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On the Spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$[1]

Xianling Fan[2] and Dun Zhao

Department of Mathematics, Lanzhou University, Lanzhou 730000, People's Republic of China

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In this paper we present some basic results on the generalized Lebesgue spaces $L^{p(x)}(\Omega)$ and generalized Lebesgue–Sobolev spaces $W^{m,p(x)}(\Omega)$. These results provide the necessary framework for the study of variational problems and elliptic equations with non-standard $p(x)$-growth conditions. © 2001 Academic Press

Key Words: generalized Lebesgue space; Nemytsky operator; imbedding; density.

The study of variational problems with nonstandard growth conditions is a new topic developed in recent years [2–8, 20]. $p(x)$-growth conditions can be regarded as a very important class of nonstandard growth conditions. In this paper we present some basic theory of the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. Most of the results are similar to those for Lebesgue spaces $L^{p}(\Omega)$ and Sobolev spaces $W^{m,p}(\Omega)$, but the Sobolev-like imbedding theorem and result on density are new; they show the essential difference between $W^{m,p(x)}(\Omega)$ and $W^{m,p}(\Omega)$. These results provide the required framework for the study of problems with $p(x)$-growth conditions.

Throughout this paper, for simplicity, we take Lebesgue measure in $\mathbf{R}^n$, and denote by meas Ω the measure of $\Omega \subset \mathbf{R}^n$; all functions appearing in this paper are assumed to be real.

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[2] E-mail: fanxl@lzu.edu.cn.

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1. THE SPACE $L^{p(x)}(\Omega)$

Let $\Omega \subset R^n$ be a measurable subset and meas $\Omega > 0$. We write

$$E = \{u : u \text{ is a measurable function in } \Omega\}.$$

Elements in E that are equal to each other almost everywhere are considered as one element.

Let $p \in E$. In the following discussion we always assume that $u \in E$ and write

$$\varphi(x, s) = s^{p(x)}, \qquad \forall x \in \Omega, s \geq 0, \tag{1}$$

$$\rho(u) = \rho_{p(x)}(u) = \int_\Omega \varphi(x, |u|)\, dx = \int_\Omega |u(x)|^{p(x)}\, dx, \tag{2}$$

$$L^{p(x)}(\Omega) = \left\{u \in E : \lim_{\lambda \to 0^+} \rho(\lambda u) = 0\right\}, \tag{3}$$

$$L_0^{p(x)}(\Omega) = \{u \in E : \rho(u) < \infty\}, \tag{4}$$

$$L_1^{p(x)}(\Omega) = \{u \in E : \forall \lambda > 0, \rho(\lambda u) < \infty\}, \tag{5}$$

and

$$L_+^\infty(\Omega) = \left\{u \in L^\infty(\Omega) : \operatorname{ess}\inf_\Omega u \geq 1\right\}. \tag{6}$$

It is easy to see that the function φ defined above belongs to the class Φ, which is defined in [18, p. 33], i.e., φ satisfies the following two conditions:

1) For all $x \in \Omega$, $\varphi(x, \cdot)$: $[0, \infty) \to \mathbf{R}$ is a non-decreasing continuous function with $\varphi(x, 0) = 0$ and $\varphi(x, s) > 0$ whenever $s > 0$; $\varphi(x, s) \to \infty$ when $s \to \infty$.

2) For every $s \geq 0$, $\varphi(\cdot, s) \in E$.

Obviously, φ is convex in s.

In view of the definition in [18, p. 1], ρ is a convex modular over E, i.e., ρ: $E \to [0, \infty]$ verifies the following properties (a)–(c):

(a) $\rho(u) = 0 \Leftrightarrow u = 0$;

(b) $\rho(-u) = \rho(u)$;

(c) $\rho(\alpha u + \beta v) \leq \alpha\rho(u) + \beta\rho(v)$, $\forall u, v \in E$, $\forall \alpha, \beta \geq 0$, $\alpha + \beta = 1$,

and thus by [18], $L^{p(x)}(\Omega)$ is a Nakano space, which is a special kind of Musielak–Orlicz space. $L_0^{p(x)}(\Omega)$ is a kind of generalized Orlicz class. It is easy to see that $L^{p(x)}(\Omega)$ is a linear subspace of E, and $L_0^{p(x)}(\Omega)$ is a convex subset of $L^{p(x)}(\Omega)$. In general we have

$$L_1^{p(x)}(\Omega) \subset L_0^{p(x)}(\Omega) \subset L^{p(x)}(\Omega).$$

By the properties of $\varphi(x, s)$ we also have

$$L^{p(x)}(\Omega) = \{u \in E : \exists\lambda > 0, \rho(\lambda u) < \infty\}.$$

THEOREM 1.1. *The following two conditions are equivalent*:

1) $p \in L_+^\infty(\Omega)$.
2) $L_1^{p(x)}(\Omega) = L^{p(x)}(\Omega)$.

Proof. 1) $\Rightarrow$ 2) is obvious.

2) $\Rightarrow$ 1). If 1) is not true, then we can take a sequence $\{I_m\}$ of disjoint subsets of Ω with positive measure such that

$$p(x) > m \qquad \text{for } x \in I_m.$$

Choosing an increasing sequence $\{u_m\} \subset (0, \infty)$ such that $u_m \to \infty$ as $m \to \infty$, we can find k_m satisfying the inequality

$$\int_{I_m} u_{k_m}^{p(x)}\, dx \geq \frac{1}{2^m}.$$

By the absolute continuity of integral, we can shrink I_m to Ω_m such that

$$\int_{\Omega_m} u_{k_m}^{p(x)}\, dx = \frac{1}{2^m}.$$

Denote by $\chi_{\Omega_m}(x)$ the characteristic function of Ω_m, i.e.,

$$\chi_{\Omega_m}(x) = \begin{cases} 1, & \text{if } x \in \Omega_m \\ 0, & \text{if } x \notin \Omega_m. \end{cases}$$

if we write

$$u_0(x) = \sum_{m=1}^{\infty} u_{k_m} \chi_{\Omega_m}(x),$$

then we have

$$\int_\Omega |u_0(x)|^{p(x)}\, dx = \sum_{n=1}^{\infty} \int_{\Omega_n} u_{k_n}^{p(x)}\, dx = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1,$$

$$\int_\Omega |2u_0(x)|^{p(x)}\, dx = \sum_{n=1}^{\infty} \int_{\Omega_n} 2^{p(x)} u_{k_n}^{p(x)}\, dx > \sum_{n=1}^{\infty} 2^n \int_{\Omega_n} u_{k_n}^{p(x)}\, dx = \infty;$$

thus we have $u_0 \in L^{p(x)}(\Omega)$, but $u_0 \notin L_1^{p(x)}(\Omega)$. This contradicts condition (2), and we complete the proof. ∎

From now on we only consider the case where $p \in L_+^\infty(\Omega)$, i.e.,

$$1 \le p^- =: \operatorname*{ess\ inf}_{x\in\Omega} p(x) \le \operatorname*{ess\ sup}_{x\in\Omega} p(x) =: p^- < \infty. \tag{7}$$

For simplicity we write $E_\rho = L^{p(x)}(\Omega) = L_0^{p(x)}(\Omega) = L_1^{p(x)}(\Omega)$, and we call $L^{p(x)}(\Omega)$ generalized Lebesgue spaces. By [18, p. 7], we can introduce the norm $\|u\|_{L^{p(x)}(\Omega)}$ on E_ρ (denoted by $\|u\|_\rho$) as

$$\|u\|_\rho = \inf\left\{\lambda > 0 : \rho\left(\frac{u}{\lambda}\right) \le 1\right\},$$

and $(E_\rho, \|u\|_\rho)$ becomes a Banach space.

