

Some Further Generalizations of the Hyers–Ulam–Rassias Stability of Functional Equations¹

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Submitted by Themistocles M. Rassias

Received May 11, 2000

In this paper we study the Hyers–Ulam–Rassias stability theory by considering the cases where the approximate remainder ϕ is defined by

$$f(x * y) - f(x) - f(y) = \phi(x, y) \quad (\forall x, y \in G), \quad (1)$$

$$f(x * y) - g(x) - h(y) = \phi(x, y) \quad (\forall x, y \in G), \quad (2)$$

$$2f((x * y)^{1/2}) - f(x) - f(y) = \phi(x, y) \quad (\forall x, y \in G), \quad (3)$$

where $(G, *)$ is a certain kind of algebraic system, E is a real or complex Hausdorff topological vector space, and f, g, h are mappings from G into E . We prove theorems for the Hyers–Ulam–Rassias stability of the above three kinds of functional equations and obtain the corresponding error formulas. © 2001 Academic Press

Key Words: Hyers–Ulam–Rassias stability; Cauchy equation; Pexider equation; Jensen equation; approximate remainder.

1. INTRODUCTION

Throughout this paper, we denote by G a certain kind of algebraic system and by E a real or complex Hausdorff topological vector space. By \mathbf{N} and \mathbf{Z} we denote the sets of positive integers and of integers, respectively. e stands for the unit (which satisfies $x * e = e * x = x$ for all $x \in G$)

¹Supported by the National Science Foundation of China (19971046), the Doctoral Programme Foundation of Institution of Higher Education, and the Foundation of Fujian Educational Committee (JA99154).



of G (if it exists), while it is θ instead of e if G is an abelian group. A mapping $T: G' \rightarrow E$ (G' with the property that $G' \subseteq G$ such that $x * y \in G'$ for all $x, y \in G'$) is called additive on G' if $T(x * y) = T(x) + T(y)$ for all $x, y \in G'$. Let f, g, h be mappings from G into E . We refer to the equations

$$f(x * y) - f(x) - f(y) = \theta \quad (\forall x, y \in G), \quad (4)$$

$$f(x * y) - g(x) - h(y) = \theta \quad (\forall x, y \in G), \quad (5)$$

$$2f((x * y)^{1/2}) - f(x) - f(y) = \theta \quad (\forall x, y \in G) \quad (6)$$

as a Cauchy equation, a Pexider equation, and a Jensen equation, respectively. ϕ in (1)–(3) is called the approximate remainder of the corresponding functional equation. The stability of these functional equations is called Hyers–Ulam–Rassias stability.

In 1940, S. M. Ulam [18] proposed the following problem for the stability of Cauchy equations:

Let G be a group and let E be a metric group with the metry $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h: G \rightarrow E$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G$, then there exists a homomorphism $H: G \rightarrow E$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G$?

In 1941, D. H. Hyers [3] answered this question in the affirmative when G and E are Banach spaces. In 1978, Rassias [12] generalized the result of Hyers. The result of the stability of Cauchy equations was further generalized by Rassias [13], Rassias and Šemrl [14], Găvruta [2], and Jung [7].

J. Rätz [11] considered the stability of Cauchy equations under the assumption that G and E are a power-associative groupoid and a sequentially complete topological vector space, respectively. The case of the stability of Pexider equations was generalized by J. Chmieliński and Tabor [1] and Kil-Woung Jun et al. [6]. The stability problems of Jensen equations can be found in [8–10]. For more theories of the Hyers–Ulam–Rassias stability of all kinds of functional equations, we refer the reader to [15–17].

In this paper, still using the direct method, we obtain some generalizations of the above theorems by considering the approximate remainders ϕ .

2. STABILITY OF CAUCHY EQUATIONS

Throughout this section let G be a power-associative groupoid and let f satisfy (1). A set G is called a power-associative groupoid if G is a nonempty set with a binary relation $x * y \in G$ such that the left powers satisfy $x^{m+n} = x^m * x^n$ for all $m, n \in \mathbb{N}$ and all $x \in G$. Left powers are defined by $x^1 = x$, $x^{m+1} = x * x^m$, $m \in \mathbb{N}$.

A subset B of E is called ideally convex if for any bounded sequence $\{x_n\} \subseteq B$ and sequence $\{\lambda_n\} \subseteq (0, +\infty)$ with $\sum_{n=1}^{\infty} \lambda_n = 1$, the series $\sum_{n=1}^{\infty} \lambda_n x_n$ either converges to an element of B or does not converge at all.

THEOREM 1. *If there exists $p \in \mathbf{N} \setminus \{1\}$ such that f satisfies (1) and*

$$f((x * y)^{p^n}) = f(x^{p^n} * y^{p^n}) \quad (\forall x, y \in G), \quad (\text{T.1.1})$$

then

$$\lim_{n \rightarrow \infty} \frac{\phi(x^{p^n}, y^{p^n})}{p^n} = \theta \quad (\forall x, y \in G), \quad (\text{T.1.2})$$

$$\lim_{n \rightarrow \infty} \frac{1}{p^n} \sum_{k=1}^{p^n-1} \phi(x^k, x) \quad \text{or} \quad \lim_{n \rightarrow \infty} \frac{1}{p^n} \sum_{k=1}^{p^n-1} \phi(x, x^k) \in E \quad (\forall x \in G), \quad (\text{T.1.3})$$

if and only if the limit $T(x) = \lim_{n \rightarrow \infty} f(x^{p^n})/p^n$ exists for any $x \in G$, and T is additive. In this case, the equality

$$T(x) = f(x) + \lim_{n \rightarrow \infty} \frac{1}{p^n} \sum_{k=1}^{p^n-1} \phi(x^k, x) = f(x) + \lim_{n \rightarrow \infty} \frac{1}{p^n} \sum_{k=1}^{p^n-1} \phi(x, x^k) \quad (\text{T.1.4})$$

holds for all $x \in G$.

