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Periodic boundary value problems for the second order functional differential equations [☆]

Wei Ding ^{a,*}, Maoan Han ^a, Jurang Yan ^b

^a *Department of Mathematics, Shanghai Jiaotong University, Shanghai 200240, PR China*

^b *Department of Mathematics, Shanxi University Taiyuan, Shanxi 030006, PR China*

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Abstract

This paper considers the existence of extreme solutions of the periodic boundary value problems for second order functional differential equations. A new concept of lower and upper solutions is introduced. And we present a new comparison principle. Meanwhile, we extend previous results.

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Keywords: Functional differential equation; Monotone iterative technique; Lower and upper solutions; Periodic boundary value problem

1. Introduction

Theory of functional differential equations has become an important aspect of differential equations. Periodic boundary value problems (PBVP) for first and second order differential equations have drawn much attention, see [1–6] and the referees therein. Recently, J.J. Nieto and R. Rodríguez-López [1,2] introduce a new concept of lower and

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* Corresponding author.

E-mail address: dingwei@sjtu.edu.cn (W. Ding).

upper solutions, they consider PBVP for the following first order functional differential equations:

$$\begin{cases} y'(t) = f(t, y(t), y(\theta(t))), & t \in J = [0, T], \\ y(0) = y(T). \end{cases}$$

In this paper, we extend the concept of lower and upper solutions for the second order functional differential equations,

$$\begin{cases} -y''(t) = f(t, y(t), y(\theta(t))), & t \in J = [0, T], \\ y(0) = y(T), \quad y'(0) = y'(T), \end{cases} \quad (1)$$

where $f \in C(J \times R^2, R)$, $0 \leq \theta(t) \leq t$, $t \in J$.

Remarks.

- (i) If $\theta(t) = t$, then Eq. (1) is an ordinary differential equation which has been studied in many papers.
- (ii) Equation (1) can be regarded as retarded differential equation just by considering

$$\theta(t) = \begin{cases} t - r, & \text{if } t \geq r, \\ 0, & \text{if } t < r. \end{cases}$$

The paper is organized as follows. In Section 2, we first show the new concept of lower and upper solutions, then establish a new comparison principle. In Section 3, by using the method of upper and lower solutions and the monotone iterative technique, we obtain the existence of extreme solutions for PBVP (1).

2. Some lemmas

Let $E = C(J, R) \cap C^2(J, R)$ with norm

$$\|y(t)\|_E = \max \left\{ \|y(t)\|, \|y'(t)\| : \|y(t)\| = \sup_{t \in J} |y(t)|, \|y'(t)\| = \sup_{t \in J} |y'(t)| \right\},$$

then E is a Banach space.

In the following, we denote $c(t) = \min\{t, T - t\}$, $t \in J$.

Definition 2.1. A function $\alpha_0 \in E$ is called a lower solution of PBVP (1) if

$$-\alpha_0''(t) \leq f(t, \alpha_0(t), \alpha_0(\theta(t))) - a(t), \quad t \in J, \quad \alpha_0(0) = \alpha_0(T),$$

where

$$a(t) = \begin{cases} 0, & \text{if } \alpha_0'(0) \geq \alpha_0'(T), \\ [Mc(t) + Nc(\theta(t))][\alpha_0'(T) - \alpha_0'(0)], & \text{if } \alpha_0'(0) < \alpha_0'(T). \end{cases}$$

Definition 2.2. A function $\beta_0 \in E$ is called an upper solution of PBVP (1) if

$$-\beta_0''(t) \geq f(t, \beta_0(t), \beta_0(\theta(t))) + b(t), \quad t \in J, \quad \beta_0(0) = \beta_0(T),$$

where

$$b(t) = \begin{cases} 0, & \text{if } \beta'_0(0) \leq \beta'_0(T), \\ [Mc(t) + Nc(\theta(t))][\beta'_0(0) - \beta'_0(T)], & \text{if } \beta'_0(0) > \beta'_0(T). \end{cases}$$

Remark 2.1. The definition of classical lower and upper solutions makes reference to the case $\alpha'_0(0) \geq \alpha'_0(T)$, $\beta'_0(0) \leq \beta'_0(T)$.