It is not hard to see that under condition (7), ρ satisfies

(d) $\rho(u+v) \le 2^{p^+}(\rho(u) + \rho(v))$; $\forall u, v \in E_\rho$.

(e) For $u \in E_\rho$, if $\lambda > 1$, we have

$$\rho(u) \le \lambda\rho(u) \le \lambda^{p^-}\rho(u) \le \rho(\lambda u) \le \lambda^{p^+}\rho(u),$$

and if $0 < \lambda < 1$, we have

$$\lambda^{p^+}\rho(u) \le \rho(\lambda u) \le \lambda^{p^-}\rho(u) \le \lambda\rho(u) \le \rho(u).$$

(f) For every fixed $u \in E_\rho \setminus \{0\}$, $\rho(\lambda u)$ is a continuous convex even function in λ, and it increases strictly when $\lambda \in [0, \infty)$

By property (f) and the definition of $\|\cdot\|_\rho$, we have

THEOREM 1.2. *Let $u \in E_\rho \setminus \{0\}$; then $\|u\|_\rho = a$ if and only if $\rho(\frac{u}{a}) = 1$.*

The norm $\|u\|_\rho$ is in close relation with the modular $\rho(u)$. We have

THEOREM 1.3. *Let $u \in E_\rho$; then*

1) $\|u\|_\rho < 1\ (= 1;\ > 1) \Leftrightarrow \rho(u) < 1\ (= 1;\ > 1)$;
2) *If* $\|u\|_\rho > 1$, *then* $\|u\|_\rho^{p^-} \le \rho(u) \le \|u\|_\rho^{p^+}$;
3) *If* $\|u\|_\rho < 1$, *then* $\|u\|_\rho^{p^+} \le \rho(u) \le \|u\|_\rho^{p^-}$.

Proof. From (f) and Theorem 1.2 we can obtain 1). We only prove 2) below, as the proof of 3) is similar. Assume that $\|u\|_\rho = a > 1$, by Theorem 1.2, $\rho(\frac{u}{a}) = 1$. Notice that $\frac{1}{a} < 1$, by (e). We have

$$\frac{1}{a^{p^+}}\rho(u) \le \rho\left(\frac{u}{a}\right) = 1 \le \frac{1}{a^{p^-}}\rho(u),$$

so we obtain 2). ∎

THEOREM 1.4. *Let $u, u_k \in E_\rho$, $k = 1, 2, \ldots$. Then the following statements are equivalent to each other*:

1) $\lim_{k\to\infty} \|u_k - u\|_\rho = 0$;
2) $\lim_{k\to\infty} \rho(u_k - u) = 0$;
3) u_k *converges to* u *in* Ω *in measure and* $\lim_{k\to\infty} \rho(u_k) = \rho(u)$.

Proof. The equivalence of 1) and 2) can be obtained from Theorem 1.6 in [18] and the property e) of ρ stated above. Now we prove the equivalence of 2) and 3).

If 2) holds, i.e.,

$$\lim_{k\to\infty} \int_\Omega |u_k - u|^{p(x)}\, dx = 0,$$

then it is easy to see that u_k converges to u in Ω in measure; thus $|u_k|^{p(x)}$ converges to $|u|^{p(x)}$ in measure. Using the inequality

$$|u_k|^{p(x)} \le 2^{p^+ - 1}\left(|u_k - u|^{p(x)} + |u|^{p(x)}\right)$$

and using the Vitali convergence theorem of integral we deduce that $\rho(u_k) \to \rho(u)$, so 3) holds.

On the other hand, if 3) holds, we can deduce that $|u_k - u|^{p(x)}$ converges to 0 in Ω in measure. By the inequality

$$|u_k - u|^{p(x)} \le 2^{p^+ - 1}\left(|u_k|^{p(x)} + |u|^{p(x)}\right)$$

and condition $\rho(u_k) \to \rho(u)$, we get $\lim_{k\to\infty} \rho(u_k - u) = 0$. ∎

For arbitrary $u \in L^{p(x)}(\Omega)$, let

$$u_n(x) = \begin{cases} u(x), & \text{if } |u(x)| \le n; \\ 0, & \text{if } |u(x)| > n. \end{cases}$$

It is easy to see that

$$\lim_{n\to\infty} \rho\big(u_n(x) - u(x)\big) = 0,$$

so by Theorem 1.4 we get

THEOREM 1.5. *The set of all bounded measurable functions over* Ω *is dense in* $(L^{p(x)}(\Omega), \|\cdot\|_\rho)$.

For every fixed $s \ge 0$, under condition (7), the function $\varphi(\cdot, s)$ is local integral in Ω; thus by Theorem 7.7 and 7.10 in [18], we get

THEOREM 1.6. *The space* $(L^{p(x)}(\Omega), \|\cdot\|_\rho)$ *is separable.*

By Theorem 7.6 in [18] we have

THEOREM 1.7. *The set S consisting of all simple integral functions over Ω is dense in the space* $(L^{p(x)}(\Omega), \|\cdot\|_\rho)$.

When $\Omega \subset \mathbf{R}^n$ is an open subset, for every element in S, we can approximate it in the means of norm $\|\cdot\|_\rho$ by the elements in $C_0^\infty(\Omega)$ through the standard method of mollifiers, so we have

THEOREM 1.8. *If $\Omega \subset \mathbf{R}^n$ is an open subset, then $C_0^\infty(\Omega)$ is dense in the space* $(L^{p(x)}(\Omega), \|\cdot\|_\rho)$.

We now discuss the uniform convexity of $L^{p(x)}(\Omega)$.
First we give the following conclusion:

LEMMA 1.9. *Let $p(x) > 1$ be bounded. Then $\varphi(x, s) = s^{p(x)}$ is strongly convex with respect to s; i.e., for arbitrary $a \in (0,1)$, there is $\delta(a) \in (0,1)$ such that for all $s \geq 0$ and $b \in [0, a]$, the inequality*

$$\varphi\left(x, \frac{1+b}{2}s\right) \leq (1-\delta(a)) \frac{\varphi(a,s) + \varphi(x,bs)}{2} \tag{8}$$

holds.

Proof. We rewrite (8) as

$$\left(\frac{1+b}{2}\right)^{p(x)} \leq (1-\delta(a)) \frac{1+b^{p(x)}}{2}.$$

It is easy to see that for almost all $x \in \Omega$ and $b \in [0,1)$, we always have $(\frac{1-b}{2})^{p(x)} < (1+b^{p(x)})/2$. Let

$$\theta_x(t) = \left(\frac{1+t}{2}\right)^{p(x)} \Big/ \frac{1+t^{p(x)}}{2}.$$

It is not hard to prove that for almost all $x \in \Omega$, $\theta(t)$ increases strictly in $[0,1)$. We only need to prove that the inequality $\theta_x(a) \leq 1 - \delta(a)$ holds. If this is not so, then we can find a sequence $\{x_n\}$ of points in Ω such that $\lim_{n\to\infty} \theta_{x_n}(a) = 1$; thus we can choose a convergence subsequence $p(x_{n_j})$ of $p(x_n)$ that still verifies $\lim_{n\to\infty} \theta_{x_{n_j}}(a) = 1$. Setting $p^* = \lim_{n_j\to\infty} p(x_{n_j}) \in [p^-, p^+]$, we get $(\frac{1+a}{2})^{p^*} = (1+a^{p^*})/2$, which is a contradiction. Thus we must have $\sup_{x\in\Omega} \theta(a) < 1$; i.e., there is $\delta(a) \in (0,1)$ such that for almost all $x \in \Omega$, we have $\theta(a) \leq 1 - \delta(a)$. This completes the proof. ∎

By Lemma 1.8 and Theorem 11.6 in [18], we can get immediately

THEOREM 1.10. *If $p^- > 1$, $p^+ < \infty$, then $L^{p(x)}(\Omega)$ is uniform convex and thus is reflexive.*

Now we give an imbedding result.

