

# An intersection theorem in $L$ -convex spaces with applications<sup>☆</sup>

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## Abstract

A new intersection theorem is obtained in  $L$ -convex spaces without linear structure. As its applications, a fixed point theorem, a maximal element theorem, a coincidence theorem, some new minimax inequalities and a saddle point theorem are given in  $L$ -convex spaces. Our results generalize many known theorems in the literature.

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## 1. Introduction

In [16, Theorem 2], Zhang and Ma establish the following intersection theorem.

**Theorem A** (Zhang and Ma [16]). *Let  $E$  and  $F$  be Hausdorff topological vector spaces, let  $X \subset E$ ,  $Y \subset F$  be nonempty convex subsets, and let  $A$  be a subset of  $X \times Y$  such that*

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- (i) for each  $x \in X$ , the set  $\{y \in Y: (x, y) \notin A\}$  is convex or empty;
- (ii) for each  $y \in Y$ , there exists a closed subset  $X_y \subset X$  such that the set  $\{x \in X: (x, y) \in A\} \subset X_y$ .

Suppose that there exists a subset  $B$  of  $A$  and a compact convex subset  $K$  of  $X$  such that  $B$  is closed in  $X \times Y$  and

- (iii) for each  $y \in Y$ , the set  $\{x \in K: (x, y) \in B\}$  is nonempty and convex. Then

$$\bigcap_{y \in Y} X_y \cap K \neq \emptyset.$$

In this paper, we obtain a new intersection theorem in  $L$ -convex spaces, which generalizes and improves the above theorem. As its applications, we give a fixed point theorem, a maximal element theorem, a coincidence theorem, some new minimax inequalities and a saddle point theorem in  $L$ -convex spaces. Our results generalize many well-known results, see for example [3,6–10,15,16].

## 2. Preliminaries

Throughout this paper, all topological spaces are assumed to be Hausdorff.

Let  $X$  be a set. We shall denote by  $2^X$  the family of all subsets of  $X$ , by  $\langle X \rangle$  the family of nonempty finite subsets of  $X$ . Let  $\Delta_n$  denote the standard  $n$ -dimensional simplex with vertices  $e_0, \dots, e_n$ . For any  $A \in \langle X \rangle$ , we denote by  $|A|$  the cardinality of  $A$ .

In 1998, Ben-Ei-Mechaiekh et al. [2] introduced the following notion of  $L$ -convex space:

An  $L$ -convexity structure on a topological space  $X$  is given a set-valued mapping  $\Gamma: \langle X \rangle \rightarrow 2^X$  with nonempty values such that for each  $A \in \langle X \rangle$  with  $|A| = n + 1$ , there exists a continuous mapping  $\varphi_A: \Delta_n \rightarrow \Gamma(A)$  such that  $B \in \langle A \rangle$  with  $|B| = J + 1$  implies  $\varphi_A(\Delta_J) \subset \Gamma(B)$ , where  $\Delta_J$  denotes the face of  $\Delta_n$  corresponding to  $B$ .

The pair  $(X, \Gamma)$  is called an  $L$ -convex space, and a set  $M \subset X$  is said to be  $L$ -convex if for each  $A \in \langle M \rangle$ ,  $\Gamma(A) \subset M$ . For a nonempty subset  $E$  of  $X$ , we define the  $L$ -convex hull of  $E$ , denoted by  $L - \text{co } E$ , as

$$L - \text{co } E = \bigcap \{D \subset X: E \subset D \text{ and } D \text{ is } L\text{-convex}\}.$$

Clearly,  $L - \text{co } E$  is  $L$ -convex and is the smallest  $L$ -convex set containing  $E$ .

Particular forms of  $L$ -convex spaces can be found in [4] and references therein.

Let  $(X, \Gamma)$  be an  $L$ -convex space,  $Y$  be a topological space, and  $f: X \times Y \rightarrow \mathbf{R}$  be a real-valued function. For each  $y \in Y$ ,  $f(x, y)$  is said to be  $L$ -quasiconcave (respectively  $L$ -quasiconvex) in  $x$  if the set  $\{x \in X: f(x, y) > t\}$  (respectively  $\{x \in X: f(x, y) < t\}$ ) is  $L$ -convex for all  $t \in \mathbf{R}$ .

A topological space  $X$  is said to be acyclic if all of its reduced Čech homology groups over rationals vanish. In particular, any contractible space is acyclic, and thus any non-empty convex or star-shaped set is acyclic. A nonempty subset  $D$  of a topological space  $X$  is said to be compactly closed (respectively compactly open) in  $X$  if for each compact subset  $C \subset X$ ,  $D \cap C$  is closed (respectively open) in  $C$ . The compact closure and the compact interior of  $D$  (see [5]) are defined by

$$\begin{aligned}\text{ccl } D &= \bigcap \{G : D \subset G \text{ and } G \text{ is compactly closed in } X\}, \quad \text{and} \\ \text{cint } D &= \bigcup \{G : G \subset D \text{ and } G \text{ is compactly open in } X\},\end{aligned}$$

respectively. It is easy to see that  $\text{ccl } D$  (respectively  $\text{cint } D$ ) is compactly closed (respectively compactly open) in  $X$  and for each nonempty compact subset  $C$  of  $X$  with  $D \cap C \neq \emptyset$ , we have  $(\text{ccl } D) \cap C = \text{cl}_C(C \cap D)$  and  $(\text{cint } D) \cap C = \text{int}_C(C \cap D)$ , where  $\text{cl}_C(C \cap D)$  and  $\text{int}_C(C \cap D)$  denote the closure and the interior of  $C \cap D$  in  $C$ , respectively.

