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J. Math. Anal. Appl. 317 (2006) 442–447
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Multiple positive solutions of singular second-order m-point boundary value problems ☆

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Received 27 July 2004
Available online 10 February 2006
Submitted by J. Henderson

Abstract

The existence of multiple positive solutions about the singular second-order m-point boundary value problem

$$\varphi''(x) + h(x) f\big(\varphi(x)\big) = 0, \quad 0 < x < 1,$$

subject to some m-point boundary value conditions is considered. $h(x)$ is allowed to be singular at $x = 0$ and $x = 1$.

Keywords: Second-order singular equation; m-Point boundary value problem; Multiple positive solutions

Recently, the existence of multiple positive solutions for nonlinear ordinary differential equations has been studied extensively, and we refer the reader to [2,3,5,6,8,9]. For the most part, each of the papers on the existence of triple positive solutions makes an application of the fixed point theorems by Leggett and Williams [7] and Avery [1]. In this paper, we apply the Avery Five Functional Fixed Point Theorem to obtain the existence of multiple positive solutions to the singular second-order m-point boundary value problem

$$\varphi''(x) + h(x) f\big(\varphi(x)\big) = 0, \quad 0 < x < 1, \tag{1}$$

☆ The project supported by the National Natural Science Foundation of China (10371013, 10371066).
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.08.020

subject to the boundary value conditions

$$\varphi(0) = 0, \qquad \varphi(1) = \sum_{i=1}^{m-2} a_i \varphi(\xi_i), \tag{2}$$

$$\varphi(0) = \sum_{i=1}^{m-2} a_i \varphi(\xi_i), \qquad \varphi(1) = 0, \tag{3}$$

$$\varphi'(0) = 0, \qquad \varphi(1) = \sum_{i=1}^{m-2} a_i \varphi(\xi_i), \tag{4}$$

$$\varphi(0) = \sum_{i=1}^{m-2} a_i \varphi(\xi_i), \qquad \varphi'(1) = 0, \tag{5}$$

respectively, where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $a_i \in [0,\infty)$ and $h(x)$ may be singular at $x = 0$ and $x = 1$. For the theory of cones in Banach spaces we refer to [4].

α is said to be a nonnegative concave functional on a cone P of a real Banach space E, if $\alpha : P \to [0,\infty)$ and $\alpha(tx + (1-t)y) \geqslant t\alpha(x) + (1-t)\alpha(y)$ for all $x, y \in P$ and $t \in [0,1]$. β is said to be a nonnegative convex functional on a cone P of a real Banach space E, if $\beta : P \to [0,\infty)$ and $\beta(tx + (1-t)y) \leqslant t\alpha(x) + (1-t)\alpha(y)$ for all $x, y \in P$ and $t \in [0,1]$.

Let γ, β, θ be nonnegative continuous concave functionals on P and α, ψ be nonnegative continuous convex functionals on P. For nonnegative real numbers l, a, b, d and c, define the convex sets

$$\begin{aligned}
&P(\gamma, c) = \bigl\{x \in P\colon\ \gamma(x) \leqslant c\bigr\},\\
&P(\gamma, \alpha, a, c) = \bigl\{x \in P\colon\ a \leqslant \alpha(x),\ \gamma(x) \leqslant c\bigr\},\\
&Q(\gamma, \beta, d, c) = \bigl\{x \in P\colon\ \beta(x) \leqslant d,\ \gamma(x) \leqslant c\bigr\},\\
&P(\gamma, \theta, \alpha, a, b, c) = \bigl\{x \in P\colon\ a \leqslant \alpha(x),\ \theta(x) \leqslant b,\ \gamma(x) \leqslant c\bigr\},\\
&Q(\gamma, \beta, \psi, l, d, c) = \bigl\{x \in P\colon\ l \leqslant \psi(x),\ \beta(x) \leqslant d,\ \gamma(x) \leqslant c\bigr\}.
\end{aligned}$$

The following theorem is the Five Functional Fixed Point Theorem [1], a generalization of the Leggett–Williams Fixed Point Theorem.

