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Continuous Selections for Multivalued Mappings with Closed Convex Images and Applications[1]

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Some existence theorems of continuous selections for multivalued mappings with closed convex images are proved. As some simple applications, we give some results on fixed points and differential inclusions. The results presented in this paper generalize some recent results. © 2000 Academic Press

Key Words: multivalued mapping; continuous selection; *r*-lower semicontinuous; partial lower semicontinuous; closed convex set; fixed point; differential inclusion.

1. INTRODUCTION

Let X be a topological space, Y a metric space, $C(Y) = \{a \in 2^Y : A$ is closed and convex$\}$, where 2^Y is the set of all nonempty subsets of Y. Suppose $A \subset Y$ and denote $B(A, \varepsilon)$ as the set $\{x \in Y : d(x, A) < \varepsilon\}$ where $d(x, A) = \inf_{y \in A} d(x, y)$.

A continuous mapping $f : X \to Y$ is called a *continuous selection* of a multivalued mapping $F : X \to 2^Y$ if $f(x) \in F(x)$ for all $x \in X$.

$F : X \to 2^Y$ is called *lower semicontinuous* (l.s.c.) at $x_0 \in X$, if for each $y_0 \in F(x_0)$ and $\varepsilon > 0$, there exists a neighborhood $N(x_0)$ of x_0 s.t. $B(y_0, \varepsilon) \cap F(x) \neq \emptyset$ for all $x \in N(x_0)$. F is called *lower semicontinuous* if F is l.s.c. at each point of X.

In 1956, E. Michael [1] established the following continuous selection theorem.

THEOREM A. *Let X be a paracompact space and Y a Banach space. Then every l.s.c. mapping $F : X \to C(Y)$ admits a continuous selection.*

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Since then, this result has been generalized in many aspects, such as from l.s.c. cases to various other kinds of l.s.c. cases, and from convex valued to non-convex valued cases (see, e.g., [2, 3, 6–10]). In 1985, F. S. De Blasi and J. Myjak [3] introduced weakly Hausdorff lower semicontinuity. That is, a mapping $F : X \to 2^Y$ is called *weakly Hausdorff lower semicontinuous* (H_w-l.s.c.) at $x_0 \in X$ if given $\varepsilon > 0$ and a neighborhood V of x_0, there exist a neighborhood U $(U \subset V)$ of x_0 and a point $x' \in U$ s.t. $F(x') \subset \bigcap_{x \in U} B(F(x), \varepsilon)$. If $x' = x$, we call F *Hausdorff lower semicontinuous* (H-l.s.c.) at x_0. F is *weakly Hausdorff lower semicontinuous* (resp. *Hausdorff lower semicontinuous*) if F is H_w-l.s.c. (resp. H-l.s.c.) at every point of X. Obviously, a H-l.s.c. mapping is l.s.c. and H_w-l.s.c. A continuous selection theorem was given in [3] as the following.

THEOREM B. *Let X be a paracompact space and Y a Banach space. Then every H_w-l.s.c. mapping admits a continuous selection.*

Since a H_w-l.s.c. mapping $F : X \to 2^Y$ is not necessarily l.s.c, and vice versa, Theorem A and Theorem B do not follow from each other.

In Section 2 of this paper, we establish a continuous selection theorem which is a generalization of both Theorem A and Theorem B. We also investigate the continuous selection of a class of multivalued mappings, say partial lower semicontinuous mappings, in Section 3. In addition, as some simple applications, we give a fixed point theorem and a theorem about differential inclusions, which generalize some known results.

2. THE CONTINUOUS SELECTIONS OF r-LOWER SEMICONTINUOUS MAPPINGS

Let X be a topological space and Y a metric space. $F : X \to 2^Y$ is called *almost lower semicontinuous* (a.l.s.c.) at $x_0 \in X$ (see [2]) if for each $\varepsilon > 0$, there exists a neighborhood $N(x_0)$ of x_0 s.t. $\bigcap_{x \in N(x_0)} B(F(x), \varepsilon) \neq \emptyset$. F is called *almost lower semicontinuous* if F is a.l.s.c. at every point of X.