Let  $X$  and  $Y$  be two topological spaces. Let  $T : X \rightarrow 2^Y$  be a set-valued mapping. The set-valued mappings  $T^{-1} : Y \rightarrow 2^X$  are defined, respectively, by  $T^{-1}(y) = \{x \in X : y \in T(x)\}$ ,  $T^*(y) = \{x \in X : y \notin T(x)\}$  for each  $y \in Y$ .  $T$  is said to be upper (respectively lower) semicontinuous if for each open (respectively closed) subset  $G$  of  $Y$ , the set  $\{x \in X : T(x) \subset G\}$  is open (respectively closed) in  $X$ .

**Lemma 1** (Aubin and Ekeland [1]).

- (i) Let  $X$  be a topological space and  $Y$  be a compact topological space. Suppose that  $T : X \rightarrow 2^Y$  is a set-valued mapping with the graph  $\{(x, y) \in X \times Y : y \in T(x)\}$  of  $T$  is closed in  $X \times Y$ . Then  $T$  is upper semicontinuous.
- (ii) Let  $X$  and  $Y$  be two topological spaces, and  $F : X \rightarrow 2^Y$  be an upper semicontinuous set-valued mapping with closed values. Then the graph of  $F$  is closed in  $X \times Y$ .

**Definition 1** (Ding [5]). Let  $X$  and  $Y$  be two topological spaces,  $F : X \rightarrow 2^Y$  be a set-valued mapping.

- (i)  $F$  is called transfer compactly closed-valued if for each  $x \in X$  and for each nonempty compact subset  $C$  of  $Y$  with  $F(x) \cap C \neq \emptyset$ ,  $y \notin F(x) \cap C$  implies that there exists an  $x'$  such that  $y \notin \text{cl}_C(F(x') \cap C)$ .
- (ii)  $F$  is called transfer compactly open-valued if for each  $x \in X$  and for each nonempty compact subset  $C$  of  $Y$  with  $F(x) \cap C \neq \emptyset$ ,  $y \in F(x) \cap C$  implies that there exists an  $x'$  such that  $y \in \text{int}_C(F(x') \cap C)$ .

By the above definition and [11, Lemma 2.1], we see that the following lemma holds.

**Lemma 2** (Lin et al. [11]).

- (i)  $F : X \rightarrow 2^Y$  is transfer compactly open-valued if and only if  $\bigcup_{x \in X} F(x) = \bigcup_{x \in X} \text{cint } F(x)$ ;

- (ii)  $F : X \rightarrow 2^Y$  is transfer compactly open-valued if and only if  $G : X \rightarrow 2^Y$  defined by  $G(x) = Y \setminus F(x)$  for each  $x \in X$ , is transfer compactly closed-valued.

**Definition 2** (Ding [5]). Let  $X$  and  $Y$  be two topological spaces, and  $f : X \times Y \rightarrow \mathbf{R}$  be a real-valued function. For some  $\lambda \in \mathbf{R}$ ,  $f(x, y)$  is said to be  $\lambda$ -transfer compactly lower (respectively upper) semicontinuous in  $y$  if for each compact subset  $C$  of  $Y$  and for each  $y \in C$ , there exists an  $x \in X$  such that  $f(x, y) > \lambda$  (respectively  $f(x, y) < \lambda$ ) implies that there exists a relatively open neighborhood  $N(y)$  of  $y$  in  $C$  and an  $x' \in X$  such that  $f(x', z) > \lambda$  (respectively  $f(x', z) < \lambda$ ) for all  $z \in N(y)$ .

**Lemma 3** (Ding [5]). For some  $\lambda \in \mathbf{R}$ ,  $f : X \times Y \rightarrow \mathbf{R}$  is  $\lambda$ -transfer compactly lower relatively (respectively upper) semicontinuous in  $y$  if and only if the set-valued mapping  $F : X \rightarrow 2^Y$  defined by  $F(x) = \{y \in Y : f(x, y) \leq \lambda\}$  (respectively  $F(x) = \{y \in Y : f(x, y) \geq \lambda\}$ ) for each  $x \in X$ , is transfer compactly closed-valued.

**Lemma 4** (Shioji [13]). Let  $\Delta_n$  be a standard  $n$ -dimensional simplex with the Euclidean topology and  $X$  be a compact topological space. Let  $\varphi : X \rightarrow \Delta_n$  be a single-valued continuous mapping and  $T : \Delta_n \rightarrow 2^X$  be an upper semicontinuous set-valued mapping such that  $T(\mu)$  is nonempty compact acyclic subset of  $X$  for each  $\mu \in \Delta_n$ . Then there exists  $\mu^* \in \Delta_n$  such that  $\mu^* \in \varphi(T(\mu^*))$ .

### 3. Main results

Our main result is the following Theorem 1 which is needed in this paper.

**Theorem 1.** Let  $X$  be a topological space,  $(Y, \Gamma)$  be an  $L$ -convex space,  $C : Y \rightarrow 2^X$  be a set-valued mapping, and let  $M, N$  be two subsets of  $X \times Y$ . Suppose the following conditions are fulfilled:

- (i)  $C$  is transfer compactly closed-valued;  
 (ii) for each  $y \in Y$ , the set  $\{x \in X : (x, y) \in N\} \subset C(y)$ .

Suppose also that there exists a subset  $P$  of  $M$  and a compact subset  $K$  of  $X$  such that  $P$  is closed in  $X \times Y$ , and

- (iii) for each  $x \in K$  and  $A \in \langle \{y \in Y : (x, y) \notin N\} \rangle$ ,  $\Gamma(A) \subset \{y \in Y : (x, y) \notin M\}$ ;  
 (iv) for each  $y \in Y$ , the set  $\{x \in K : (x, y) \in P\}$  is nonempty acyclic.

