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Stability of Jensen equations in the space of generalized functions

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Abstract

Making use of heat kernel, we prove stabilities of the Jensen and Jensen–Pexider equations in a space of generalized functions like the spaces of tempered distributions and Fourier hyperfunctions.

Keywords: Gauss transforms; Heat kernel; Distributions; Jensen equation; Stability

1. Introduction

In 1941 D.H. Hyers showed a stability theorem for the Cauchy equation which was motivated by S.M. Ulam [19]:

Theorem 1.1 [10]. *Let $f: E_1 \to E_2$ with E_1, E_2 Banach spaces, be ε-additive, that is, f satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon$$

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for all $x, y \in E_1$. *Then there exists a unique function* $L : E_1 \to E_2$ *such that*

$$L(x+y) = L(x) + L(y)$$

and

$$\| f(x) - L(x) \| \leqslant \varepsilon$$

for all $x, y \in E_1$.

The above stability theorem was firstly generalized by T.M. Rassias [16] in 1978. Since then, stability theorems of many other functional equations have been proposed by many authors in [3,8,11–13,15,16,18].

In this paper we consider the stabilities of the Jensen equation

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\|_{L^\infty} \leqslant \varepsilon \tag{1.1}$$

and Jensen–Pexider equation

$$\left\| 2f\left(\frac{x+y}{2}\right) - g(x) - h(y) \right\|_{L^\infty} \leqslant \varepsilon \tag{1.2}$$

in spaces of generalized functions such as tempered distributions or Fourier hyperfunctions. Note that inequalities (1.1) and (1.2) themselves make no sense in the space of generalized functions.

As in [1–3,6,7], making use of the pullbacks of generalized functions we reformulate the inequality (1.1) and (1.2) to the space of generalized functions as follows:

Let A, P_1 and P_2 be the functions

$$A(x, y) = x + y, \qquad P_1(x, y) = x, \quad \text{and} \quad P_2(x, y) = y, \quad x, y \in \mathbb{R}^n.$$

Then the inequalities (1.1) and (1.2) can be naturally extended as

$$\|2u \circ A/2 - u \circ P_1 - u \circ P_2\| \leqslant \varepsilon, \tag{1.3}$$

$$\|2u \circ A/2 - v \circ P_1 - w \circ P_2\| \leqslant \varepsilon, \tag{1.4}$$

where $u \circ A/2$, $u \circ P_1$, and $u \circ P_2$ are the pullbacks of u by $A/2$, P_1, and P_2, respectively, and $\|v\| \leqslant \epsilon$ means that $|\langle v, \varphi\rangle| \leqslant \epsilon \|\varphi\|_{L^1}$ for all test functions φ.

As results, we prove that every solution u of the inequality (1.3) can be written uniquely in the form

$$u = a \cdot x + c + h(x), \quad a \in \mathbb{C}^n,\ c \in \mathbb{C},$$

where $h(x)$ is a bounded measurable function with $\|h\|_{L^\infty} \leqslant \frac{3}{2}\varepsilon$.

Also, every solution u, v, and w of the inequality (1.4) can be written uniquely in the form

$$u = a \cdot x + c + h_1(x), \qquad v = a \cdot x + c_1 + h_2(x), \qquad w = a \cdot x + c_2 + h_3(x),$$

where $a \in \mathbb{C}^n$, c, c_1, and c_2 are some complex numbers and $h_1(x)$, $h_2(x)$, and $h_3(x)$ are bounded measurable functions satisfying $\|h_1(x)\|_{L^\infty} \leqslant \frac{3}{2}\varepsilon$, $\|h_2(x)\|_{L^\infty} \leqslant 4\varepsilon$, $\|h_3(x)\|_{L^\infty} \leqslant 4\varepsilon$.

2. Distributions and hyperfunctions

We introduce some space of generalized functions such as the space $\mathcal{S}'$ of tempered distributions and the space $\mathcal{F}'$ of Fourier hyperfunctions which is a natural generalization of $\mathcal{S}'$. Here we use the multi-index notation, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of non-negative integers and $\partial_j = \partial/\partial x_j$.