THEOREM 1.11. *Let* meas $\Omega < \infty$, *$p_1(x)$, $p_2(x) \in E$, and let condition* (7) *be satisfied. Then the necessary and sufficient condition for $L^{p_2(x)}(\Omega) \subset L^{p_1(x)}(\Omega)$ is that for almost all $x \in \Omega$ we have $p_1(x) \le p_2(x)$, and in this case, the imbedding is continuous.*

Proof. Let $p_1(x) \le p_2(x)$. Then

$$\theta_x(t) = \left(\frac{1+t}{2}\right)^{p(x)} \bigg/ \frac{1+t^{p(x)}}{2},$$

and we deduce that $L^{p_2(x)}(\Omega) \subset L^{p_1(x)}(\Omega)$. From Theorem 8.5 in [18] we know that the imbedding is continuous. On the other hand, if $L^{p_2(x)}(\Omega) \subset L^{p_1(x)}(\Omega)$, from Theorem 8.5 in [18], there exists a positive constant K and a non-negative integrable function $f(x)$ over Ω such that

$$s^{p_1(x)} \le Ks^{p_2(x)} + f(x), \qquad \forall s \ge 0, x \in \Omega.$$

If $p_1(x) \le p_2(x)$ is not true, then there exists a subset A of Ω with positive measure such that $p_1(x) > p_2(x)$ for $x \in A$. By the non-negative integrability of $f(x)$, we can find a subset $B \subset A$ such that for some positive constant M, $f(x) \le M$ whenever $x \in B$, and at the same time the inequality $s^{p_1(x)} \le Ks^{p_2(x)} + f(x)$ holds; i.e., for arbitrary $s \ge 0$, when $x \in B$, there holds

$$s^{p_1(x)-p_2(x)} \le K + Ms^{-p_2(x)}.$$

Let $s \to \infty$. We get a contradiction, and this ends the proof. ∎

The norm $\|\cdot\|_\rho$ of $L^{p(x)}(\Omega)$ defined before is usually called the Luxembury norm. We can introduce another norm $|||\cdot|||_\rho$ as

$$|||\cdot|||_\rho = \inf_{\lambda > 0} \lambda\left(1 + \rho\left(\frac{u}{\lambda}\right)\right). \tag{9}$$

This is called the Amemiya norm. The above two norms are equivalent; they satisfy

$$\|u\|_\rho \le |||u|||_\rho \le 2\|u\|_\rho, \qquad \forall u \in L^{p(x)}(\Omega).$$

A simple calculation shows that if $p(x) = p$ is a constant and we write

$$\|u\|_{L^{p(\Omega)}} = \left(\int_\Omega |u(x)|^p \, dx\right)^{1/p},$$

then we have

$$\|u\|_\rho = \|u\|_{L^p(\Omega)}, \qquad |||\, u \,|||_\rho = 2\|u\|_{L^p(\Omega)}.$$

If $p^- > 1$, we can also introduce the so-called Orlicz norm as

$$\|u\|'_\rho = \|u\|'_{L^{p(x)}(\Omega)} = \sup_{\rho_{q(x)}(v) \le 1} \left| \int_\Omega u(x) v(x)\, dx \right|,$$

and we have

$$\|u\|_\rho \le \|u\|'_\rho \le 2\|u\|_\rho, \qquad \forall u \in L^{p(x)}(\Omega),$$

so $\|u\|'_\rho$ is equivalent to $\|u\|_\rho$ and $|||\, u \,|||_\rho$. For the norm $\|u\|_\rho$, we have the Hölder inequality [18, p. 87]

$$\left| \int_\Omega u(x) v(x)\, dx \right| \le \|u\|_{\rho_{p(x)}} \|v\|'_{\rho_{q(x)}}, \qquad \forall u(x) \in L^{p(x)}(\Omega),$$

$$v(x) \in L^{q(x)}(\Omega),$$

and therefore we have

$$\left| \int_\Omega u(x) v(x)\, dx \right| \le 2\|u\|_{\rho_{p(x)}} \|v\|_{\rho_{q(x)}}, \qquad \forall u(x) \in L^{p(x)}(\Omega),$$

$$v(x) \in L^{q(x)}(\Omega),$$

where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$.

DEFINITION 1.12. Let $u \in L^{p(x)}(\Omega)$, let $D \subset \Omega$ be a measurable subset, and let χ_D be the characteristic function of E. If

$$\lim_{\text{meas } D \to 0} \|u(x) \chi_D(x)\|_\rho = 0,$$

then we say that u is absolutely continuous with respect to norm $\|\cdot\|_\rho$.

THEOREM 1.13. *$u \in L^{p(x)}(\Omega)$ is absolutely continuous with respect to norm $\|\cdot\|_\rho$.*

Proof. As

$$L^{p(x)}(\Omega) = \{u \in E : \forall \lambda > 0, \rho(\lambda u) < \infty\}$$

for arbitrary $\varepsilon > 0$, we have $\rho(\frac{u}{\varepsilon}) < \infty$. Let

$$u_n(x) = \begin{cases} u(x), & \text{if } |u(x)| \le n, \\ 0, & \text{if } |u(x)| > n. \end{cases}$$

Then by Theorem 1.5, we can take N such that

$$\|u - u_N\|_\rho \leq \frac{\varepsilon}{2}.$$

Because $u_N(x)$ is bounded, we can find $\delta > 0$ such that when meas $D < \delta$, we have

$$\|u_N(x)\chi_D(x)\|_\rho < \frac{\varepsilon}{2},$$

and thus we get

$$\|u(x)\chi_D(x)\|_\rho \leq \|(u - u_N(x))\chi_D(x)\|_\rho + \|u_N(x)\chi_D(x)\|_\rho < \varepsilon.$$

∎

Let $\alpha \in E$ and $0 < a \leq \alpha(x) \leq b < \infty$, where a and b are positive constants. Setting φ_α: $\Omega \times \mathbf{R}^+ \to \mathbf{R}^+$ as

$$\varphi_\alpha(x, s) = \alpha(x)\varphi(x, s) = \alpha(x)s^{p(x)},$$

similar to the definition of ρ and E_ρ, let

$$\rho_\alpha(u) = \int_\Omega \varphi_\alpha(x, |u(x)|)\, dx,$$

and

$$E_{\rho_\alpha} = \left\{u \in E : \lim_{\lambda \to 0^+} \rho_\alpha(\lambda u) = 0\right\}.$$

By

$$a\varphi(x, s) \leq \varphi_\alpha(x, s) \leq b\varphi(x, s),$$

and

$$a\rho(u) \leq \rho_\alpha(u) \leq b\rho(u),$$

we have $E_{\rho_\alpha} = E_\rho = L^{p(x)}(\Omega)$. If we define the norm $\|\cdot\|_{\rho_\alpha}$ of E_ρ as before,

$$\|u\|_{\rho_\alpha} = \inf\left\{\lambda > 0 : \rho_\alpha\left(\frac{u}{\lambda}\right) \leq 1\right\}, \tag{10}$$

it is easy to see that $\|\cdot\|_{\rho_\alpha}$ and $\|\cdot\|_\rho$ are equivalent norms on E_ρ.

Let us begin to discuss the conjugate space of $L^{p(x)}(\Omega)$, i.e., the space $(L^{p(x)}(\Omega))^*$ consisting of all continuous linear functionals over $L^{p(x)}(\Omega)$.