DEFINITION 2.1. Letting $r \in (0,1)$, $F : X \to 2^Y$ is r-lower semicontinuous (r-l.s.c.) at x_0 if F is a.l.s.c. at x_0 and the following statement is true:

(P) For every $\varepsilon > 0$ and every neighborhood $N(x_0)$ of x_0, if $y_0 \in \bigcap_{x \in N(x_0)} B(F(x), \varepsilon)$, there exists a neighborhood $N(x_1)$ of every $x_1 \in N(x_0)$ s.t.

$$y_0 \in B\left(\bigcap_{x \in N(x_1)} B(F(x), r\varepsilon), \varepsilon \right). \tag{2.1}$$

F is r-lower semicontinuous if F is r-l.s.c. at each point of X.

If $r \geq 1$ in Definition 2.1, we may choose $N(x_1) = N(x_0)$ and then the statement (P) is self-evident. That is why we focus on the case $r \in (0, 1)$.

PROPOSITION 2.1. *If $F : X \to 2^Y$ is l.s.c. or H_w-l.s.c., F is r-l.s.c. for each $r \in (0, 1)$.*

Proof. Obviously, F is l.s.c. or H_w-l.s.c. implies that F is a.l.s.c. Let $r \in (0, 1)$, $x_0 \in X$, $\varepsilon > 0$, and $N(x_0)$ be a neighborhood of x_0 s.t.

$$y_0 \in \bigcap_{x \in N(x_0)} B(F(x), \varepsilon). \tag{2.2}$$

By (2.2), we may find a point $y_1 \in F(x_1) \cap B(y_0, \varepsilon)$ for every $x_1 \in N(x_0)$. If F is l.s.c., there exists a neighborhood $N(x_1)$ of x_1 s.t. $y_1 \in \bigcap_{x \in N(x_1)} B(F(x), r\varepsilon)$. Then (2.1) holds. Thus F is r-l.s.c.

If F is H_w-l.s.c., since $N(x_0)$ is a neighborhood of every $x_1 \in N(x_0)$, there exist a neighborhood $N(x_1)$ of x_1 $(N(x_1) \subset N(x_0))$ and a point $x' \in N(x_1)$ s.t.

$$F(x') \subset \bigcap_{x \in N(x_1)} B(F(x), r\varepsilon).$$

At the same time, (2.2) implies that $y_0 \in B(F(x'), \varepsilon)$. Then (2.1) holds, so that F is r-l.s.c. The proof is complete.

The converse of Proposition 2.1 is not true. For example, $F : R \to 2^R$ is defined as

$$F(x) = \begin{cases} [0, +\infty), & \text{if } x \geq 0, \\ (-\infty, 0], & \text{if } x < 0. \end{cases}$$

It is easy to verify that F is not l.s.c. or H_w-l.s.c. at $x = 0$, but F is r-l.s.c. for each $r \in (0, 1)$.

Now we are in a position to give our main result in this paper.

THEOREM 2.1. *Let X be a paracompact space and Y a Banach space. Then every r.l.s.c. mapping $F : X \to C(Y)$ admits a continuous selection, where $r \in (0, 1)$.*

Proof. We construct a sequence of continuous mappings $f_n : X \to Y$ such that

(i) For each $x \in X$, $f_n(x) \in B(F(x), r^n)$, $n = 1, 2, \ldots,$ and

$$f_n(x) = \sum_{i \in I_n} \psi_{n,i}(x) y_{n,i},$$

where $\{\psi_{n,i}\}_{i \in I_n}$ is a partition of unity, and subordinate to some locally finite open cover $\{\Omega_{n,i}\}_{i \in I_n}$ of X. For each $i \in I_n$, $\Omega_{n,i}$ is a subset of a neighborhood $N(x_{n,i})$ of some point $x_{n,i}$ s.t.

$$y_{n,i} \in \bigcap_{x \in N(x_{n,i})} B(F(x), r^n).$$

(ii) For each $x \in X$, $\|f_n(x) - f_{n-1}(x)\| \leq r^{n-2}$, $n = 2, 3, \dots$.