Proof. (Necessity). For every $k \in \mathbf{N}$, putting $y = x^k$ in (1), we obtain that

$$f(x^{k+1}) - f(x) - f(x^k) = \phi(x, x^k) \quad (\forall x \in G). \quad (\text{T.1.5})$$

Adding the n formulas together in (T.1.5) from 1 to n , we conclude that

$$f(x^{n+1}) - (n+1)f(x) = \sum_{k=1}^n \phi(x, x^k) \quad (\forall x \in G). \quad (\text{T.1.6})$$

Since $\phi(x, y) = \phi(y, x)$ for all $x, y \in G$, $\phi(x, x^k) = \phi(x^k, x)$ for all $x \in G$ and all $k \in \mathbf{N}$. This yields two limit formulas in (T.1.3) that converge or diverge simultaneously, and they are equal when they converge. Thus, by (T.1.3), we may assume that

$$\lim_{n \rightarrow \infty} \frac{1}{p^n} \sum_{k=1}^{p^n-1} \phi(x, x^k) = \lim_{n \rightarrow \infty} \frac{1}{p^n} \sum_{k=1}^{p^n-1} \phi(x^k, x) = \eta(x) \quad (\forall x \in G).$$

With p^n in place of $n + 1$ in (T.1.6), and dividing by p^n , we have

$$\frac{f(x^{p^n})}{p^n} - f(x) = \frac{1}{p^n} \sum_{k=1}^{p^n-1} \phi(x^k, x) \quad (\forall x \in G, n \in \mathbf{N}). \quad (\text{T.1.7})$$

Let $n \rightarrow \infty$ in (T.1.7) to obtain

$$T(x) = \lim_{n \rightarrow \infty} \frac{f(x^{p^n})}{p^n} = f(x) + \eta(x) \quad (\forall x \in G).$$

To show that T is additive, replace x with x^{p^n} and y with y^{p^n} in (1), then divide by p^n to obtain, by (T.1.1),

$$\frac{f((x * y)^{p^n})}{p^n} - \frac{f(x^{p^n})}{p^n} - \frac{f(y^{p^n})}{p^n} = \frac{\phi(x^{p^n}, y^{p^n})}{p^n} \quad (\forall x, y \in G, n \in \mathbf{N}).$$

Consequently, the left side of the above equality tends to θ as $n \rightarrow \infty$ by (T.1.2). Thus it follows that

$$T(x * y) = T(x) + T(y) \quad (\forall x, y \in G).$$

The proof of the sufficiency is straightforward. It leads to the asserted result. ■

COROLLARY 1. *Let E be sequentially complete. Set*

$$B(x) = \text{co}\left(\{\theta\} \cup \{\phi(x^i, x^j)\}_{i,j=1}^{\infty}\right) \quad (\forall x \in G).$$

If there exists $p \in \mathbf{N} \setminus \{1\}$ such that f satisfies (1) and (T.1.1), ϕ satisfies (T.1.2), and $B(x)$ are bounded for all $x \in G$, then there exists a unique additive mapping $T: G \rightarrow E$ such that

$$T(x) - f(x) \in \bar{B}^s(x) \quad (\forall x \in G), \quad (\text{C.1.1})$$

where $\text{co}(A)$ is the convex hull of A , and \bar{A}^s denotes the sequential closure of A .

In particular, if E is locally convex, then the boundedness of $B(x)$ can be replaced by the boundedness of $\{\phi(x^i, x^j)\}_{i,j=1}^{\infty}$.

Proof. First, we show that (T.1.3) holds. Let $x \in G$. By the definition of $B(x)$, $B(x^n) \subseteq B(x)$ for all $n \in \mathbf{N}$ and all $x \in G$. Since $B(x)$ is convex, which contains θ , $B(x)$ is a starlike set (i.e., $tB(x) \subseteq B(x)$ for any $t \in (0, 1]$). It is easy to see that $\lim_{n \rightarrow \infty} \frac{1}{p^n} \sum_{k=1}^{p^n-1} \phi(x^k, x)$ exists for every $x \in G$.

Indeed, from (T.1.7), we conclude that for any $m \geq n (m, n \in \mathbf{N})$ and any $x \in G$,

$$\begin{aligned} & \frac{f(x^{p^m})}{p^m} - \frac{f(x^{p^n})}{p^n} \\ &= \frac{1}{p^n} \left[\frac{f((x^{p^n})^{p^{m-n}})}{p^{m-n}} - f(x^{p^n}) \right] = \frac{1}{p^n} \left[\frac{1}{p^{m-n}} \sum_{k=1}^{p^{m-n}-1} \phi((x^{p^n})^k, x^{p^n}) \right] \\ &= \frac{1}{p^n} \left[\frac{1}{p^{m-n}-1} \sum_{k=1}^{p^{m-n}-1} \frac{p^{m-n}-1}{p^{m-n}} \phi((x^{p^n})^k, x^{p^n}) \right] \\ &\in \frac{1}{p^n} B(x^{p^n}) \subseteq \frac{1}{p^n} B(x). \end{aligned}$$

By the hypothesis of boundedness of $B(x)$, $\{f(x^{p^n})/p^n\}$ is a Cauchy sequence of E . Since E is sequentially complete, $\{f(x^{p^n})/p^n\}$ converges to an element of E . This implies that

$$\lim_{n \rightarrow \infty} \frac{1}{p^n} \sum_{k=1}^{p^n-1} \phi(x^k, x) = \lim_{n \rightarrow \infty} \frac{1}{p^n} \sum_{k=1}^{p^n-1} \phi(x, x^k)$$

exists for all $x \in G$.

