Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
J. Math. Anal. Appl. 315 (2006) 144–153
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

# Existence of multiple positive solutions for one-dimensional $p$-Laplacian ☆

Youyu Wang [a,*], Chengmin Hou [b]

[a] *Department of Mathematics, Beijing Institute of Technology, Beijing 100081, PR China*
[b] *Department of Mathematics, Yanbian University, Yanji 133000, PR China*

Received 12 March 2004
Available online 4 November 2005
Submitted by P.G.L. Leach

**Abstract**

In this paper we consider the multipoint boundary value problem for one-dimensional $p$-Laplacian

$$\big(\phi_p(u')\big)' + f(t,u) = 0, \quad t \in (0,1),$$

subject to the boundary value conditions:

$$\phi_p\big(u'(0)\big) = \sum_{i=1}^{n-2} a_i \phi_p\big(u'(\xi_i)\big), \qquad u(1) = \sum_{i=1}^{n-2} b_i u(\xi_i),$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{n-2} < 1$, and $a_i, b_i$ satisfy $a_i, b_i \in [0,\infty]$, $0 < \sum_{i=1}^{n-2} a_i < 1$, and $\sum_{i=1}^{n-2} b_i < 1$. Using a fixed point theorem for operators on a cone, we provide sufficient conditions for the existence of multiple (at least three) positive solutions to the above boundary value problem.


*Keywords:* Multiple positive solutions; Boundary value problems; One-dimensional $p$-Laplacian

☆ The project is supported by the National Natural Science Foundation of China (10371006).
* Corresponding author.
*E-mail address:* wang_youyu@sohu.com (Y. Wang).

0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2005.09.085

## 1. Introduction

In this paper we study the existence of multiple positive solutions to the boundary value problem (BVP) for the one-dimensional $p$-Laplacian

$$\big(\phi_p(u')\big)' + f(t,u) = 0, \quad t \in (0,1), \tag{1.1}$$

$$\phi_p\big(u'(0)\big) = \sum_{i=1}^{n-2} a_i \phi_p\big(u'(\xi_i)\big), \qquad u(1) = \sum_{i=1}^{n-2} b_i u(\xi_i), \tag{1.2}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{n-2} < 1$ and $a_i, b_i, f$ satisfy

$(H_1)$ $a_i, b_i \in [0,\infty]$ satisfy $0 < \sum_{i=1}^{n-2} a_i < 1$, and $\sum_{i=1}^{n-2} b_i < 1$;
$(H_2)$ $f \in C([0,1] \times [0,\infty), [0,\infty))$.

The study of multipoint boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [1]. Since then there has been much current attention focused on the study of nonlinear multipoint boundary value problems, see [2–4,7].

In recent papers [5,6] the authors have investigated the following BVP for one-dimensional $p$-Laplacian:

$$\big(\phi_p(u')\big)' + a(t) f(t,u) = 0, \quad t \in (0,1), \tag{1.3}$$

$$u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i), \tag{1.4}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{n-2} < 1$ and $\sum_{i=1}^{m-2} a_i \xi_i < 1$, and

$$\big(\phi_p(u')\big)' + f(t,u) = 0, \quad t \in (0,1), \tag{1.5}$$

$$u'(0) = \sum_{i=1}^{m-2} b_i u'(\xi_i), \qquad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i), \tag{1.6}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{n-2} < 1$ and $a_i, b_i$ satisfy $0 < \sum_{i=1}^{m-2} a_i < 1$, $\sum_{i=1}^{m-2} b_i < 1$.

In paper [5] the authors claim that:

> It is easy to check that system (1.3) and (1.4) has a solution $u = u(t)$ if and only if $u$ solves the operator equation
>
> $$u(t) = -\int_0^t \phi_q\left(\int_0^s a(\tau) f\big(\tau, u(\tau)\big)\, d\tau\right) ds - t\,\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q(\int_0^s a(\tau) f(\tau, u(\tau))\, d\tau)\, ds}{1 - \sum_{i=1}^{m-2} a_i \xi_i}$$

$$+\,t\,\frac{\int_0^1 \phi_q(\int_0^s a(\tau)f(\tau,u(\tau))\,d\tau)\,ds}{1-\sum_{i=1}^{m-2} a_i\xi_i},$$

where $\phi_q(s)$ is the inverse function to $\phi_p(s)$.