Definition 2.1 [4,17]. We denote by $\mathcal{S}$ or $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all infinitely differentiable functions φ in $\mathbb{R}^n$ satisfying

$$\|\varphi\|_{\alpha,\beta} = \sup_x \bigl|x^\alpha \partial^\beta \varphi(x)\bigr| < \infty \tag{2.1}$$

for all $\alpha, \beta \in \mathbb{N}_0^n$, equipped with the topology defined by the seminorms $\|\cdot\|_{\alpha,\beta}$. The elements of $\mathcal{S}$ are called rapidly decreasing functions and the elements of the dual space $\mathcal{S}'$ are called tempered distributions.

Imposing growth conditions on $\|\cdot\|_{\alpha,\beta}$ in (2.1) Sato and Kawai introduced the space $\mathcal{F}$ of test functions for the Fourier hyperfunctions as follows:

Definition 2.2 [5]. We denote by $\mathcal{F}$ or $\mathcal{F}(\mathbb{R}^n)$ the Sato space of all infinitely differentiable function φ in $\mathbb{R}^n$ such that

$$\|\varphi\|_{h,k} = \sup_{x\in\mathbb{R}^n,\ \alpha\in\mathbb{N}_0^n} \frac{|\partial^\alpha \varphi(x)| \exp k|x|}{h^{|\alpha|}\alpha!} < \infty \tag{2.2}$$

for some $h, k > 0$. We say that $\varphi_j \to 0$ as $j \to \infty$ if $\|\varphi_j\|_{h,k} \to 0$ as $j \to \infty$ for some h, $k > 0$, and denote by $\mathcal{F}'$ the strong dual of $\mathcal{F}$ and call its elements Fourier hyperfunctions.

It can be verified that (2.2) is equivalent to

$$\|\varphi\|_{\alpha,\beta} = \sup_x \bigl|x^\alpha \partial^\beta \varphi(x)\bigr| \leqslant C A^{|\alpha|} B^{|\beta|} \alpha! \beta!$$

for some positive constant A, B, and C depending only on φ. It is easy to see the following topological inclusion:

$$\mathcal{F} \hookrightarrow \mathcal{S}, \qquad \mathcal{S}' \hookrightarrow \mathcal{F}'.$$

From now on a test function means an element in the Schwartz space $\mathcal{S}$ or the Sato space $\mathcal{F}$ and a generalized function means a tempered distribution or a Fourier hyperfunction.

We employ the n-dimensional heat kernel, this is, the fundamental solution $E(x,t)$ of the heat operator $\partial_t - \Delta_x$ in $\mathbb{R}_x^n \times \mathbb{R}_t^+$ given by

$$E_t(x) = E(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp\bigl(-|x|^2/4t\bigr), & t > 0, \\ 0, & t \leqslant 0. \end{cases}$$

The semigroup property

$$(E_s * E_t)(x) = E_{s+t}(x) \tag{2.3}$$

of the heat kernel will be very useful later.

Since for each $t > 0$, $E(\cdot,t)$ belongs to the Sato space $\mathcal{F}$ and the Schwartz space $\mathcal{S}$, the convolution

$$Gu(x,t) = (u * E)(x,t) = u_y\big(E(x-y,t)\big), \quad x \in \mathbb{R}^n,\ t > 0$$

is well defined for each generalized function u, which is called the Gauss transform of u.

As a matter of fact it is shown in [14] that the Gauss transform $Gu(x,t)$ of u is a C^∞ solution of the heat equation and $Gu(x,t)$ converges to u as $t \to 0^+$ in the following sense of generalized functions: for all test function φ,

$$\big\langle Gu(\cdot,t),\varphi\big\rangle = \int Gu(x,t)\varphi(x)\,dx \to \langle u,\varphi\rangle \quad \text{as } t \to 0^+.$$

3. Main theorems

In this section we consider the stabilities of Jensen equation (1.3) and Jensen–Pexider equation (1.4). We refer to [9, Chapter VI], for pullbacks of generalized functions. As a matter of fact, the pullbacks $u \circ \frac{A}{2}$, $u \circ P_1$ and $u \circ P_2$ involved in (1.3) and (1.4) can be written in a more transparent way as follows:

$$\begin{aligned}
&\big\langle u \circ A/2, \varphi(x,y)\big\rangle = \Big\langle u_x, 2^n \int \varphi(2x-y,y)\,dy\Big\rangle,\\
&\big\langle u \circ P_1, \varphi(x,y)\big\rangle = \Big\langle u_x, \int \varphi(x,y)\,dy\Big\rangle,\\
&\big\langle u \circ P_2, \varphi(x,y)\big\rangle = \Big\langle u_y, \int \varphi(x,y)\,dx\Big\rangle
\end{aligned}$$