Now we are in the position to establish comparison theorems corresponding to the new concept of lower and upper solutions.

Lemma 2.1. Assume that $y \in E$ satisfy

$$\begin{cases} -y''(t) + My(t) + Ny(\theta(t)) \leq 0, & t \in J, \\ y(0) = y(T), \quad y'(0) \geq y'(T), \end{cases}$$

where constants $M > 0$, $N \geq 0$, and they satisfy

$$T^2(M + N) \leq 1. \quad (2)$$

Then $y(t) \leq 0$ for all $t \in J$.

Proof. Suppose, to the contrary, that $y(t) > 0$ for some $t \in J$. It is enough to consider the following cases:

- (i) there exists a $\bar{t} \in J$, such that $y(\bar{t}) > 0$, and $y(t) \geq 0$ for all $t \in J$;
- (ii) there exist $t^*, t_* \in J$, such that $y(t^*) > 0$, $y(t_*) < 0$.

Case (i). We have $-y''(t) \leq -My(t) - Ny(\theta(t)) \leq 0$, which implies $y'(t)$ is nondecreasing in $t \in J$. Thus $y'(0) \leq y'(T)$. However, $y'(0) \geq y'(T)$, hence $y'(0) = y'(T)$, which implies $y'(t) = \text{constant}$, for all $t \in J$. Therefore, $0 = -y''(t) \leq -My(\bar{t}) < 0$, a contradiction.

Case (ii). First, we suppose $y(T) > 0$. There exists $t^* \in [0, T]$, such that

$$y(t^*) = \max_{t \in [0, T]} y(t) > 0, \quad y'(t^*) = 0.$$

If $y(T) = \max_{t \in [0, T]} y(t)$, then we choose $t^* = T$.

Denote $\bar{t} \in (0, t^*)$, such that $y(\bar{t}) = \min_{t \in (0, t^*)} y(t) < 0$.

Hence for $t \in (0, t^*)$,

$$-y''(t) \leq -My(t) - Ny(\theta(t)) \leq -(M + N)y(\bar{t}). \quad (3)$$

Integrate (3) from t to t^* , $t \in (\bar{t}, t^*)$, then

$$-y'(t^*) + y'(t) \leq -(M + N)(t^* - t)y(\bar{t}) \leq -T(M + N)y(\bar{t}). \quad (4)$$

Integrate (4) from \bar{t} to t^* , we obtain

$$y(t^*) - y(\bar{t}) \leq -(M + N)T^2y(\bar{t}).$$

Hence,

$$0 < y(t^*) \leq y(\bar{t}) - (M + N)T^2 y(\bar{t}),$$

which implies that $(M + N)T^2 > 1$, a contradiction.

For the case $y(T) \leq 0$, the proof is similar and thus we omit it. This completes the proof. \square

Lemma 2.2. Assume that $y \in E$ satisfy

$$\begin{cases} -y''(t) + My(t) + Ny(\theta(t)) + [Mc(t) + Nc(\theta(t))][y'(T) - y'(0)] \leq 0, & t \in J, \\ y(0) = y(T), & y'(0) < y'(T), \end{cases}$$

where constants $M > 0$, $N \geq 0$, and they satisfy (2).

Then $y(t) \leq 0$ for all $t \in J$.

Proof. Let

$$u(t) = y(t) + c(t)[y'(0) - y'(T)], \quad t \in J,$$

then $u(t) \geq y(t)$ for all $t \in J$, and

$$u'(t) = y'(t) + \begin{cases} y'(T) - y'(0), & 0 \leq t \leq T/2, \\ y'(0) - y'(T), & T/2 \leq t \leq T, \end{cases}$$

and $u''(t) = y''(t)$, $t \in J$.