In fact this statement is not true. It is also easy to check that $u$ does not solve Eq. (1.3).
In paper [6] the authors also claim that:

It is easy to check that system (1.5) and (1.6) has a solution $u=u(t)$ if and only if $u$ solves the operator equation

$$\begin{aligned}
u(t) &= -\int_0^t \phi_q\left(\int_0^s f\big(\tau,u(\tau)\big)\,d\tau\right)ds + t\,\frac{\sum_{i=1}^{m-2} b_i\phi_q(\int_0^{\xi_i} f(\tau,u(\tau))\,d\tau)}{\sum_{i=1}^{m-2} b_i-1}\\
&\quad+\frac{1}{1-\sum_{i=1}^{m-2} a_i}\left(\int_0^1 \phi_q\left(\int_0^s f\big(\tau,u(\tau)\big)\,d\tau\right)ds\right.\\
&\quad\left.-\sum_{i=1}^{m-2} a_i\int_0^{\xi_i}\phi_q\left(\int_0^s f\big(\tau,u(\tau)\big)\,d\tau\right)ds\right)\\
&\quad-\frac{1}{1-\sum_{i=1}^{m-2} a_i}\cdot\frac{\sum_{i=1}^{m-2} b_i\phi_q(\int_0^{\xi_i} f(\tau,u(\tau))\,d\tau)}{\sum_{i=1}^{m-2} b_i-1}\left(1-\sum_{i=1}^{m-2} a_i\xi_i\right).
\end{aligned}$$

Unfortunately this statement is also wrong. So the conclusions of [5,6] should be reconsidered. The aim of this paper is to show the existence of multiple positive solutions to the BVP (1.1) and (1.2).

By the positive solution of (1.1) and (1.2) one means a function, $u(t)$, which is positive on $0<t<1$ and satisfies the differential equation (1.1) and the boundary conditions (1.2).

To obtain positive solutions of (1.1) and (1.2) the following fixed point theorem in cones is fundamental.

**Lemma 1.1** [8,9]. *Let $K$ be a cone in a Banach space $X$. Let $D$ be an open bounded subset of $X$ with $D_K=D\cap K\neq\emptyset$ and $\overline{D}_K\neq K$. Assume that $A:\overline{D}_K\to K$ is a compact map such that $x\neq Ax$ for $x\in\partial D_K$. Then the following results hold*:

(1) *If $\|Ax\|\leqslant\|x\|$, $x\in\partial D_K$, then $i_K(A,D_K)=1$.*
(2) *If there exists $e\in K\backslash\{0\}$ such that $x\neq Ax+\lambda e$ for all $x\in\partial D_K$ and all $\lambda>0$, then $i_K(A,D_K)=0$.*
(3) *Let $U$ be open in $X$ such that $\overline{U}\subset D_K$. If $i_K(A,D_K)=1$ and $i_K(A,U_K)=0$, then $A$ has a fixed point in $D_K\backslash\overline{U}_K$. The same result holds if $i_K(A,D_K)=0$ and $i_K(A,U_K)=1$.*

## 2. The preliminary lemmas

In this paper we assume that $(H_1)$ and $(H_2)$ hold. $\phi_q(s)$ is the inverse function to $\phi_p(s)$.

