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Entropy solutions to a strongly degenerate anisotropic convection–diffusion equation with application to utility theory

A.L. Amadori and R. Natalini *

Istituto per le Applicazioni del Calcolo "Mauro Picone," Consiglio Nazionale delle Ricerche, Viale del Policlinico 137, I-00161 Roma, Italy

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Abstract

We study the deterministic counterpart of a backward–forward stochastic differential utility, which has recently been characterized as the solution to the Cauchy problem related to a PDE of degenerate parabolic type with a conservative first order term. We first establish a local existence result for strong solutions and a continuation principle, and we produce a counterexample showing that, in general, strong solutions fail to be globally smooth. Afterward, we deal with discontinuous entropy solutions, and obtain the global well posedness of the Cauchy problem in this class. Eventually, we select a sufficient condition of geometric type which guarantees the continuity of entropy solutions for special initial data. As a byproduct, we establish the existence of an utility process which is a solution to a backward–forward stochastic differential equation, for a given class of final utilities, which is relevant for financial applications.

Keywords: Degenerate parabolic problems; Conservation laws; Entropy solutions; Financial mathematics; Utility models

1. Introduction

In the present work, we investigate a basic model of anisotropic convection–diffusion equation

* Corresponding author.
E-mail address: r.natalini@iac.cnr.it (R. Natalini).

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$$\partial_t u = \partial^2_{xx} u + \partial_y f(u), \quad (x, y) \in \mathbb{R}^2,\ t > 0, \tag{1.1}$$

with an initial condition at $t = 0$,

$$u(\cdot, 0) = u_0, \quad \text{on } \mathbb{R}^2. \tag{1.2}$$

We shall make use of the standard assumptions:

(A1) $f \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$,
(A2) $u_0 \in L^\infty(\mathbb{R}^2)$.

This kind of problems arises in the framework of stochastic models for the utility function, which has been extensively developed since the work by Duffie and Epstein [9]. For instance, Antonelli et al. [1] proposed to describe the utility function by means of a nonlinear backward–forward stochastic differential equation. The first problem is to establish the existence of solutions; they proposed to use the four step scheme by Ma et al. [19] in order to relate this problem to the study of a deterministic partial differential equation and obtained a convection–diffusion equation on $\mathbb{R}^2 \times (0, T)$ of the following type:

$$\partial_t u = \frac{1}{2}\sigma^2 \partial^2_{xx} u + \mu \partial_x u + \partial_y f(u) - \gamma y \partial_y u - \beta u + w, \tag{1.3}$$

where $\sigma, \mu, \gamma, \beta$ are fixed parameters, f is a convex function of u (possibly depending also by x, t), and w is a smooth function of x, y, t. The source term w and the initial condition u_0 are either decreasing or increasing with respect to y, according to the particular economic effect that should be captured.

Eventually, the existence of an utility process is obtained whenever the related Cauchy problem admits a solution which is, at least, Lipschitz-continuous with respect to both x and y. Unfortunately, this fact does not hold in general. For instance, if $w, \beta, \gamma, \mu = 0$, $f(u) = u^2/2$, and u_0 only depends by y, the solutions to (1.3), (1.2) is of the form $u(x, y, t) = \tilde{u}(y, t)$, where $\tilde{u}$ solves the Burger's equation $\partial_t \tilde{u} + \tilde{u}\partial_y \tilde{u} = 0$. It is well known that this problem does not admit, in general, continuous solutions for all time $t > 0$, in spite of the smoothness of the initial datum. In the present paper we show that the presence of the diffusion term is not sufficient to avoid this loss of continuity. In order to take heed to the main effect of the nonlinear term in conservative form, we study the simplified equation (1.1).

Let us now review the main related existing results. In [11] Escobedo et al. proposed a notion of solutions, possibly discontinuous, inspired by the entropy solutions introduced for first order equations by Kruzhkov [16]. They defined an entropy solution to (1.1) as a function $u \in \mathcal{C}(0, T; L^1(\mathbb{R}^2)) \cap L^\infty((0, T) \times \mathbb{R}^2)$ such that

$$\int_0^T \int_{\mathbb{R}^2} -|u - k|\big(\partial_t \varphi - \partial^2_{xx}\varphi\big) + \operatorname{sgn}(u - k)\big(f(u) - f(k)\big)\partial_y \varphi \leqslant \int_0^T \int_{\mathbb{R}^2} \operatorname{sgn}(u - k) k'' \varphi$$

for all smooth functions k of x and $\varphi \in \mathcal{D}(\mathbb{R}^2 \times (0, T))$, $\varphi \geqslant 0$. Next, they obtain the well posedness of the Cauchy problem in the class $L^1 \cap L^\infty$ for all times (see [11, Theorem 1]). However, this well posedness result is not significant for the financial applications: since the solution to the convection–diffusion equation is not sufficiently smooth to apply Ito's lemma, there is no way to deduce the existence of a solution to the former backward–forward stochastic differential equation. Furthermore this notion of solution is too weak to give a geometric description of the possible shocks, in the spirit of the classical result by Oleinik [20].

In [2], Antonelli and Pascucci used the viscosity solutions approach (see, for instance, [6]) to prove the local existence of solutions. Global existence was not obtained, since the crucial quasi monotonicity property with respect to u does not hold for such equation.

A completely different flavor inherits the interior regularity result obtained by Citti et al. [5] in the particular case $f(u) = u^2/2$. By making use of hypoelliptic operators' techniques, they established that any classical solution is indeed of class C^∞ in every open set where $\partial_x u \neq 0$. Unfortunately, the assumption that u is a priori of class C^1 is essential in the proof of this result, so that it may not be applied to the entropy solutions, which existence has been proved by Escobedo et al. [11], or to the solutions given in the present paper.

The paper is organized as follows. In Section 2 we propose a strong notion for solution and, by a compactness procedure, we obtain in Theorem 2.6 local existence and a continuation principle, stating that this smooth solution does exist until a discontinuity in the y-direction comes forth. We also produce a counterexample showing that discontinuities may arise in finite time, even starting from smooth and compactly supported initial data. This result goes into the opposite direction compared with the one in [5]: the assumptions u_0 smooth and $\partial_x u \neq 0$ are not sufficient to prevent the appearance of discontinuities.

In Section 3 we investigate a weaker notion of solution, possibly discontinuous, to achieve global existence. The estimates obtained in Section 2 show that the solutions produced in [11] have an additional regularity property, namely that $\partial_x u \in L^2(0, T; L^2_{\mathrm{loc}}(\mathbb{R}^2))$. On the other hand, assuming this regularity property enables to establish uniqueness by using less entropy tests. This leads us to a notion of entropy solution which is substantially different from the one in [11], because we ask a priori that $\partial_x u$ is locally square integrable, but we reduce the number of entropy tests. This is more in the spirit of the results by Carrillo [4] for some different nonlinear degenerate problems. The well posedness of the Cauchy problem in the class L^∞ is established by Theorem 3.6, even though our entropy solutions coincide with the ones of [11] (when both exist), the uniqueness classes are distinct. Besides, by taking advantage of the property $\partial_x u \in L^2(0, T; L^2_{\mathrm{loc}}(\mathbb{R}^2))$, Theorem 3.8 characterizes entropic shocks by virtue of a Rankine–Hugoniot–Oleinik type condition, inspired by the analogous result concerning scalar conservation laws. Eventually we obtain a sufficient condition of geometric type which guarantees the Lipschitz continuity of solutions.

Finally, we give an application to utility theory in Section 4.

2. Local strong solutions and a continuation principle

Definition 2.1. A *strong solution* to problem (1.1)–(1.2) in the time interval $(0, T)$ is a distributional solution to (1.1), $u \in L^2(0, T; L^2(\mathbb{R}^2)) \cap L^\infty(\mathbb{R}^2 \times (0, T))$, such that $\partial^2_{xx} u, \partial_t u \in L^2(0, T; L^2(\mathbb{R}^2))$ and

$$\lim_{t\to 0}\int_{\mathbb{R}^2}\left|u(t)-u_0\right|^2\,dx\,dy=0.$$

In order to obtain existence for such solutions, we approximate (1.1)–(1.2) with a more regular problem

$$\partial_t u=\Delta^\varepsilon u+\partial_y f_\varepsilon(u),\quad (x,y)\in\mathbb{R}^2,\ t>0, \tag{2.1}$$

$$u(\cdot,0)=u_{0,\varepsilon},\quad \text{on } \mathbb{R}^2, \tag{2.2}$$

where Δ^ε is the linear uniformly elliptic operator

$$\Delta^\varepsilon u=\partial^2_{xx}u+\varepsilon\partial^2_{yy}u$$

and $f_\varepsilon=f*\chi_\varepsilon$, $u_{0,\varepsilon}=u_0*\chi_\varepsilon$ are the mollified functions of f and u_0, respectively. Next, we take that u_0 belongs to some H^m and we pursue the compactness of $\{u_\varepsilon\}$ by making use of iterated energy estimates.