We suppose that $p(x)$ satisfies condition 7 and $p^- > 1$. By the definition in [18, p. 33] $\varphi(x, s) = s^{p(x)}$ belongs to the class Φ, and for $x \in \Omega$, φ is convex in s and satisfies

$$(0):\ \lim_{s\to 0^+} \frac{\varphi(x,s)}{s} = 0;$$

$$(\infty):\ \lim_{s\to \infty} \frac{\varphi(x,s)}{s} = \infty.$$

Let $\varphi_p(x,s) = \frac{1}{p(x)} s^{p(x)}$. Then φ_p also belongs to the class Φ. Writing

$$\rho_p(u) = \int_\Omega \varphi_p(x, |u(x)|)\, dx,$$

$$\|u\|_{\rho_p} = \inf\left\{\lambda > 0 : \rho_p\left(\frac{u}{\lambda}\right) \le 1\right\},$$

$\|u\|_{\rho_p}$ is an equivalent norm on $L^{p(x)}(\Omega)$. Obviously, the Young's conjugative function of φ_p is

$$\varphi_p^*(x,s) = \frac{1}{q(x)} s^{q(x)},$$

where $q(x)$ is the conjugative function of $p(x)$, i.e., $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. It is obvious that $(\varphi_p^*)^* = \varphi_p$, and q^-, q^+ are conjugative numbers of p^+, p^- respectively. In particular, we have $q^- > 1$ and $q^+ < \infty$. Writing

$$\rho_p^*(v) = \int_\Omega \frac{1}{q(x)} |v(x)|^{q(x)}\, dx = \int_\Omega \varphi_p^*(x, |v(x)|)\, dx;$$

$$E_{\rho_p}^* = \left\{v \in E : \lim_{\lambda\to 0^+} \rho_p^*(\lambda v) = 0\right\},$$

we have

$$E_{\rho_p}^* = L^{q(x)}(\Omega) = L_0^{q(x)}(\Omega) = \left\{v \in E : \int_\Omega |v(x)|^{q(x)}\, dx < \infty\right\}.$$

By Corollary 13.14 and Theorem 13.17 in [18] we have

THEOREM 1.14. $(L^{p(x)}(\Omega))^* = L^{q(x)}(\Omega)$, *i.e.*,

1°) *For every* $v \in L^{q(x)}(\Omega)$, *f defined by*

$$f(u) = \int_\Omega u(x) v(x)\, dx, \qquad \forall u \in L^{p(x)}(\Omega), \tag{11}$$

is a continuous linear functional over $L^{p(x)}(\Omega)$.

2°) *For every continuous linear functional f on $L^{p(x)}(\Omega)$, there is a unique element $v \in L^{q(x)}(\Omega)$ such that f is exactly defined by* (11)

From Theorem 1.14 we can also deduce that when $p^- > 1$, $p^+ < \infty$, the space $L^{p(x)}(\Omega)$ is reflexive.

We know that for Banach space $(X, \|\cdot\|)$, the norm $\|\cdot\|'$ on its conjugate space X^* is usually defined by the formulation

$$\|x^*\|' = \sup\{\langle x^*, x\rangle : \|x\| \leq 1\}, \tag{12}$$

where $x^* \in X^*$, $\langle x^*, x\rangle = x^*(x)$, and the inequality

$$|\langle x^*, x\rangle| \leq \|x^*\|', \|x\|, \qquad \forall x \in X, x^* \in X^* \tag{13}$$

holds.

It is obvious that the norm $\|\cdot\|'$ on X^* depends on the norm $\|\cdot\|$ on X.

Now we take $X = L^{p(x)}(\Omega)$, then $X^* = L^{q(x)}(\Omega)$. For $v \in X^*$ and $u \in X$,

$$\langle u, v\rangle = \int_\Omega u(x)v(x)\,dx. \tag{14}$$

If we use the norm $\|\cdot\|_{\rho_p}$ on X, then according to Theorem 13.11 in [18], we have

$$\|v\|_{\rho_p^*} \leq \|v\|'_{\rho_p^*}, \qquad \forall v \in X^*. \tag{15}$$

An interesting question we are concerned with is the relation between the prime norm $\|\cdot\|_{L^{q(x)}(\Omega)}$ of X^* and the norm $\|\cdot\|'_\rho$ of X^* when X is equipped with norm $\|\cdot\|_\rho$. It is well known that when $p(x)$ is a constant $p \in (1, \infty)$, the two norms defined above are exactly the same. Here we give

THEOREM 1.15. *Under the above assumptions, for arbitrary $v \in L^{q(x)}(\Omega)$, we have*

$$\|v\|_{L^{q(x)}(\Omega)} \leq \|v\|'_\rho \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right)\|v\|_{L^{q(x)}(\Omega)}. \tag{16}$$

Proof. For $v \in L^{q(x)}(\Omega)$, $u \in L^{p(x)}(\Omega)$, setting $\|v\|_{L^{q(x)}(\Omega)} = a$, $\|u\|_{L^{p(x)}(\Omega)} = b \leq 1$,

$$\begin{aligned}\int_\Omega \frac{u(x)}{b}\cdot\frac{v(x)}{a}\,dx &\leq \int_\Omega \frac{1}{p(x)}\left|\frac{u(x)}{b}\right|^{p(x)} dx + \int_\Omega \frac{1}{q(x)}\left|\frac{v(x)}{a}\right|^{q(x)} dx \\ &\leq \frac{1}{p^-}\int_\Omega\left|\frac{u(x)}{b}\right|^{p(x)} dx + \frac{1}{q^-}\int_\Omega\left|\frac{v(x)}{a}\right|^{q(x)} dx \\ &= \frac{1}{p^-} + \frac{1}{q^-}.\end{aligned}$$

So we get

$$\int_\Omega u(x)v(x)\,dx \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right)ab \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right)a,$$

and then

$$\|v\|'_\rho \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right)\|v\|_{L^{q(x)}(\Omega)}.$$

On the other hand, for $v \in L^{q(x)}(\Omega)$ with $\|v\|_{L^{q(x)}(\Omega)} = a$,

$$u(x) = \left|\frac{v(x)}{a}\right|^{q(x)-1} \operatorname{sgn} v(x).$$

Then

$$|u(x)|^{p(x)} = \left|\frac{v(x)}{a}\right|^{q(x)};$$

thus $u(x) \in L^{p(x)}(\Omega)$ and $\|u\|_{L^{p(x)}(\Omega)} = 1$. So

$$\int_\Omega u(x)v(x)\,dx = \int_\Omega a\left|\frac{v(x)}{a}\right|^{q(x)} dx = a = \|v\|_{L^{q(x)}(\Omega)}.$$

This equality means that $\|v\|'_\rho \geq \|v\|_{L^{q(x)}(\Omega)}$. The proof is completed. ∎

This theorem can be regarded as a generalization of conclusion (15).

The importance of Nemytsky operators from $L^{p_1}(\Omega)$ to $L^{p_2}(\Omega)$ is well known. Here we give the basic properties of Nemytsky operators from $L^{p_1(x)}(\Omega)$ to $L^{p_2(x)}(\Omega)$.