**Lemma 2.1.** *The BVP* (1.1) *and* (1.2) *has a solution $u(t)$ if and only if $u(t)$ solves the equation*

$$u(t) = -\int_0^t \phi_q\left(\int_0^s f\big(\tau, u(\tau)\big)\,d\tau - \widetilde{A}\right) ds + \widetilde{B}, \tag{2.1}$$

*where*

$$\widetilde{A} = -\frac{\sum_{i=1}^{n-2} a_i \int_0^{\xi_i} f(\tau, u(\tau))\,d\tau}{1 - \sum_{i=1}^{n-2} a_i},$$

$$\widetilde{B} = \frac{\int_0^1 \phi_q(\int_0^s f(\tau, u(\tau))\,d\tau - \widetilde{A})\,ds - \sum_{i=1}^{n-2} b_i \int_0^{\xi_i} \phi_q(\int_0^s f(\tau, u(\tau))\,d\tau - \widetilde{A})\,ds}{1 - \sum_{i=1}^{n-2} b_i}.$$

**Lemma 2.2.** *The solution of BVP* (1.1) *and* (1.2) *satisfies* $u(t) \geqslant 0$, $t \in [0, 1]$.

**Proof.** Let

$$\varphi(s) = \phi_q\left(\int_0^s f\big(\tau, u(\tau)\big)\,d\tau - \widetilde{A}\right).$$

Since

$$\int_0^s f\big(\tau, u(\tau)\big)\,d\tau - \widetilde{A} = \int_0^s f\big(\tau, u(\tau)\big)\,d\tau + \frac{\sum_{i=1}^{n-2} a_i \int_0^{\xi_i} f(\tau, u(\tau))\,d\tau}{1 - \sum_{i=1}^{n-2} a_i} \geqslant 0,$$

then $\varphi(s) \geqslant 0$.

According to Lemma 2.1, we get

$$u(0) = \widetilde{B} = \frac{\int_0^1 \varphi(s)\,ds - \sum_{i=1}^{n-2} b_i \int_0^{\xi_i} \varphi(s)\,ds}{1 - \sum_{i=1}^{n-2} b_i} = \int_0^1 \varphi(s)\,ds + \frac{\sum_{i=1}^{n-2} b_i \int_{\xi_i}^1 \varphi(s)\,ds}{1 - \sum_{i=1}^{n-2} b_i} \geqslant 0$$

and

$$u(1) = -\int_0^1 \varphi(s)\,ds + \widetilde{B} = -\int_0^1 \varphi(s)\,ds + \int_0^1 \varphi(s)\,ds + \frac{\sum_{i=1}^{n-2} b_i \int_{\xi_i}^1 \varphi(s)\,ds}{1 - \sum_{i=1}^{n-2} b_i}$$
$$= \frac{\sum_{i=1}^{n-2} b_i \int_{\xi_i}^1 \varphi(s)\,ds}{1 - \sum_{i=1}^{n-2} b_i} \geqslant 0.$$

If $t \in (0, 1)$, we have

$$u(t) = -\int_0^t \varphi(s)\,ds + \frac{1}{1 - \sum_{i=1}^{n-2} b_i}\left[\int_0^1 \varphi(s)\,ds - \sum_{i=1}^{n-2} b_i \int_0^{\xi_i} \varphi(s)\,ds\right]$$
$$\geqslant -\int_0^1 \varphi(s)\,ds + \frac{1}{1 - \sum_{i=1}^{n-2} b_i}\left[\int_0^1 \varphi(s)\,ds - \sum_{i=1}^{n-2} b_i \int_0^{\xi_i} \varphi(s)\,ds\right]$$

$$
\begin{aligned}
&= \frac{\sum_{i=1}^{n-2} b_i}{1-\sum_{i=1}^{n-2} b_i} \int_0^1 \varphi(s)\,ds - \frac{1}{1-\sum_{i=1}^{n-2} b_i} \sum_{i=1}^{n-2} b_i \int_0^{\xi_i} \varphi(s)\,ds \\
&= \frac{1}{1-\sum_{i=1}^{n-2} b_i} \sum_{i=1}^{n-2} b_i \int_{\xi_i}^1 \varphi(s)\,ds \\
&\geqslant 0.
\end{aligned}
$$

So $u(t) \geqslant 0$, $t \in [0,1]$. $\square$

**Lemma 2.3.** *The solution of BVP* (1.1) *and* (1.2) *satisfies*

$$\inf_{t\in[0,1]} u(t) \geqslant \gamma_1 \|u\|,$$

*where*

$$\gamma_1 = \frac{\sum_{i=1}^{n-2} b_i(1-\xi_i)}{1-\sum_{i=1}^{n-2} b_i\xi_i}.$$