Step 1. Construct f_1 satisfying (i). For each $x_0 \in X$, since F is a.l.s.c., there exist a neighborhood $N(x_0)$ of x_0 and a point $y_0 \in Y$ s.t. $y_0 \in \bigcap_{x \in N(x_0)} B(F(x), r)$. Since X is paracompact, the open cover $\{N(x) : x \in X\}$ of X has a locally finite refinement $\{\Omega_{1,i}\}_{i \in I_1}$. Then $\Omega_{1,i} \subset N(x_{1,i})$ for each $i \in I_1$, where $N(x_{1,i})$ is a neighborhood of some point $x_{1,i}$, and $y_{1,i} \in \bigcap_{x \in N(x_{1,i})} B(F(x), r)$. Suppose that $\{\psi_{1,i}\}_{i \in I_1}$ is a partition of unity and subordinate to $\{\Omega_{1,i}\}_{i \in I_1}$. Set

$$f_1 = \sum_{i \in I_1} \psi_{1,i}(x) y_{1,i}.$$

It is trivial to check that f_1 satisfies all our requirements.

Step 2. Supposing that we have f_n satisfies (i) up to $n = k$ and construct f_{k+1} satisfying (i) and (ii). Set $I_k(x_0) = \{i \in I_k : x_0 \in \Omega_{k,i}\}$ for each $x_0 \in X$. Then $I_k(x_0)$ is a finite set, and $y_{k,i} \in \bigcap_{x \in N(x_{k,i})} B(F(x), r^k)$ for each $i \in I_k(x_0)$. Since F is r-l.s.c., for each $i \in I_k(x_0)$, there exist a neighborhood $N_i(x_0)$ of x_0 and a point $y_i \in B(y_{k,i}, r^k)$ s.t. for all $x \in N_i(x_0)$

$$y_i \in B\big(F(x), r^{k+1}\big). \tag{2.3}$$

Let $N'(x_0) = \bigcap_{i \in I_k(x_0)} N_i(x_0)$, $y_0 = \sum_{i \in I_k(x_0)} \psi_{k,i}(x_0) y_i$. Then (2.3) holds for all $x \in N'(x_0)$ and $i \in I_k(x_0)$. Since $B(F(x), r^{k+1})$ is convex, for all $x \in N'(x_0)$ we have

$$y_0 = \sum_{i \in I_k(x_0)} \psi_{k,i}(x_0) y_i = \sum_{i \in I_k} \psi_{k,i}(x_0) y_i \in B\big(F(x), r^{k+1}\big). \tag{2.4}$$

Since $y_i \in B(y_{k,i}, r^k)$,

$$\begin{aligned} \| y_0 - f_k(x_0) \| &= \Big\| \sum_{i \in I_k} \psi_{k,i}(x_0) y_i - \sum_{i \in I_k} \psi_{k,i}(x_0) y_{k,i} \Big\| \\ &\leq \sum_{i \in I_k} \psi_{k,i}(x_0) \| y_i - y_{k,i} \| \\ &< r^k. \end{aligned}$$

On the other hand, since $f_k(x)$ is continuous, there exists a neighborhood $N''(x_0)$ of x_0 s.t. for all $x \in N''(x_0)$

$$\| f_k(x_0) - f_k(x) \| < r^{k-1} - r^k.$$

Thus, for each $x \in N''(x_0)$ we have

$$\| y_0 - f_k(x) \| < r^{k-1}. \tag{2.5}$$

Let $N(x_0) = N'(x_0) \cap N''(x_0)$. Then y_0 satisfies both (2.4) and (2.5) for each $x \in N(x_0)$. Since X is paracompact, the open cover $\{N(x_0) : x_0 \in X\}$ of X has a locally finite refinement $\{\Omega_{k+1,i}\}_{i \in I_{k+1}}$. Then for each $i \in I_{k+1}$, there exist a neighborhood $N(x_{k+1,i})$ of some point $x_{k+1,i}$ and a point $y_{k+1,i}$ s.t. $\Omega_{k+1,i} \subset N(x_{k+1,i})$ and $y_{k+1,i}$ satisfies

$$y_{k+1,i} \in \bigcap_{x \in N(x_{k+1,i})} B\big(F(x), r^{k+1}\big) \tag{2.6}$$

and

$$y_{k+1,i} \in \bigcap_{x \in N(x_{k+1,i})} B\big(f_k(x), r^{k-1}\big). \tag{2.7}$$