We shall show that $\bar{B}(x)$ is ideally convex. In fact, first, we can claim that $\bar{B}(x)$ is convex by [19, Theorem 4-2-12]. Furthermore, we note that any closed convex set is ideally convex. This implies that $\bar{B}(x)$ is ideally convex. Since $B(x)$ is a starlike convex set,

$$\frac{1}{p^n} \sum_{k=1}^{p^n-1} \phi(x^k, x) = \frac{1}{p^n-1} \sum_{k=1}^{p^n-1} \frac{p^n-1}{p^n} \phi(x, x^k) \in B(x) \quad (\forall x \in G).$$

Moreover,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{p^n} \sum_{k=1}^{p^n-1} \phi(x^k, x) &= \lim_{n \rightarrow \infty} \frac{1}{p^n-1} \sum_{k=1}^{p^n-1} \frac{p^n-1}{p^n} \phi(x^k, x) \in \bar{B}(x) \\ & \quad (\forall x \in G). \end{aligned}$$

Clearly,

$$\lim_{n \rightarrow \infty} \frac{1}{p^n} \sum_{k=1}^{p^n-1} \phi(x^k, x) = \lim_{n \rightarrow \infty} \frac{1}{p^n} \sum_{k=1}^{p^n-1} \phi(x, x^k) \in \bar{B}^s(x) \quad (\forall x \in G).$$

Hence (C.1.1) holds by Theorem 1.

Now we show the uniqueness of T . Suppose that $U: G \rightarrow E$ is another additive mapping that satisfies $U(x) - f(x) \in \bar{B}^s(x)$ for all $x \in G$. Then it follows from (C.1.1) that

$$\begin{aligned} U(x) - T(x) &= \frac{1}{n}(U(x^n) - T(x^n)) \\ &= \frac{1}{n}(U(x^n) - f(x^n) + f(x^n) - T(x^n)) \\ &\in \frac{1}{n}(\bar{B}^s(x^n) - \bar{B}^s(x^n)) \\ &\subseteq \frac{1}{n}(\bar{B}^s(x) - \bar{B}^s(x)) \quad (\forall x \in G). \end{aligned}$$

But $\bar{B}(x)$ is bounded by the boundedness of $B(x)$. Consequently, we conclude that $U(x) - T(x) \rightarrow \theta$ as $n \rightarrow \infty$. Thus it follows that $U(x) = T(x)$ for all $x \in G$.

Finally, note that if E is also locally convex, then the boundedness of $B(x)$ as a sequel to $\{\phi(x^i, x^j)\}_{i,j=1}^\infty$ is bounded. This concludes the proof. ■

THEOREM 2. *If (T.1.1) holds for $p = 2$, then*

$$\lim_{n \rightarrow \infty} \frac{\phi(x^{2^n}, y^{2^n})}{2^n} = \theta \quad (\forall x, y \in G), \quad (\text{T.2.1})$$

$$\sum_{k=1}^{\infty} \frac{\phi(x^{2^{k-1}}, x^{2^{k-1}})}{2^k} = \eta(x) \in E \quad (\forall x \in G), \quad (\text{T.2.2})$$

if the only if the limit $T(x) = \lim_{n \rightarrow \infty} f(x^{2^n})/2^n$ exists for any $x \in G$, and T is additive. In this case, we have $T(x) = f(x) + \eta(x)$ for any $x \in G$.

Proof. We need only show the necessity. Put $y = x$ in (1) to obtain

$$\frac{1}{2}f(x^2) - f(x) = \frac{1}{2}\phi(x, x) \quad (\forall x \in G). \quad (\text{T.2.3})$$

Assume that

$$\frac{1}{2^n}f(x^{2^n}) - f(x) = \sum_{k=1}^n \frac{\phi(x^{2^{k-1}}, x^{2^{k-1}})}{2^k} \quad (\forall x \in G) \quad (\text{T.2.4})$$

holds for a certain $n \in \mathbf{N}$. Then for all $x \in G$,

$$\begin{aligned} \frac{1}{2^{n+1}}f(x^{2^{n+1}}) - f(x) &= \frac{1}{2} \left[\frac{f((x^2)^{2^n})}{2^n} - f(x^2) \right] + \frac{1}{2}f(x^2) - f(x) \\ &= \frac{1}{2} \sum_{k=1}^n \frac{\phi((x^2)^{2^{k-1}}, (x^2)^{2^{k-1}})}{2^k} + \frac{1}{2}\phi(x, x) \\ &= \sum_{k=1}^{n+1} \frac{\phi(x^{2^{k-1}}, x^{2^{k-1}})}{2^k}, \end{aligned}$$

and so (T.2.4) holds for any $n \in \mathbf{N}$ and any $x \in G$ by induction.

In the same way as in the proof Theorem 1, we achieve the result. \blacksquare

COROLLARY 2. *Let E be sequentially complete. If (T.1.1) and (T.2.1) hold, and $B(x) = \text{co}(\{\theta\} \cup \{\phi(x^{2^i}, x^{2^i})\}_{i=1}^\infty)$ are bounded for all $x \in G$, then there exists a unique additive mapping $T: G \rightarrow E$ such that*

$$T(x) - f(x) \in \bar{B}^s(x) \quad (\forall x \in G). \quad (\text{C.2.1})$$

When E is also locally convex, the boundedness of $B(x)$ can be replaced by the boundedness of $\{\phi(x^{2^i}, x^{2^i})\}_{i=1}^\infty$.

Remark 1. Corollary 1 is a generalization of the result of J. Rätz [11], and Theorem 2 is just a generalization of the result of Gävrutä [2].