Hence

$$\begin{aligned} -u''(t) + Mu(t) + Nu(\theta(t)) &= -y''(t) + My(t) + Mc(t)[y'(0) - y'(T)] \\ &\quad + Ny(\theta(t)) + Nc(\theta(t))[y'(0) - y'(T)] \leq 0. \end{aligned}$$

$$u(0) = y(0) = y(T) = u(T),$$

$$u'(0) = y'(T) > y'(0) = u'(T).$$

Hence by Lemma 2.1, $u(t) \leq 0$ in all $t \in J$, which implies that $y(t) \leq 0$, $t \in J$. So we complete the proof. \square

Consider the PBVP

$$\begin{cases} -y''(t) + My(t) + Ny(\theta(t)) = \sigma(t), & t \in J, \\ y(0) = y(T), y'(0) = y'(T). \end{cases} \quad (5)$$

where $M > 0$, $N \geq 0$ are constants and $\sigma(t) \in C(J, R)$.

Lemma 2.3. $y \in E$ is a solution of (5) if and only if $y \in C(J, R)$ is a solution of the integral equation

$$y(t) = \int_0^T G(t, s)[\sigma(s) - Ny(\theta(s))] ds, \quad (6)$$

where

$$G(t, s) = [2\sqrt{M}(e^{\sqrt{M}T} - 1)]^{-1} \begin{cases} e^{\sqrt{M}(T-t+s)} + e^{\sqrt{M}(t-s)}, & 0 \leq s < t \leq T, \\ e^{\sqrt{M}(T+t-s)} + e^{\sqrt{M}(s-t)}, & 0 \leq t \leq s \leq T. \end{cases}$$

Proof. Suppose that $y(t)$ is a solution of (5), then

$$\begin{aligned} y''(t) - My(t) &= -[\sigma(t) - Ny(\theta(t))], \\ [e^{-2\sqrt{M}t}(e^{\sqrt{M}t}y(t))]' &= -Me^{-\sqrt{M}t}y(t) + e^{-\sqrt{M}t}y''(t) \\ &= -e^{-\sqrt{M}t}[\sigma(t) - Ny(\theta(t))]. \end{aligned}$$

Let

$$u(t) = e^{-2\sqrt{M}t}(e^{\sqrt{M}t}y(t))',$$

then

$$u'(t) = -e^{-\sqrt{M}t}[\sigma(t) - Ny(\theta(t))].$$

Integrating the above equation from 0 to t , $t \in J$ yields

$$u(t) - u(0) = - \int_0^t e^{-\sqrt{M}s} [\sigma(s) - Ny(\theta(s))] ds.$$

Hence,

$$(e^{\sqrt{M}t}y(t))' = e^{2\sqrt{M}t} \left\{ u(0) - \int_0^t e^{-\sqrt{M}s} [\sigma(s) - Ny(\theta(s))] ds \right\}.$$

Denote

$$v(t) = e^{\sqrt{M}t}y(t),$$

then

$$v'(t) = e^{2\sqrt{M}t} \left\{ u(0) - \int_0^t e^{-\sqrt{M}s} [\sigma(s) - Ny(\theta(s))] ds \right\}. \quad (7)$$

Integrating (7) from 0 to t ,

$$\begin{aligned} v(t) &= v(0) + \int_0^t e^{2\sqrt{M}s} \left\{ u(0) - \int_0^s e^{-\sqrt{M}l} [\sigma(l) - Ny(\theta(l))] dl \right\} ds \\ &= v(0) + \frac{1}{2\sqrt{M}} \left\{ u(0)(e^{2\sqrt{M}t} - 1) + \int_0^t e^{\sqrt{M}s} [\sigma(s) - Ny(\theta(s))] ds \right. \\ &\quad \left. - e^{2\sqrt{M}t} \int_0^t e^{-\sqrt{M}s} [\sigma(s) - Ny(\theta(s))] ds \right\}. \end{aligned}$$