Then

$$\bigcap_{y \in Y} C(y) \cap K \neq \emptyset.$$

**Proof.** We show that the family  $\{\text{ccl } C(y) \cap K : y \in Y\}$  has the finite intersection property. If there exists  $A = \{y_0, \dots, y_n\} \in \langle Y \rangle$  such that  $\bigcap_{y \in A} \text{ccl } C(y) \cap K = \emptyset$ , then it follows that

$$K \subset \bigcup \{(X \setminus \text{ccl } C(y)) \cap K : y \in A\}.$$

By the definition of an  $L$ -convex space, there exists a continuous mapping  $g : \Delta_n \rightarrow \Gamma(A)$  such that for each  $B = \{y_{i_0}, \dots, y_{i_k}\} \in \langle A \rangle$ , we have

$$g(\Delta_k) \subset \Gamma(B). \quad (1)$$

Since each  $\text{ccl } C(y)$  is compactly closed, we can assume that there exists a continuous partition of unity  $\{\beta_0, \dots, \beta_n\}$  subordinated to the open covering  $\{(X \setminus \text{ccl } C(y)) \cap K : y \in A\}$ , that is, for each  $i \in \{0, \dots, n\}$ ,  $\beta_i : K \rightarrow [0, 1]$  is continuous such that for each  $x \in K$ ,  $\sum_{i=0}^n \beta_i(x) = 1$  and for each  $i \in \{0, \dots, n\}$ ,  $\beta_i(x) = 0$  for  $x \notin (X \setminus \text{ccl } C(y_i)) \cap K$ . In other words, for each  $i \in \{0, \dots, n\}$ ,  $\beta_i(x) \neq 0$  implies that

$$x \in (X \setminus \text{ccl } C(y_i)) \cap K \subset \{x \in X : (x, y_i) \notin N\}. \quad (2)$$

Define a mapping  $\varphi : K \rightarrow \Delta_n$  by

$$\varphi(x) = \sum_{i=0}^n \beta_i(x) e_i \quad \text{for all } x \in K.$$

Then clearly  $\varphi$  is continuous and for each  $x \in K$ ,  $\varphi(x) \in \Delta_{|J(x)|-1}$ ,  $\{y_i : i \in J(x)\} \in \langle \{y \in Y : (x, y) \notin N\} \rangle$  by (2), where  $J(x) = \{i \in \{0, \dots, n\} : \beta_i(x) \neq 0\}$ . We define  $f : K \rightarrow Y$  as follows:

$$f(x) = g(\varphi(x)) \quad \text{for all } x \in K.$$

By (iii) and (1),  $f(x) = g(\varphi(x)) \in g(\Delta_{|J(x)|-1}) \subset \Gamma(y_i : i \in J(x)) \subset \{y \in Y : (x, y) \notin M\}$  for all  $x \in K$ . This shows that

$$(x, f(x)) \in M \quad \text{for all } x \in K. \quad (3)$$

On the other hand, we define a set-valued mapping  $G : Y \rightarrow 2^K$  by

$$G(y) = \{x \in K : (x, y) \in P\} \quad \text{for all } y \in Y.$$

By (iv),  $G(y)$  is nonempty acyclic for all  $y \in Y$ . Since  $P$  is closed in  $X \times Y$ , it is easy to see that each  $G(y)$  is closed in  $K$  and the graph of  $G$  is closed in  $Y \times K$ ; thus,  $G$  is an upper semicontinuous set-valued mapping defined on  $Y$ . Consequently, so is the mapping  $F : \Delta_n \rightarrow 2^K$  defined by  $F(\mu) = G(g(\mu))$  for all  $\mu \in \Delta_n$ . By Lemma 4, there exists a point  $\bar{\mu} \in \Delta_n$  such that  $\bar{\mu} \in \varphi(F(\bar{\mu})) = \varphi(G(g(\bar{\mu})))$ , and so there is a point  $\bar{x} \in G(g(\bar{\mu})) \subset K$  such that  $\bar{\mu} = \varphi(\bar{x})$ . Let  $\bar{y} = g(\bar{\mu})$ , then  $\bar{y} = g(\varphi(\bar{x})) = f(\bar{x})$  and  $\bar{x} \in G(\bar{y})$ , i.e.,

$$(\bar{x}, f(\bar{x})) = (\bar{x}, \bar{y}) \in P \subset M.$$

This contradicts (3). Therefore the family  $\{\text{ccl } C(y) \cap K : y \in Y\}$  has the finite intersection property. Since  $K$  is compact and each  $\text{ccl } C(y)$  is compactly closed, we must have  $\bigcap_{y \in Y} \text{ccl } C(y) \cap K \neq \emptyset$ . Now we show that  $\bigcap_{y \in Y} C(y) \cap K = \bigcap_{y \in Y} \text{ccl } C(y) \cap K$ . Clearly,  $\bigcap_{y \in Y} C(y) \cap K \subset \bigcap_{y \in Y} \text{ccl } C(y) \cap K$ . If  $\bigcap_{y \in Y} \text{ccl } C(y) \cap K \not\subset \bigcap_{y \in Y} C(y) \cap K$ ,

then there exists an  $x \in \bigcap_{y \in Y} \text{ccl} C(y) \cap K = \bigcap_{y \in Y} \text{cl}_K(C(y) \cap K)$  and a  $y \in Y$  such that  $x \notin C(y) \cap K$ . Since  $C$  is transfer compactly closed-valued, there exists  $y'$  such that  $x \notin \text{cl}_K(C(y') \cap K)$ , which is a contradiction. Hence  $\bigcap_{y \in Y} C(y) \cap K = \bigcap_{y \in Y} \text{ccl} C(y) \cap K \neq \emptyset$ . This completes the proof of Theorem 1.  $\square$

### Remark 1.

- (1) Note that the following two conditions are special cases of condition (iii) of Theorem 1:
  - (iii)(a) for each  $x \in K$ ,  $L - \text{co}\{y \in Y: (x, y) \notin N\} \subset \{y \in Y: (x, y) \notin M\}$ ;
  - (iii)(b)  $M \subset N$  and for each  $x \in K$ ,  $\{y \in Y: (x, y) \notin M\}$  is  $L$ -convex.
- (2) Theorem 1 generalizes Theorem A and [15, Theorem 1] in several ways.