Let $p_1, p_2 \in L^\infty_+(\Omega)$. We denote by ρ_1, ρ_2 the modular corresponding to p_1 and p_2, respectively. Let $g(x, u)$ ($x \in \Omega$, $u \in \mathbf{R}$) be a Caracheodory function, and G is the Nemytsky operator defined by g, i.e., $(Gu)(x) = g(x, u(x))$. We have

THEOREM 1.16. *If G maps $L^{p_1(x)}(\Omega)$ into $L^{p_2(x)}(\Omega)$, then G is continuous and bounded, and there is a constant $b \geq 0$ and a non-negative function $a \in L^{p_2(x)}(\Omega)$ such that for $x \in \Omega$ and $u \in \mathbf{R}$, the following inequality holds*:

$$g(x,u) \leq a(x) + b|u|^{p_1(x)/p_2(x)}. \tag{17}$$

On the other hand, if g satisfies (17), *then G maps $L^{p_1(x)}(\Omega)$ into $L^{p_2(x)}(\Omega)$, and thus G is continuous and bounded.*

First we give

LEMMA 1.17. *If the operator G maps a ball $B_r(0) \subset L^{p_1(x)}(\Omega)$ into $L^{p_2(x)}(\Omega)$, then G maps all of $L^{p_1(x)}(\Omega)$ into $L^{p_2(x)}(\Omega)$. Here, we denote by $B_r(0)$ the ball with radius r and center at the origin* 0.

Proof. We may assume that $g(x,0) = 0$. Otherwise we can consider $g(x,s) - g(x,0)$ instead. Let $u \in L^{p_1(x)}(\Omega)$. By the absolute continuity of the norm $\|\cdot\|_\rho$, we can divide Ω into the union of disjoint subsets $\Omega_i (i \in I)$ such that

$$\|u(x)\chi_{\Omega_i}(x)\|_\rho < r,$$

where $\chi_{\Omega_i}(x)$ is the characteristic function of Ω_i. Therefore we have

$$u(x) = \sum_{i \in I} u(x)\chi_{\Omega_i}(x).$$

Writing $u_i(x) = u(x)\chi_{\Omega_i}(x)$, then $u_i \in B_r(0) \subset L^{p_1(x)}(\Omega)$ and

$$Gu = \sum_i Gu_i.$$

By the assumption, $Gu_i \in L^{p_2(x)}(\Omega)$, and thus we obtain $Gu \in L^{p_2(x)}(\Omega)$. ∎

Proof of Theorem 1.16. We need only prove G that is continuous at 0 when $g(x,0) = 0$. If this is not true, we can find a sequence $\{u_n(x)\} \subset L^{p_1(x)}(\Omega)$ $(n = 1, 2, \dots)$ satisfies

$$\lim_{n \to \infty} \|u_n\|_{\rho_1} = 0,$$

but

$$\|Gu_n\|_{\rho_2} > \varepsilon_0,$$

where ε_0 is some positive constant. Without loss of generality we can suppose that $\|u_n\|_{\rho_1} \le 1$; thus by Theorem 1.3 we have

$$\rho_1(u_n) \le \|u_n\|_{\rho_1}. \tag{18}$$

and therefore

$$\lim_{n \to \infty} \int_\Omega |u_n|^{p_1(x)}\, dx = 0.$$

For $v \in L^1(\Omega)$, we now define

$$(Hv)(x) = h(x, v(x)) = |G(\operatorname{sgn} v(x)|v(x)|^{1/p_1(x)})|^{p_2(x)}, \tag{19}$$

where h: $\Omega \times \mathbf{R} \to \mathbf{R}$, defined by $h(x,s) = |G(\operatorname{sgn} s|s|^{1/p_1(x)})|^{p_2(x)}$. Then H maps $L^1(\Omega)$ into $L^1(\Omega)$, and thus H is continuous at 0 ([19]). Writing

$$v_n(x) = \operatorname{sgn} u_n(x)|u_n(x)|^{p_1(x)}, \tag{20}$$

then

$$\lim_{n\to\infty} \|v_n\|_{L^1(\Omega)} = 0,$$

and thus

$$\lim_{n\to\infty} \|Hv_n\|_{L^1(\Omega)} = 0.$$

We get

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega |Hv_n| dx &= \lim_{n\to\infty} \int_\Omega |G(\operatorname{sgn} u_n(x)|u_n(x)|)|^{p_2(x)}\, dx \\ &= \lim_{n\to\infty} \int_\Omega |Gu_n|^{p_2(x)}\, dx \\ &= 0. \end{aligned}$$

By Theorem 1.4, in $L^{p(x)}(\Omega)$, $u_n (n = 1, 2, \dots)$ coverage to u in modular iff u_n coverage to u in norm, we have

$$\lim_{n\to\infty} \|Gu_n\|_{\rho_2} = 0.$$

This contradicts $\|Gu_n\|_{\rho_2} > \varepsilon_0$, and we have proved the continuity of G.

Let A be a bounded set in $L^{p_1(x)}(\Omega)$, i.e., for arbitrary $u(x) \in A$, $\|u\|_{\rho_1}$ is uniform bounded, so by Theorem 1.3, A is bounded in modular. For $v(x) \in L^1(\Omega)$ let H be defined as above; then H: $L^1(\Omega) \to L^1(\Omega)$ and thus H is bounded. For $u(x) \in A$, $\operatorname{sgn} u(x)|u(x)|^{p_1(x)} \in L^1(\Omega)$ and $\|\operatorname{sgn} u(x)|u(x)|^{p_1(x)}\|_{L^1(\Omega)} = \rho_1(u)$ is uniformly bounded. There is a constant $K > 0$ such that

$$\|H(\operatorname{sgn} u(x)|u(x)|^{p_1(x)})\|_{L^1(\Omega)} \le K,$$

i.e., we have

$$\int_\Omega |Gu|^{p_2(x)}\, dx \le K. \tag{21}$$

Inequality (21) shows that $G(A)$ is bounded in modular. Again from (21) we know that $G(A)$ is bounded in norm.

Now if (17) holds, we let $u(x) \in L^{p_1(x)}(\Omega)$. It is obvious that $a(x) + b|u|^{p_1(x)/p_2(x)} \in L^{p_2(x)}(\Omega)$. Therefore

$$\int_\Omega |Gu(x)|^{p_2(x)}\, dx \le \int_\Omega |a(x) + b|u(x)|^{p_1(x)/p_2(x)}|^{p_2(x)}\, dx < \infty,$$

and thus G maps $L^{p_1(x)}(\Omega)$ into $L^{p_2(x)}(\Omega)$.

On the other hand, if G maps $L^{p_1(x)}(\Omega)$ into $L^{p_2(x)}(\Omega)$, for $v \in L^1(\Omega)$, as H: $L^1(\Omega) \to L^1(\Omega)$, we can assert that there is a constant $b_1 \geq 0$ and function $a_1 \geq 0$, $a_1 \in L^1(\Omega)$ such that

$$|(Hv)(x)|| \leq a_1(x) + b_1|v(x)|,$$

for $u \in L^{p_1(x)}(\Omega)$. Let $v(x) = \operatorname{sgn} u(x)|u(x)|^{p_1(x)}$; then $v \in L^1(\Omega)$ and thus

$$|(Hv)(x)| = |(Gu)(x)|^{p_2(x)} \leq a_1(x) + b_1|u(x)|^{p_1(x)},$$

as $p_2(x) \geq 1$. From (17) we can deduce that

$$\begin{aligned}|(Gu)(x)| &\leq \left(a_1(x) + b_1|u|^{p_1(x)}\right)^{1/p_2(x)}\\ &\leq a_1(x)^{1/p_2(x)} + b_1^{1/p_2(x)}|u|^{p_1(x)/p_2(x)}\\ &\leq a(x) + b|u|^{p_1(x)/p_2(x)},\end{aligned}$$

where $a(x) = a_1(x)^{1/p_2(x)} \geq 0$, $a(x) \in L^{p_2(x)}(\Omega)$, and $b = b_1^{1/p_2(x)}$. We conclude the proof. ∎

As an application, we give an example.