for all test functions $\varphi(x,y)$ defined on $\mathbb{R}^{2n}$. It will be verified that by convolving $E_t(x)E_s(y)$ in each side of the inequalities (1.3) and (1.4) we have the following inequalities for smooth functions

$$\left\| 2f\left(\frac{x+y}{2}, \frac{t+s}{4}\right) - f(x,t) - f(y,s)\right\|_{L^\infty} \leqslant \varepsilon, \tag{3.1}$$

$$\left\| 2f\left(\frac{x+y}{2}, \frac{t+s}{4}\right) - g(x,t) - h(y,s)\right\|_{L^\infty} \leqslant \varepsilon \tag{3.2}$$

for $x, y \in \mathbb{R}^n$, $t, s > 0$. From now on we call a function L defined on a group is called *an additive function* if it satisfies the Cauchy equation

$$L(x+y) = L(x) + L(y).$$

We first consider the inequalities (3.1) and (3.2).

Lemma 3.1. *Let $f : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ be a continuous function satisfying the inequality*

$$\left\| 2f\left(\frac{x+y}{2}, \frac{t+s}{4}\right) - f(x,t) - f(y,s)\right\|_{L^\infty} \leqslant \varepsilon.$$

Then there exist a unique additive function $L(x)$ defined on $\mathbb{R}^n$ and a constant c such that

$$\left\| f(x,t) - L(x) - c \right\|_{L^\infty} \leqslant \frac{3}{2}\varepsilon.$$

Proof. Let $F(x,t) = 2f(x/2, t/4)$. Then we have

$$\left\| F(x+y, t+s) - f(x,t) - f(y,s) \right\|_{L^\infty} \leqslant \varepsilon. \tag{3.3}$$

Putting $y = 0$ in (3.3), it is easy to see that $c := \limsup_{s\to 0^+} f(0,s)$ exists. Thus putting $y = 0$ and letting $s \to 0^+$ in (3.3) we obtain that

$$\left\| F(x,t) - f(x,t) - c \right\|_{L^\infty} \leqslant \varepsilon. \tag{3.4}$$

Now it follows from (3.3) and (3.4) that

$$\left\| F(x+y, t+s) - F(x,t) - F(y,s) + 2c \right\|_{L^\infty} \leqslant 3\varepsilon. \tag{3.5}$$

Thus it follows from Theorem 1.1, there exists a unique mapping $L(x,t)$ satisfying

$$L(x+y, t+s) - L(x,t) - L(y,s) = 0, \tag{3.6}$$

$$\left\| F(x,t) - L(x,t) - 2c \right\|_{L^\infty} \leqslant 3\varepsilon. \tag{3.7}$$

Replacing x, t in (3.7) by $2x, 4t$ and dividing by 2 in the result we have in view of (3.6)

$$\left\| f(x,t) - L(x,2t) - c \right\|_{L^\infty} \leqslant \frac{3}{2}\varepsilon. \tag{3.8}$$

Applying the triangle inequalities in the inequalities (3.4), (3.7), and (3.8) it follows that

$$\left\| L(x,2t) - L(x,t) \right\|_{L^\infty} \leqslant \frac{11}{2}\varepsilon. \tag{3.9}$$

Putting $x = 0$ in (3.9) and using additive property of $L(0,t)$ we have

$$\left\| L(0,t) \right\|_{L^\infty} \leqslant \frac{11}{2}\varepsilon$$

and hence

$$L(0,t) = 0 \quad \text{for all } t > 0.$$

Putting $y = 0$ in (3.6) we have $L(x, t+s) = L(x,t)$ for all $x \in \mathbb{R}^n$, t, $s > 0$ and hence $L(x,t)$ is independent of $t > 0$. Thus we obtain the result from Eq. (3.7). □

Lemma 3.2. *Let $f, g, h : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ be continuous functions satisfying the inequality*

$$\left\| 2f\left(\frac{x+y}{2}, \frac{t+s}{4}\right) - g(x,t) - h(y,s) \right\|_{L^\infty} \leqslant \varepsilon.$$

Then there exist a unique additive function $L(x,t)$ and constants c, c_1, and c_2 such that

$$\left\| f(x,t) - L(x,2t) - c \right\|_{L^\infty} \leqslant \frac{3}{2}\varepsilon, \qquad \left\| g(x,t) - L(x,t) - c_1 \right\|_{L^\infty} \leqslant 4\varepsilon,$$

$$\left\| h(x,t) - L(x,t) - c_2 \right\|_{L^\infty} \leqslant 4\varepsilon,$$

where $c = \frac{1}{2}(c_1 + c_2)$.