Thus

$$y(t) = \frac{1}{2\sqrt{M}} e^{-\sqrt{M}t} \left\{ 2\sqrt{M}v(0) + u(0)(e^{2\sqrt{M}t} - 1) \right. \\ \left. + \int_0^t e^{\sqrt{M}s} [\sigma(s) - Ny(\theta(s))] ds \right. \\ \left. - e^{2\sqrt{M}t} \int_0^t e^{-\sqrt{M}s} [\sigma(s) - Ny(\theta(s))] ds \right\}.$$

Note that

$$v(0) = y(0), \quad u(0) = \sqrt{M}y(0) + y'(0).$$

Thus for $t \in J$, we obtain

$$y(t) = \frac{1}{2\sqrt{M}} e^{-\sqrt{M}t} \left\{ 2\sqrt{M}y(0) + (\sqrt{M}y(0) + y'(0))(e^{2\sqrt{M}t} - 1) \right. \\ \left. + \int_0^t e^{\sqrt{M}s} [\sigma(s) - Ny(\theta(s))] ds - e^{2\sqrt{M}t} \int_0^t e^{-\sqrt{M}s} [\sigma(s) - Ny(\theta(s))] ds \right\} \\ = \frac{1}{2\sqrt{M}} \left\{ (\sqrt{M}y(0) - y'(0))e^{-\sqrt{M}t} + (\sqrt{M}y(0) + y'(0))e^{\sqrt{M}t} \right. \\ \left. + e^{-\sqrt{M}t} \int_0^t e^{\sqrt{M}s} [\sigma(s) - Ny(\theta(s))] ds - e^{\sqrt{M}t} \int_0^t e^{-\sqrt{M}s} (\sigma(s) \right. \\ \left. - Ny(\theta(s))) ds \right\}. \quad (8)$$

Hence

$$y'(t) = \frac{1}{2} \left\{ -(\sqrt{M}y(0) - y'(0))e^{-\sqrt{M}t} + (\sqrt{M}y(0) + y'(0))e^{\sqrt{M}t} \right. \\ \left. - e^{-\sqrt{M}t} \int_0^t e^{\sqrt{M}s} [\sigma(s) - Ny(\theta(s))] ds \right. \\ \left. - e^{\sqrt{M}t} \int_0^t e^{-\sqrt{M}s} [\sigma(s) - Ny(\theta(s))] ds \right\}.$$

In view of that $y(0) = y(T)$, $y'(0) = y'(T)$, we have

$$\sqrt{M}y(0) - y'(0) = (e^{\sqrt{M}T} - 1)^{-1} \left\{ \int_0^T e^{\sqrt{M}s} [\sigma(s) - Ny(\theta(s))] ds \right\}, \quad (9)$$

$$\sqrt{M}y(0) + y'(0) = (e^{\sqrt{M}T} - 1)^{-1} \left\{ e^{\sqrt{M}T} \int_0^T e^{-\sqrt{M}s} [\sigma(s) - Ny(\theta(s))] ds \right\}. \quad (10)$$

Substituting (9), (10) into (8), for $t \in J$, we obtain

$$y(t) = \int_0^T G(t, s) [\sigma(s) - Ny(\theta(s))] ds$$

i.e., $y(t)$ is also the solution of (6).

On the other hand, assume $y(t)$ is a solution of (6).