**Theorem 2.** Let  $(X, \Gamma)$  be an  $L$ -convex space,  $T : X \rightarrow 2^X$  be a set-valued mapping, and let  $M, N$  be two subsets of  $X \times X$ . Suppose the following conditions are fulfilled:

- (i)  $T$  is transfer compactly open-valued;
- (ii) for each  $y \in X$ , the set  $T(y) \subset \{x \in X: (x, y) \notin N\}$ ;
- (iii) for each  $A \in \langle X \rangle$ ,  $\bigcup_{y \in \Gamma(A)} \{x \in X: (x, y) \in M\} \subset \bigcup_{y \in A} \{x \in X: (x, y) \in N\}$ .

Suppose also that there exists a subset  $P$  of  $M$  and a compact subset  $K$  of  $X$  such that  $P$  is closed in  $X \times X$ , and

- (iv) for each  $x \in K$ ,  $T^{-1}(x) \neq \emptyset$ ;
- (v) for each  $y \in X$ , the set  $\{x \in K: (x, y) \in P\}$  is nonempty acyclic.

Then there exists  $y_0 \in X$  such that  $y_0 \in T(y_0)$ .

**Proof.** If the conclusion of Theorem 2 is false, then for each  $y \in X$ ,  $y \notin T(y)$ . Define a set-valued mapping  $C : X \rightarrow 2^X$  by

$$C(y) = \{x \in X: x \notin T(y)\} \quad \text{for all } y \in X.$$

Then it follows from (i) and (ii) that  $C : X \rightarrow 2^X$  is transfer compactly closed-valued and for each  $y \in X$ ,  $\{x \in X: (x, y) \in N\} \subset C(y)$ . For each  $x \in K$ , let  $A \in \langle \{y \in X: (x, y) \notin N\} \rangle$ , then  $A \subset X \setminus \{y \in X: (x, y) \in N\}$ , that is,  $A \cap \{y \in X: (x, y) \in N\} = \emptyset$ . Therefore  $x \notin \bigcup_{y \in A} \{x \in X: (x, y) \in N\}$ . By (iii),  $x \notin \bigcup_{y \in \Gamma(A)} \{x \in X: (x, y) \in M\}$ . This shows that  $\Gamma(A) \cap \{y \in X: (x, y) \in M\} = \emptyset$ . Therefore  $\Gamma(A) \subset \{y \in Y: (x, y) \notin M\}$ . Then it follows from Theorem 1 that  $\bigcap_{y \in X} C(y) \cap K \neq \emptyset$ . Take any  $x^* \in \bigcap_{y \in X} C(y) \cap K$ , then for each  $y \in X$ ,  $x^* \in C(y) \cap K$ . It follows that  $x^* \notin T(y)$  for each  $y \in X$ , that is, for each  $y \in X$ ,  $y \notin T^{-1}(x^*)$ . Therefore  $T^{-1}(x^*) = \emptyset$ , which contradicts (iv). Therefore there exists  $y_0 \in X$  such that  $y_0 \in T(y_0)$ .  $\square$

As a consequence of Theorem 2, we establish the following maximal element theorem, which is equivalent to Theorem 2.

**Theorem 3.** Let  $(X, \Gamma)$  be an  $L$ -convex space,  $T : X \rightarrow 2^X$  be a set-valued mapping, and let  $M, N$  be two subsets of  $X \times X$ . Suppose the following conditions are fulfilled:

- (i)  $T$  is transfer compactly open-valued;
- (ii) for each  $y \in X$ , the set  $T(y) \subset \{x \in X : (x, y) \notin N\}$ ;
- (iii) for each  $A \in \langle X \rangle$ ,  $\bigcup_{y \in \Gamma(A)} \{x \in X : (x, y) \in M\} \subset \bigcup_{y \in A} \{x \in X : (x, y) \in N\}$ .

Suppose also that there exists a subset  $P$  of  $M$  and a compact subset  $K$  of  $X$  such that  $P$  is closed in  $X \times X$ , and

- (iv) for each  $y \in X$ ,  $y \notin T(y)$ ;
- (v) for each  $y \in X$ , the set  $\{x \in K : (x, y) \in P\}$  is nonempty acyclic.

Then there exists  $x_0 \in K$  such that  $T^{-1}(x_0) = \emptyset$ .

**Remark 2.** In view of Lemma 2, condition (i) of Theorem 2 and Theorem 3 can be replaced by the following condition:

- (i)(a)  $\bigcup_{x \in X} T(x) = \bigcup_{x \in X} \text{cint } T(x)$ .

By using Theorem 1, we obtain the following coincidence theorem.

**Theorem 4.** Let  $X$  be a topological space,  $K$  be a compact subset of  $X$ ,  $(Y, \Gamma)$  be an  $L$ -convex space. Let  $C, F : Y \rightarrow 2^X$  and  $H, T : X \rightarrow 2^Y$  be four set-valued mappings satisfying the following conditions:

- (i)  $C$  is transfer compactly closed-valued and for each  $x \in K$ ,  $C^*(x) \neq \emptyset$ ;
- (ii) the graph of  $F$  is closed in  $Y \times X$ ;
- (iii) for each  $y \in Y$ ,  $\{x \in X : y \notin H(x)\} \subset C(y)$ ;
- (iv) for each  $x \in K$  and  $A \in \{\{y \in Y : y \in H(x)\}\}$ ,  $\Gamma(A) \subset \{y \in Y : y \in T(x)\}$ ;
- (v) for each  $y \in Y$ ,  $F(y) \cap K$  is nonempty acyclic.

Then there exists a point  $x_0 \in X$  and a point  $y_0 \in Y$  such that  $x_0 \in F(y_0)$  and  $y_0 \in T(x_0)$ .