EXAMPLE. Let Ω be a measurable set in R^n and $\operatorname{meas}(\Omega) < \infty$, f: $\Omega \times \mathbf{R} \to \mathbf{R}$ is a Caratheodory function satisfying the condition

$$f(x,u) \leq a(x) + b|u|^{p(x)},$$

where $p(x) \in L^\infty_+(\Omega)$, $a(x) \in L^1(\Omega)$, $a(x) \geq 0$, $b \geq 0$ is a constant. Then the functional

$$J(u) = \int_\Omega f(x,u(x))\,dx$$

defined on $L^{p(x)}(\Omega)$ is continuous and J is uniformly bounded on a bounded set in $L^{p(x)}(\Omega)$.

2. THE SPACE $W^{m,p(x)}(\Omega)$

In this section we will give some basic results on the generalized Lebesgue–Sobolev space $W^{m,p(x)}(\Omega)$, where Ω is a bounded domain of $\mathbf{R}^n$ and m is a positive integer, $p \in L^\infty_+(\Omega)$. $W^{m,p(x)}(\Omega)$ is defined as

$$W^{m,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq m\right\}.$$

$W^{m,p(x)}(\Omega)$ is a special class of so-called generalized Orlicz–Sobolev spaces. Some elementary conceptions and results of the general case can be found in Hudzik's papers [9–17]. From [11] we know that $W^{m,p(x)}(\Omega)$ can be equipped with the norm $\|u\|_{W^{m,p(x)}(\Omega)}$ as Banach spaces, where

$$\|u\|_{W^{m,p(x)}(\Omega)} = \sum_{|\alpha| \le m} \|D^{\alpha} u\|_{L^{p(x)}(\Omega)}.$$

According to [17] and Theorem 1.10 in Section 1, we already have

THEOREM 2.1. *$W^{m,p(x)}(\Omega)$ is separable and reflexive.*

An immediate consequence of Theorem 1.7 is

THEOREM 2.2. *Assume that $p_1(x), p_2(x) \in L_+^{\infty}(\Omega)$. If $p_1(x) \le p_2(x)$, then $W^{m,p_2(x)}(\Omega)$ can be imbedded into $W^{m,p_1(x)}(\Omega)$ continuously.*

Now let us generalize the well-known Sobolev imbedding theorem of $W^{m,p}(\Omega)$ to $W^{m,p(x)}(\Omega)$. We have

THEOREM 2.3. *Let $p, q \in C(\overline{\Omega})$ and $p, q \in L_+^{\infty}(\Omega)$. Assume that*

$$mp(x) < n, \quad q(x) < \frac{np(x)}{n - mp(x)}, \qquad \forall x \in \overline{\Omega}.$$

Then there is a continuous and compact imbedding $W^{m,p(x)}(\Omega) \to L^{q(x)}(\Omega)$.

Proof. For positive constant r with $mr < n$, denote

$$r^* = \frac{nr}{n - mr}.$$

Under the assumptions it is easy to see that for arbitrary $x \in \overline{\Omega}$, we can find a neighborhood U_x in $\overline{\Omega}$ such that

$$q^+(U_x) < (p^-(U_x))^*,$$

where $p^-(U_x) = \inf\{p(y) : y \in U_x\}$, $q^-(U_x) = \sup\{q(y) : y \in U_x\}$. Now $\{U_x\}_{x \in \overline{\Omega}}$ is an open covering of compact set $\overline{\Omega}$. Choosing a finite sub-covering $\{U_i : i = 1, 2, \dots, s\}$ and denoting

$$p_i^- = p^-(U_i), \qquad q_i^+ = q^+(U_i),$$

it is obvious that if $u \in W^{m,p(x)}(\Omega)$ then $u \in W^{m,p(x)}(U_i)$, and thus from Theorem 2.2, $u \in W^{m,p_i^-}(U_i)$. Therefore by the well-known Sobolev imbedding theorem [1] we have continuous and compact imbedding,

$$W^{m,p_i^-}(U_i) \to L^{q_i^+}(U_i).$$

According to Theorem 1.7, there is a continuous imbedding

$$L^{q_i^+}(U_i) \to L^{q(x)}(U_i),$$

so for every U_i, $i = 1, 2, \dots, s$, we have $u \in L^{q(x)}(U_i)$ and therefore $u \in L^{q(x)}(\Omega)$. We can now assert that $W^{m,p(x)}(\Omega) \subset L^{q(x)}(\Omega)$, and the imbedding is continuous and compact.

Remark 2.4. We do not known whether we have the imbedding

$$W^{m,p(x)}(\Omega) \to L^{p^*(x)}(\Omega),$$

but if the assumption on $p(x)$ is not satisfied, we cannot have it.

EXAMPLE. Let $\Omega = \{x = (x_1, x_2)\colon 0 < x_1 < 1, 0 < x_2 < 1\} \subset \mathbf{R}^2$, $p(x) = 1 + x_2$, $u(x) = (2 + x_2)^{1/(1+x_2)}$; then we have $u(x) \in W^{1,p(x)}(\Omega)$ and $p^*(x) = 2(1 - x_2)/(1 - x_2)$. It is easy to test that $u \notin L^{p^*(x)}(\Omega)$.

Let us turn to the problem of density.

DEFINITION 2.5. We define $W_0^{m,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{m,p(x)}(\Omega)$ and $W^{m,p(x)} = W^{m,p(x)} \cap W_0^{m,1}(\Omega)$.

It is well known that when $p(x)$ is a constant p on Ω, we have $W_0^{m,p}(\Omega) = \mathring{W}^{m,p}(\Omega)$, and in this case $C^\infty(\Omega)$ is dense in $W^{m,p}(\Omega)$. For the general function $p(x)$, from the definition we have $W_0^{m,p(x)}(\Omega) \subset \mathring{W}^{m,p(x)}(\Omega)$, and $\mathring{W}^{m,p(x)}(\Omega)$ is a closed linear subspace of $W^{m,p(x)}(\Omega)$. In general, $\mathring{W}^{m,p(x)}(\Omega) \neq W_0^{m,p(x)}(\Omega)$. Zhikov showed the following. Let $\Omega = \{x = (x_1, x_2) \in \mathbf{R}^2 : |x| < 1\}$, $1 < \alpha_1 < 2 < \alpha_2$. If we define

$$p(x) = \begin{cases} \alpha_1, & \text{if } x_1 x_2 > 0 \\ \alpha_2, & \text{if } x_1 x_2 < 0, \end{cases}$$

then

$$\mathring{W}^{1,p(x)}(\Omega) \neq W_0^{1,p(x)}(\Omega).$$

This example also shows that $C^\infty(\Omega)$ is not dense in $W^{1,p(x)}(\Omega)$.

The identity

$$W_0^{m,p(x)}(\Omega) = \mathring{W}^{m,p(x)}(\Omega)$$

means that $C_0^\infty(\Omega)$ is dense in $(\mathring{W}^{m,p(x)}(\Omega), \|\cdot\|_{W^{m,p(x)}(\Omega)})$. As Musielak pointed out in [18], for Orlicz–Sobolev spaces, the problem of density is very complicated. But by the method of Fan [3, 4], we can get

THEOREM 2.6. *If Ω is a bounded open set in $\mathbf{R}^n$ with a Lipschitz boundary $p \in L_+^\infty(\Omega)$ and $p(x)$ satisfies condition $(F{-}Z)$ on $\overline{\Omega}$, i.e., there is* *a constant* $L > 0$ *such that*

$$-|p(x) - p(y)|\log|x-y| \le L, \qquad \forall x, y \in \overline{\Omega}, \tag{22}$$

then

1) $C^\infty(\Omega)$ *is dense in* $W^{m,p(x)}(\Omega)$.
2) $\mathring{W}^{m,p(x)}(\Omega) = W_0^{m,p(x)}(\Omega)$.

Proof. Essentially the proof can be found in [3]; Zhikov improved the proof later. For completion we write it out here.