3. STABILITY OF PEXIDER EQUATIONS

Throughout this section let G be a power-associative groupoid with a unit e . f satisfies (2) and

$$f((x * y)^{2^n}) = f(x^{2^n} * y^{2^n}) \quad (\forall x, y \in G, n \in \mathbf{N}). \quad (\text{T.3.1})$$

Let $\phi_{(g,h)}$ be the approximate reminder of the Pexider equation with respect to $g, h: G \rightarrow E$.

THEOREM 3. *ϕ satisfies*

$$\lim_{n \rightarrow \infty} \frac{\phi(x^{2^n}, y^{2^n})}{2^n} = \theta \quad (\forall x, y \in G) \quad (\text{T.3.2})$$

and

$$\sum_{k=1}^{\infty} \frac{\phi(x^{2^{k-1}}, x^{2^{k-1}}) - \phi(x^{2^{k-1}}, e) - \phi(e, x^{2^{k-1}})}{2^k} = \eta(x) \in E$$

$$(\forall x \in G) \quad (\text{T.3.3})$$

if and only if the limit $T(x) = \lim_{n \rightarrow \infty} f(x^{2^n})/2^n$ exists for any $x \in G$, and T is additive. In addition, we have

$$T(x) - f(x) = \eta(x) - g(e) - h(e) \quad (\forall x \in G), \quad (\text{T.3.4})$$

$$T(x) - g(x) = \eta(x) - g(e) + \phi(x, e) \quad (\forall x \in G), \quad (\text{T.3.5})$$

$$T(x) - h(x) = \eta(x) - h(e) + \phi(e, x) \quad (\forall x \in G). \quad (\text{T.3.6})$$

Moreover, T is independent of g, h , whose ϕ satisfies (T.3.2) and (T.3.3).

Proof. We have only to show the necessity. In (2), set $y = x$ to obtain

$$f(x^2) - g(x) - h(x) = \phi(x, x) \quad (\forall x \in G). \quad (\text{T.3.7})$$

Put $y = e$ in (2) to obtain

$$f(x) - g(x) = h(e) + \phi(x, e) \quad (\forall x \in G). \quad (\text{T.3.8})$$

Putting $x = e$ with x in place of y in (2), we have

$$f(x) - h(x) = g(e) + \phi(e, x) \quad (\forall x \in G). \quad (\text{T.3.9})$$

By induction, we can show that

$$\frac{f(x^{2^n})}{2^n} - f(x) = \sum_{k=1}^n \frac{\psi(x^{2^{k-1}})}{2^k} \quad (\forall x \in G, \forall n \in \mathbf{N}), \quad (\text{T.3.10})$$

where $\psi(x) = -g(e) - h(e) + \phi(x, x) - \phi(x, e) - \phi(e, x)$.

Indeed, for $n = 1$, it follows from (T.3.7)–(T.3.9) that for all $x \in G$,

$$\begin{aligned} \frac{1}{2}f(x^2) - f(x) &= \frac{1}{2}[f(x^2) - g(x) - h(x)] + \frac{1}{2}[g(x) - f(x)] \\ &\quad + \frac{1}{2}[h(x) - f(x)] \\ &= \frac{1}{2}[\phi(x, x) - h(e) - \phi(x, e) - g(e) - \phi(e, x)] \\ &= \frac{1}{2}\psi(x). \end{aligned}$$

Assume that (T.3.10) holds for a certain n . Then

$$\begin{aligned}
 \frac{1}{2^{n+1}}f(x^{2^{n+1}}) - f(x) &= \frac{1}{2} \left[\frac{f((x^2)^{2^n})}{2^n} - f(x^2) \right] + \frac{1}{2}f(x^2) - f(x) \\
 &= \frac{1}{2} \sum_{k=1}^n \frac{\psi((x^2)^{2^{k-1}})}{2^k} + \frac{1}{2}\psi(x) \\
 &= \sum_{k=1}^n \frac{\psi(x^{2^k})}{2^{k+1}} + \frac{1}{2}\psi(x) \\
 &= \sum_{k=1}^{n+1} \frac{\psi(x^{2^{k-1}})}{2^k} \quad (\forall x \in G),
 \end{aligned}$$

and so (T.3.10) holds for $n+1$.

Moreover, we conclude from (T.3.3) that

$$\begin{aligned}
 &\sum_{k=1}^{\infty} \frac{\psi(x^{2^{k-1}})}{2^k} \\
 &= \sum_{k=1}^{\infty} \frac{-g(e) - h(e) + \phi(x^{2^{k-1}}, x^{2^{k-1}}) - \phi(x^{2^{k-1}}, e) - \phi(e, x^{2^{k-1}})}{2^k} \\
 &= -g(e) - h(e) + \eta(x) \quad (\forall x \in G).
 \end{aligned}$$

Letting $n \rightarrow \infty$ in (T.3.10), for all $x \in G$ we achieve

$$\begin{aligned}
 T(x) &= \lim_{n \rightarrow \infty} \frac{f(x^{2^n})}{2^n} = f(x) + \sum_{k=1}^{\infty} \frac{\psi(x^{2^{k-1}})}{2^k} \\
 &= f(x) - g(e) - h(e) + \eta(x).
 \end{aligned}$$

This implies that (T.3.4) holds.

It easily follows from (T.3.8) and (T.3.9) that

$$\lim_{n \rightarrow \infty} \frac{g(x^{2^n})}{2^n} = \lim_{n \rightarrow \infty} \frac{h(x^{2^n})}{2^n} = \lim_{n \rightarrow \infty} \frac{f(x^{2^n})}{2^n} = T(x) \quad (\forall x \in G)$$

by (T.3.2). Replacing x with x^{2^n} and y with y^{2^n} in (2) and then dividing by 2^n , by (T.3.1) we get for any $x \in G$ and any $n \in \mathbb{N}$

$$\frac{f((x * y)^{2^n})}{2^n} - \frac{g(x^{2^n})}{2^n} - \frac{h(y^{2^n})}{2^n} = \frac{\phi(x^{2^n}, y^{2^n})}{2^n}.$$

Letting $n \rightarrow \infty$, by (T.3.2) we conclude that $T(x * y) = T(x) + T(y)$ ($\forall x, y \in G$). From (T.3.8), (T.3.9), and (T.3.4), we get that (T.3.5) and (T.3.6) hold.