The standard theory of quasilinear parabolic equations guarantees that, for each fixed $\varepsilon>0$, problem (2.1)–(2.2) has a unique classical solution u_ε. In addition it is not hard to obtain the following uniform estimates.

Lemma 2.1. *We assume that f satisfies* (A1) *and that $u_0\in L^2(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$. Then, for all $T>0$, problem* (2.1)–(2.2) *admits a unique solution $u_\varepsilon\in L^2(0,T;H^1(\mathbb{R}^2))\cap L^\infty((0,T)\times\mathbb{R}^2)$. Moreover $u\in L^2(0,T;H^\infty(\mathbb{R}^2))$, $\partial^k_{t^k}u_\varepsilon$ is bounded and continuous from $(0,T)$ to $H^\infty(\mathbb{R}^2)$ for all integers $k\geqslant 0$, and the following uniform estimates hold*:

$$\left\|u_\varepsilon(t)\right\|_p\leqslant\|u_0\|_p\quad \textit{for almost all } t\in(0,T), \tag{2.3}$$

for all $p\in[2,\infty]$, and

$$\int_0^T\left\|\partial_x u_\varepsilon(t)\right\|_2^2\,dt\leqslant\|u_0\|_2^2. \tag{2.4}$$

Proof. The existence and the smoothness of u_ε, together with the uniform estimates (2.3), may be obtained by arguing as in [14, Section II.3]. Estimates of $\partial_x u_\varepsilon$ immediately come from the energy estimate. Indeed, by multiplying (2.1) by u_ε, integrating over $\mathbb{R}^2\times(0,T)$ and applying the Green's formula, we obtain

$$\frac{1}{2}\int_{\mathbb{R}^2}\big(u_\varepsilon(T)\big)^2\,dx\,dy+\int_0^T\int_{\mathbb{R}^2}\big[(\partial_x u_\varepsilon)^2+(\sqrt{\varepsilon}\partial_y u_\varepsilon)^2\big]\,dx\,dy\,dt$$
$$\leqslant\frac{1}{2}\int_{\mathbb{R}^2}u_{0,\varepsilon}^2\,dx\,dy-\int_0^T\int_{\mathbb{R}^2}f_\varepsilon(u_\varepsilon)\partial_y u_\varepsilon\,dx\,dy\,dt.$$

However, the last term is equal to zero, since it may be written in the conservation form

$$\int_0^T \int_{\mathbb{R}^2} \partial_y F(u_\varepsilon)\,dx\,dy\,dt$$

with $F(u) = \int_0^u f_\varepsilon(v)\,dv \in L^1(\mathbb{R}^2 \times (0,T))$. □

Remark 2.2. The same arguments of the proof of estimate (2.4) give that $\{\sqrt{\varepsilon}\partial_y u_\varepsilon\}$ is bounded in $L^2(\mathbb{R}^2 \times (0,T))$.

A relevant consequence of the uniform estimate (2.4) is that the solution of the original equation (1.1) is expected to be smooth with respect to the variable x, no matter what kind of topology is chosen to pass into the limit. Besides, such property does not depend from the global L^2-norm of the initial condition u_0. This may be seen by making use of a localization technique which goes up to De Giorgi [7]. Let us set, for all $r > 0$, $\mathcal{S}_r = \{(x,y)\colon |x| < r\}$; the L^2-norm of $\partial_x u_\varepsilon$ in the strip $\mathcal{S}_r$ may be estimated as follows.

Corollary 2.3. *Under the same assumption of Lemma* 2.1, *for all* $r > 0$ *we have*

$$\int_0^T \int_{\mathcal{S}_r} |\partial_x u_\varepsilon|^2\,dx\,dy\,dt \leqslant \int_{\mathcal{S}_{2r}} |u_{0,\varepsilon}|^2\,dx\,dy + 4\int_0^T \int_{\mathcal{S}_{2r}\setminus\mathcal{S}_r} |u_\varepsilon|^2\,dx\,dy\,dt.$$

Proof. In order to attain a local energy estimate, we fix $r > 0$ and we take the cut-off function

$$\alpha(x) = \begin{cases} 1 & \text{if } |x| \leqslant r, \\ 2r - |x| & \text{if } r \leqslant |x| \leqslant 2r, \\ 0 & \text{if } |x| \geqslant 2r. \end{cases}$$

By multiplying (2.1) by $u_\varepsilon \alpha^2$ and by arguing as in the proof of (2.4) we obtain

$$\int_0^T \int_{\mathbb{R}^2} (\partial_x u_\varepsilon)^2\,dx\,dy\,dt \leqslant \frac{1}{2}\int_{\mathbb{R}^2} u_{0,\varepsilon}^2 \alpha^2\,dx\,dy - 2\int_0^T \int_{\mathbb{R}^2} u_\varepsilon \partial_x u_\varepsilon \alpha\alpha'\,dx\,dy\,dt,$$

and then the conclusion follows after estimating the last term on the right-hand side by means of the Cauchy–Schwartz inequality. □

We introduce a functional space which is well fitting with the structure of Eq. (1.1),

$$\mathbb{X}^{m+1} = \bigl\{h \in H^m(\mathbb{R}^2)\colon\ \partial_x h \in H^m(\mathbb{R}^2)\bigr\},$$

which is a Banach space endowed with the norm

$$\|h; \mathbb{X}^{m+1}\| = \Biggl(\|h\|_{m,2}^2 + \sum_{|\alpha|=m} \|D^\alpha \partial_x h\|_2^2\Biggr)^{1/2}.$$

Next we look for the compactness of $\{u_\varepsilon\}$ in $\mathbb{X}^{m+1}$ by iterating the energy estimate (2.4). Since such procedure guarantees a gain of regularity only with respect to the variable x, the behavior with respect to y must rather be postulated. To this aim a crucial observation is that, whenever $\partial_y u_0 \in L^\infty(\mathbb{R}^2)$, then $\partial_y u_\varepsilon(t)$ is bounded in $L^\infty(\mathbb{R}^2)$ for some positive time t, uniformly with respect to ε.

Lemma 2.4. *Suppose that $f \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ and that $u_0 \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ with $\partial_y u_0 \in L^\infty(\mathbb{R}^2)$. Then, for all $N > \|\partial_y u_0\|_\infty$, there is a positive time T_N such that $\|\partial_y u_\varepsilon(\cdot\,,t)\|_\infty \leqslant N$ for all $t \in (0, T_N)$ and $\varepsilon > 0$.*

Proof. Let us take a cut-off function $\beta \in \mathcal{C}^\infty(\mathbb{R})$, $\beta \geqslant 0$,

$$\beta(r) = \begin{cases} r & \text{if } |r| \leqslant 1, \\ 0 & \text{if } |r| \geqslant 2, \end{cases}$$

bounded with all its derivatives by a constant B; we set $\beta_N(r) = N\beta(r/N)$ for all $N > 0$. An easy computation shows that $|\beta_N| \leqslant BN$, $|\beta'_N| \leqslant B$.