**Proof.** Clearly $u'(t) = -\varphi(t) \leqslant 0$. This implies that

$$\|u\| = u(0), \qquad \min_{t\in[0,1]} u(t) = u(1).$$

It is easy to see that $u'(t_2) \leqslant u'(t_1)$ for any $t_1, t_2 \in [0,1]$ with $t_1 \leqslant t_2$. Hence $u'(t)$ is a decreasing function on $[0,1]$. This means that the graph of $u'(t)$ is concave down on $(0,1)$.

For each $i \in \{1, 2, \dots, n-2\}$ we have

$$\frac{u(\xi_i)-u(1)}{1-\xi_i} \geqslant \frac{u(0)-u(1)}{1},$$

i.e.,

$$u(\xi_i) - \xi_i u(1) \geqslant (1-\xi_i)u(0)$$

so that

$$\sum_{i=1}^{n-2} b_i u(\xi_i) - \sum_{i=1}^{n-2} b_i \xi_i u(1) \geqslant \sum_{i=1}^{n-2} b_i(1-\xi_i)u(0)$$

and, with the boundary condition $u(1) = \sum_{i=1}^{n-2} b_i u(\xi_i)$, we have

$$u(1) \geqslant \frac{\sum_{i=1}^{n-2} b_i(1-\xi_i)}{1-\sum_{i=1}^{n-2} b_i\xi_i} u(0).$$

This completes the proof. $\square$

**Lemma 2.4.** *Let*

$$K = \Big\{ u\colon\ u \in C[0,1],\ u \geqslant 0,\ \min_{0\leqslant t\leqslant 1} \geqslant \gamma \|u\| \Big\},$$

*where* $\gamma = \gamma_1\gamma_2$, $\gamma_1$ *is defined in Lemma* 2.3 *and*

$$\gamma_2=\left[\sum_{i=1}^{n-2} b_i\left(1+\frac{\sum_{i=1}^{n-2} a_i\xi_i}{1-\sum_{i=1}^{n-2} a_i}\right)^q-\sum_{i=1}^{n-2} b_i\left(\xi_i+\frac{\sum_{i=1}^{n-2} a_i\xi_i}{1-\sum_{i=1}^{n-2} a_i}\right)^q\right]$$
$$\div\left[\left(1+\frac{\sum_{i=1}^{n-2} a_i\xi_i}{1-\sum_{i=1}^{n-2} a_i}\right)^q-\sum_{i=1}^{n-2} b_i\left(\xi_i+\frac{\sum_{i=1}^{n-2} a_i\xi_i}{1-\sum_{i=1}^{n-2} a_i}\right)^q\right].$$

*Define operator* $A: K\to K$,

$$(Au)(t):=-\int_0^t \phi_q\left(\int_0^s f\big(\tau,u(\tau)\big)\,d\tau-\widetilde{A}\right)ds+\widetilde{B}\big(=u(t)\big),\quad u(t)\in K. \tag{2.2}$$

*Then* $A: K\to K$ *is completely continuous.*

**Proof.** It is easy to check that $0<\gamma_1,\gamma_2<1$ and $K$ is a cone in $C[0,1]$. According to Lemma 2.3, we easily obtain

$$Au\geqslant 0 \quad\text{and}\quad \inf_{t\in[0,1]}(Au)(t)\geqslant \gamma_1\|Au\|>\gamma\|Au\| \quad \text{for } u\in K.$$

This means that $AK\subset K$. It is easy to see that $A: K\to K$ is completely continuous. □

We define

$$K_\rho=\big\{x\in K\colon\ \|x\|<\rho\big\},$$
$$\Omega_\rho=\Big\{x\in K\colon\ \min_{0\leqslant t\leqslant 1} x(t)<\gamma\rho\Big\}=\Big\{x\colon\ x\in C[0,1],\ x\geqslant 0,\ \gamma\|x\|\leqslant \min_{0\leqslant t\leqslant 1} x(t)<\gamma\rho\Big\}.$$