Let T and T' be additive mappings with respect to (g, h) and (g', h') , which satisfy (T.3.4)–(T.3.6), respectively. To show that $T = T'$, we observe that for any $x \in G$ and all $n \in \mathbb{N}$,

$$\begin{aligned} T(x) - T'(x) &= \frac{1}{2^n} [T(x^{2^n}) - T'(x^{2^n})] \\ &= \frac{1}{2^n} [T(x^{2^n}) - f(x^{2^n}) + g(e) + h(e) - g(e) - h(e) \\ &\quad + f(x^{2^n}) - T'(x^{2^n}) - g'(e) - h'(e) + g'(e) + h'(e)] \\ &= \frac{1}{2^n} [\eta_{(g, h)}(x^{2^n}) - \eta_{(g', h')}(x^{2^n}) + C], \end{aligned}$$

where $C = -g(e) - h(e) + g'(e) + h'(e)$.

We need only show $\lim_{n \rightarrow \infty} \eta(x^{2^n})/2^n = \theta$ for all $x \in G$. Indeed, by (T.3.3) we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\eta(x^{2^n})}{2^n} &= \lim_{n \rightarrow \infty} \frac{1}{2^n} \sum_{k=1}^{\infty} \frac{\phi((x^{2^n})^{2^{k-1}}, (x^{2^n})^{2^{k-1}}) - \phi((x^{2^n})^{2^{k-1}}, e) - \phi(e, (x^{2^n})^{2^{k-1}})}{2^k} \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \frac{\phi(x^{2^{n+k-1}}, x^{2^{n+k-1}}) - \phi(x^{2^{n+k-1}}, e) - \phi(e, x^{2^{n+k-1}})}{2^{n+k}} \\ &= \lim_{n \rightarrow \infty} \sum_{k=n+1}^{\infty} \frac{\phi(x^{2^{k-1}}, x^{2^{k-1}}) - \phi(x^{2^{k-1}}, e) - \phi(e, x^{2^{k-1}})}{2^k} = \theta \quad (\forall x \in G). \end{aligned}$$

This completes the proof. \blacksquare

For abbreviation we set

$$A(x, y) = A_{(g, h)}(x, y) = \left\{ \phi_{(g, h)}(x^{2^i}, y^{2^i}) \right\}_{i=1}^{\infty} \quad (\forall x, y \in G),$$

$$B(x, y) = B_{(g, h)}(x, y) = \text{co}(\{\theta\} \cup A_{(g, h)}(x, y)) \quad (\forall x, y \in G).$$

$\phi_{(g, h)}$ is said to have property (C.B) if for any $x \in G$ $B_{(g, h)}(x, x)$, $B_{(g, h)}(x, e)$ and $B_{(g, h)}(e, x)$ are bounded, and (T.3.2) holds. We denote $\mathcal{F} = \{(g, h): \phi_{(g, h)} \text{ has property (C.B)}\}$.

If E is also locally convex, the boundedness of $B_{(g, h)}(x, y)$ can be replaced by the boundedness of $A_{(g, h)}(x, y)$.

COROLLARY 3. *Let E be sequentially complete. If $(g, h) \in \mathcal{F}$, then there exists a unique additive mapping $T: G \rightarrow E$ such that*

$$T(x) - f(x) + g(e) + h(e) \in \bar{B}^s(x, x) - \bar{B}^s(x, e) - \bar{B}^s(e, x) \quad (\forall x \in G), \quad (\text{C.3.1})$$

$$T(x) - g(x) + g(e) \in \bar{B}^s(x, x) - \bar{B}^s(x, e) - \bar{B}^s(e, x) + B(x, e) \quad (\forall x \in G), \quad (\text{C.3.2})$$

$$T(x) - h(x) + h(e) \in \bar{B}^s(x, x) - \bar{B}^s(x, e) - \bar{B}^s(e, x) + B(e, x) \quad (\forall x \in G). \quad (\text{C.3.3})$$

Proof. As in the proof Corollary 1, we see that $\bar{B}(x, x)$, $\bar{B}(x, e)$, and $\bar{B}(e, x)$ are ideally convex for any $x \in G$. It follows that for all $x \in G$,

$$\begin{aligned} \eta(x) &= \sum_{k=1}^{\infty} \frac{\phi(x^{2^{k-1}}, x^{2^{k-1}}) - \phi(x^{2^{k-1}}, e) - \phi(e, x^{2^{k-1}})}{2^k} \\ &\in \bar{B}^s(x, x) - \bar{B}^s(x, e) - \bar{B}^s(e, x). \end{aligned}$$

By the definition of $B(x, y)$, we have $B(x^{2^n}, y^{2^n}) \subseteq B(x, y)(\forall x, y \in G, \forall n \in \mathbb{N})$.

In a manner analogous to that of Corollary 1, we can complete the proof. ■

COROLLARY 4. *If E is sequentially complete, then there exists a unique additive mapping $T: G \rightarrow E$ such that for any $x \in G$ and any $(g, h) \in \mathcal{F}$*

$$T(x) - f(x) + g(e) + h(e) \in B_{(g, h)}(x), \quad (\text{C.4.1})$$

$$T(x) - g(x) + g(e) \in B_{(g, h)}(x) + B_{(g, h)}(x, e), \quad (\text{C.4.2})$$

$$T(x) - h(x) + h(e) \in B_{(g, h)}(x) + B_{(g, h)}(e, x), \quad (\text{C.4.3})$$

where $B_{(g, h)}(x) \stackrel{\text{def}}{=} \bar{B}_{(g, h)}^s(x, x) - \bar{B}_{(g, h)}^s(x, e) - \bar{B}_{(g, h)}^s(e, x)$.