Next, let us denote by $u_{\varepsilon,N}$ the global solution to the parabolic Cauchy problem $\partial_t u - \Delta^\varepsilon u = f'_\varepsilon(u)\beta_N(\partial_y u)$, with initial condition (2.2). Now, $\partial_y u_{\varepsilon,N}$ solves the linear equation with bounded coefficients $\partial_t v = \Delta^\varepsilon v + f'_\varepsilon(u)\beta'_N(\partial_y u)\partial_y v + f''_\varepsilon(u)\beta_N(\partial_y u)v$, with initial condition $\partial_y u_{\varepsilon,N}(0) = \partial_y u_{0,\varepsilon}$; then the maximum principle yields

$$\big\|\partial_y u_{\varepsilon,N}(t)\big\|_\infty \leqslant \|\partial_y u_0\|_\infty \exp\big(BN\big\|f''(u_0)\big\|_\infty t\big)$$

for all $\varepsilon > 0$. In particular, $\|\partial_y u_{\varepsilon,N}(t)\|_\infty \leqslant N$ on a suitable interval $(0, T_N)$, where T_N does not depend on ε. Finally, $\beta_N(\partial_y u_{\varepsilon,N}) = \partial_y u_{\varepsilon,N}$ in $(0, T_N)$ and then $u_{\varepsilon,N}$ is indeed the unique solution to (2.1)–(2.2) in $(0, T_N)$. □

Until $\partial_y u_\varepsilon$ stays bounded, it is not hard to obtain uniform bounds for higher order derivatives of u_ε.

Proposition 2.5 (Uniform estimates in $\mathbb{X}^{m+1}$). *We suppose that there exists $m \geqslant 2$ such that f and u_0 satisfy*

(A^{m}1) $f \in W^{m+1,\infty}_{\mathrm{loc}}(\mathbb{R})$,
(A^{m}2) $u_0 \in L^\infty(\mathbb{R}^2) \cap H^m(\mathbb{R}^2)$, $\partial_y u_0 \in L^\infty(\mathbb{R}^2)$.

Then, the solutions u_ε of (2.1)–(2.2) *satisfy the following regularity properties*:

$$\partial^k_{t^k} u_\varepsilon \in L^2_{\mathrm{loc}}\big(0,\infty; H^{m+1-2k}(\mathbb{R}^2)\big) \cap \mathcal{C}\big(0,\infty; H^{m-2k}(\mathbb{R}^2)\big), \quad 0 \leqslant k \leqslant m/2,$$

$$\partial^k_{t^k} u_\varepsilon \in L^2_{\mathrm{loc}}\big(0,T; L^2(\mathbb{R}^2)\big), \quad 2k = m+1.$$

Furthermore for all $N > \|\partial_y u_0\|_\infty$ and $T_N > 0$ which verifies the conclusion of Lemma 2.4, *the following estimates hold*:

$$\int_0^{T_N} \big\|u_\varepsilon(t); \mathbb{X}^{m+1}\big\|^2\, dt \leqslant C_1, \tag{2.5}$$

$$\sup_{t\in(0,T_N)} \left\| u_\varepsilon(t); H^m(\mathbb{R}^2) \right\| \leqslant C_2, \tag{2.6}$$

$$\int_0^{T_N} \left\| \partial_t u_\varepsilon(t); H^{m-1}(\mathbb{R}^2) \right\|^2 dt \leqslant C_3, \tag{2.7}$$

$$\sup_{t\in(0,T_N)} \left\| \partial_t u_\varepsilon(t); H^{m-2}(\mathbb{R}^2) \right\| \leqslant C_4 \tag{2.8}$$

for all $\varepsilon > 0$. *Here, the constants* $C_1, \dots, C_4$ *only depend by* N, *by the norm of* u_0 *in* H^m *and by the norm of* f *in* $W^{m+1,\infty}(-\|u_0\|_\infty, \|u_0\|_\infty)$.

Proof. The existence of a solution of (2.1)–(2.2) with the regularity properties stated by the first part of the claim is standard (see, for instance, [18]). Estimates (2.5) and (2.6) may be obtained by iterating the energy estimate used in the proof of Lemma 2.4 for all the derivatives of u till order m and by making use of the fact that $\|\partial_y u_\varepsilon(t)\|_\infty \leqslant N$ for all $t \in [0, T_N]$. In particular, the equality $\partial_t u = \Delta_\varepsilon u + f'(u)\partial_y u$ holds almost everywhere, so that estimates (2.7) and (2.8) are a straightforward consequence of (2.5) and (2.6). □

The existence of a strong solution is attained by means of a relative compactness result in the spaces $L^2(0, T; \mathbb{X}^{m+1})$ and $\mathcal{C}(0, T; H^m)$ for the global solutions u_ε of the problems with viscosity (2.1)–(2.2).

Theorem 2.6 (Local existence and continuation principle). *Under the same assumptions* $(\mathrm{A}^m 1)$ *and* $(\mathrm{A}^m 2)$ *of the previous lemma, there exists a time* $T > 0$ *such that* (1.1)–(1.2) *has a strong solution* $u \in L^2(0, T; \mathbb{X}^{m+1}(\mathbb{R}^2)) \cap L^\infty(0, T; H^m(\mathbb{R}^2))$, *with* $u \in \mathcal{C}(0, T; H^m(\mathbb{R}^2))$ *and* $\partial_t u \in L^2(0, T; H^{m-1}(\mathbb{R}^2))$. *Moreover, let* T^* *be the maximal time of existence for a strong solution, then* $T^* < \infty$ *if and only if* $\lim_{t\nearrow T^*} \|\partial_y u(t)\|_\infty = \infty$.

Proof. First, we remember that, thanks to Lemma 2.4, there exists a time $T > 0$ such that $\{\partial_y u_\varepsilon(t)\}$ is uniformly bounded in $L^\infty(\mathbb{R}^2)$ for all $t \in (0, T)$. So, Proposition 2.5 leads to the conclusion by arguing as follows.

By virtue of (2.5) and (2.7), $\{u_\varepsilon\}$ and $\{\partial_t u_\varepsilon\}$ are equibounded in $L^2(0, T; \mathbb{X}^{m+1})$ and in $L^2(0, T; H^{m-1}(\mathbb{R}^2))$, respectively. Hence, because $\mathbb{X}^{m+1}$ is compactly embedded in $\mathbb{X}^m$, a general compactness result (see, for instance, [21]) states that $\{u_\varepsilon\}$ is relatively compact in $L^2(0, T; \mathbb{X}^m)$. So, there is an extracted sequence from $\{u_\varepsilon\}$ converging strongly in $L^2(0, T; \mathbb{X}^m)$ and weakly in $L^2(0, T; \mathbb{X}^{m+1})$ to a function u. Trivially u solves Eq. (1.1) in the sense of distributions and consequently $\partial_t u \in L^2(0, T; H^{m-1}(\mathbb{R}^2))$.

Afterward, thanks to (2.6) and (2.8), $\{u_\varepsilon\}$ and $\{\partial_t u_\varepsilon\}$ are equibounded in $L^\infty(0, T; H^m(\mathbb{R}^2))$ and in $L^\infty(0, T; L^2(\mathbb{R}^2))$, respectively. So, another general compactness result (which may also be found in [21]) guarantees that there is an extracted sequence from $\{u_\varepsilon\}$ converging strongly in $L^\infty(0, T; H^{m-1}(\mathbb{R}^2))$ to a function $u \in \mathcal{C}(0, T; H^{m-1}(\mathbb{R}^2)) \cap L^\infty(0, T; H^m(\mathbb{R}^2))$. In particular, u takes the initial datum u_0 in the sense of Definition 2.1. On the other hand, since $\partial_t u \in L^2(0, T; H^{m-1}(\mathbb{R}^2))$ from the first step, we have by a general interpolation result (see, for instance, [18]) that u is bounded and continuous from $(0, T)$ to $H^m(\mathbb{R}^2)$.

Finally, if by contradiction $\|\partial_y(t)\|_\infty$ is bounded for $t \leqslant T^*$, Lemma 2.4 guarantees that $\partial_y u_\varepsilon$ are uniformly bounded in $L^\infty(0, T_1; \mathbb{R}^2)$ for some $T_1 > T^*$. Therefore arguing as above one can prove that there exists a strong solution untill $T_1 > T^*$, which contradicts the maximality of T^*. $\square$

We do not discuss directly here the problem of uniqueness for strong solutions. Actually, in the next section we shall prove the uniqueness for entropy solutions and we shall notice that any strong solution is, in particular, an entropy solution. Hence, the uniqueness for strong solutions is attained. By now, we prefer to further investigate the possible continuation of strong solutions.