1) For simplicity we assume that the domain Ω is star-shaped (with respect to the origin). For the more general case, one can write the proof similarly according to [3]. Let $u \in W^{m,p(x)}(\Omega)$. We denote by $u_\varepsilon \in C^\infty(\overline{\Omega})$ the typical mollifier of u; i.e., u_ε is defined as

$$u_\varepsilon = \varepsilon^{-n} \int_\Omega \rho\left(\frac{x-y}{\varepsilon}\right) u(y)\, dy. \tag{23}$$

It suffices to prove

$$u_\varepsilon \to u \text{ in } W^{1,p(x)}(\Omega), \qquad \text{as } \varepsilon \to 0.$$

Denote $\sigma(\varepsilon) = 1/\log \frac{1}{\varepsilon}$. From (22) it follows that for $x \in \overline{\Omega}$,

$$|u_\varepsilon(x)|^{p(x)-L\sigma(\varepsilon)} \le \int_{|y-x|\le\varepsilon} |u(y)|^{p(x)-L\sigma(\varepsilon)} \varepsilon^{-n} \rho\left(\frac{y-x}{\varepsilon}\right) dy.$$

Noticing that $p(x) - L\sigma(\varepsilon) \le p(y)$, for every $s \in (0,1)$ we have

$$\begin{aligned} |u_\varepsilon(x)|^{p(x)-L\sigma(\varepsilon)} &\le \int_{|u(y)|<s} |u(y)|^{p(x)-L\sigma(\varepsilon)} \varepsilon^{-n} \rho\left(\frac{y-x}{\varepsilon}\right) dy \\ &\quad + \int_{|u(y)|\ge\varepsilon} |u(y)|^{p(x)-L\sigma(\varepsilon)} \varepsilon^{-n} \rho\left(\frac{y-x}{\varepsilon}\right) dy \\ &\le s + s^{-2L\sigma(\varepsilon)} \int_{|y-x|\le\varepsilon} |u(y)|^{p(y)} \varepsilon^{-n} \rho\left(\frac{y-x}{\varepsilon}\right) dy. \end{aligned} \tag{24}$$

From (24) it follows that

$$\begin{aligned} &\int_\Omega |u_\varepsilon(x)|^{p(x)-L\sigma(\varepsilon)}\, dx \\ &\quad \le s|\Omega| + s^{-2L\sigma(\varepsilon)} \int_\Omega \left(\int_{|y-x|\le\varepsilon} |u(y)|^{p(y)} \varepsilon^{-n} \rho\left(\frac{y-x}{\varepsilon}\right) dy \right) dx \\ &\quad \le s|\Omega| + s^{-2L\sigma(\varepsilon)} \int_\Omega \left(\int_{\mathbf{R}^n} \left(\varepsilon^{-n} \rho\left(\frac{y-x}{\varepsilon}\right) dx \right) |u(y)| dy \right. \\ &\quad = s|\Omega| + s^{-2L\sigma(\varepsilon)} \int_\Omega u(y)^{p(y)}\, dy. \end{aligned}$$

Let $\varepsilon > 0$ be given. Choosing $s \in (0,1)$ such that $s|\Omega| < \varepsilon$, then

$$\int_\Omega |u_\varepsilon(x)|^{p(x)-L\sigma(\varepsilon)}\,dx \le \varepsilon + s^{-2L\sigma(\varepsilon)}\int_\Omega |u(x)|^{p(x)}\,dx.$$

and hence

$$\overline{\lim_{\varepsilon\to 0}}\int_\Omega |u_\varepsilon(x)|^{p(x)-L\sigma(\varepsilon)}\,dx \le \varepsilon + \int_\Omega |u(x)|^{p(x)}\,dx. \tag{25}$$

By the arbitrariness of $\varepsilon > 0$ we obtain

$$\overline{\lim_{\varepsilon\to 0}}\int_\Omega |u_\varepsilon(x)|^{p(x)-L\sigma(\varepsilon)}\,dx \le \int_\Omega |u(x)|^{p(x)}\,dx. \tag{26}$$

By (23), (26), and Fatou's lemma we have

$$\lim_{\varepsilon\to 0}\int_\Omega |u_\varepsilon(x)|^{p(x)-L\sigma(\varepsilon)}\,dx = \int_\Omega |u(x)|^{p(x)}\,dx. \tag{27}$$

By (23) and the Hölder inequality we can deduce that for $x \in \overline{\Omega}$,

$$\begin{aligned}
|u_\varepsilon(x)| &\le \int_{|y-x|<\varepsilon} |u(y)|\varepsilon^{-n}\rho\left(\frac{y-x}{\varepsilon}\right)dy \\
&\le \left(\int_\Omega |u(y)|^{p^-}\,dy\right)^{1/p^-}\left(\int_{\mathbf{R}^n}\left|\varepsilon^{-n}\rho\left(\frac{y-x}{\varepsilon}\right)\right|^{p^{-\prime}}dy\right)^{1/p^{-\prime}} \\
&\le c_1\left(\int_{\mathbf{R}^n}|\varepsilon^{-n}\rho(z)|^{p^{-\prime}}\varepsilon^n\,dz\right)^{1/p^{-\prime}} \\
&= c_1\varepsilon^{-n(1-1/p^{-\prime})}\left(\int_{\mathbf{R}^n}|\rho(z)|^{p^{-\prime}}\,dz\right)^{1/p^{-\prime}} \\
&= c_1c_2\varepsilon^{-n/p^-},
\end{aligned} \tag{28}$$

where $1/p^- + 1/p^{-\prime} = 1$, $c_1 = (\int_\Omega |u(y)|^{p^-}\,dy)^{1/p^-}$, and $c_2 = (\int_{\mathbf{R}^n}|\rho(z)|^{p^{-\prime}}\,dz)^{1/p^{-\prime}}$.

From (28) it follows that for $x \in \Omega$,

$$|u_\varepsilon(x)|^{L\sigma(\varepsilon)} \le (c_1c_2)^{L\sigma(\varepsilon)}\varepsilon^{-\sigma(\varepsilon)Ln(1/p^-)} = \tau(\varepsilon).$$

It is easy to see that $\tau(\varepsilon) \to (\frac{1}{e})^{nL/p^-} \le 1$ as $\varepsilon \to 0$, and therefore

$$\begin{aligned}
\int_\Omega |u_\varepsilon(x)|^{p(x)}\,dx &= \int_\Omega |u_\varepsilon(x)|^{p(x)-L\sigma(\varepsilon)}|u_\varepsilon(x)|^{L\sigma(\varepsilon)}\,dx \\
&= \tau(\varepsilon)\int_\Omega |u_\varepsilon(x)|^{p(x)-L\sigma(\varepsilon)}\,dx.
\end{aligned} \tag{29}$$

From (29) and (27) it follows that

$$\overline{\lim_{\varepsilon\to 0}} \int_\Omega |u_\varepsilon(x)|^{p(x)}\,dx \le \int_\Omega |u(x)|^{p(x)}\,dx. \tag{30}$$

By (30), (23), and Fatou's lemma we obtain

$$\lim_{\varepsilon\to 0} \int_\Omega |u_\varepsilon(x)|^{p(x)}\,dx = \int_\Omega |u(x)|^{p(x)}\,dx. \tag{31}$$

From (31) and (23) we get

$$\lim_{\varepsilon\to 0} \int_\Omega |u_\varepsilon(x) - u(x)|^{p(x)}\,dx = 0. \tag{32}$$

From (23) it is easy to see that

$$D_i u_\varepsilon = (D_i u)_\varepsilon, \tag{33}$$

where $D_i = \partial/\partial x_i$, $i = 1, 2, \dots, n$.