Proof. Corollary 3 asserts that there is a unique additive mapping T that satisfies (C.4.1)–(C.4.3) for any $(g, h) \in \mathcal{F}$.

We shall show the uniqueness of T . Let T and T' be additive mappings with respect to (g, h) and (g', h') in \mathcal{F} , respectively. To show that $T = T'$,

we observe that for any $x \in G$ and all $n \in \mathbf{N}$,

$$\begin{aligned}
 T(x) - T'(x) &= \frac{1}{2^n} [T(x^{2^n}) - T'(x^{2^n})] \\
 &= \frac{1}{2^n} [T(x^{2^n}) - f(x^{2^n}) + g(e) + h(e) - g(e) - h(e) \\
 &\quad + f(x^{2^n}) - T'(x^{2^n}) - g'(e) - h'(e) + g'(e) + h'(e)] \\
 &\in \frac{1}{2^n} [B_{(g,h)}(x^{2^n}) - B_{(g',h')}(x^{2^n}) + C] \\
 &\subseteq \frac{1}{2^n} [B_{(g,h)}(x) - B_{(g',h')}(x) + C],
 \end{aligned}$$

where $C = -g(e) - h(e) + g'(e) + h'(e)$. Because $B_{(g,h)}(x)$ and $B_{(g',h')}(x)$ are bounded, letting $n \rightarrow \infty$, we conclude that $T(x) = T'(x)$ for all $x \in G$. ■

COROLLARY 5. *Let B be a bounded convex subset of E which contains θ , where E is sequentially complete. Then there is a unique additive mapping $T: G \rightarrow E$ such that for any $x \in G$ and any $(g, h) \in \mathcal{F}$*

$$T(x) - f(x) + g(e) + h(e) \in \bar{B}^s - 2\bar{B}^s, \quad (\text{C.5.1})$$

$$T(x) - g(x) + g(e) \in 2\bar{B}^s - 2\bar{B}^s, \quad (\text{C.5.2})$$

$$T(x) - h(x) + h(e) \in 2\bar{B}^s - 2\bar{B}^s, \quad (\text{C.5.3})$$

where $\mathcal{F}_B = \{(g, h): \phi_{(g,h)}(x, y) \in B \text{ for all } x, y \in G\}$.

If E is also locally convex and A is a bounded set of E , then (C.5.1)–(C.5.3) hold for all $(g, h) \in \mathcal{F}_A$ and $x \in G$, where $B = \text{co}(\{\theta\} \cup A)$.

Remark 2. Theorem 3 shows that we have succeeded here in giving a generalization of the result of [6].

4. STABILITY OF JENSEN EQUATIONS

Throughout this section let $(G, *)$ be a power-associative groupoid with a unit element e and inverse elements. By x^{-1} we denote the inverse element of x in G (which satisfies $x^{-1} * x = x * x^{-1} = e$). Let G_0 be a subset of G such that $x^n \in G_0$ for any $n \in \mathbf{Z}$ and all $x \in G_0$. We assume

that if $x \in G_0 \setminus \{e\}$, then $x^2 \neq e$, $x^3 \neq e$. If there exists $z \in G_0$ such that $x * y = z^2$ for $x, y \in G_0$, then we can state that symbolically $(x * y)^{1/2} = z$. In particular, $x * y = [(x * y)^2]^{1/2}$ for all $x, y \in G_0$ with $x * y \in G_0$.

Let $f: G_0 \rightarrow E$ satisfy (3)

$$f\left(\left[(x * y)^{1/2}\right]^{3^n}\right) = f\left((x^{3^n} * y^{3^n})^{1/2}\right) \quad (\text{T.4.1})$$

and

$$f\left(\left[(x * y)^2\right]^{1/2}\right) = f\left((x^2 * y^2)^{1/2}\right) \quad (\text{T.4.2})$$

for any $x, y \in G_0 \setminus \{e\}$ with $(x * y)^{1/2} \in G_0$.

THEOREM 4. $\phi: G_0 \setminus \{e\} \times G_0 \setminus \{e\} \rightarrow E$ satisfies

$$\lim_{n \rightarrow \infty} \frac{\phi(x^{3^n}, y^{3^n})}{3^n} = \theta \quad (\forall x, y \in G_0 \setminus \{e\}), \quad (\text{T.4.3})$$

$$\sum_{k=1}^{\infty} \frac{\phi(x^{3^{k-1}}, x^{-3^{k-1}}) - \phi(x^{-3^{k-1}}, x^{3^k})}{3^k} = \eta(x) \in E \quad (\forall x \in G_0 \setminus \{e\}), \quad (\text{T.4.4})$$

if and only if the limit $T(x) = \lim_{n \rightarrow \infty} \phi(x^{3^n})/3^n$ exists for all $x \in G_0 \setminus \{e\}$, and T is additive in the sense that $T(x * y) = T(x) + T(y)$ for all $x, y \in G_0 \setminus \{e\}$. In addition, we have

$$T(x) - f(x) = \eta(x) - f(e) \quad (\forall x \in G_0 \setminus \{e\}). \quad (\text{T.4.5})$$

Proof. We shall show only the necessity. In (3), take $y = x^{-1}$ to get

$$2f(e) - f(x) - f(x^{-1}) = \phi(x, x^{-1}) \quad (\forall x \in G_0 \setminus \{e\}). \quad (\text{T.4.6})$$