2.1. A first order blow up result

Here we give a counterexample showing that, even starting from a smooth and compactly supported initial datum, the strong solution fails to be continuous after a finite time. This fact emphasizes the analogy between convection–diffusion equations and first order conservation laws; so, in the next section we shall give a weaker notion of solution, which takes into account the presence of discontinuities.

By now, we consider the particular case when $f(u) = u^2/2$ is the flux function of Burger's equation and the initial data are of type

$$u_0(x, y) = y v_0(x, y), \quad v_0 \in C_0^\infty(\mathbb{R}^2),\ v_0 \geqslant 0. \tag{2.9}$$

Let us set the "initial mass" and the "initial energy," respectively, along the direction $y = 0$ as

$$\mathcal{F}_0 = \frac{1}{2} \int_{\mathbb{R}} \big(v_0(x, 0)\big)^2 \, dx,$$

$$\mathcal{E}_0 = \frac{1}{3} \int_{\mathbb{R}} \big(v_0(x, 0)\big)^3 \, dx - \frac{1}{2} \int_{\mathbb{R}} \big(\partial_x v_0(x, 0)\big)^2 \, dx.$$

Provided that $\mathcal{E}_0 > 0$, the solution has a shock in the y-direction at a finite time. Furthermore, since the blow up time is estimated from above by an explicit function of $\mathcal{F}_0$ and $\mathcal{E}_0$, for all fixed time $T > 0$ there exists a smooth and compactly supported initial datum which develops a discontinuity within the time T.

Proposition 2.7. *Let u be the strong solution of* (1.1)–(1.2) *with $f(u) = u^2/2$ and u_0 given by* (2.9). *If $\mathcal{E}_0 > 0$, then u fails to be continuous within a finite time, namely we have* $\sup\{\|\partial_y u(t)\|_\infty\colon t \in (0, T^*)\} = \infty$ *for $T^* = 2\mathcal{F}_0/3\mathcal{E}_0$.*

Proof. We suppose by contradiction that $\|\partial_y u(t)\|_\infty < \infty$ for all $t \in (0, T^*)$. Then Theorem 2.6 guarantees that there exists $T > T^*$ such that problem (1.1)–(1.2) has a solution $u \in \mathcal{C}^\infty(\mathbb{R}^2 \times (0, T))$.

By setting $a(x, t) = \partial_y u(x, 0, t)$, $(x, t) \mapsto u(x, 0, t)$ is the classical solution to the linear parabolic equation $\partial_t \hat{u} = \partial_{xx}^2 \hat{u} + a\hat{u}$ with homogeneous initial condition. Hence, $u(x, 0, t) = 0$ for all (x, t). It follows that there exists $v \in \mathcal{C}^\infty(\mathbb{R}^2 \times (0, T))$ such that

$u(x, y, t) = yv(x, y, t)$. In particular $\partial_y u(x, 0, t) = v(x, 0, t)$, thus our assumption by contradiction implies that $\sup\{\|v(\cdot, 0, t)\|_\infty\colon t \in (0, T^*)\} < \infty$. In addition $(x, t) \mapsto v(x, 0, t)$ is the classical solution to

$$\begin{cases} \partial_t w = \partial^2_{xx} w + w^2, \\ w(x, 0) = v_0(x, 0). \end{cases} \tag{2.10}$$

Since v_0 has compact support, standard comparison arguments give that $v(\cdot, 0, \cdot) \in L^\infty(0, T^*; L^p(\mathbb{R}))$ for all $p \in [1, \infty]$. Eventually we reach a contradiction by showing that $\|v(\cdot, 0, t)\|_2 \geqslant \sqrt{2\mathcal{F}_0}/(1 - t/T^*)$, which is plainly implied by

$$\partial_t \sqrt{\frac{\mathcal{F}_0}{\mathcal{F}(t)}} \leqslant -\frac{1}{T^*}, \tag{2.11}$$

where $\mathcal{F}(t) = (1/2)\|v(\cdot, 0, t)\|_2^2$. The proof of (2.11) is quite technical and requests the auxiliary functions

$$\mathcal{E}(t) = \frac{1}{3}\big\|v(\cdot, 0, t)\big\|_3^3 - \frac{1}{2}\big\|\partial_x v(\cdot, 0, t)\big\|_2^2, \qquad \mathcal{G}(t) = \frac{3}{2}\frac{\mathcal{E}(t)}{\mathcal{F}(t)^{3/2}}.$$

In force of (2.10), $\mathcal{F}'(t) \geqslant 3\mathcal{E}(t)$, $\mathcal{E}'(t) \geqslant 0$, and $\mathcal{E}'(t)\mathcal{F}(t) \geqslant (1/2)\mathcal{F}'(t)$. In particular $\mathcal{G}'(t) \geqslant 0$, so that $\mathcal{G}(t) \geqslant \mathcal{G}(0) = 1/\sqrt{\mathcal{F}_0}T^*$. Finally we obtain (2.11) by computing

$$\partial_t \sqrt{\frac{\mathcal{F}_0}{\mathcal{F}(t)}} = -\frac{1}{2}\frac{\sqrt{\mathcal{F}_0}\mathcal{F}'(t)}{\mathcal{F}(t)^{3/2}} = -\sqrt{\mathcal{F}_0}\mathcal{G}(t)\frac{\mathcal{F}'(t)}{3\mathcal{E}(t)} \leqslant -\sqrt{\mathcal{F}_0}\mathcal{G}(0) = -\frac{1}{T^*}. \qquad \square$$

We recall that the first blowup result for problem (2.10) is due to [12], while the use of energy norms to establish global nonexistence has been introduced by [15]. The class of initial data (2.9) has been used in [10] to show that classical solutions for the unsteady Prandtl's equation do not exist for all times, in general. Concerning Eq. (1.1), the result is somewhat stronger, because by taking advantage of the continuation principle stated by Theorem 2.6 we are able to establish an effective blowup of first order derivative.

3. Entropy approach

In view of the blowup result stated by Proposition 2.7, the class of strong solutions has to be enlarged, avoiding to impose the continuity with respect to y, in order to obtain the existence of solutions for all time $t > 0$. On the other hand a criterion of choice among distributional solutions is needed to guarantee uniqueness. Besides, the new notion of solution must be consistent with the classical one: to this end we still construct the entropy solution as the limit of the classical solutions of the regularized problems (2.1)–(2.2), but according to a weaker topology. In view of this fact and of the uniform estimates of $\partial_x u_\varepsilon$ obtained in Corollary 2.3, it seems natural to impose as the standing regularity of an entropy solution that $\partial_x u(t)$ belongs to $L^2_{\mathrm{loc}}(\mathbb{R}^2)$. This brings to the following definition.

Definition 3.1. A distributional solution to (1.1), $u \in L^\infty(\mathbb{R}^2 \times (0,T))$, is an *entropy solution* for the Cauchy problem (1.1)–(1.2) in the time interval $(0,T)$ if $\partial_x u \in L^2(0,T; L^2_{\mathrm{loc}}(\mathbb{R}^2))$,

$$\operatorname*{ess\,lim}_{t\to 0} \int_K \big|u(t) - u_0\big|\,dx\,dy = 0$$

for all compact subsets K of $\mathbb{R}^2$, and

$$\int_0^T \int_{\mathbb{R}^2} -|u-k|\partial_t\varphi + \partial_x|u-k|\partial_x\varphi + \operatorname{sgn}(u-k)\big(f(u)-f(k)\big)\partial_y\varphi \leqslant 0 \tag{3.1}$$

for all real constant k and all smooth functions $\varphi \in \mathcal{C}^1(\mathbb{R}^2 \times (0,T))$ with $\varphi \geqslant 0$ and with compact support.

Remark 3.1. It is an easy exercise to show that any strong solution in the sense of Definition 2.1 is indeed an entropy solution, according to Definition 3.1.

Definition 3.1 and [11, Definition in Section 1] cannot be directly compared. Actually, the solutions in [11] are not solutions according to Definition 3.1, because they do not satisfy $\partial_x u \in L^2(0,T; L^2_{\mathrm{loc}}(\mathbb{R}^2))$, so that they cannot be checked against the entropy criterion (3.1). On the other hand, solutions according to Definition 3.1 satisfy the entropy criterion [11, (EC)] only for constant k, so that they do not fulfill [11, Definition in Section 1]. Although, the constructed solutions happen to coincide when both exist (see Corollary 3.7, later on).