**Proof.** Now we define  $G : X \rightarrow 2^Y$  by

$$G(x) = \{y \in Y : y \in F^{-1}(x)\} \quad \text{for all } x \in X.$$

By (ii),  $G$  has a closed graph. Let  $P = \{(x, y) \in X \times Y : y \in G(x)\}$ ,  $M = \{(x, y) \in X \times Y : y \notin T(x)\}$  and  $N = \{(x, y) \in X \times Y : y \notin H(x)\}$ . By (iii), for each  $y \in Y$ ,  $\{x \in X : (x, y) \in N\} \subset C(y)$ . By (iv), for each  $x \in K$  and  $A \in \{\{y \in Y : (x, y) \notin N\}\}$ ,  $\Gamma(A) \subset \{y \in Y : (x, y) \notin M\}$ . By (v), the set

$$\begin{aligned} \{x \in K : (x, y) \in P\} &= \{x \in X : y \in G(x)\} \cap K \\ &= G^{-1}(y) \cap K \\ &= \{x \in X : x \in F(y)\} \cap K \\ &= F(y) \cap K \end{aligned}$$

is nonempty acyclic. If the conclusion of Theorem 4 is false, then  $T(x) \cap G(x) = \emptyset$  for all  $x \in X$ . Therefore  $P \subset M$ . Then it follows from Theorem 1 that  $\bigcap_{y \in Y} C(y) \cap K \neq \emptyset$ . Therefore there exists  $x^* \in K$  such that  $C^*(x^*) = \emptyset$ . This contradicts (i) and completes the proof.  $\square$

### Remark 3.

- (1) Theorem 4 generalizes [15, Theorem 7] and thus [3, Theorem 1], [10, Theorem 1] and [14, Theorem 1] in several ways.
- (2) In view of Lemma 1, condition (ii) of Theorem 4 can be replaced by the following condition:
  - (ii)(a)  $F$  is an upper semicontinuous set-valued mapping with closed values.

## 4. Applications to minimax inequalities

**Theorem 5.** Let  $X$  be a topological space,  $K$  be a compact subset of  $X$ ,  $(Y, \Gamma)$  be an  $L$ -convex space, and let  $e, f, g, h : X \times Y \rightarrow \mathbf{R}$  be four real-valued functions satisfying the following conditions:

- (i) for each  $(x, y) \in X \times Y$ ,  $e(x, y) \leq f(x, y)$ ,  $g(x, y) \leq h(x, y)$ ;
- (ii)  $e(x, y)$  is 0-transfer compactly lower semicontinuous in  $x$ ;
- (iii) for each  $x \in K$  and  $A \in \{\{y \in Y : f(x, y) > 0\}\}$ ,  $\Gamma(A) \subset \{y \in Y : g(x, y) > 0\}$ ;
- (iv)  $h(x, y)$  is lower semicontinuous on  $X \times Y$  and for each  $y \in Y$ , the set  $\{x \in K : h(x, y) \leq 0\}$  is nonempty acyclic.

Then there exists  $x_0 \in K$  such that  $e(x_0, y) \leq 0$  for each  $y \in Y$ .

**Proof.** Define  $C : Y \rightarrow 2^X$  as follows:

$$C(y) = \{x \in X : e(x, y) \leq 0\} \quad \text{for all } y \in Y.$$

Let  $M = \{(x, y) \in X \times Y : g(x, y) \leq 0\}$ ,  $N = \{(x, y) \in X \times Y : f(x, y) \leq 0\}$  and  $P = \{(x, y) \in X \times Y : h(x, y) \leq 0\}$ . By (i), for each  $y \in Y$ ,  $\{x \in X : (x, y) \in N\} \subset C(y)$ . By (ii) and Lemma 3,  $C$  is transfer compactly closed-valued. By (iii), for each  $x \in K$  and  $A \in \{\{y \in Y : (x, y) \notin N\}\}$ ,  $\Gamma(A) \subset \{y \in Y : (x, y) \notin M\}$ . By (i) and (iv),  $P \subset M$ ,  $P$  is closed in  $X \times Y$  and for each  $y \in Y$ ,  $\{x \in K : (x, y) \in P\}$  is nonempty acyclic. Then it follows from Theorem 1 that  $\bigcap_{y \in Y} C(y) \cap K \neq \emptyset$ , that is, there exists  $x_0 \in K$  such that  $e(x_0, y) \leq 0$  for each  $y \in Y$ .  $\square$

The following two corollaries follow immediately from Theorem 5.

**Corollary 1.** Let  $X$  be a topological space,  $K$  be a compact subset of  $X$ ,  $(Y, \Gamma)$  be an  $L$ -convex space, and let  $f, g : X \times Y \rightarrow \mathbf{R}$  be two real-valued functions satisfying the following conditions:



- (i)  $f(x, y)$  is 0-transfer compactly lower semicontinuous in  $x$ ;
- (ii) for each  $x \in K$  and  $A \in \langle \{y \in Y: f(x, y) > 0\} \rangle$ ,  $\Gamma(A) \subset \{y \in Y: g(x, y) > 0\}$ ;
- (iii)  $g(x, y)$  is lower semicontinuous on  $X \times Y$  and for each  $y \in Y$ , the set  $\{x \in K: g(x, y) \leq 0\}$  is nonempty acyclic.

Then there exists  $x_0 \in K$  such that  $f(x_0, y) \leq 0$  for each  $y \in Y$ .

**Corollary 2.** Let  $X$  be a topological space,  $K$  be a compact subset of  $X$ ,  $(Y, \Gamma)$  be an  $L$ -convex space, and let  $f: X \times Y \rightarrow \mathbf{R}$  be a real-valued function satisfying the following conditions:

- (i) for each  $x \in K$ ,  $\{y \in Y: f(x, y) > 0\}$  is  $L$ -convex;
- (ii)  $f(x, y)$  is lower semicontinuous on  $X \times Y$  and for each  $y \in Y$ , the set  $\{x \in K: f(x, y) \leq 0\}$  is nonempty acyclic.