Suppose that $\{\psi_{k+1,i}\}_{i \in I_{k+1}}$ is a partition of unity and subordinate to $\{\Omega_{k+1,i}\}_{i \in I_{k+1}}$. Set

$$f_{k+1}(x) = \sum_{i \in I_{k+1}} \psi_{k+1,i}(x) y_{k+1,i}.$$

Obviously, f_{k+1} is well-defined and continuous. For each $x_0 \in X$, (2.6) implies $f_{k+1}(x_0) \in B(F(x_0), r^{k+1})$, and (2.7) implies $\|f_{k+1}(x_0) - f_k(x_0)\| < r^{k-1}$. Hence f_{k+1} satisfies (i) and (ii).

Now we have constructed the continuous sequence $\{f_n\}$ satisfying (i) and (ii) by induction. By (ii), $\{f_n\}$ is a uniformly Cauchy sequence, and therefore converges uniformly to a continuous $f : X \to Y$. It follows from (i) that $f(x) \in F(x)$ for every $x \in X$. This completes the proof.

Remark 2.1. By Proposition 2.1, we know that Theorem 2.1 is a generalization of both Theorem A and Theorem B.

3. CONTINUOUS SELECTIONS FOR PARTIAL LOWER SEMICONTINUOUS MAPPINGS AND APPLICATIONS

DEFINITION 3.1. Let X be a topological space and Y a Banach space. $F : X \to 2^Y$ is a partial lower semicontinuous (p.l.s.c.) at $x_0 \in X$, if $F^{(1)}(x_0) = \{y_0 \in Y :$ for each $\varepsilon > 0$, there exists a neighborhood $N(x_0)$ of

x_0 s.t. $y_0 \in \bigcap_{x \in N(x_0)} B(F(x), \varepsilon)\} \neq \emptyset$. F is partial lower semicontinuous if F is p.l.s.c. at every point of X. $F^{(1)}$ is called the main mapping of F. Generally, for each $n \in N$, F is partial lower semicontinuous of order n if $F^{(n-1)}$ is p.l.s.c. (where $F^{(0)}$ denotes F), and the main mapping $F^{(n)}$ of $F^{(n-1)}$ is called the main mapping of order n of F.

PROPOSITION 3.1. (1) *If F is p.l.s.c., $F^{(1)}$ is closed-valued and $F^{(1)}(x) \subset \overline{F(x)}$ for each $x \in X$.*

(2) *If F is convex-valued p.l.s.c., $F^{(1)}$ is convex-valued.*

(3) *If F is closed-valued, F is l.s.c. if and only if F is p.l.s.c. and $F = F^{(1)}$.*

(4) *If F has a continuous selection f, F is p.l.s.c. of order n for each $n \in N$ and f is a continuous selection of $F^{(n)}$.*

Proof. By the definition, it is easy to verify that (1), (2), and (3) hold. To prove (4), suppose f is a continuous selection of F. Given $x_0 \in X$ and $\varepsilon > 0$, let $y_0 = f(x_0)$. Then $f(x_0) \in \bigcap_{x \in N(x_0)} B(f(x), \varepsilon)$ for some neighborhood $N(x_0)$ of x_0. Thus $y_0 = f(x_0) \in \bigcap_{x \in N(x_0)} B(F(x), \varepsilon)$, which shows that F is p.l.s.c. at x_0 and $f(x_0) \in F^{(1)}(x_0)$. Since x_0 is arbitrary, F is p.l.s.c. and f is a continuous selection of $F^{(1)}$. Take $F^{(1)}$ in the place of F in the above discussion. We know that $F^{(1)}$ is p.l.s.c. and f is a continuous selection of $F^{(2)}$. Repeat this discussion finite times. Then we get that F is p.l.s.c. of order n for each $n \in N$, and f is a continuous selection of $F^{(n)}$. This completes the proof.