Theorem 1. *Suppose there exist $c > 0$ and $m > 0$ such that $\alpha(x) \leqslant \beta(x)$ and $\|x\| \leqslant m\gamma(x)$ for all $x \in P(\gamma, c)$. Let $A : P(\gamma, c) \to P(\gamma, c)$ be a completely continuous operator. If there exist nonnegative real numbers a, b, d and l with $0 < d < a$ such that*

(i) $\{x \in P(\gamma, \theta, \alpha, a, b, c)\colon\ \alpha(x) > a\} \neq \emptyset$ *and* $\alpha(Ax) > a$ *for* $x \in P(\gamma, \theta, \alpha, a, b, c)$;
(ii) $\{x \in Q(\gamma, \beta, \psi, l, d, c)\colon\ \beta(x) < d\} \neq \emptyset$ *and* $\beta(Ax) < d$ *for* $x \in Q(\gamma, \beta, \psi, l, d, c)$;
(iii) $\alpha(Ax) > a$ *for* $x \in P(\gamma, \alpha, a, c)$ *with* $\theta(Ax) > b$;
(iv) $\beta(Ax) < d$ *for* $x \in Q(\gamma, \beta, d, c)$ *with* $\psi(Ax) < l$,

then A has at least three fixed points x_1, x_2 and x_3 in $P(\gamma, c)$ with $\beta(x_1) < d$, $a < \alpha(x_2)$, $d < \beta(x_3)$ and $\alpha(x_3) < a$.

In Banach space $C[0,1]$ in which the norm is defined by $\|\varphi\| = \max_{0\leqslant x\leqslant 1} |\varphi(x)|$, we set

$$P = \{\varphi \in C[0,1]\colon\ \varphi(x) \geqslant 0,\ x \in [0,1]\},$$

then P is a cone in $C[0,1]$. φ is called to be a positive solution of (1)–(i) if $\varphi \in C[0,1] \cap C^2(0,1)$, $\varphi(x) > 0$, $x \in (0,1)$ and satisfies (1)–(i) ($i = 2,3,4,5$).

Suppose that

(H$_1$) $\sum_{i=1}^{m-2} a_i < 1$.

(H$_2$) $h : (0,1) \to [0,+\infty)$ is continuous, $h(x) \not\equiv 0$, and

$$\int_0^1 h(x)\,dx < +\infty. \tag{6}$$

(H$_3$) $f : [0,+\infty) \to [0,+\infty)$ is continuous.

We first consider (1)–(2) and let

$$k(x,y) = \begin{cases} x(1-y), & 0 \leqslant x \leqslant y \leqslant 1, \\ y(1-x), & 0 \leqslant y \leqslant x \leqslant 1. \end{cases} \tag{7}$$

Denote under (H$_1$)

$$K(x,y) = k(x,y) + x\left(1 - \sum_{i=1}^{m-2} a_i\xi_i\right)^{-1} \sum_{i=1}^{m-2} a_i k(\xi_i, y), \quad 0 \leqslant x,\ y \leqslant 1. \tag{8}$$

Set

$$(A\varphi)(x) = \int_0^1 K(x,y)h(y)f\big(\varphi(y)\big)\,dy, \quad x \in [0,1]. \tag{9}$$

It follows from [10] that if (H$_1$)–(H$_3$) are satisfied, $A : P \to P$ is completely continuous and that if A has a fixed point $\varphi \neq \theta$, then φ is the positive solution of (1)–(2).

Let $\delta = \sum_{i=1}^{m-2} a_i$. It is easy to see that $\forall x,\ y \in [0,1]$,

$$0 \leqslant k(x,y) \leqslant K(x,y) \leqslant \frac{1}{4} + \frac{\delta}{1-\delta} \triangleq \sigma_1. \tag{10}$$

Take $\tau \in (0, \frac{1}{2})$ such that $h(x) \not\equiv 0$, $x \in [\tau, 1-\tau]$. Obviously,

$$k(x,y) \geqslant \tau k(y,y), \quad x \in [\tau, 1-\tau],\ y \in [0,1], \tag{11}$$

and $k(x,x) \geqslant \tau(1-\tau)$, $\forall x \in [\tau, 1-\tau]$. Denote

$$\sigma_2 = \frac{1}{\tau^2(1-\tau)}, \qquad h_0 = \int_0^1 h(x)\,dx, \qquad h_\tau = \int_\tau^{1-\tau} h(x)\,dx. \tag{12}$$

It is easy to see that $\sigma_1 \geqslant \frac{1}{4}$, $\sigma_2 > 8$, $h_0 h_\tau^{-1} \geqslant 1$, thus $h_0 h_\tau^{-1} \sigma_1 \sigma_2 > 2$.