Then there exists  $x_0 \in K$  such that  $f(x_0, y) \leq 0$  for each  $y \in Y$ .

**Theorem 6.** Theorems 1 and 5 are equivalent.

**Proof.** We have seen that Theorem 1 implies Theorem 5. Now we prove that Theorem 5 implies Theorem 1. For each  $(x, y) \in X \times Y$ , we define  $e, f, g, h: X \times Y \rightarrow \mathbf{R}$  by

$$\begin{aligned} e(x, y) &= \begin{cases} 0, & x \in C(y), \\ 1, & x \notin C(y), \end{cases} & f(x, y) &= \begin{cases} 0, & (x, y) \in N, \\ 1, & (x, y) \notin N, \end{cases} \\ g(x, y) &= \begin{cases} 0, & (x, y) \in M, \\ 1, & (x, y) \notin M, \end{cases} & h(x, y) &= \begin{cases} 0, & (x, y) \in P, \\ 1, & (x, y) \notin P. \end{cases} \end{aligned}$$

By the definition of  $e$ , for each  $y \in Y$ ,  $\{x \in X: e(x, y) \leq 0\} = \{x \in X: x \in C(y)\}$ . Then by (i) and Lemma 3,  $e$  is 0-transfer compactly lower semicontinuous in  $x$ . By (ii), for each  $(x, y) \in X \times Y$ ,  $e(x, y) \leq f(x, y)$ . By  $P \subset M$ ,  $g(x, y) \leq h(x, y)$  for all  $(x, y) \in X \times Y$ . By (iii) and the definitions of  $f$  and  $g$ , we know that for each  $x \in K$  and  $A \in \langle \{y \in Y: f(x, y) > 0\} \rangle$ ,  $\Gamma(A) \subset \{y \in Y: g(x, y) > 0\}$ . By (iv) and the definition of  $h$ , the set  $\{x \in K: h(x, y) \leq 0\}$  is nonempty acyclic. It follows from Theorem 5 that there exists  $x_0 \in K$  such that  $e(x_0, y) \leq 0$  for all  $y \in Y$ , that is,  $\bigcap_{y \in Y} C(y) \cap K \neq \emptyset$ .  $\square$

**Theorem 7.** Let  $X$  be a topological space,  $(Y, \Gamma)$  be an  $L$ -convex space, and let  $e, f, g, h: X \times Y \rightarrow \mathbf{R}$  be four real-valued functions. Let  $\beta = \inf_{K \in \bar{K}} \sup_{y \in Y} \min_{x \in K} h(x, y)$ , where  $\bar{K} = \{K \subset X: K \text{ is compact subset of } X\}$ . Suppose the following conditions are fulfilled:

- (i) for each  $(x, y) \in X \times Y$ ,  $e(x, y) \leq f(x, y)$ ,  $g(x, y) \leq h(x, y)$ ;
- (ii) for each  $t > \beta$ ,  $e(x, y)$  is  $t$ -transfer compactly lower semicontinuous in  $x$ ;
- (iii) for each  $t > \beta$ ,  $x \in X$  and  $A \in \langle \{y \in Y: f(x, y) > t\} \rangle$ ,  $\Gamma(A) \subset \{y \in Y: g(x, y) > t\}$ ;
- (iv)  $h(x, y)$  is lower semicontinuous on  $X \times Y$  and the set  $\{x \in K: h(x, y) < t\}$  is acyclic or empty for each  $t > \beta$ ,  $K \in \bar{K}$  and  $y \in Y$ . Then

$$\inf_{x \in X} \sup_{y \in Y} e(x, y) \leq \inf_{K \in \bar{K}} \sup_{y \in Y} \min_{x \in K} h(x, y). \quad (4)$$

If  $X$  is compact, then

$$\inf_{x \in X} \sup_{y \in Y} e(x, y) \leq \sup_{y \in Y} \min_{x \in X} h(x, y). \quad (5)$$

**Proof.** We can assume that the right-hand side of (4) is not  $+\infty$ . If the conclusion of Theorem 7 is false, then there is a real number  $t$  such that

$$\inf_{x \in X} \sup_{y \in Y} e(x, y) > t > \inf_{K \in \bar{K}} \sup_{y \in Y} \min_{x \in K} h(x, y).$$

For the above  $t$ , we define  $C : Y \rightarrow 2^X$  as follows:

$$C(y) = \{x \in X : e(x, y) \leq t\} \quad \text{for all } y \in Y.$$

Let  $M = \{(x, y) \in X \times Y : g(x, y) \leq t\}$ ,  $N = \{(x, y) \in X \times Y : f(x, y) \leq t\}$  and  $P = \{(x, y) \in X \times Y : h(x, y) \leq t\}$ . By (i), for each  $y \in Y$ ,  $\{x \in X : (x, y) \in N\} \subset \{x \in X : e(x, y) \leq t\} = C(y)$ . By (ii),  $C : Y \rightarrow 2^X$  is transfer compactly closed-valued. By (iii), we know that for each  $x \in K$  and  $A \in \{y \in Y : (x, y) \notin N\}$ ,  $\Gamma(A) \subset \{y \in Y : (x, y) \notin M\}$ . It is easy to verify that  $P$  is closed in  $X \times Y$  and  $P \subset M$ . Let  $K$  be a compact subset of  $X$  such that

$$t > \sup_{y \in Y} \min_{x \in K} h(x, y).$$

Then for any  $y \in Y$ , the set  $\{x \in K : h(x, y) \leq t\}$  is nonempty and we know the set  $\{x \in K : h(x, y) \leq t\} = \bigcap_{\varepsilon > 0} \{x \in K : h(x, y) < t + \varepsilon\}$  is acyclic (this follows from the continuity of Čech homology. See, e.g., McClendon [12]). Thus, by Theorem 1,

$$\bigcap_{y \in Y} C(y) \cap K \neq \emptyset,$$

that is, there exists  $x_0 \in K$  such that

$$e(x_0, y) \leq t \quad \text{for all } y \in Y.$$

Hence

$$\inf_{x \in X} \sup_{y \in Y} e(x, y) \leq t.$$

This contradicts the choice of  $t$ . Therefore (4) is proved.  $\square$

**Remark 4.** Theorem 7 generalizes [16, Theorem 3] from topological vector spaces to topological spaces and  $L$ -convex spaces.