**Lemma 2.5** [9]. *$\Omega_\rho$ has the following properties*:

(a) *$\Omega_\rho$ is open relative to $K$.*
(b) *$K_{\gamma\rho}\subset\Omega_\rho\subset K_\rho$.*
(c) *$x\in\partial\Omega_\rho$ if and only if* $\min_{0\leqslant t\leqslant 1} x(t)=\gamma\rho$.
(d) *If $x\in\partial\Omega_\rho$, then $\gamma\rho\leqslant x(t)\leqslant\rho$ for $t\in[0,1]$.*

Now for convenience we introduce the following notations. Let

$$f_{\gamma\rho}^{\rho}=\min\left\{\min_{t\in[0,1]}\frac{f(t,u)}{\phi_p(\rho)}\colon\ u\in[\gamma\rho,\rho]\right\},$$
$$f_0^{\rho}=\max\left\{\max_{t\in[0,1]}\frac{f(t,u)}{\phi_p(\rho)}\colon\ u\in[0,\rho]\right\},$$
$$f^{\alpha}=\lim_{u\to\alpha}\sup\max_{t\in[0,1]}\frac{f(t,u)}{\phi_p(u)},$$
$$f_{\alpha}=\lim_{u\to\alpha}\inf\min_{t\in[0,1]}\frac{f(t,u)}{\phi_p(u)}\quad (\alpha:=\infty \text{ or } 0^+),$$
$$\frac{1}{m}=\frac{1}{q}\cdot\frac{1}{1-\sum_{i=1}^{n-2} b_i}\left[\left(1+\frac{\sum_{i=1}^{n-2} a_i\xi_i}{1-\sum_{i=1}^{n-2} a_i}\right)^q-\sum_{i=1}^{n-2} b_i\left(\xi_i+\frac{\sum_{i=1}^{n-2} a_i\xi_i}{1-\sum_{i=1}^{n-2} a_i}\right)^q\right],$$

$$\frac{1}{M}=\frac{1}{q}\cdot\frac{1}{1-\sum_{i=1}^{n-2}b_i}\left[\sum_{i=1}^{n-2}b_i\left(1+\frac{\sum_{i=1}^{n-2}a_i\xi_i}{1-\sum_{i=1}^{n-2}a_i}\right)^q-\sum_{i=1}^{n-2}b_i\left(\xi_i+\frac{\sum_{i=1}^{n-2}a_i\xi_i}{1-\sum_{i=1}^{n-2}a_i}\right)^q\right].$$

**Remark 2.1.** By $(H_1)$ it is easy to see that $0<m$, $M<\infty$ and $M\gamma=M\gamma_1\gamma_2=\gamma_1 m<m$.

**Lemma 2.6.** *If $f$ satisfies the condition*

$$f_0^\rho\leqslant\phi_p(m)\quad\text{and}\quad x\neq Ax\quad\text{for } x\in\partial K_\rho,\tag{2.3}$$

*then $i_K(A,K_\rho)=1$.*

**Proof.** By (2.2) and (2.3), we have for $u(t)\in\partial K_\rho$,

$$\begin{aligned}\int_0^s f\big(\tau,u(\tau)\big)\,d\tau-\widetilde{A}&=\int_0^s f\big(\tau,u(\tau)\big)\,d\tau+\frac{\sum_{i=1}^{n-2}a_i\int_0^{\xi_i}f(\tau,u(\tau))\,d\tau}{1-\sum_{i=1}^{n-2}a_i}\\&\leqslant\phi_p(m)\phi_p(\rho)s+\frac{\sum_{i=1}^{n-2}a_i\xi_i}{1-\sum_{i=1}^{n-2}a_i}\phi_p(m)\phi_p(\rho)\\&=\phi_p(m)\phi_p(\rho)\left(s+\frac{\sum_{i=1}^{n-2}a_i\xi_i}{1-\sum_{i=1}^{n-2}a_i}\right)\end{aligned}$$

so that

$$\varphi(s)=\phi_q\left(\int_0^s f\big(\tau,u(\tau)\big)\,d\tau-\widetilde{A}\right)\leqslant m\rho\left(s+\frac{\sum_{i=1}^{n-2}a_i\xi_i}{1-\sum_{i=1}^{n-2}a_i}\right)^{q-1}.$$