Theorem 2. *Suppose that* (H$_1$)–(H$_3$) *are satisfied. If there exist* $0 < d < a < c$ ($c > h_0 h_\tau^{-1}\sigma_1\sigma_2 a$) *such that*

$$f(u) < dh_0^{-1}\sigma_1^{-1}, \quad \forall 0 \leqslant u \leqslant d, \tag{13}$$

$$f(u) \leqslant \sigma_2 h_\tau^{-1}\varepsilon_0 a, \quad \forall d \leqslant u \leqslant c, \tag{14}$$

$$f(u) \geqslant \sigma_2 h_\tau^{-1} a, \quad \forall a \leqslant u \leqslant c, \tag{15}$$

where $\varepsilon_0 \geqslant 1$ *with* $\varepsilon_0 h_0 h_\tau^{-1}\sigma_1\sigma_2 a < c$, *then* (1)–(2) *has at least three nonnegative solutions in which there are at least two positive solutions.*

Proof. Let $l = 0$, $m = 1$ and for $\varphi \in P$,

$$\alpha(\varphi) = \min_{\tau \leqslant x \leqslant 1-\tau} \varphi(x), \qquad \psi(\varphi) \equiv 0, \qquad \gamma(\varphi) = \beta(\varphi) = \theta(\varphi) = \|\varphi\|.$$

It is easy to see that γ, β, θ are nonnegative continuous convex functionals on P and α, ψ are nonnegative continuous concave functionals on P. Moreover, $\alpha(\varphi) \leqslant \beta(\varphi)$ and $\|\varphi\| \leqslant m\gamma(\varphi)$ for $\varphi \in P$. In the following we shall show that the conditions in Theorem 1 are satisfied with $b = c$.

Clearly we have

$$\{\varphi \in P(\gamma, \theta, \alpha, a, b, c)\colon\ \alpha(\varphi) > a\} = \Big\{\varphi \in P\colon\ \min_{\tau \leqslant x \leqslant 1-\tau} \varphi(x) > a,\ \|\varphi\| \leqslant c\Big\} \neq \emptyset.$$

From (11), (12) and (15) we have that for $\varphi \in P(\gamma, \theta, \alpha, a, b, c)$,

$$\alpha(A\varphi) = \min_{\tau \leqslant x \leqslant 1-\tau} (A\varphi)(x) = \min_{\tau \leqslant x \leqslant 1-\tau} \int_0^1 K(x,y)h(y)f\big(\varphi(y)\big)\,dy$$

$$\geqslant \tau \int_\tau^{1-\tau} k(y,y)h(y)f\big(\varphi(y)\big)\,dy > \tau^2(1-\tau)\int_\tau^{1-\tau} h(y)\cdot\sigma_2 h_\tau^{-1} a\,dy = a. \tag{16}$$

Therefore (i) holds.

It is obvious that

$$\{\varphi \in Q(\gamma, \beta, \psi, l, d, c)\colon\ \beta(\varphi) < d\} = \{\varphi \in P\colon\ \|\varphi\| < d\} \neq \emptyset.$$

It follows from (10) and (13) that for $\varphi \in Q(\gamma, \beta, \psi, l, d, c)$,

$$\beta(A\varphi) = \|A\varphi\| = \max_{0 \leqslant x \leqslant 1} \int_0^1 K(x,y)h(y)f\big(\varphi(y)\big)\,dy \leqslant \sigma_1 \int_0^1 h(y)f\big(\varphi(y)\big)\,dy < d.$$

Thus (ii) is true.

Since

$$P(\gamma, \alpha, a, c) = \Big\{\varphi \in P\colon\ \min_{\tau \leqslant x \leqslant 1-\tau} \varphi(x) \geqslant a,\ \|\varphi\| \leqslant c\Big\},$$

we have $\alpha(A\varphi) > a$ for $\varphi \in P(\gamma, \alpha, a, c)$ according to (16). This means that (iii) is satisfied.

As far as (iv) is concerned, we notice that $\psi(A\varphi) < l = 0$ is impossible.