Replacing x with x^{-1} and y with x^3 , we obtain

$$2f(x) - f(x^{-1}) - f(x^3) = \phi(x^{-1}, x^3) \quad (\forall x \in G_0 \setminus \{e\}). \quad (\text{T.4.7})$$

By (T.4.6) and (T.4.7), we conclude that for all $x \in G_0 \setminus \{e\}$,

$$\begin{aligned} \frac{1}{3}f(x^3) - f(x) &= \frac{1}{3}[f(x^3) + f(x^{-1}) - 2f(x) \\ &\quad - f(x^{-1}) - f(x) + 2f(e) - 2f(e)] \\ &= \frac{1}{3}[-\phi(x^{-1}, x^3) + \phi(x, x^{-1}) - 2f(e)] = \frac{1}{3}u(x), \end{aligned}$$

where $u(x) = \phi(x, x^{-1}) - \phi(x^{-1}, x^3) - 2f(e)$. If

$$\frac{1}{3^n} f(x^{3^n}) - f(x) = \sum_{k=1}^n \frac{u(x^{3^{k-1}})}{3^k} \quad (\forall x \in G_0 \setminus \{e\}) \quad (\text{T.4.8})$$

holds for a certain n , then for every $x \in G_0 \setminus \{e\}$,

$$\begin{aligned} \frac{1}{3^{n+1}} f(x^{3^{n+1}}) - f(x) &= \frac{1}{3} \left[\frac{1}{3^n} f((x^3)^{3^n}) - f(x^3) \right] + \frac{1}{3} f(x^3) - f(x) \\ &= \frac{1}{3} \sum_{k=1}^n \frac{u((x^3)^{3^{k-1}})}{3^k} + \frac{1}{3} u(x) \\ &= \sum_{k=1}^n \frac{u(x^{3^k})}{3^{k+1}} + \frac{1}{3} u(x) = \sum_{k=1}^{n+1} \frac{u(x^{3^{k-1}})}{3^k}, \end{aligned}$$

and so (T.4.8) holds for any $n \in \mathbb{N}$ and any $x \in G_0 \setminus \{e\}$ by induction.

From (T.4.4) we compute that for every $x \in G_0 \setminus \{e\}$,

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{u(x^{3^{k-1}})}{3^k} &= \sum_{k=1}^{\infty} \frac{\phi(x^{3^{k-1}}, x^{-3^{k-1}}) - \phi(x^{-3^{k-1}}, x^{3^k})}{3^k} - \sum_{k=1}^{\infty} \frac{2f(e)}{3^k} \\ &= \eta(x) - f(e). \end{aligned}$$

This implies that $\{f(x^{3^n})/3^n\}$ converges in E . That is, $T(x) = \lim_{n \rightarrow \infty} \{f(x^{3^n})/3^n\}$ exists for any $x \in G_0 \setminus \{e\}$. Moreover, $T(x) - f(x) = \eta(x) - f(e)$ holds for any $x \in G_0 \setminus \{e\}$ by (T.4.8).

Now we show that T is additive in several steps.

Step 1. From (T.4.1) and (3), we have

$$\frac{2f\left(\left[(x * y)^{1/2}\right]^{3^n}\right)}{3^n} - \frac{f(x^{3^n})}{3^n} - \frac{f(y^{3^n})}{3^n} = \frac{\phi(x^{3^n}, y^{3^n})}{3^n}$$

for any $x, y \in G_0 \setminus \{e\}$ with $(x * y)^{1/2} \in G_0$.

Letting $n \rightarrow \infty$, by (T.4.3), we obtain

$$2T((x * y)^{1/2}) = T(x) + T(y) \quad (\text{T.4.9})$$

for any $x, y \in G_0 \setminus \{e\}$ with $(x * y)^{1/2} \in G_0$.

Step 2. The definition of T implies that for any $x \in G_0 \setminus \{e\}$ and any $n \in \mathbb{N}$,

$$T(x^{3^n}) = \lim_{m \rightarrow \infty} \frac{f((x^{3^n})^{3^m})}{3^m} = \lim_{m \rightarrow \infty} \frac{f(x^{3^{n+m}})}{3^{n+m}} 3^n = 3^n T(x).$$

Step 3. For any $x \in G_0 \setminus \{e\}$, we obtain by (T.4.4), that

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \frac{\eta(x^{3^n})}{3^n} \\
 &= \lim_{n \rightarrow \infty} \frac{1}{3^n} \sum_{k=1}^{\infty} \frac{\phi((x^{3^n})^{3^{k-1}}, (x^{3^n})^{-3^{k-1}}) - \phi((x^{3^n})^{-3^{k-1}}, (x^{3^n})^{3^k})}{3^k} \\
 &= \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \frac{\phi(x^{3^{n+k-1}}, x^{-3^{n+k-1}}) - \phi(x^{-3^{n+k-1}}, x^{3^{n+k}})}{3^{n+k}} \\
 &= \lim_{n \rightarrow \infty} \sum_{k=n+1}^{\infty} \frac{\phi(x^{3^{k-1}}, x^{-3^{k-1}}) - \phi(x^{-3^{k-1}}, x^{3^k})}{3^k} = \theta.
 \end{aligned}$$