Definition 3.1 seems more natural, because it asks for less entropy tests by taking advantage of a regularity property, $\partial_x u \in L^2(0,T; L^2_{\mathrm{loc}}(\mathbb{R}^2))$, which directly comes from the viscosity procedure. Moreover, the information $\partial_x u \in L^2(0,T; L^2_{\mathrm{loc}}(\mathbb{R}^2))$ enables to discuss the shocks in Section 3.2.

Like in the case of scalar hyperbolic conservation laws, the entropic approach investigate the compactness of $\{u_\varepsilon\}$ with respect to the topology of $W^{1,1}_{\mathrm{loc}}$. Since the final equation (1.1) involves higher order derivatives with respect to x, it is requested that the initial datum satisfies a narrowest regularity assumption with respect to x. We list here some uniform estimates that may be obtained by arguing as in [11].

Lemma 3.2. *We assume that f satisfies* (A1) *and that $u_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap BV(\mathbb{R}^2)$. Then*

$$\big\|u_\varepsilon(t)\big\|_p \leqslant \|u_0\|_p \quad \textit{for almost all } t \in (0,T), \tag{3.2}$$

for all $p \in [1,\infty]$ and the mass is preserved:

$$\int_{\mathbb{R}^2} u_\varepsilon(t)\,dx\,dy = \int_{\mathbb{R}^2} u_0\,dx\,dy. \tag{3.3}$$

Moreover $\partial_y u_\varepsilon$ is uniformly bounded in $L^\infty(0,T; L^1(\mathbb{R}^2))$ with

$$\big\|\partial_y u_\varepsilon(t)\big\|_1 \leqslant TV u_0. \tag{3.4}$$

If, in addition, $\partial_x u_0 \in BV(\mathbb{R}^2)$*, then*

$$\left\|\partial_t u_\varepsilon(t)\right\|_1 \leqslant C\,TV(u_0) + TV(\partial_x u_0). \tag{3.5}$$

This uniform estimates, together with the one obtained in Corollary 2.3, allow us to obtain the existence of entropy solutions for smooth initial data by a well-understood compactness technique.

Proposition 3.3 (Existence with smooth data). *We suppose that* f *satisfies* (A1) *and that* $u_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap BV(\mathbb{R}^2)$, $\partial_x u_0 \in BV_x(\mathbb{R}^2)$. *Then for all* $T > 0$ *problem* (1.1)–(1.2) *has an entropy solution* $u \in L^\infty(\mathbb{R}^2 \times (0,T)) \cap \mathcal{C}(0,T; L^1(\mathbb{R}^2))$. *Such solution is the limit in* $L^1(\mathbb{R}^2 \times (0,T))$ *and almost everywhere of the solutions* u_ε *of the regularized problems* (2.1)–(2.2), *up to an extracted sequence. In addition* u *verifies estimates* (3.2) *and* (3.3), $\partial_x u$ *satisfies* (2.4), *and*

$$TV\big(u(t)\big) \leqslant TV(u_0), \tag{3.6}$$

$$\left\|u(t_1) - u(t_2)\right\|_1 \leqslant \big[(M + C)TV_y(u_0) + TV_x(\partial_x u_0)\big]|t_1 - t_2|. \tag{3.7}$$

Proof. The convergence of $\{u_\varepsilon\}$ to a distributional solution $u \in \mathcal{C}(0,T; L^1_{\mathrm{loc}}(\mathbb{R}^2))$ satisfying (3.2), (3.3), and (3.6) has been proved in [11, Section 2]. Moreover (2.4) implies that $\partial_x u \in L^2(\mathbb{R}^2 \times (0,T))$. Lastly, one may check that u satisfies the entropy criterion (3.1) by approximating u with the smooth functions u_ε, by integrating by parts separately on the two sets $\{u_\varepsilon > k\}$ and $\{u_\varepsilon < k\}$, and by taking advantage of Remark 2.2 when passing to the limit. □

Indeed, the hypotheses about u_0 of Proposition 3.3 are quite strong and may be removed by an elementary procedure of approximation. To this end, a crucial result is the contraction property of entropy solutions in L^1. So, we delay the discussion of this extension to next paragraph.

3.1. Existence and uniqueness of entropy solutions

Our main result shall be the existence and uniqueness of an entropy solution of problem (1.1)–(1.2) for all initial data u_0 belonging to $L^\infty(\mathbb{R}^2)$. The scheme of the proof is the usual one: we first obtain a contraction property in L^1 for entropy solutions; as a first consequence, we obtain uniqueness of entropy solutions. Next, we use this property to improve Proposition 3.3 obtaining the existence of entropy solution for any initial data verifying (A2), by approximating them with smooth ones.

We begin by stating a differential inequality for the difference of two solutions, that is obtained by (3.1) via the standard technique of doubling variables, which goes up to Kruzhkov [16].

Lemma 3.4. *Let u, v be two entropy solutions of* (1.1); *then*

$$\int_0^T \int_{\mathbb{R}^2} -|u-v|\partial_t\varphi + \partial_x|u-v|\partial_x\varphi + \operatorname{sgn}(u-v)\big(f(u)-f(v)\big)\partial_y\varphi \leqslant 0 \tag{3.8}$$

for all smooth functions $\varphi \in \mathcal{C}^1(\mathbb{R}^2 \times (0,T))$ with compact support.

A relevant consequence of the weak inequality (3.8) is the contraction property stated by (3.9), that is the corner of the proof of uniqueness of solution according to any Kruzhkov type technique. For any given $T, r > 0$, we set α_r the classical solution of the backward heat equation

$$\begin{cases} \partial_t\alpha + \partial_{xx}^2\alpha = 0, & (x,t) \in \mathbb{R} \times (0,T), \\ \alpha(x,T) = \begin{cases} 1, & |x| \leqslant r, \\ 0, & |x| > r. \end{cases} \end{cases}$$

Proposition 3.5 (Uniqueness of entropy solutions). *Let u, v be two entropy solutions of* (1.1) *and $M = \max\{\|f'(u)\|_\infty, \|f'(v)\|_\infty\}$. Then for all $r, s, T > 0$ we have*

$$\int_s^s \int_{-r}^r |u-v|(T)\,dx\,dy \leqslant \int_{-s-MT}^{s+MT} \int_{\mathbb{R}} |u_0-v_0|\alpha_r(x,0)\,dx\,dy. \tag{3.9}$$

In particular, the Cauchy problem (1.1)–(1.2) *has at most one entropy solution.*

Proof. Inequality (3.9) follows by (3.8) by a careful choice of the test function φ. We approximate the heavy side function by the smooth one

$$H_\delta(s) = \int_{-\infty}^s \chi_\delta(\tau)\,d\tau,$$

where χ_δ stands for the standard one dimensional mollifier. Afterwards, we choose three different positive parameters $\tau > \rho > \delta$ and we approximate the functions $I_{\{|y| \leqslant s+M(T-t)\}}$ and $I_{[\tau,T]}$ by means of

$$\beta_\delta(y,t) = 1 - H_\delta\big(|y| - s - M(T-t)\big),$$
$$\theta_{\rho\tau}(t) = H_\rho(t-\tau) - H_\rho(t-T).$$

Now, we are ready to write (3.8) using $\varphi = \alpha_r\beta_\delta\theta_{\rho\tau}$ as a test function: remembering that $\partial_t\alpha_r + \partial_{xx}^2\alpha_r = 0$ pointwise and that $\operatorname{sgn}(u-v)[f(u)-f(v)] \geqslant -M|u-v|$ almost everywhere we obtain

$$-\int_0^T \int_{\mathbb{R}^2} |u-v|\alpha_r\beta_\delta\theta'_{\rho\tau}\,dx\,dy\,dt \leqslant 0.$$

Extracting the limit as δ, ρ, τ go to zero (in this order) yields the conclusion. □

The local estimate (3.9) is the well posedness of the Cauchy problem (1.1)–(1.2) with the only assumption that $u_0 \in L^\infty$, by an approximation argument.