$$\begin{aligned} G'_t(t, s) &= [2\sqrt{M}(e^{\sqrt{M}T} - 1)]^{-1} \\ &\times \begin{cases} \sqrt{M}(-e^{\sqrt{M}(T-t+s)} + e^{\sqrt{M}(t-s)}), & 0 \leq s < t \leq T, \\ \sqrt{M}(e^{\sqrt{M}(T+t-s)} - e^{\sqrt{M}(s-t)}), & 0 \leq t < s \leq T, \end{cases} \\ &\doteq G^*(t, s); \end{aligned}$$

and

$$\begin{aligned} G''_t(t, s) &= G^{*'}_t(t, s) = -\sqrt{M}[2(e^{\sqrt{M}T} - 1)]^{-1} \\ &\times \begin{cases} e^{\sqrt{M}(T-t+s)} + e^{\sqrt{M}(t-s)}, & 0 \leq s < t \leq T, \\ e^{\sqrt{M}(T+t-s)} + e^{\sqrt{M}(s-t)}, & 0 \leq t < s \leq T. \end{cases} \\ &= MG(t, s). \end{aligned}$$

Then by elementary calculus, we have

$$-y''(t) + My(t) + Ny(\theta(t)) = \sigma(t).$$

It is easy to see $G(0, s) = G(T, s)$, for $s \in J$, then

$$y(0) = y(T), \quad y'(0) = y'(T), \quad y(t) = y(0).$$

This completes the proof. \square

Lemma 2.4. Assume that constants $\min\{M, \sqrt{M}\} > N \geq 0$. Then Eq. (5) has a unique solution y in E .

Proof. For any $y \in E$, define an operator F by

$$(Fy)(t) = \int_0^T G(t, s) [\sigma(s) - Ny(\theta(s))] ds,$$

where G is given by Lemma 2.3. Then $Fy \in E$, and

$$(Fy)'(t) = \int_0^T G'_t(t, s) [\sigma(s) - Ny(\theta(s))] ds.$$

For any $x, y \in E_0$,

$$\begin{aligned} & \| (Fx)(t) - (Fy)(t) \|_E \\ &= \left\| \int_0^T G(t, s) [\sigma(s) - Nx(\theta(s))] ds - \int_0^T G(t, s) [\sigma(s) - Ny(\theta(s))] ds \right\| \\ &= \left\| \int_0^T G(t, s) [Nx(\theta(s)) - Ny(\theta(s))] ds \right\| \\ &\leq \frac{N}{M} \|x - y\|. \end{aligned}$$

Similarly,

$$\begin{aligned} & \| (Fx)'(t) - (Fy)'(t) \|_{PC} \\ &= \left\| \int_0^T G'_t(t, s) [\sigma(s) - Nx(\theta(s))] ds - \int_0^T G'_t(t, s) [\sigma(s) - Ny(\theta(s))] ds \right\| \\ &= \left\| \int_0^T G'_t(t, s) [-Nx(\theta(s)) + Ny(\theta(s))] ds \right\| \\ &\leq \frac{N}{\sqrt{M}} \|x - y\|. \end{aligned}$$

By Banach fixed point theorem, F has a unique fixed point $y^* \in E$, by Lemma 2.3, y^* is also the unique solution of (5). This completes the proof. \square

3. Main result

In this section, we establish the existence theorem of (1) by method of upper and lower solutions coupled with monotone iterative technique.

For $\alpha_0, \beta_0 \in E$, we write $\alpha_0 \leq \beta_0$ if $\alpha_0(t) \leq \beta_0(t)$ for all $t \in J$. In such a case, we denote

$$[\alpha_0, \beta_0] = \{y \in E, \alpha_0(t) \leq y(t) \leq \beta_0(t), t \in J\}.$$

Now we are in the position to establish the main result.

Theorem. *Let the following conditions hold:*

(H₁) The functions α_0, β_0 are lower and upper solutions of PBVP (1), respectively, such that

$$\alpha_0(t) \leq \beta_0(t).$$

(H₂) The function f satisfies

$$f(t, x, y) - f(t, \bar{x}, \bar{y}) \geq -M(x - \bar{x}) - N(y - \bar{y}),$$

for $\alpha_0(t) \leq \bar{x}(t) \leq x(t) \leq \beta_0(t)$, $\alpha_0(\theta(t)) \leq \bar{y}(t) \leq y(t) \leq \beta_0(\theta(t))$, $t \in J$.