Step 4. For any $x \in G_0 \setminus \{e\}$, we compute, by (T.4.1), (T.4.5), and (T.4.10), that

$$\begin{aligned}
 2T(x^2) - 4T(x) &= 2T(x^2) - T(x^3) - T(x) \\
 &= \frac{1}{3^n} \left[2T((x^2)^{3^n}) - T((x^3)^{3^n}) - T(x^{3^n}) \right] \\
 &= \frac{1}{3^n} \left[2T((x^2)^{3^n}) - 2f((x^2)^{3^n}) \right. \\
 &\quad \left. - T((x^3)^{3^n}) + f((x^3)^{3^n}) \right. \\
 &\quad \left. - T(x^{3^n}) + f(x^{3^n}) + 2f\left(\left[(x^3 * x)^{1/2}\right]^{3^n}\right) \right. \\
 &\quad \left. - f((x^3)^{3^n}) - f(x^{3^n}) \right] \\
 &= \frac{1}{3^n} \left[2\eta((x^2)^{3^n}) - 2f(e) - \eta((x^3)^{3^n}) \right. \\
 &\quad \left. + f(e) - \eta(x^{3^n}) + f(e) + \phi(x^{3^{n+1}}, x^{3^n}) \right] \\
 &= \frac{1}{3^n} \left[2\eta((x^2)^{3^n}) - \eta((x^3)^{3^n}) \right. \\
 &\quad \left. - \eta(x^{3^n}) + \phi(x^{3^{n+1}}, x^{3^n}) \right].
 \end{aligned}$$

Letting $n \rightarrow \infty$, we claim, by Step 3 and (T.4.3), that

$$T(x^2) = 2T(x) \quad (\forall x \in G_0 \setminus \{e\}). \quad (\text{T.4.11})$$

Finally, for any $x, y \in G_0 \setminus \{e\}$ with $x * y \in G_0$, we obtain, by (T.4.2), (T.4.9), and (T.4.11), that

$$\begin{aligned} T(x * y) &= T\left(\left[(x * y)^2\right]^{1/2}\right) = T\left(\left[(x^2 * y^2)\right]^{1/2}\right) \\ &= \frac{1}{2}(T(x^2) + T(y^2)) = T(x) + T(y). \end{aligned}$$

■

For abbreviation we denote

$$\begin{aligned} B(x^{-1}, x) &= \text{co}\left(\{\theta\} \cup \left\{\phi(x^{-3^i}, x^{3^{i+1}})\right\}_{i=1}^{\infty}\right) \quad (\forall x \in G_0 \setminus \{e\}), \\ B(x, x^{-1}) &= \text{co}\left(\{\theta\} \cup \left\{\phi(x^{3^i}, x^{-3^i})\right\}_{i=1}^{\infty}\right) \quad (\forall x \in G_0 \setminus \{e\}). \end{aligned}$$

COROLLARY 6. *Suppose that E is sequentially complete and (T.4.3) holds. If $B(x^{-1}, x)$ and $B(x, x^{-1})$ are bounded for any $x \in G_0 \setminus \{e\}$, then there exists a unique additive mapping $T: G_0 \rightarrow E$ such that*

$$T(x) - f(x) + f(e) \in \bar{B}^s(x^{-1}, x) - \bar{B}^s(x, x^{-1}) \quad (\forall x \in G_0 \setminus \{e\}). \quad (\text{C.6.1})$$

If E is also locally convex, then the boundedness of $\{\phi(x^{-3^i}, x^{3^{i+1}})\}_{i=1}^{\infty}$ and $\{\phi(x^{3^i}, x^{-3^i})\}_{i=1}^{\infty}$ ensures the boundedness of $B(x^{-1}, x)$ and $B(x, x^{-1})$, respectively.

Proof. Note that

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\phi(x^{-3^{k-1}}, x^{3^k})}{3^k} &= \sum_{k=1}^{\infty} \frac{(2/3)^k \phi(x^{-3^{k-1}}, x^{3^k})}{2^k} \quad (\forall x \in G_0 \setminus \{e\}), \\ \sum_{k=1}^{\infty} \frac{\phi(x^{3^{k-1}}, x^{-3^k})}{3^k} &= \sum_{k=1}^{\infty} \frac{(2/3)^k \phi(x^{3^{k-1}}, x^{-3^k})}{2^k} \quad (\forall x \in G_0 \setminus \{e\}). \end{aligned}$$

In the same way as in the proof Corollary 1, we may show the result. ■

Remark 3. If G is an abelian group and E is a Banach space, then Theorem 4 is a generalization of the result of [9].

Now we localize some conditions by the following theorem.

THEOREM 5. *Let G be a real topological vector space. If mapping $T: G \rightarrow E$ satisfies that there exists a θ -neighborhood U such that*

$$T(x + y) = T(x) + T(y) \quad (\text{T.5.1})$$

whenever $y - x \in U$ for all $x, y \in G$, then T is additive.

Proof. We first may show by induction that

$$T\left(\frac{x}{2^n}\right) = \frac{1}{2^n}T(x) \quad (\forall x \in G \text{ and } \forall n \in \mathbf{N}). \quad (\text{T.5.2})$$

Next let $x, y \in G$. We have only to consider the situation when $y - x \notin U$ for all $x, y \in G$. By the absorbance of the neighborhood of zero, there exists $N \in \mathbf{N}$ such that $(y - x)/2^N \in U$. From (T.5.1) and (T.5.2) we get

$$T(x + y) = 2^N T\left(\frac{x + y}{2^N}\right) = 2^N T\left(\frac{x}{2^N}\right) + 2^N T\left(\frac{y}{2^N}\right) = T(x) + T(y)$$

for all $x, y \in G$ with $y - x \notin U$.

Finally, for all $x, y \in G$, (T.5.1) holds (i.e., T is additive). ■

Remark 4. If G is a real topological vector space, then, by Theorem 5, the conditions (T.1.1), (T.1.2), (T.2.1), (T.3.1), (T.3.2), and (T.4.1)–(T.4.3) hold as long as $y - x \in U$ for some neighborhood of zero U in G and any $x, y \in G$. In this case, the operation $*$ is a usual addition $+$.

ACKNOWLEDGMENTS

I express my deep gratitude to my Ph.D. advisor, Professor Ding Guanggui, for his advice. In addition, I express my appreciation to Professor Themistocles M. Rassias, whose research introduced me to this field of mathematics.

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