Proof. Letting $F(x,t) = 2f(x/2, t/4)$ we have the following inequality for the Pexider equation

$$\big\|F(x+y, t+s) - g(x,t) - h(y,s)\big\|_{L^\infty} \leqslant \varepsilon. \tag{3.10}$$

Let $x = y = 0$ in (3.10). Then it follows from the continuity of the functions F that there exist sequences t_n, $n = 1, 2, \ldots$, and s_n, $n = 1, 2, \ldots$, of positive numbers which tend to 0 as $n \to \infty$ such that

$$c_1 := \lim_{n\to\infty} g(0, t_n), \qquad c_2 := \lim_{n\to\infty} h(0, s_n)$$

exist. Letting $y = 0$ and $s = s_n \to 0^+$ in (3.10) we have

$$\big\|F(x,t) - g(x,t) - c_2\big\|_{L^\infty} \leqslant \varepsilon. \tag{3.11}$$

Similarly, we have

$$\big\|F(x,t) - h(x,t) - c_1\big\|_{L^\infty} \leqslant \varepsilon. \tag{3.12}$$

Thus it follows from (3.10), (3.11), and (3.12) that

$$\big\|F(x+y, t+s) - F(x,t) - F(y,s) + c_1 + c_2\big\|_{L^\infty} \leqslant 3\varepsilon.$$

Thus there exists a unique function $L(x,t)$ such that

$$L(x+y, t+s) - L(x,t) - L(y,s) = 0, \tag{3.13}$$

$$\big\|F(x,t) - L(x,t) - c_1 - c_2\big\|_{L^\infty} \leqslant 3\varepsilon. \tag{3.14}$$

From (3.11), (3.12), and (3.14) we have

$$\big\|g(x,t) - L(x,t) - c_1\big\|_{L^\infty} \leqslant 4\varepsilon, \tag{3.15}$$

$$\big\|h(x,t) - L(x,t) - c_2\big\|_{L^\infty} \leqslant 4\varepsilon. \tag{3.16}$$

Replacing x, t by $2x, 4t$ in (3.14) and dividing by 2 we have

$$\big\|f(x,t) - L(x,2t) - c\big\|_{L^\infty} \leqslant \frac{3}{2}\varepsilon, \tag{3.17}$$

where $c = \frac{1}{2}(c_1 + c_2)$. This completes the proof. □

We now state and prove the stabilities of the Jensen equation and the Jensen–Pexider equation in the spaces of tempered distributions and Fourier hyperfunctions.

Theorem 3.3. *Let u be a tempered distribution or a Fourier hyperfunction satisfying*

$$\|2u \circ A/2 - u \circ P_1 - u \circ P_2\| \leqslant \varepsilon. \tag{3.18}$$

Then u can be written uniquely in the form

$$u = a \cdot x + c + h(x), \quad a \in \mathbb{C}^n,\ c \in \mathbb{C},$$

where $h(x)$ is a bounded measurable function such that $\|h\|_{L^\infty} \leqslant \frac{3}{2}\varepsilon$.

Proof. Convolving in the each side of (3.18) the tensor product $E_t(x)E_s(y)$ of n-dimensional heat kernels, we have in view of (2.3)

$$\begin{aligned}
&\big[(2u\circ A/2 - u\circ P_1 - u\circ P_2) * \big(E_t(x)E_s(y)\big)\big](\xi,\eta)\\
&\quad= \big[(2u\circ A/2) * \big(E_t(x)E_s(y)\big)\big](\xi,\eta) - \big[(u\circ P_1) * \big(E_t(x)E_s(y)\big)\big](\xi,\eta)\\
&\qquad - \big[(u\circ P_2) * \big(E_t(x)E_s(y)\big)\big](\xi,\eta)\\
&\quad:= \mathbf{I} - \mathbf{II} - \mathbf{III},\\
&\mathbf{I} = \big\langle 2u\circ A/2,\, E_t(\xi - x)E_s(\eta - y)\big\rangle = 2^{n+1}\Big\langle u_x,\, \int E_t(\xi - 2x + y)E_s(\eta - y)\,dy\Big\rangle\\
&\quad= 2^{n+1}\big\langle u_x,\, (E_t * E_s)(\xi + \eta - 2x)\big\rangle = 2^{n+1}\big\langle u_x,\, E_{t+s}(\xi + \eta - 2x)\big\rangle\\
&\quad= 2^{n+1}\Big\langle u_x,\, 2^{-n}E_{(t+s)/4}\Big(\frac{\xi+\eta}{2} - x\Big)\Big\rangle = 2(u * E)\Big(\frac{\xi+\eta}{2}, \frac{t+s}{4}\Big).
\end{aligned}$$