Theorem 3.6 (Well posedness). *We suppose that f and u_0 satisfy* (A1) *and* (A2). *Then, for all $T > 0$ the Cauchy problem* (1.1)–(1.2) *has a unique entropy solution u. In addition u satisfies*

$$\|u(t)\|_\infty \leqslant \|u_0\|_\infty, \tag{3.10}$$

$$\int_0^T \int_{y_0-\rho}^{y_0+\rho} \int_{x_0-r}^{x_0+r} |\partial_x u|^2 \, dx \, dy \, dt \leqslant Cr(\rho + MT)\|u_0\|_\infty^2 \tag{3.11}$$

for all $(x_0, y_0) \in \mathbb{R}^2$, and $r, s > 0$.

Proof. Let $M = \|f'(u_0)\|_\infty$. First, we cut the function u_0 in the y direction by means of

$$u_0^s = \begin{cases} u_0 & \text{if } |y| \leqslant s + MT, \\ 0 & \text{elsewhere.} \end{cases}$$

Next, we introduce an index n to define a sequence $u_{0,n}^s \in \mathcal{C}^1(\mathbb{R}^2)$ with compact support contained in the strip $\mathbb{R} \times [-s - 2MT, s + 2MT]$ such that $\|u_{0,n}^s\|_\infty \leqslant \|u_0\|_\infty$ and $u_{0,n}^s \to u_0^s$ in $L^1(\mathcal{S}_R)$ for all $R > 0$. Because $u_{0,n}^s$ satisfies the assumptions of Proposition 3.3, there exists an entropy solution u_n^s to (1.1) with $u_n^s(0) = u_{0,n}^s$. Now (3.9) guarantees that

$$\int_{-\rho}^{\rho} \int_{-r}^{r} |u_n^s - u_m^s|(t) \, dx \, dy \leqslant \int_{-\rho-Mt}^{\rho+Mt} \int_{\mathbb{R}} |u_{0,n}^s - u_{0,m}^s| \alpha_r(x) \, dx \, dy$$
$$\leqslant \|u_{0,n}^s - u_{0,m}^s; L^1(\mathcal{S}_R)\| + 4\sqrt{\pi}\|u_0\|_\infty e^{-(R-r)^2}$$

for all $\rho, r, R > 0$. Hence u_n^s converges strongly in $\mathcal{C}(0, T; L^1_{\mathrm{loc}}(\mathbb{R}^2))$ to a function u^s, which in addition is bounded by $\|u_0\|_\infty$. Furthermore u_n^s tends to u^s in $L^p(0, T; L^p_{\mathrm{loc}}(\mathbb{R}^2))$ for all $p \in [1, \infty)$, and $u^s(t) = 0$ outside the strip $\mathbb{R} \times [-s - 2MT, s + 2MT]$.

In order to check that u^s is indeed the entropy solution with $u^s(0) = u_0^s$, it suffices to check that $\partial_x u_n^s$ weakly converges to $\partial_x u^s$: it easily follows by Corollary 2.3.

Eventually estimate (3.9) guarantees that $u^s = u^{s'}$ on $\mathbb{R} \times [-s, s]$ for all $s' > s$; thus the function u obtained by gluing together the u^s turns out to satisfy $\partial_x u \in L^2(0, T; L^2_{\mathrm{loc}}(\mathbb{R}^2))$ and then it is the entropy solution of (1.1)–(1.2).

Next, estimate (3.10) is immediately implied by the construction procedure. In order to check (3.11), we may suppose without loss of generality that $(x_0, y_0) = (0, 0)$. By cost ruction we have that

$$\int_0^T \int_{-s}^{s} \int_{-r}^{r} |\partial_x u|^2 \, dx \, dy \, dt \leqslant \int_0^T \int_{\mathcal{S}_r} |\partial_x u^s|^2 \, dx \, dy \, dt.$$

Hence applying Corollary 2.3 to u_n^s and extracting the limit as $n \to \infty$ gives the thesis. □

We end this section by establishing the coincidence of the entropy solutions in [11] and the ones in the present paper, in the common existence domain.

Corollary 3.7. *Let $u_0 \in L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. Then the entropy solution constructed in* [11, Theorem 1] *is the solution according to Definition* 3.1.

Proof. In force of the uniform estimate established in Corollary 2.3, one easily obtains that the solution u constructed in [11, Theorem 1] satisfies $\partial_x u \in L^2(0,T; L^2_{\mathrm{loc}}(\mathbb{R}^2))$, indeed. Next, integrating by parts the entropy criterion [11, (EC)] gives that (3.1) holds true for any constant k. Therefore u is a solution according to Definition 3.1, and the uniqueness result by Proposition 3.9 gives the thesis. □

3.2. Characterization of entropic shocks

We now deduce from the entropy criterion (3.1) a characterization of admissible discontinuities. First of all, we define what is meant by "shock" for a function of three variables (x, y, t), i.e., a discontinuity across a two dimensional surface. Since in general entropy solutions only belong to L^1_{loc}, we need to use the notion of approximate limit.

For all $z_0 = (x_0, y_0, t_0) \in \mathbb{R}^2 \times (0,\infty)$, $\eta > 0$, and $v \in \mathbb{R}^3 \setminus \{0\}$, we introduce the notations

$$B_\eta(z_0) = \bigl\{z \in \mathbb{R}^3\colon\ |z - z_0| < \eta\bigr\},$$
$$B^\pm_\eta(z_0, v) = \bigl\{z \in B_\eta(z_0)\colon\ (z - z_0)\cdot v \gtrless 0\bigr\}.$$

Definition 3.2. Any function u has a *shock* at a point $z_0 = (x_0, y_0, t_0)$ in the direction $v \in \mathbb{R}^3 \setminus \{0\}$ if there exist two real numbers $u^+ \neq u^-$ such that

$$\lim_{\eta\to 0} \frac{2}{|B_\eta(z_0)|} \int\limits_{B^\pm_\eta(z_0,v)} \bigl|u(z) - u^\pm\bigr| = 0.$$

An entropy solution to (1.1) may not have arbitrary shocks. Actually, such discontinuities may occur only in the y-direction, besides the values of u at the two sides of the surface of discontinuity must satisfy the same restrictions as well as for scalar conservation laws.

Theorem 3.8 (Entropic shocks). *Let u be an entropy solution to* (1.1) *and let us suppose that it has a shock at the point $z_0 = (x_0, y_0, t_0)$ in the direction v. Then*

$$v = (0, \lambda, 1), \tag{3.12}$$
$$f(u^+) - f(u^-) = -\lambda[u^+ - u^-], \tag{3.13}$$
$$\operatorname{sgn}(u^+ - u^-)\bigl[\alpha f(u^+) + (1-\alpha) f(u^-) - f\bigl(\alpha u^+ + (1-\alpha)u^-\bigr)\bigr] \geqslant 0 \tag{3.14}$$

for all $\alpha \in (0, 1)$.

Proof. We define the piecewise constant function

$$U(z) = \begin{cases} u^+, & z \cdot v > 0, \\ u^-, & z \cdot v < 0. \end{cases}$$

By definition of shock, the rescaled function $u_\eta(z) = u(z_0 + \eta z)$ converges to U in $L^1(B_1(0))$. Moreover there exists a constant C such that for all $\eta < t_0$ we have

$$\left\| \partial_x u_\eta; L^2\big(B_1(0)\big)\right\|^2 \leqslant \frac{1}{\eta} \int_0^T \int_{-\eta}^{\eta} \int_{-\eta}^{\eta} |\partial_x u|^2 \, dx \, dy \, dt \leqslant C \|u_0\|_\infty^2,$$

by (3.11). Hence, up to an extracted sequence, $\partial_x u_\eta$ weakly converges to some $w \in L^2(B_1(0))$. As a consequence U has distributional derivative with respect to x equal to w. Since U is piecewise constant, we deduce that $w = 0$ indeed. In particular U does not depend by x and v is orthogonal to the x-axis.