Therefore,

$$\begin{aligned}(Au)(t)&\leqslant\widetilde{B}\\&=\frac{1}{1-\sum_{i=1}^{n-2}b_i}\left(\int_0^1\varphi(s)\,ds-\sum_{i=1}^{n-2}b_i\int_0^{\xi_i}\varphi(s)\,ds\right)\\&\leqslant m\rho\frac{1}{q}\cdot\frac{1}{1-\sum_{i=1}^{n-2}b_i}\left[\left(1+\frac{\sum_{i=1}^{n-2}a_i\xi_i}{1-\sum_{i=1}^{n-2}a_i}\right)^q-\sum_{i=1}^{n-2}b_i\left(\xi_i+\frac{\sum_{i=1}^{n-2}a_i\xi_i}{1-\sum_{i=1}^{n-2}a_i}\right)^q\right]\\&=\rho.\end{aligned}$$

This implies that $\|Au\|\leqslant\|u\|$ for $u(t)\in\partial K_\rho$. By Lemma 1.1(1), we have $i_K(A,K_\rho)=1$. □

**Lemma 2.7.** *If $f$ satisfies the condition*

$$f_{\gamma\rho}^\rho\geqslant\phi_p(M\gamma)\quad\text{and}\quad x\neq Ax\quad\text{for } x\in\partial\Omega_\rho,\tag{2.4}$$

*then $i_K(A,\Omega_\rho)=0$.*

**Proof.** Let $e(t)\equiv1$ for $t\in[0,1]$. Then $e\in\partial K_1$. We claim that

$$u\neq Au+\lambda e,\quad u\in\partial\Omega_\rho,\ \lambda>0.$$

In fact, if not, there exist $u_0\in\partial\Omega_\rho$ and $\lambda_0>0$ such that $u_0=Au_0+\lambda_0e$.

By (2.2), (2.4) and Lemma 2.5(d), we have for $t \in [0, 1]$,

$$\begin{aligned}\int_0^s f\big(\tau, u_0(\tau)\big)\, d\tau - \widetilde{A} &= \int_0^s f\big(\tau, u_0(\tau)\big)\, d\tau + \frac{\sum_{i=1}^{n-2} a_i \int_0^{\xi_i} f(\tau, u_0(\tau))\, d\tau}{1 - \sum_{i=1}^{n-2} a_i} \\ &\geqslant \phi_p(M\gamma)\phi_p(\rho)s + \frac{\sum_{i=1}^{n-2} a_i \xi_i}{1 - \sum_{i=1}^{n-2} a_i}\phi_p(M\gamma)\phi_p(\rho) \\ &= \phi_p(M\gamma)\phi_p(\rho)\left(s + \frac{\sum_{i=1}^{n-2} a_i \xi_i}{1 - \sum_{i=1}^{n-2} a_i}\right)\end{aligned}$$

so that

$$\varphi_0(s) = \phi_q\left(\int_0^s f\big(\tau, u_0(\tau)\big)\, d\tau - \widetilde{A}\right) \geqslant M\gamma\rho\left(s + \frac{\sum_{i=1}^{n-2} a_i \xi_i}{1 - \sum_{i=1}^{n-2} a_i}\right)^{q-1}.$$