Similarly, we have

$$\mathbf{II} = Gu(\xi, t), \qquad \mathbf{III} = Gu(\eta, s).$$

Thus the inequality (3.18) is converted to

$$\Big\| 2Gu\Big(\frac{x+y}{2}, \frac{t+s}{4}\Big) - Gu(x,t) - Gu(y,s)\Big\|_{L^\infty} \leqslant \varepsilon.$$

By Lemma 3.1 there exist a unique additive function $L(x)$ and constant c such that

$$\big\|Gu(x,t) - L(x) - c\big\|_{L^\infty} \leqslant \frac{3}{2}\varepsilon. \tag{3.19}$$

Since the Gauss transform Gu is a smooth function, and $L(x,t)$ is obtained by the uniform limit of the sequence $2^{-n}Gu(2^n x, 2^n t)$ as seen in the proof of Theorem 1.1 L must be a continuous function and $L(x) = a\cdot x$ for some $a \in \mathbb{C}^n$. By letting $t \to 0^+$ in (3.19), it follows that

$$\|u - a\cdot x - c\| \leqslant \frac{3}{2}\varepsilon. \tag{3.20}$$

Now the inequality (3.20) implies that $h(x) := u - a\cdot x - c$ belongs to $(L^1)' = L^\infty$. Thus all the solutions u can be written uniquely in the form $u = a\cdot x + c + h(x)$, where $\|h\|_{L^\infty} \leqslant \frac{3}{2}\varepsilon$. This completes the proof. □

Theorem 3.4. *Let u, v and w be tempered distributions or Fourier hyperfunctions satisfying*

$$\|2u\circ A/2 - v\circ P_1 - w\circ P_2\| \leqslant \varepsilon. \tag{3.21}$$

Then u, v and w can be written uniquely in the form

$$u = a\cdot x + c + h_1(x), \qquad v = a\cdot x + c_1 + h_2(x), \qquad w = a\cdot x + c_2 + h_3(x),$$

where $a \in \mathbb{C}^n$, $c, c_1, c_2 \in \mathbb{C}$ with $c = \frac{1}{2}(c_1 + c_2)$ and $h_1(x)$, $h_2(x)$, and $h_3(x)$ are bounded measurable functions such that $\|h_1(x)\|_{L^\infty} \leqslant \frac{3}{2}\varepsilon$, $\|h_2(x)\|_{L^\infty} \leqslant 4\varepsilon$, $\|h_3(x)\|_{L^\infty} \leqslant 4\varepsilon$.

Proof. As in the proof of Theorem 3.3, by convolving in (3.21) the tensor product $E_t(x)E_s(y)$ of heat kernels, the inequality (3.21) is converted to

$$\left\| 2Gu\left(\frac{x+y}{2}, \frac{t+s}{4}\right) - Gv(x,t) - Gw(y,s)\right\|_{L^\infty} \leqslant \varepsilon.$$

By Lemma 3.2 there exists an additive function $L(x,t)$ such that

$$\left\|Gu(x,t) - L(x,2t) - c\right\|_{L^\infty} \leqslant 3\varepsilon, \tag{3.22}$$

$$\left\|Gv(x,t) - L(x,t) - c_1\right\|_{L^\infty} \leqslant 4\varepsilon, \tag{3.23}$$

$$\left\|Gw(x,t) - L(x,t) - c_2\right\|_{L^\infty} \leqslant 4\varepsilon, \tag{3.24}$$

where $c = \frac{1}{2}(c_1 + c_2)$.

Since $L(x,t)$ is obtained by the uniform limit of the sequence $2^{-n}Gu(2^n x, 2^n t)$ as seen in the proof of Theorem 1.1 L is a continuous function. Thus we must have $L(x,t) = a \cdot x + bt$ for some $a \in \mathbb{C}^n$, $b \in \mathbb{C}$. Letting $t \to 0^+$ in (3.22), (3.23), and (3.24) we get the result. □

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