Now U is an entropy solution to the scalar conservation law $\partial_t U - \partial_y f(U) = 0$ on the set $\Omega = \{(y,t) \in \mathbb{R} \times (0,T)\colon\ |y - y_0|^2 + |t - t_0|^2 < 1/2\}$. Indeed, U satisfies the entropy condition for scalar conservation laws because for all constant k and for all smooth functions $\psi \in \mathcal{C}_0^\infty(\Omega)$, $\xi \in \mathcal{C}_0^\infty(x_0 - 1/2, x_0 + 1/2)$, $\psi, \xi \geqslant 0$ we have

$$\begin{aligned} &\int_{\mathbb{R}} \left(\int_0^T \int_{\mathbb{R}} \big[-|U - k| \partial_t \psi + \operatorname{sgn}(U - k)\big(f(U) - f(k)\big)\partial_y \psi \big] \, dy \, dt \right) \xi(x) \, dx \\ &\quad = \lim_{\eta \to 0} \int_0^T \int_{\mathbb{R}^2} \big[-|u_\eta - k| \partial_t(\xi\psi) + \operatorname{sgn}(u_\eta - k)\big(f(u_\eta) - f(k)\big)\partial_y(\xi\psi)\big] \, dx \, dy \, dt \\ &\quad \leqslant \lim_{\eta \to 0} \int_0^T \int_{\mathbb{R}^2} |u_\eta - k| \partial_{xx}^2 (\xi\psi) \, dx \, dy \, dt. \end{aligned}$$

Finally, the conclusion follows from the standard theory for entropy solutions of scalar conservation laws (see, for instance, [14]). □

Notice that conditions (3.13) and (3.14) are very similar (and play the same role) of the well-known Rankine–Hugoniot condition and Oleinik condition, respectively, for first order conservation laws.

If we knew a priori that u is piecewise smooth, Theorem 3.8 would provide an easy characterization of entropy solutions. Roughly speaking, a function which is smooth almost everywhere, apart from some surfaces across which it may jump, is an entropy solution if and only if it solves (1.1) almost everywhere and it has admissible shocks (according to Theorem 3.8) across the surfaces of discontinuity. To be more precise, let us first state what we mean by “piecewise smooth.”

Definition 3.3. A function $u \in L^\infty(\mathbb{R}^2 \times (0,T))$ with $\partial_x u \in L^2(0,T; L^2_{\mathrm{loc}}(\mathbb{R}^2))$ is *piecewise smooth* if there exist finitely many disjoint surfaces $\mathcal{J}_n$ of class $\mathcal{C}^1$ such that

(i) According to the two dimensional Hausdorff measure, almost every point outside the surfaces $\mathcal{J}_n$ has a neighborhood where u and $\partial_x u$ are Lipschitz continuous;
(ii) Every point z_0 inside the surface $\mathcal{J}_n$ has a neighborhood V such that
 - $\mathcal{J}_n \cap V$ has a local parametrization of type

$$\mathcal{J}_n \cap V = \big\{\big(x(t,s), y(t,s), t\big) \in \mathbb{R}^2 \times (0,T)\colon\ (t,s) \in (a_n, b_n) \times (c_n, d_n)\big\};$$

 - The restrictions of u to the subsets

$$V^{\pm} = \big\{\zeta \in V\colon\ \zeta = z + t\nu_n(z),\ z \in \mathcal{J}_n \cap V,\ t \gtrless 0\big\}$$

 are Lipschitz continuous.

Here, $\nu_n(z)$ is the normal vector to $\mathcal{J}_n$ at the point z.

Theorem 3.9 (Characterization of piecewise smooth entropy solutions). *Let $u \in L^\infty(\mathbb{R}^2 \times (0,T))$, with $\partial_x u \in L^2(0,T; L^2_{\mathrm{loc}}(\mathbb{R}^2))$ be a piecewise smooth function. Then u is an entropy solution of* (1.1) *if and only if*

(a) *u satisfies Eq.* (1.1) *almost everywhere on $\mathbb{R}^2 \times (0,T) \setminus \bigcup_n \mathcal{J}_n$;*
(b) *$\mathcal{J}_n$ is part of a cylinder parallel to the x axis, i.e., any point in $\mathcal{J}_n$ has a neighborhood V such that*

$$\mathcal{J}_n \cap V = \big\{\big(x, \lambda_n(t), t\big)\colon\ (t,x) \in (a_n, b_n) \times (c_n, d_n)\big\}$$

with $\lambda_n \in \mathcal{C}^1(a_n, b_n)$;
(c) *u has an admissible shock across the surface $\mathcal{J}_n$, namely for all $(x, \lambda_n(t), t) \in \mathcal{J}_n$ u has a shock in the direction $(0, \dot\lambda_n(t), 1)$ fulfilling condition* (3.13) *and* (3.14).

Proof. By miming the arguments of [3, Theorem 4.2] and remembering Theorem 3.8, one easily obtains that u is a distributional solution if and only if items (a)–(c, 3.13) hold. It remains to check that u is an entropy solution if and only if (c) and (3.14) holds true.

But (c, 3.14) is necessary in force of Theorem 3.8. In order to check that it is sufficient, i.e., that it guarantees that u satisfies the entropy criterion (3.1), let us fix $k \in \mathbb{R}$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2 \times (0,T)$, $\varphi \geqslant 0$. By integrating by parts separately in the four sets obtained by intersecting the two sides of $\mathcal{J}_n$ with $\{u > k\}$ and $\{u < k\}$, and recalling (3.13), we obtain

$$\int_0^T \int_{\mathbb{R}^2} \big[-|u-k|\partial_t\varphi + \partial_x|u-k|\partial_x\varphi + \operatorname{sgn}(u-k)\big(f(u) - f(k)\big)\partial_y\varphi\big]\,dx\,dy\,dt$$
$$\leqslant -\int_{a_n}^{b_n} \int_{\{k \in I\}} \operatorname{sgn}(u^+ - u^-)\big[f(u^+) + f(u^-) - 2f(k) + \dot\lambda(u^+ - u^- - 2k)\big] \times \varphi\big(x, \lambda_n(t), t\big)\,dt,$$

where I stands for the segment between u^+ and u^-. Finally, the term on the right-hand side is nonpositive thanks to (3.14). □

Theorem 3.9 allows to easily check if a piecewise smooth distributional solution is an entropy solution or not. Therefore it provides as a byproduct examples of discontinuous solutions showing that the partial diffusion in the direction x does not bring any smoothing effect in the direction y, in the framework of entropy solutions. Indeed, the presence of the diffusion term $\partial^2_{xx}u$ may not avoid the propagation of the discontinuities in the y variable, nor obstruct any movement of the eventual plane of discontinuity.

Example 3.10. Take $f(u) = u^2/2$ and

$$u_0(x, y) = \begin{cases} -v_0(x) + C, & y \leqslant 0, \\ v_0(x) + C, & y > 0, \end{cases}$$

where v_0 is a smooth strictly positive function and $C \geqslant 0$. Now the solution to (1.1)–(1.2) is

$$u(x, y, t) = \begin{cases} -v(x, t), & y \leqslant Ct, \\ v(x, t), & y > Ct, \end{cases}$$

where v is the solution of the heat equation with initial datum v_0. Because v is strictly positive by strong maximum principle, u jumps across the plane $\{y = Ct\}$ for all t.

3.3. A geometrical condition for regularity of entropy solutions

We now establish that the well-known Oleinik condition for first order conservation law guarantees continuity of solutions also for problem (1.1)–(1.2).

Proposition 3.11. *Under the following assumptions*:

(i) $f \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ *is uniformly convex,*
(ii) u_0 *is nonincreasing with respect to* y,

for all $t > 0$ *the solution* $u(t)$ *of* (1.1)–(1.2) *is nonincreasing and Lipschitz continuous respect to* y, *uniformly with respect to* x.

Proof. We denote by u_ε the classical solution to the uniformly parabolic Cauchy problem (2.1)–(2.2). Because u_ε converges pointwise almost everywhere to u, it is sufficient to show that $-1/\operatorname{ess\,inf}(f'')t \leqslant \partial_y u_\varepsilon \leqslant 0$ for all ε. But $\partial_y u_\varepsilon$ is a classical solutions to $\partial_t v = \Delta^\varepsilon v + f'_\varepsilon(u_\varepsilon)\partial_y v + f''_\varepsilon(u_\varepsilon)v^2$, where $f''_\varepsilon \geqslant 0$ in force of the convexity of f. Hence the conclusion follows by standard comparison arguments. □

An easy consequence of Proposition 3.11, coupled with the continuation principle stated in Theorem 2.6, is the indefinite continuation of strong solutions. Indeed, it also guarantees some pointwise regularity of the strong solution u.