Therefore,

$$\begin{aligned}u_0(t) &= Au_0(t) + \lambda_0 e \\ &\geqslant -\int_0^1 \varphi_0(s)\, ds + \frac{1}{1 - \sum_{i=1}^{n-2} b_i}\left(\int_0^1 \varphi_0(s)\, ds - \sum_{i=1}^{n-2} b_i \int_0^{\xi_i} \varphi_0(s)\, ds\right) + \lambda_0 \\ &= \frac{\sum_{i=1}^{n-2} b_i}{1 - \sum_{i=1}^{n-2} b_i}\int_0^1 \varphi_0(s)\, ds - \frac{\sum_{i=1}^{n-2} b_i}{1 - \sum_{i=1}^{n-2} b_i}\int_0^{\xi_i} \varphi_0(s)\, ds + \lambda_0 \\ &\geqslant M\gamma\rho\frac{1}{q} \cdot \frac{1}{1 - \sum_{i=1}^{n-2} b_i}\left[\sum_{i=1}^{n-2} b_i\left(1 + \frac{\sum_{i=1}^{n-2} a_i \xi_i}{1 - \sum_{i=1}^{n-2} a_i}\right)^q \right. \\ &\quad \left. - \sum_{i=1}^{n-2} b_i\left(\xi_i + \frac{\sum_{i=1}^{n-2} a_i \xi_i}{1 - \sum_{i=1}^{n-2} a_i}\right)^q\right] + \lambda_0 \\ &= \gamma\rho + \lambda_0.\end{aligned}$$

This implies that $\gamma\rho \geqslant \gamma\rho + \lambda_0$ which is a contradiction. Hence by Lemma 1.1(2) it follows that $i_K(A, \Omega_\rho) = 0$. □

## 3. Main results

We now give our results on the existence of multiple positive solutions of BVP (1.1) and (1.2).

**Theorem 3.1.** *Assume that one of the following conditions holds*:

$(H_3)$ *There exist* $\rho_1, \rho_2, \rho_3 \in (0, \infty)$ *with* $\rho_1 < \gamma\rho_2$ *and* $\rho_2 < \rho_3$ *such that*

$$f_0^{\rho_1} \leqslant \phi_p(m), \qquad f_{\gamma\rho_2}^{\rho_2} \geqslant \phi_p(M\gamma), \qquad x \neq Ax \quad \text{for } x \in \partial\Omega_{\rho_2} \quad \text{and} \\ f_0^{\rho_3} \leqslant \phi_p(m).$$

$(H_4)$ *There exist* $\rho_1, \rho_2, \rho_3 \in (0, \infty)$ *with* $\rho_1 < \rho_2 < \gamma\rho_3$ *such that*

$$f_{\gamma\rho_1}^{\rho_1} \geqslant \phi_p(M\gamma), \qquad f_0^{\rho_2} \leqslant \phi_p(m), \qquad x \neq Ax \quad \text{for } x \in \partial K_{\rho_2} \quad \text{and}$$
$$f_{\gamma\rho_3}^{\rho_3} \geqslant \phi_p(M\gamma).$$

*Then system* (1.1) *and* (1.2) *has two positive solutions in* $K$. *Moreover, if in* $(H_3)$ $f_0^{\rho_1} \leqslant \phi_p(m)$ *is replaced by* $f_0^{\rho_1} < \phi_p(m)$, *then system* (1.1) *and* (1.2) *has a third positive solution* $u_3 \in K_{\rho_1}$.

The proof is similar to that given for Theorem 2.10 in [9]. We omit it here.

As a special case of Theorem 3.1 we obtain the following result.

**Corollary 3.1.** *If there exists* $\rho > 0$ *such that one of the following conditions holds*:

$(H_5)$ $0 \leqslant f^0 < \phi_p(m)$, $f_{\gamma\rho}^{\rho} \geqslant \phi_p(M\gamma)$, $x \neq Ax$ *for* $x \in \partial\Omega_\rho$ *and* $0 \leqslant f^\infty < \phi_p(m)$,
$(H_6)$ $\phi_p(M) < f_0 \leqslant \infty$, $f_0^{\rho} \leqslant \phi_p(m)$, $x \neq Ax$ *for* $x \in \partial K_\rho$ *and* $\phi_p(M) < f_\infty \leqslant \infty$,

*then* (1.1) *and* (1.2) *has two positive solutions in* $K$.