By Theorem 1, A has at least three fixed points φ_1, φ_2 and φ_3 in P satisfying

$$\|\varphi_1\| < d, \qquad \min_{\tau \leqslant x \leqslant 1-\tau} \varphi_2(x) > a, \qquad d < \|\varphi_3\|, \qquad \min_{\tau \leqslant x \leqslant 1-\tau} \varphi_3(x) < a.$$

Therefore, (1)–(2) has at least three nonnegative solutions in which there are at least two positive solutions. □

Example 1. Let $h(x) = \frac{1}{\sqrt{x(1-x)}}$,

$$f(u) = \begin{cases} \frac{1}{5\pi}(3-u), & 0 \leqslant u < 1, \\ \frac{2}{5\pi}(319u - 318), & 1 \leqslant u < 2, \\ \frac{4}{15\pi}(u + 478), & u \geqslant 2. \end{cases}$$

Then the singular three-point boundary value problem

$$\begin{cases} \varphi''(x) + h(x) f(\varphi(x)) = 0, & 0 < x < 1, \\ \varphi(0) = 0, \quad \varphi(1) = \frac{1}{2}\varphi(\eta), & 0 < \eta < 1, \end{cases} \tag{17}$$

has at least three positive solutions.

Proof. Obviously, $h(x)$ is singular at both $x = 0$ and $x = 1$ and (H_2) is satisfied by the properties of Euler integral.

Choose $\tau = \frac{1}{4}$ and take $d = 1,\ a = 2,\ c = 242,\ \varepsilon_0 = \frac{3}{2}$ in Theorem 2. It is easy to verify that the conditions in Theorem 2 are satisfied. By Theorem 2 and $f(0) > 0$, we know that the singular three-point boundary value problem (17) has at least three positive solutions. □

For the case subject to the other boundary value conditions, we obtain the same existence results of multiple positive solutions as in Theorem 2. In the following we point out the differences briefly.

For the case subject to the boundary value condition (3), let

$$K_1(x, y) = k(x, y) + (1 - x)\left(1 - \sum_{i=1}^{m-2} a_i(1 - \xi_i)\right)^{-1} \sum_{i=1}^{m-2} a_i k(\xi_i, y), \quad 0 \leqslant x,\ y \leqslant 1, \tag{18}$$

where $k(x, y)$ is defined by (7).

For the case subject to the boundary value condition (4), let

$$k^*(x, y) = \begin{cases} 1 - y, & 0 \leqslant x \leqslant y \leqslant 1, \\ 1 - x, & 0 \leqslant y \leqslant x \leqslant 1, \end{cases} \tag{19}$$

$$K_2(x, y) = k^*(x, y) + \left(1 - \sum_{i=1}^{m-2} a_i\right)^{-1} \sum_{i=1}^{m-2} a_i k^*(\xi_i, y), \quad 0 \leqslant x,\ y \leqslant 1. \tag{20}$$

Take $\sigma_1 = 1 + \frac{\delta}{1-\delta},\ \sigma_2 = \frac{1}{\tau^2}$.

For the case subject to the boundary value condition (5), let

$$k^{**}(x, y) = \begin{cases} x, & 0 \leqslant x \leqslant y \leqslant 1, \\ y, & 0 \leqslant y \leqslant x \leqslant 1, \end{cases} \tag{21}$$

$$K_3(x, y) = k^{**}(x, y) + \left(1 - \sum_{i=1}^{m-2} a_i\right)^{-1} \sum_{i=1}^{m-2} a_i k^{**}(\xi_i, y), \quad 0 \leqslant x,\ y \leqslant 1. \tag{22}$$

Take $\sigma_1 = 1 + \frac{\delta}{1-\delta},\ \sigma_2 = \frac{1}{\tau^2}$.

Example 2. Let $h(x) = \frac{1}{\sqrt{x(1-x)}}$,

$$f(u) = \begin{cases} \frac{1}{12\pi}(4-u), & 0 \leqslant u < 1, \\ \frac{1}{4\pi}(383u - 382), & 1 \leqslant u < 2, \\ \frac{1}{6\pi}(u + 574), & u \geqslant 2. \end{cases}$$

Then the singular three-point boundary value problem

$$\begin{cases} \varphi''(x) + h(x) f(\varphi(x)) = 0, & 0 < x < 1, \\ \varphi'(0) = 0, \quad \varphi(1) = \frac{1}{2}\varphi(\eta), & 0 < \eta < 1, \end{cases} \tag{23}$$

has at least three positive solutions.

Proof. Choose $\tau = \frac{1}{4}$ and take $d = 1,\ a = 2,\ c = 290,\ \varepsilon_0 = \frac{3}{2}$ in Theorem 2 for the version that is in the case of (1)–(4). It is easy to verify that the conditions in the theorem are satisfied. By Theorem 2 and $f(0) > 0$, we know that the singular three-point boundary value problem (23) has at least three positive solutions. □

Acknowledgments

The authors express their gratitude to the referees for their valuable comments and suggestions.

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