**Corollary 3.** Let  $X$  be a topological space,  $(Y, \Gamma)$  be an  $L$ -convex space, and let  $e, f, g, h : X \times Y \rightarrow \mathbf{R}$  be four real-valued functions. Let  $\beta = \inf_{K \in \bar{K}} \sup_{y \in Y} \min_{x \in K} h(x, y)$ , where  $\bar{K} = \{K \subset X : K \text{ is compact subset of } X\}$ . Suppose the following conditions are fulfilled:

- (i) for each  $(x, y) \in X \times Y$ ,  $e(x, y) \leq f(x, y)$ ,  $g(x, y) \leq h(x, y)$ ;
- (ii) for each  $t > \beta$ ,  $e(x, y)$  is  $t$ -transfer compactly lower semicontinuous in  $x$ ;
- (iii) for each  $t > \beta$  and  $x \in X$ ,  $L - \text{co}\{y \in Y : f(x, y) > t\} \subset \{y \in Y : g(x, y) > t\}$ ;

- (iv)  $h(x, y)$  is lower semicontinuous on  $X \times Y$  and the set  $\{x \in K: h(x, y) < t\}$  is acyclic or empty for each  $t > \beta$ ,  $K \in \bar{K}$  and  $y \in Y$ . Then

$$\inf_{x \in X} \sup_{y \in Y} e(x, y) \leq \inf_{K \in \bar{K}} \sup_{y \in Y} \min_{x \in K} h(x, y).$$

If  $X$  is compact, then

$$\inf_{x \in X} \sup_{y \in Y} e(x, y) \leq \sup_{y \in Y} \min_{x \in X} h(x, y).$$

**Corollary 4.** Let  $X$  be a topological space,  $(Y, \Gamma)$  be an  $L$ -convex space, and let  $e, f, g, h: X \times Y \rightarrow \mathbf{R}$  be four real-valued functions. Let  $\beta = \inf_{K \in \bar{K}} \sup_{y \in Y} \min_{x \in K} h(x, y)$ , where  $\bar{K} = \{K \subset X: K \text{ is compact subset of } X\}$ . Suppose the following conditions are fulfilled:

- (i) for each  $(x, y) \in X \times Y$ ,  $e(x, y) \leq f(x, y) \leq g(x, y) \leq h(x, y)$ ;
- (ii) for each  $t > \beta$ ,  $e(x, y)$  is  $t$ -transfer compactly lower semicontinuous in  $x$ ;
- (iii) for each  $x \in X$ ,  $g(x, y)$  is  $L$ -quasiconcave in  $y$ ;
- (iv)  $h(x, y)$  is lower semicontinuous on  $X \times Y$  and the set  $\{x \in K: h(x, y) < t\}$  is acyclic or empty for each  $t > \beta$ ,  $K \in \bar{K}$  and  $y \in Y$ . Then

$$\inf_{x \in X} \sup_{y \in Y} e(x, y) \leq \inf_{K \in \bar{K}} \sup_{y \in Y} \min_{x \in K} h(x, y).$$

If  $X$  is compact, then

$$\inf_{x \in X} \sup_{y \in Y} e(x, y) \leq \sup_{y \in Y} \min_{x \in X} h(x, y).$$

**Corollary 5.** Let  $X$  be a topological space,  $(Y, \Gamma)$  be an  $L$ -convex space, and let  $f: X \times Y \rightarrow \mathbf{R}$  be a real-valued function. Let  $\beta = \inf_{K \in \bar{K}} \sup_{y \in Y} \min_{x \in K} f(x, y)$ , where  $\bar{K} = \{K \subset X: K \text{ is compact subset of } X\}$ . Suppose the following conditions are fulfilled:

- (i) for each  $x \in X$ ,  $f(x, y)$  is  $L$ -quasiconcave in  $y$ ;
- (ii)  $f(x, y)$  is lower semicontinuous on  $X \times Y$  and the set  $\{x \in K: f(x, y) < t\}$  is acyclic or empty for each  $t > \beta$ ,  $K \in \bar{K}$  and  $y \in Y$ . Then

$$\inf_{x \in X} \sup_{y \in Y} f(x, y) = \inf_{K \in \bar{K}} \sup_{y \in Y} \min_{x \in K} f(x, y).$$

If  $X$  is compact, then

$$\min_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \min_{x \in X} f(x, y).$$

**Remark 5.** Corollary 5 generalizes [8, Theorem 4] in several ways.

**Theorem 8.** Let  $Y$  be a topological space,  $K$  be a compact subset of  $Y$ ,  $(X, \Gamma)$  be an  $L$ -convex space, and let  $e, f, g: X \times Y \rightarrow \mathbf{R}$  be three real-valued functions such that

- (i) for each  $(x, y) \in X \times Y$ ,  $f(x, y) \leq g(x, y)$ ;
- (ii) for each  $t > \sup_{y \in Y} \inf_{x \in X} g(x, y)$ ,  $y \in Y$  and  $A \in \{\{x \in X: f(x, y) < t\}, \Gamma(A) \subset \{x \in X: e(x, y) < t\}\}$ ;
- (iii) for each  $t > \sup_{y \in Y} \inf_{x \in X} g(x, y)$ ,  $g(x, y)$  is  $t$ -transfer compactly upper semicontinuous in  $y$ .