Corollary 3.12. *Under the same hypotheses of Proposition* 3.11, *for all* $t > 0$ *and almost every* y *the solution* u *of problem* (1.1)–(1.2) *is continuously differentiable with respect to* x. *Moreover for all* $\delta > 0$, u *is Lipschitz continuous as a function of* x, y *and Holder continuous with exponent* $1/2$ *as a function of* t *on* $\mathbb{R}^2 \times (\delta, T)$.

Proof. We denote by Y the set of $y \in \mathbb{R}$ such that $u_0(\cdot, y) \in L^\infty(\mathbb{R})$ and $\partial_y u(\cdot, y, \cdot) \in L^\infty(\mathbb{R} \times (0, T))$. Thanks to Proposition 3.11, $\mathbb{R} \setminus Y$ has zero measure. Moreover, for all $y \in Y$, $v^y(x, t) = u(x, y, t)$ solves the heat equation with source $f'(u(x, y, t))\partial_y u(x, y, t) \in L^\infty(\mathbb{R} \times (0, T))$ and initial datum $u_0(\cdot, y) \in L^\infty(\mathbb{R})$. By classical potential theory (see, for instance, [8]) v^y is continuous and continuously differentiable respect to x, and $|\partial_x v^y(x, t)| \leqslant O(\delta, \|f'(u_0)\|_\infty, \operatorname{ess\,inf}(f''(u_0)))$ for all $(x, t) \in \mathbb{R} \times (\delta, T)$. Hence, $u \in L^\infty(\delta, T, W^{1,\infty}(\mathbb{R}^2))$. In addition, [17, Theorem 1] yields that u is continuous also with respect to t.

Now we may read (1.1) as a linear equation $\partial_t u = \partial^2_{xx} u + a\partial_y u$, where $a(x, y, t) = f'(u(x, y, t))$ is bounded and continuous. Lastly, approximating u with the classical solutions of the linear and uniformly parabolic equation $\partial_t v = \Delta^\varepsilon v + a\partial_y v$, and applying to all of them [13, Theorem 1], gives the Holder continuity of u with respect to t. □

4. An application to utility theory

We conclude this paper by showing how the stated results may be applied to utility theory, providing a new outcome that may not be obtained in the classical framework. We begin by recalling the standard notion of utility process taking into account the habit toward consumption. It is assigned as the solution to the backward stochastic differential equation

$$V_t = \mathcal{E}\left[\int_t^T \left[w(c_\tau, y_\tau, \tau) - \beta V_\tau\right] d\tau + w_T(c_T, y_T) \middle| \mathcal{F}_t\right], \tag{4.1}$$

where w, w_T are deterministic functions standing for the instantaneous and for the final utility from consumption, respectively, and β is an updating factor. The processes c_t (consumption) and y_t (habit level of consumption) are commonly assumed to be described by forward stochastic differential equations of type

$$c_t = c_0 \exp\left(\int_0^t \mu\, d\tau + \int_0^t \sigma\, dW_\tau\right), \tag{4.2}$$

$$y_t = y_0 + \int_0^t \left[a(c_\tau, \tau) - \gamma y_\tau\right] d\tau. \tag{4.3}$$

A more detailed analysis of agents' decisions under risk put into light that the habit formation itself is influenced by the utility experienced in the past. Recently, Antonelli et al. [1] proposed to describe the habit formation as

$$y_t = y_0 + \int_0^t \left[a(c_\tau, V_\tau, \tau) - \gamma y_\tau\right] d\tau, \tag{4.4}$$

where the deterministic function a takes into account the effect of the past consumption and of the conditional expected utility levels that the agent experienced in the past about the future consumption plan. It is usually increasing, i.e., high consumption and utility experienced in the past positively affects the present habit. This pattern captures the "disappointment effect" if the agent's instantaneous and final utilities w, w_T are decreasing with respect to y: the higher the standard of living is, the lower the utility from consumption results. Instead, increasing w and w_T model, with respect to y, model "anticipation effect": high expected utility in the past generates a positive expectation for the future and the agent is inclined to appreciate the actual consumption rate.

Now, the recursive utility is defined as the solution of the nonlinear backward–forward stochastic equation (4.1)–(4.4). In the same paper, Antonelli et al. proposed to use the four step scheme by Ma et al. [19] in order to relate this problem to the study of a deterministic partial differential equation. They assumed that there exists a deterministic function of three variables $u \in L^\infty(0,T; W^{1,\infty}(\mathbb{R}^2))$ such that $V_t = u(\log c_t, y_t, T-t)$ and they showed that u solves an anisotropic convection–diffusion equation of type (1.3) for

$$f(x,u,t) = \int_0^u a(e^x, v, T-t)\,dv,$$

and that it satisfies the initial condition (1.2) for $u_0(x,y) = w_T(e^x, y)$. Eventually, the existence of an utility process is obtained whenever the Cauchy problem (1.3), (1.2) admits a solution u which is, at least, Lipschitz-continuous with respect to both x and y.

Let us present one example where the existence of an utility function providing a new type of preferences order is achieved.

Example 4.1. Take the backward–forward utility (4.1)–(4.4) with

$$w(c,y,\tau) = w_0(c_\tau) + \alpha y_\tau, \qquad a(c,V,\tau) = \delta V.$$

The related differential equation is

$$\partial_t u = \frac{1}{2}\sigma^2 \partial^2_{xx} u + \mu \partial_x u + (\delta u - \gamma y)\partial_y u - \beta u + w(e^x) + \alpha y.$$

The existence and uniqueness of the entropy solution is a straightforward extension of Theorem 3.3. Following the line of the proof of Proposition 3.11, we denote by u_ε the classical solution of the regularized problem obtained by adding the term $\varepsilon \partial^2_{yy} u$ to the equation. Thus, $\partial_y u_\varepsilon$ is a classical solution to

$$\partial_t v = \frac{1}{2}\sigma^2 \partial^2_{xx} v + \varepsilon \partial^2_{yy} v + \mu \partial_x v + (\delta u_\varepsilon - \gamma y)\partial_y v + \delta v^2 - (\beta+\gamma) v + \alpha$$

with initial condition $v(x,y,0) = \partial_y w_T(e^x, y)$. Now, in the case of disappointment effect, i.e., if both α and $\partial_y w_T$ are nonpositive, comparison principle yields that $u(t)$ is nonincreasing and Lipschitz continuous with respect to y, so that Corollary 3.12 still holds and $u \in L^\infty(0,T; W^{1,\infty}(\mathbb{R}^2))$ for arbitrary T.

Besides, in the case of anticipation effect, i.e., if both α and $\partial_y w_T$ are nonnegative, comparison principle yields that $u(t)$ is nondecreasing and Lipschitz continuous with respect to y if

$$4\alpha\delta \leqslant (\beta+\gamma)^2 \quad \text{and} \quad \partial_y w_T \leqslant \frac{\beta+\gamma}{2\delta}\left(1 + \sqrt{1 - 4\alpha\delta/(\beta+\gamma)^2}\right).$$

In this case Corollary 3.12 still holds and $u \in L^\infty(0, T; W^{1,\infty}(\mathbb{R}^2))$ for arbitrary T. On the contrary, if this condition is violated, the entropy solution may become discontinuous after a finite time: it is always the case, for instance, if $w = 0$. In any case, if there is no contribution from final utility, i.e., if $w_T = 0$, and if $\alpha \leqslant (\beta + \gamma)^2/4\delta$, the backward–forward differential utility is well defined for any horizon T.

As pointed out in [1], this pattern reduces to the standard expected utility if $\alpha = 0$, while it models disappointment effect if $\alpha < 0$, or anticipation effect if $\alpha > 0$. The two consumption processes

$$c_t^1 = C_1 \quad \text{and} \quad c_t^2 = \begin{cases} 0, & t \leqslant \frac{T}{2}, \\ C_2, & t > \frac{T}{2} \text{ with prob. } \pi, \\ 0, & t > \frac{T}{2} \text{ with prob. } 1 - \pi, \end{cases}$$

are ordinally equivalent under the standard expected utility if $w(C_1) = (\pi/2)w(C_2)$. Now, they are no longer equivalent if $\alpha \neq 0$, in particular, c_t^1 is better than c_t^2 in the case of disappointment effect.

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