\tau P_1 z_1 \dot{w}(t,l) \\
&\quad + z_2^\tau (A_2^\tau P_2 + P_2 A_2) z_2 + 2\tilde{b}_2^\tau P_2 z_2 \dot{w}'(t,l) \\
&= -\gamma_1 \dot{w}^2(t,l) - \gamma_2 [\dot{w}'(t,l)]^2 - \epsilon_1 z_1^\tau Q_1 z_1 - \epsilon_2 z_2^\tau Q_2 z_2 \\
&\quad - \left[\sqrt{\tilde{d}_1 - \gamma_1}\, \dot{w}(t,l) - z_1^\tau q_1 \right]^2 - \left[\sqrt{\tilde{d}_2 - \gamma_2}\, \dot{w}'(t,l) - z_2^\tau q_2 \right]^2 .
\end{aligned}$$

It follows easily that $\dot{E}_1(t) \le 0$.

In order to investigate the stability of the feedback control system (4.9), we now consider a Lyapunov functional as follows:

$$V_1(t) = 2(1-\epsilon) t E_1(t) + 2\rho \int_0^l r\dot{w}(t,r) w'(t,r)\, dr \qquad (0 < \epsilon < 1).$$

LEMMA 4.6. *Let* $\beta(t) = \int_0^l r\dot{w}(t,r)w'(t,r)\,dr$. *Then there exist the positive constants* α_i, $i = 0, 1, 2, \dots, 5$, *such that the following inequalities hold*:

$$\begin{aligned}
|\beta(t)| &\le \alpha_0 E_1(t) \qquad (t \ge 0), \\
\frac{d\beta(t)}{dt} &\le -\alpha_1 E_1(t) + \alpha_2 \dot{w}^2(t,l) + \alpha_2 [\dot{w}'(t,l)]^2 \\
&\qquad + \alpha_4 z_1^\tau Q_1 z_1 + \alpha_5 z_2^\tau Q_2 z_2 .
\end{aligned}$$

Proof. By definition of $\beta(t)$ and integration by parts, we have

$$\begin{aligned}
|\beta(t)| &\le l \int_0^l (w')^2 (\dot{w})^2\, dr \\
&\le (l^3/EI) \int_0^l EI(w'')^2\, dr + (l/\rho) \int_0^l \rho \dot{w}^2\, dr \\
&\le c_0 E_1(t),
\end{aligned}$$

where $c_0 = \max\{l^3/EI, l/\rho\}$, and

$$\begin{aligned}\frac{d\beta(t)}{dt} &= \int_0^l r\ddot{w}w'\,dr + \int_0^l r\dot{w}\dot{w}'\,dr \\ &= -\frac{1}{2}\int_0^l (\dot{w})^2\,dr - \frac{3EI}{2\rho}\int_0^l (w'')^2\,dr + \frac{l}{2}\dot{w}^2(t,l) \\ &\quad - \frac{lEI}{2\rho}\left[\tilde{c}_2^\tau z_2 + \tilde{d}_2\dot{w}'(t,l)\right]^2 \\ &\quad - \frac{1}{\rho}w'(t,l)\left\{l\left[\tilde{c}_1^\tau z_1 + \tilde{d}_1\dot{w}(t,l)\right] - \left[\tilde{c}_2^\tau z_2 + \tilde{d}_2\dot{w}'(t,l)\right]\right\}.\end{aligned}$$

THEOREM 4.7. *The dynamical feedback control system* (4.1) *and* (4.7) *about the flexible robot is exponentially asymptotically stable.*

Proof. From Lemma 4.6 we can easily see that

$$\left[2(1-\epsilon)t - c_0\right]E_1(t) \le V_1(t) \le \left[2(1-\epsilon)t + c_0\right]E_1(t) \tag{4.11}$$

and

$$\begin{aligned}\dot{V}_1(t) &= 2(1-\epsilon)t\dot{E}_1(t) + 2(1-\epsilon)E_1(t) + 2\rho\dot{\beta}(t) \\ &\le \left[-2(1-\epsilon)\gamma_1 t + 2\rho\alpha_2\right]\dot{w}^2(t,l) \\ &\quad + \left[-2(1-\epsilon)\gamma_2 t - 2\rho\alpha_3\right]\left[\dot{w}'(t,l)\right]^2 \\ &\quad + \left[-2(1-\epsilon)\epsilon_1 t + 2\rho\alpha_4\right]z_1^2 Q_1 z_1 \\ &\quad + \left[-2(1-\epsilon)\epsilon_2 t + 2\rho\alpha_5\right]z_2^\tau Q_2 z_2 - \epsilon E_1(t) \\ &\quad - 2(1-\varepsilon)t\left\{\left[\sqrt{\tilde{d}_1 - \gamma_1}\,\dot{w}(t,l) - \tilde{z}_1^\tau q_1\right]^2 \right. \\ &\qquad\qquad \left. + \left[\sqrt{\tilde{d}_2 - \gamma_2}\,\dot{w}'(t,l) - \tilde{z}_2^\tau q_2\right]^2\right\}.\end{aligned}$$

It is obvious that if $t \ge T_1$ and T_1 is large enough, then we have that $\dot{V}_1(t) \le 0$. Due to (4.11) we know that

$$E_1(t) \le V_1(T_2)/\left(2(1-\epsilon)t - c_0\right), \qquad t \ge T_2,$$

where $T_2 = \max\{T_1, \rho c_0/(1-\epsilon)\}$. However, this implies that $V_1(T_2) \le \infty$, and, for t large enough, we have $E_1(t) = O(1/t)$. Thus,

$$\int_0^\infty E_1^2(t)\,dt < +\infty.$$

By means of the decision theorem of the solution for exponential decay [12], there exist $M_1 \geq 1$ and $\mu_1 > 0$ such that

$$E_1(t) \leq E_1(0) M_1 e^{-\mu_1 t},$$

and therefore the dynamical feedback control system (4.1) about the flexible robot is exponentially asymptotically stable, that is,

$$\|\tilde{W}(t)\|_{\mathscr{H}_1} \leq M_1 e^{-\mu_1 t}.$$

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