Using arguments similar to those above, we can prove that

$$\lim_{\varepsilon\to 0} \int_\Omega |D_i u_\varepsilon(x) - D_i u(x)|^{p(x)}\,dx = 0, \qquad i = 1, 2, \dots, n. \tag{34}$$

Thus we have proved that $C^\infty(\Omega)$ is dense in $W^{1,p(x)}(\Omega)$. Using induction for m, we can complete the proof.

The proof of 2) is similar to 1), and we omit it. ∎

THEOREM 2.7. *Let $p(x) \in C(\overline{\Omega})$. Then we can take*

$$\|u\|'_{m,p(x)} = \sum_{\alpha = m} \|\partial^\alpha u\|_{L^{p(x)}(\Omega)}$$

as an equivalence norm in the space $(\mathring{W}^{m,p(x)}(\Omega), \|\cdot\|_{W^{m,p(x)}})$; i.e., there is a positive constant C such that

$$\|\partial^\alpha u\|_{L^{p(x)}(\Omega)} \le C\|u\|'_{m,p(x)}, \qquad \forall 0 \le |\alpha| \le m, u \in \mathring{W}^{m,p(x)}(\Omega).$$

Proof. For simplicity we only give the proof for $m = 1$. It is easy to see that $\|Du\|_{L^{p(x)}(\Omega)}$ is equivalent to $\sum_{i=1}^n \|(\partial^u/\partial x_i)\|_{L^{p(x)}(\Omega)}$.

As $p^+ < \infty$, we can find $p_i(x) \in C(\overline{\Omega})$ $(i = 1, 2, \dots, s)$ such that

$$p(x) =: p_0(x) \ge p_1(x) \ge p_2(x) \ge \cdots \ge p_s(x) =: 1 \tag{35}$$

and

$$p_i(x) < p^*_{i-1}(x), \qquad i = 0, 1, \dots, s-1, \tag{36}$$

where $p^*(x) = \frac{np(x)}{n - p(x)}$. By Theorem 3.3 there are continuous imbeddings,

$$W^{1, p_{i+1}(x)}(\Omega) \to L^{p_i(x)}(\Omega), \qquad i = 0, 1, \ldots, s - 1,$$

so we can get, subsequently,

$$\begin{aligned}
\|u\|_{L^{p(x)}(\Omega)} &\le C_0\big(\|Du\|_{L^{p_1(x)}(\Omega)} + \|u\|_{L^{p_1(x)}(\Omega)}\big) \\
&\le C_0'\|Du\|_{L^{p(x)}(\Omega)} + C_0\|u\|_{L^{p_1(x)}(\Omega)} \\
\|u\|_{L^{p_1(x)}(\Omega)} &\le C_1\big(\|Du\|_{L^{p_2(x)}(\Omega)} + \|u\|_{L^{p_2(x)}(\Omega)}\big) \\
&\le C_1'\|Du\|_{L^{p(x)}(\Omega)} + C_1\|u\|_{L^{p_2(x)}(\Omega)} \\
&\cdots\cdots \\
\|u\|_{L^{p_{s-1}(x)}(\Omega)} &\le C_{s-1}\big(\|Du\|_{L^{p_s(x)}(\Omega)} + \|u\|_{L^{p_s(x)}(\Omega)}\big) \\
&\le C_{s-1}'\|Du\|_{L^{p(x)}(\Omega)} + C_{s-1}\|u\|_{L^{p_s(x)}(\Omega)} \\
\|u\|_{L^{p_s(x)}(\Omega)} &= \|u\|_{L^1(\Omega)} \le C_s\|Du\|_{L^1(\Omega)} \le C_s'\|Du\|_{L^{p(x)}(\Omega)}.
\end{aligned}$$

The last equality above is represented by the fact $u \in W_0^{1,1}(\Omega)$. Combining these inequalities, we complete the proof. ∎

Remark 2.8. In Theorem 2.6, replace $\mathring{W}^{m, p(x)}(\Omega)$ by $W_0^{m, p(x)}(\Omega)$. The conclusion is obviously true.

Remark 2.9. Condition (F−Z) is given by Fan and Zhikov [20]. It is easy to see that if $p(x) \in C^{0, \alpha}(\Omega)$ then $p(x)$ satisfies condition (F−Z).

We now point out a difference between $W_0^{m, p(x)}(\Omega)$ and $W_0^{m, p}(\Omega)$. This difference shows that in $W_0^{m, p(x)}(\Omega)$, the variational problems become very complicated. Let

$$\lambda = \inf_{0 \ne u \in W_0^{1, p(x)}(\Omega)} \frac{\int_\Omega |Du|^{p(x)}\, dx}{\int_\Omega |u|^{p(x)}\, dx}. \tag{37}$$

It is well known that when $p(x)$ is a constant p, λ (defined above) is the first eigenvalue of p-Laplace operator $-\Delta_p = -\operatorname{div}(|Du|^{p-2}Du)$. It must be a positive number. But for general $p(x)$, this is not true; λ may take 0.

EXAMPLE. Let $\Omega = (-2, 2) \subset \mathbf{R}^1$. Define

$$p(x) = \begin{cases} 3 & \text{if } 0 \le |x| \le 1; \\ 4 - |x| & \text{if } 1 \le |x| \le 2. \end{cases}$$

Then we have

$$\lambda = \inf_{0 \neq u \in W_0^{1,p(x)}(\Omega)} \frac{\int_\Omega |Du|^{p(x)}\,dx}{\int_\Omega |u|^{p(x)}\,dx} = 0.$$

Proof. Let

$$u(x) = \begin{cases} 1 & \text{if } 0 \le |x| \le 1; \\ 2 - |x| & \text{if } 1 \le |x| \le 2. \end{cases}$$

Then $u(x) \in W_0^{1,p(x)}(\Omega)$. Let us prove that for $a > 0$, there holds

$$\lim_{a \to \infty} \frac{\int_\Omega |au'(x)|^{p(x)}\,dx}{\int_\Omega |au|^{p(x)}\,dx} = 0. \tag{38}$$

In fact, we have

$$\begin{aligned} \int_\Omega |au'(x)|^{p(x)}\,dx &= 2\left(\int_0^1 0\,dx + \int_1^2 (a \cdot 1)^{4-x}\,dx\right) \\ &= 2\int_1^2 a^{4-x}\,dx = \frac{2a^2}{\log a}(a-1) \end{aligned}$$

and

$$\int_\Omega |au|^{p(x)}\,dx \ge 2\int_0^1 a^3\,dx = 2a^3.$$

The conclusion is dropped. ∎

At last we present an elementary result of the difference quotients in $W^{1,p(x)}(\Omega)$.

THEOREM 2.10. *Let $\Omega' \subset\subset \Omega$, $h < \operatorname{dist}(\Omega', \partial\Omega)$, if $u \in W^{1,p(x)}(\Omega)$, where $p(x) \in L_+^\infty(\Omega)$ satisfies condition (F–Z). Then $\Delta_h^i u(x) \in L^{p(x)}(\Omega')$ and we have*

1) $\int_{\Omega'} |\Delta_h^i u(x)|^{p(x)}\,dx \le \int_\Omega |D_i u(x)|^{p(x)}\,dx$;
2) *$\Delta_h^i u(x)$ converges strongly to $D_i u(x)$ in $L^{p(x)}(\Omega')$, where*

$$\Delta_h^i u(x) = \frac{1}{h}\big(u(x + he_i) - u(x)\big)$$

is the ith quotient of $u(x)$ (e_i denotes the unit vector of the x_i axis), $D_i u(x) = (\partial/\partial x_i)u(x)$.

The proof of Theorem 2.10 is easy and we omit it.

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