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Positive solutions of multiparameter semipositone p-Laplacian problems

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Abstract

We obtain multiple positive solutions of multiparameter semipositone p-Laplacian problems using the sub- and supersolution method and the mountain pass lemma.

Keywords: p-Laplacian problems; Semipositone; Multiparameter; Multiple positive solutions; Sub- and supersolutions; Variational methods

1. Introduction

We consider the problem

$$\begin{cases} -\Delta_p u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, is a bounded domain with the boundary $\partial\Omega \in C^2$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian of u, $1 < p < \infty$, $\lambda > 0$ and $\mu \in \mathbb{R}$ are parameters, and f and g are Carathéodory functions on $\Omega \times (0, \infty)$ such that

$$\left|f(x,t)\right| \leqslant a_1 t^{q-1} + a_2 \tag{1.2}$$

for some $1 \leqslant q < p$ and constants $a_1, a_2 \geqslant 0$,

$$f(x,t) \geqslant a_3, \quad t \geqslant t_1, \tag{1.3}$$

for some $a_3, t_1 > 0$, and g is bounded on bounded sets. We make no assumptions about the signs of $f(x,0)$ and $g(x,0)$ and hence allow the semipositone case $\lambda f(x,0) + \mu g(x,0) < 0$.

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In the semilinear case $p = 2$, Caldwell, Shivaji, and Zhu [2] studied the evolution of solution curves of (1.1) as $\lambda, \mu > 0$ vary for the ODE case $n = 1$. For some related results in the case $p = 2$ and $n \geqslant 2$ see Caldwell [1]. Here we seek weak solutions in the general quasilinear case $1 < p < \infty$, $n \geqslant 1$.

Recall that a weak solution of (1.1) is a positive function u in the Sobolev space $W_0^{1,p}(\Omega)$ satisfying

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v - \big(\lambda f(x,u) + \mu g(x,u)\big)v = 0 \quad \forall v \in W_0^{1,p}(\Omega). \tag{1.4}$$

Bounded weak solutions are of class $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha > 0$ by Lieberman [4], and every weak solution is bounded by Guedda and Véron [3] if g grows at most critically.

Our first result imposes no growth restrictions on g.

Theorem 1.1. *There is $\lambda_0 > 0$ such that for each $\lambda \geqslant \lambda_0$, there is $m(\lambda) > 0$ for which* (1.1) *has a $C^{1,\alpha}(\overline{\Omega})$ solution whenever $|\mu| \leqslant m(\lambda)$.*

Denote by

$$p^* = \begin{cases} np/(n-p), & n > p, \\ \infty, & n \leqslant p, \end{cases} \qquad G(x,t) = \int_0^t g(x,s)\,ds \tag{1.5}$$

the critical Sobolev exponent and the primitive of g, respectively.

Theorem 1.2. *Let λ_0 be as in Theorem* 1.1. *Then for each $\lambda \geqslant \lambda_0$, there is $\tilde{m}(\lambda) \in (0, m(\lambda))$ for which* (1.1) *has two $C^{1,\alpha}(\overline{\Omega})$ solutions whenever $0 < \mu \leqslant \tilde{m}(\lambda)$ in the following cases*:

(i) *g is subcritical and p-superlinear*

$$\big|g(x,t)\big| \leqslant a_4 t^{r-1} + a_5 \tag{1.6}$$

for some $1 \leqslant r < p^*$ and $a_4, a_5 \geqslant 0$ and

$$0 < \theta G(x,t) \leqslant t g(x,t), \quad t \geqslant t_2, \tag{1.7}$$

for some $\theta > p$ and $t_2 > 0$,

(ii) *$n > p$ and $g(x,t) = t^{p^*-1}$.*

Example 1.3. The problem

$$\begin{cases} -\Delta_p u = \lambda(u^{q-1} - 1) + \mu u^{r-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.8}$$

with $1 \leqslant q < p$, $r \geqslant 1$, and large $\lambda > 0$ has

(i) a solution if $|\mu|$ is small,
(ii) two solutions if $p < r \leqslant p^*$ and $\mu > 0$ is small.

2. Proofs

We begin by constructing a positive subsolution. Let $\lambda_1 > 0$ and $0 < \varphi_1 \leqslant 1$ be the first Dirichlet eigenvalue of $-\Delta_p$ on Ω and the corresponding eigenfunction, respectively. By (1.2) and (1.3), $f \geqslant -a_6$ for some $a_6 > 0$. Let

$$1 < \beta < \frac{p}{p-1}, \quad a_7 > \frac{\lambda_1 a_6 \beta^{p-1}}{a_3}, \quad c_\lambda = \left(\frac{\lambda a_6 + 1}{a_7}\right)^{\frac{1}{p-1}}, \quad \underline{u} = c_\lambda \varphi_1^\beta. \tag{2.1}$$

Lemma 2.1. *$\underline{u}$ is a subsolution of* (1.1) *for λ sufficiently large and $|\mu|$ small.*

Proof. We have

$$-\Delta_p \varphi_1^\beta = \beta^{p-1}\left(\lambda_1 \varphi_1^{\beta(p-1)} - (\beta-1)(p-1)\frac{|\nabla\varphi_1|^p}{\varphi_1^{1-(\beta-1)(p-1)}}\right). \tag{2.2}$$

Since $\varphi_1 = 0$ and $\nabla\varphi_1 \neq 0$ on $\partial\Omega$, in some neighborhood $\Omega' \subset \Omega$ of $\partial\Omega$ the right-hand side of (2.2) is $\leqslant -a_7$ and hence

$$-\Delta_p \underline{u} \leqslant -c_\lambda^{p-1} a_7 = -(\lambda a_6 + 1) \leqslant \lambda f(x, \underline{u}) - 1. \tag{2.3}$$