**Proof.** We show that $(H_5)$ implies $(H_3)$. It is easy to verify that $0 \leqslant f^0 < \phi_p(m)$ implies that there exists $\rho_1 \in (0, \gamma\rho)$ such that $f_0^{\rho_1} < \phi_p(m)$. Let $k \in (f^\infty, \phi_p(m))$. Then there exists $r > \rho$ such that $\max_{t\in[0,1]} f(t,u) \leqslant k\phi_p(u)$ for $u \in [r, \infty)$ since $0 \leqslant f^0 < \phi_p(m)$. Let

$$\beta = \max\Big\{\max_{t\in[0,1]} f(t,u)\colon\ 0 \leqslant u \leqslant r\Big\} \quad \text{and} \quad \rho_3 > \phi_q\left(\frac{\beta}{\phi_p(m) - k}\right).$$

Then we have

$$\max_{t\in[0,1]} f(t,u) \leqslant k\phi_p(u) + \beta \leqslant k\phi_p(\rho_3) + \beta < \phi_p(m)\phi_p(\rho_3) \quad \text{for } u \in [0, \rho_3].$$

This implies that $f_0^{\rho_3} < \phi_p(m)$ and $(H_3)$ holds. Similarly $(H_6)$ implies $(H_4)$. □

By an argument similar to that of Theorem 3.1 we obtain the following results.

**Theorem 3.2.** *Assume that one of the following conditions holds*:

$(H_7)$ *There exist* $\rho_1, \rho_2 \in (0, \infty)$ *with* $\rho_1 < \gamma\rho_2$ *such that* $f_0^{\rho_1} \leqslant \phi_p(m)$ *and* $f_{\gamma\rho_2}^{\rho_2} \geqslant \phi_p(M\gamma)$.
$(H_8)$ *There exist* $\rho_1, \rho_2 \in (0, \infty)$ *with* $\rho_1 < \rho_2$ *such that* $f_{\gamma\rho_1}^{\rho_1} \geqslant \phi_p(M\gamma)$ *and* $f_0^{\rho_2} \leqslant \phi_p(m)$.

*Then* (1.1) *and* (1.2) *has a positive solution in* $K$.

As a special case of Theorem 3.2 we obtain the following result.

**Corollary 3.2.** *Assume that one of the following conditions holds*:

$(H_9)$ $0 \leqslant f^0 < \phi_p(m)$ *and* $\phi_p(M) < f_\infty \leqslant \infty$.
$(H_{10})$ $0 \leqslant f^\infty < \phi_p(m)$ *and* $\phi_p(M) < f_0 \leqslant \infty$.

*Then* (1.1) *and* (1.2) *has a positive solution in* $K$.

## Acknowledgment

The author thanks the referee for his (or her) valuable suggestions regarding the original manuscript.

## References

[1] V.A. Il'in, E.I. Moiseev, Nonlocal boundary value problem of the second kind for a Sturm–Liouville operator, Differential Equations 23 (1987) 979–987.
[2] W. Feng, On an m-point boundary value problem, Nonlinear Anal. 30 (1997) 5369–5374.
[3] W. Feng, J.R.L. Webb, Solvability of m-point boundary value problem with nonlinear growth, J. Math. Anal. Appl. 212 (1997) 467–480.
[4] C.P. Gupta, A generalized multi-point boundary value problem for second order ordinary differential equations, Appl. Math. Comput. 89 (1998) 133–146.
[5] C. Bai, J. Fang, Existence of multiple positive solutions for nonlinear m-point boundary value problems, J. Math. Anal. Appl. 281 (2003) 76–85.
[6] C. Bai, J. Fang, Existence of multiple positive solutions for nonlinear m-point boundary value problems, Appl. Math. Comput. 140 (2003) 297–305.
[7] R. Ma, Existence of solutions of nonlinear m-point boundary-value problems, J. Math. Anal. Appl. 256 (2001) 556–567.
[8] D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, San Diego, CA, 1988.
[9] K.Q. Lan, Multiple positive solutions of semilinear differential equations with singularities, J. London Math. Soc. 63 (2001) 690–704.