By Proposition 3.1(3), if F is p.l.s.c. of order n, and $F^{(n)}$ is l.s.c., $F^{(n)} = F^{(n+k)}$ for every $k \in N$. By Proposition 3.1(4), if F is p.l.s.c. of order n, to discuss the continuous selections of F, it is sufficient to consider those of $F^{(n)}$. By Theorem 2.1 and Proposition 3.1, that F is r-l.s.c. for some $r \in (0,1)$ implies that F is p.l.s.c. of order n for every $n \in N$, and we can immediately obtain the following theorem which is actually Theorem 2.1 when $n = 1$.

THEOREM 3.1. *Let X be a paracompact space and Y a Banach space. $F : X \to C(Y)$ is p.l.s.c. of order n for some $n \in N$, and $F^{(n-1)}$ is r-l.s.c. for some $r \in (0,1)$. Then F admits a continuous selection.*

The following example is a modification of a counterexample (mentioned in [2]) which was kindly communicated by Professor F. Deutsch.

EXAMPLE 3.1. Suppose that $F : [0,1] \to 2^{R^2}$ is defined as

$$F(x) = \begin{cases} OA_n, & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right],\ n = 1, 2, \ldots, \\ OA_\infty, & \text{if } x = 0, \end{cases}$$

where OA_∞ is the line segment from $O := (0,1)$ to $A_\infty := (1,0)$, and OA_n from O to $A_n := (1, \frac{1}{n})$, $n = 1, 2, \dots$.

It is not difficult for us to verify that, for every $r \in (0,1)$, F is not r-l.s.c. at $x = \frac{1}{n}$ for n sufficiently large. So Theorem 2.1 cannot be applied here. But it is easy to verify that F is p.l.s.c. and

$$F^{(1)}(x) = \begin{cases} O, & \text{if } x = \frac{1}{n},\ n = 1, 2, \dots, \\ F(x), & \text{otherwise}. \end{cases}$$

By definition, it is easy to check that $F^{(1)}$ is H-l.s.c. at $x \neq 0$, and H_w-l.s.c. but not l.s.c. at $x = 0$. Thus $F^{(1)}$ is r-l.s.c. for each $r \in (0,1)$. By Theorem 3.1, F has a continuous selection.

Furthermore, it is easy to see that

$$F^{(2)}(x) = \begin{cases} O, & \text{if } x = 0, \\ F^{(1)}(x), & \text{if } x \in (0,1], \end{cases}$$

and $F^{(2)}$ is H-l.s.c., consequently l.s.c., and then $F^{(2)} = F^{(n)}$ for all $n \geq 2$.

Theorem 2.1 and Theorem 3.1 guarantee the existence of continuous selections for multivalued mappings. So the fixed point theorems concerning the continuous single-valued mappings can be naturally generalized to the multivalued case. The following theorem is one which generalizes Schauder's fixed point theorem.

THEOREM 3.2. *Let X be a Banach space and $K \subset X$ be convex compact. If $F : K \to C(K)$ is p.l.s.c. of order n for some $n \in N$, and $F^{(n-1)}$ is r-l.s.c. for some $r \in (0,1)$, then F has a fixed point in K.*

The proof of Theorem 3.2 is an immediate consequence of Theorem 3.1 and Schauder's fixed point theorem.

The following Theorem 3.3 about differential inclusions is a generalization of [4, Theorem 2.1]. The proof of Theorem 3.3 is a slight modification to that one, so we omit it here.

THEOREM 3.3. *Let $\Omega \subset R \times R^n$ be open with $(0, x_0) \in \Omega$. If $F : \Omega \to C(R^n)$ is p.l.s.c. of order n for some $n \in N$, and $F^{(n-1)}$ is r-l.s.c. for some $r \in (0,1)$, then there exists some interval $I = (\omega_-, \omega_+)$, $\omega_- < 0 < \omega_+$ and at least one continuously differential function $x : I \to R^n$, a solution to the Cauchy problem for differential inclusion*

$$x'(t) \in F(t, x(t)), \qquad x(0) = x_0.$$

Moreover either $\omega_+ = +\infty$ or the solution $x(t)$ tends to the boundary of Ω as $t \to \omega_+$ and analogously for ω_-.

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