On $\Omega \setminus \Omega'$, $\varphi_1 \geqslant a_8$ for some $a_8 > 0$ and hence

$$-\Delta_p \underline{u} \leqslant \lambda_1 (c_\lambda \beta)^{p-1} = \frac{\lambda_1(\lambda a_6 + 1)\beta^{p-1}}{a_7} \leqslant \lambda a_3 - 1 \leqslant \lambda f(x, \underline{u}) - 1 \tag{2.4}$$

for λ so large that the second inequality holds, which is possible by the choice of a_7, and $c_\lambda a_8^\beta \geqslant t_1$. Now take $|\mu|$ so small that $|\mu g(x, \underline{u})| \leqslant 1$. $\square$

Proof of Theorem 1.1. Let $s > 1/(p-q)$, ψ be the solution of

$$\begin{cases} -\Delta_p \psi = 1 & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

and $\bar{u} = \lambda^s \psi$. For λ large and $|\mu|$ small,

$$-\Delta_p \bar{u} = \lambda^{s(p-1)} \geqslant \lambda\big(a_1 \lambda^{s(q-1)} \psi^{q-1} + a_2\big) + 1 \geqslant \lambda f(x, \bar{u}) + \mu g(x, \bar{u}) \tag{2.6}$$

by (1.2) and hence $\bar{u}$ is a supersolution of (1.1), and

$$-\Delta_p \bar{u} \geqslant \frac{\lambda_1(\lambda a_6 + 1)\beta^{p-1}}{a_7} \geqslant -\Delta_p \underline{u} \tag{2.7}$$

and hence $\bar{u} \geqslant \underline{u}$ by the weak comparison principle. A standard argument now gives a solution in the order interval $[\underline{u}, \bar{u}]$. $\square$

Proof of Theorem 1.2. Let $\underline{u}$ be the subsolution constructed in Theorem 1.1.

(i) Let

$$\tilde{f}(x,t) = \begin{cases} f(x,t), & t \geqslant \underline{u}(x), \\ f(x, \underline{u}(x)), & t < \underline{u}(x), \end{cases} \qquad \tilde{g}(x,t) = \begin{cases} g(x,t), & t \geqslant \underline{u}(x), \\ g(x, \underline{u}(x)), & t < \underline{u}(x) \end{cases} \tag{2.8}$$

and consider the problem

$$\begin{cases} -\Delta_p u = \lambda \tilde{f}(x,u) + \mu \tilde{g}(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.9}$$

Weak solutions of (2.9) are $\geqslant \underline{u}$ by the weak comparison principle, and hence also solve (1.1), and coincide with the critical points of the C^1 functional

$$\Phi(u) = \int_\Omega \frac{1}{p}|\nabla u|^p - \lambda \widetilde{F}(x,u) - \mu \widetilde{G}(x,u), \quad u \in W = W_0^{1,p}(\Omega), \tag{2.10}$$

where

$$\widetilde{F}(x,t) = \int_0^t \tilde{f}(x,s)\,ds, \qquad \widetilde{G}(x,t) = \int_0^t \tilde{g}(x,s)\,ds. \tag{2.11}$$

By (1.7), Φ satisfies the Palais–Smale compactness condition (PS).

By (1.2), (1.6), and the Sobolev imbedding theorem,

$$\Phi(u) \geqslant \frac{1}{p}\|u\|^p - a_\lambda\big(\|u\|^q + 1\big) - \mu a_9\big(\|u\|^r + 1\big) \tag{2.12}$$

for some $a_\lambda, a_9 > 0$, so

$$\inf_{\partial B_R} \Phi > 0, \tag{2.13}$$

where $B_R = \{u \in W\colon\ \|u\| < R\}$, for sufficiently large $R > 0$ and small μ. Since $\Phi(0) = 0$, by weak lower semicontinuity Φ attains its minimum on $\overline{B_R}$ at a level $\leqslant 0$ and hence at a point in B_R, which then is a local minimizer.

By (1.7),

$$\widetilde{G}(x,t) \geqslant a_{10}t^\theta - a_{11} \tag{2.14}$$

for some $a_{10}, a_{11} > 0$, so $\Phi(t_3\varphi_1) < 0$ for sufficiently large $t_3 > R/\|\varphi_1\|$. Now the mountain pass lemma gives a second critical point at the level

$$c = \inf_{\gamma\in\Gamma}\ \max_{u\in\gamma([0,1])} \Phi(u) \geqslant \inf_{\partial B_R} \Phi, \tag{2.15}$$

where $\Gamma = \{\gamma \in C([0,1], W)\colon\ \gamma(0) = 0,\ \gamma(1) = t_3\varphi_1\}$ is the class of paths in W joining 0 and $t_3\varphi_1$.

(ii) Let γ_0 be the line segment joining 0 and $t_3\varphi_1$ and Φ_0 be the functional obtained by setting $\mu = 0$ in Φ. Since $\mu > 0$ and $\widetilde{G}(x,t) \geqslant 0$ for $t \geqslant 0$,

$$c \leqslant \max_{u\in\gamma_0([0,1])} \Phi(u) \leqslant \max_{u\in\gamma_0([0,1])} \Phi_0(u) = c_0. \tag{2.16}$$

By Proposition 3.4 of Silva and Xavier [5], Φ satisfies the (PS) condition at all levels $\leqslant c_0$ for sufficiently small μ, so the conclusion follows as before. □

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