(H₃) Constants $\min\{M, \sqrt{M}\} > N \geq 0$, and satisfy (2).

Then there exist monotone sequences $\{\alpha_n(t)\}, \{\beta_n(t)\} \subset E$ which converge in E to the extreme solutions of PBVP (1) in $[\alpha_0, \beta_0]$, respectively.

Proof. For any $\eta \in [\alpha_0, \beta_0]$, consider linear PBVP (5) with

$$\sigma(t) = f(t, \eta(t), \eta(\theta(t))) + M\eta(t) + N\eta(\theta(t)).$$

By Lemma 2.4, (5) has exactly one solution $y \in E$. Denote $y(t) = A\eta(t)$, then A is an operator from $[\alpha_0, \beta_0]$ to E .

We complete the proof by four steps.

Step 1. We claim that $\alpha_0 \leq A\alpha_0$, and $A\beta_0 \leq \beta_0$. The latter can be prove similarly, so we only prove $\alpha_0 \leq A\alpha_0$.

Let $\alpha_1 = A\alpha_0$, and $p = \alpha_0 - \alpha_1$. Then α_1 satisfies

$$\begin{cases} -\alpha_1''(t) + M\alpha_1(t) + N\alpha_1(\theta(t)) = f(t, \alpha_0(t), \alpha_0(\theta(t))) \\ \quad + M\alpha_0(t) + N\alpha_0(\theta(t)), & t \in J, \\ \alpha_1(0) = \alpha_1(T), & \alpha_1'(0) = \alpha_1'(T). \end{cases}$$

We finish Step 1 by two cases.

Case 1. $\alpha_0'(0) \geq \alpha_0'(T)$, then

$$a(t) = 0, -\alpha_0''(t) \leq f(t, \alpha_0(t), \alpha_0(\theta(t))).$$

As α_0 is a lower solution of (1), then for $t \in J$,

$$\begin{aligned} -p''(t) + Mp(t) + Np(\theta(t)) &\leq -\alpha_0''(t) + \alpha_1''(t) + M\alpha_0(t) - M\alpha_1(t) + N\alpha_0(\theta(t)) \\ &\quad - N\alpha_1(\theta(t)) \leq 0. \end{aligned}$$

It is easy to verify

$$p(0) = p(T), \quad p'(0) \geq p'(T).$$

Then by Lemma 2.1, $p(t) \leq 0$, which implies $\alpha_0(t) \leq A\alpha_0(t)$, i.e., $\alpha_0 \leq A\alpha_0$.

Case 2. $\alpha_0'(0) < \alpha_0'(T)$, which implies that

$$a(t) = [Mc(t) + Nc(\theta(t))][\alpha_0'(T) - \alpha_0'(0)].$$

Hence

$$-p''(t) + Mp(t) + Np(\theta(t)) + [Mc(t) + Nc(\theta(t))][p'_0(T) - p'_0(0)] \leq 0.$$

Still, we have

$$p(0) = p(T), \quad p'(0) < p'(T).$$

Then by Lemma 2.2, $p(t) \leq 0$, which implies $\alpha_0(t) \leq A\alpha_0(t)$, i.e., $\alpha_0 \leq A\alpha_0$.

Step 2. We show that $A\eta_1 \leq A\eta_2$, if $\alpha_0 \leq \eta_1 \leq \eta_2 \leq \beta_0$.

Let $\eta_1^* = A\eta_1$, $\eta_2^* = A\eta_2$ and $p = \eta_1^* - \eta_2^*$, then for $t \neq t_k$, $t \in J$, we obtain

$$\begin{aligned} -p''(t) + Mp(t) + Np(\theta(t)) &= [f(t, \eta_1(t), \eta_1(\theta(t))) + M\eta_1(t) + N\eta_1(\theta(t)) \\ &\quad - f(t, \eta_2(t), \eta_2(\theta(t))) - M\eta_2(t) - N\eta_2(\theta(t))] \\ &\leq 0 \quad (\text{by } (H_2)). \end{aligned}$$

It is easy to verify

$$p(0) = p(T), \quad p'(0) = p'(T).$$

Still by Lemma 2.1, $p(t) \leq 0$, which implies $A\eta_1 \leq A\eta_2$.