If  $T : X \rightarrow 2^Y$  is an upper semicontinuous set-valued mapping with closed values such that each  $Tx \cap K$  is nonempty acyclic, then

$$\inf_{y \in Tx} e(x, y) \leq \sup_{y \in Y} \inf_{x \in X} g(x, y). \quad (6)$$

**Proof.** If the conclusion of Theorem 8 is false, then there is a real number  $t$  such that

$$\inf_{y \in Tx} e(x, y) > t > \sup_{y \in Y} \inf_{x \in X} g(x, y). \quad (7)$$

For the above  $t$ , we define  $C : X \rightarrow 2^Y$  as follows:

$$C(x) = \{y \in Y: g(x, y) \leq t\} \quad \text{for all } x \in X.$$

Let  $M = \{(x, y) \in X \times Y: e(x, y) \geq t\}$ ,  $N = \{(x, y) \in X \times Y: f(x, y) \geq t\}$  and  $P = \{(x, y) \in X \times Y: y \in Tx\}$ . By (7),  $P \subset M$ . It is easy to check that  $M$ ,  $N$  and  $C$  satisfy (i), (ii) and (iii) of Theorem 1, and  $P$  is closed in  $X \times Y$ . For each  $x \in X$ , the set  $\{y \in K: (x, y) \in P\} = Tx \cap K$  is nonempty acyclic. Thus, by Theorem 1,  $\bigcap_{x \in X} C(x) \cap K \neq \emptyset$ , that is, there exists  $y_0 \in K$  such that

$$g(x, y_0) \geq t \quad \text{for all } x \in X.$$

Hence

$$\sup_{y \in Y} \inf_{x \in X} g(x, y) \geq t.$$

This contradicts the choice of  $t$ . Therefore (6) is proved.  $\square$

**Remark 6.** Theorem 8 generalizes [16, Theorem 8] and thus ([6, Theorem 1], [9, Theorem 1]), where  $Y$  need not to be compact.

By taking  $K = Y$ , we obtain the following corollary.

**Corollary 6.** Let  $Y$  be a compact topological space,  $(X, \Gamma)$  be an  $L$ -convex space, and let  $e, f, g : X \times Y \rightarrow \mathbf{R}$  be three real-valued functions such that

- (i) for each  $(x, y) \in X \times Y$ ,  $f(x, y) \leq g(x, y)$ ;
- (ii) for each  $t > \sup_{y \in Y} \inf_{x \in X} g(x, y)$ ,  $y \in Y$  and  $A \in \{\{x \in X: f(x, y) < t\}, \Gamma(A) \subset \{x \in X: e(x, y) < t\}\}$ ;
- (iii) for each  $t > \sup_{y \in Y} \inf_{x \in X} g(x, y)$ ,  $g(x, y)$  is  $t$ -transfer compactly upper semicontinuous in  $y$ .

If  $T : X \rightarrow 2^Y$  is an upper semicontinuous set-valued mapping such that each  $Tx$  is non-empty compact acyclic subset of  $Y$ , then

$$\inf_{y \in Tx} e(x, y) \leq \sup_{y \in Y} \inf_{x \in X} g(x, y).$$

Finally, as application of Corollary 2, we establish the following saddle point theorem.

**Theorem 9.** Let  $(X_1, \Gamma_1)$  and  $(X_2, \Gamma_2)$  be two  $L$ -convex spaces,  $K_1$  and  $K_2$  be two compact subsets of  $X_1$  and  $X_2$ , respectively. Let  $f : X_1 \times X_2 \rightarrow \mathbf{R}$  be a real-valued function satisfying the following conditions:

- (i) for each  $x \in K_1$ ,  $\{y \in X_2 : f(x, y) > 0\}$  is  $L$ -convex and for each  $y \in K_2$ ,  $\{x \in X_1 : f(x, y) < 0\}$  is  $L$ -convex;
- (ii)  $f(x, y)$  is continuous on  $X_1 \times X_2$  and for each  $y \in X_2$ , the set  $\{x \in K_1 : f(x, y) \leq 0\}$  is nonempty acyclic;
- (iii) for each  $x \in X_1$ , the set  $\{y \in K_2 : f(x, y) \geq 0\}$  is nonempty acyclic.

Then  $f$  has a saddle point  $(\bar{x}, \bar{y}) \in K_1 \times K_2$ , i.e.,  $f(\bar{x}, y) \leq f(\bar{x}, \bar{y}) \leq f(x, \bar{y})$  for all  $(x, y) \in X_1 \times X_2$ . In particular,  $\inf_{x \in X_1} \sup_{y \in X_2} f(x, y) = \sup_{y \in X_2} \inf_{x \in X_1} f(x, y)$ .

**Proof.** By (i)–(ii) and Corollary 2, there exists  $\bar{x} \in K_1$  such that  $f(\bar{x}, y) \leq 0$  for all  $y \in X_2$ . Let  $g(y, x) = -f(x, y)$  for all  $(y, x) \in X_2 \times X_1$ . Then by (i)–(iii) and Corollary 2, there exists  $\bar{y} \in K_2$  such that  $g(\bar{y}, x) \leq 0$  for all  $x \in X_1$ . It follows that

$$f(\bar{x}, y) \leq 0 \leq f(x, \bar{y}) \quad \text{for all } (x, y) \in X_1 \times X_2.$$

Hence we have

$$f(\bar{x}, y) \leq f(\bar{x}, \bar{y}) = 0 \leq f(x, \bar{y}) \quad \text{for all } (x, y) \in X_1 \times X_2,$$

and

$$\inf_{x \in X_1} \sup_{y \in X_2} f(x, y) \leq \sup_{y \in X_2} \inf_{x \in X_1} f(x, y).$$

Since  $\inf_{x \in X_1} \sup_{y \in X_2} f(x, y) \geq \sup_{y \in X_2} \inf_{x \in X_1} f(x, y)$  is always true, we have

$$\inf_{x \in X_1} \sup_{y \in X_2} f(x, y) = \sup_{y \in X_2} \inf_{x \in X_1} f(x, y). \quad \square$$

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