Step 3. We prove that PBVP (1) have solutions.

Let $\alpha_n = A\alpha_{n-1}$, $\beta_n = A\beta_{n-1}$, $n = 1, 2, \dots$. Following the first two steps, we have

$$\alpha_0 \leq \alpha_1 \leq \dots \leq \alpha_n \leq \dots \leq \beta_n \leq \dots \leq \beta_1 \leq \beta_0.$$

Obviously, each α_i , β_i ($i = 1, 2, \dots$) satisfies

$$\begin{cases} -\alpha_i''(t) + M\alpha_i(t) + N\alpha_i(\theta(t)) = f(t, \alpha_{i-1}(t), \alpha_{i-1}(\theta(t))) \\ \quad + M\alpha_{i-1}(t) + N\alpha_{i-1}(\theta(t)), & t \in J, \\ \alpha_i(0) = \alpha_i(T), \quad \alpha_i'(0) = \alpha_i'(T), \end{cases}$$

and

$$\begin{cases} -\beta_i''(t) + M\beta_i(t) + N\beta_i(\theta(t)) = f(t, \beta_{i-1}(t), \beta_{i-1}(\theta(t))) \\ \quad + M\beta_{i-1}(t) + N\beta_{i-1}(\theta(t)), & t \in J, \\ \beta_i(0) = \beta_i(T), \quad \beta_i'(0) = \beta_i'(T). \end{cases}$$

Therefore there exist y_* and y^* such that

$$\lim_{i \rightarrow +\infty} \alpha_i(t) = y_*(t), \quad \lim_{i \rightarrow +\infty} \beta_i(t) = y^*(t) \quad \text{uniformly on } t \in J.$$

Clearly, y_* , y^* satisfy PBVP (1).

Step 4. We prove y_* , y^* are extreme solutions of PBVP (1).

Let $y(t)$ be any solution of PBVP (1), which satisfies $\alpha_0(t) \leq y(t) \leq \beta_0(t)$, $t \in J$. Also suppose there exists a positive integer n such that for $t \in J$, $\alpha_n(t) \leq y(t) \leq \beta_n(t)$.

Setting $p(t) = \alpha_{n+1}(t) - y(t)$, then for $t \in J$,

$$\begin{aligned}
-p''(t) &= -\alpha''_{n+1} + y''(t) \\
&= -M\alpha_{n+1}(t) - N\alpha_{n+1}(\theta(t)) + f(t, \alpha_n(t), \alpha_n(\theta(t))) \\
&\quad + M\alpha_n(t) + N\alpha_n(\theta(t)) - f(t, y(t), y(\theta(t))) \\
&= -M\alpha_{n+1}(t) - N\alpha_{n+1}(\theta(t)) + My(t) + Ny(\theta(t)) \\
&\quad + \{f(t, \alpha_n(t), \alpha_n(\theta(t))) - f(t, y(t), y(\theta(t))) + M(\alpha_n(t) - y(t)) \\
&\quad + N(\alpha_n(\theta(t)) - y(\theta(t)))\} \\
&\leq -Mp(t) - Np(\theta(t));
\end{aligned}$$

and

$$p(0) = p(T), \quad p'(0) \geq p'(T).$$

By Lemma 2.1, we have for all $t \in [0, T]$, $p(t) \leq 0$, i.e., $\alpha_{n+1} \leq y(t)$. Similarly, we can prove $y(t) \leq \beta_{n+1}$, $t \in J$. Thus $\alpha_{n+1} \leq y(t) \leq \beta_{n+1}$, for all $t \in J$, which implies $y_*(t) \leq y(t) \leq y^*(t)$. We complete the proof. \square

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