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Strong asymptotics for Sobolev orthogonal polynomials in the complex plane

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Abstract

We obtain the strong asymptotics for the sequence of monic polynomials minimizing the norm

$$\|q\|_S = \left(\sum_{k=0}^{N} \|q^{(k)}\|_k^2\right)^{1/2},$$

where $\|\cdot\|_k$, $k = 0, \dots, N-1$, are L^2 norms with respect to measures supported on the same rectifiable Jordan closed curve or arc Γ, and $\|\cdot\|_N$ is the L^2 norm corresponding to a weight supported on Γ, which satisfies the Szegő condition, plus mass points in the unbounded connected component of $\mathbb{C} \setminus \Gamma$.

Keywords: Sobolev orthogonal polynomials; Asymptotic behavior; Szegő's condition

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1. Introduction

Let Γ be an arc or a closed rectifiable Jordan curve in the complex plane. For simplicity, we assume that the parametrization of Γ with respect to the arc length is a complex valued function f defined on an interval $[a,b]\subset\mathbb{R}$. We assume that $\Gamma\in C^{2+}$; that is, f has second derivative which satisfies the Lipschitz condition

$$\left|f''(x)-f''(y)\right|\leqslant C|x-y|^{\alpha},$$

for some constants $C,\alpha>0$ and for all $x,y\in[a,b]$.

By Ω we denote the unbounded connected component of $\overline{\mathbb{C}}\setminus\Gamma$. Throughout this paper we assume that $\{z_1,\dots,z_m\}\subset\Omega$ is a finite set of points, and $\{\mu_k\}_{k=0}^{N}$ a set of $N+1$ finite positive Borel measures supported on Γ, where μ_N is such that $d\mu_N(\xi)=\rho_N(\xi)|d\xi|$. On the vector space of polynomials $\mathbb{P}$, we consider the inner products

$$\langle p,q\rangle_k=\int_\Gamma p(\xi)\overline{q(\xi)}\,d\mu_k(\xi),\quad k=0,\dots,N-1, \tag{1}$$

$$\langle p,q\rangle_N=\int_\Gamma p(\xi)\overline{q(\xi)}\rho_N(\xi)|d\xi|+p(Z)\mathcal{A}q(Z)^*,$$
$$p(Z)=\left(p(z_1),\dots,p^{(d_1)}(z_1),p(z_2),\dots,p^{(d_2)}(z_2),\dots,p(z_m),\dots,p^{(d_m)}(z_m)\right), \tag{2}$$

and

$$\langle p,q\rangle_S=\sum_{k=0}^{N}\left\langle p^{(k)},q^{(k)}\right\rangle_k, \tag{3}$$

where $p^{(k)}$ denotes the kth derivative of p, $p(Z)^*$ is the transposed conjugate vector of $p(Z)$, and $\mathcal{A}$ is a hermitian positive definite matrix of order $M=m+\sum_{i=1}^{m}d_i$. The norms corresponding to (1), (2), and (3) on $\mathbb{P}$ are denoted $\|\cdot\|_k$, $\|\cdot\|_N$, and $\|\cdot\|_S$, respectively. (3) is called a Sobolev inner product and (2) a discrete Sobolev inner product. Notice that only derivatives of order $\geqslant N$ are evaluated at the points z_k, $k=1,\dots,m$, in (3). The nth monic orthogonal polynomial with respect to the inner product (3) is the unique polynomial Q_n of degree n and leading coefficient equal to 1, such that

$$\kappa_n=\|Q_n\|_S^2=\inf\left\{\|q\|_S^2\colon\ q(z)=z^n+\cdots\right\}. \tag{4}$$

In the past two decades, the study of the algebraic and asymptotic properties of sequences of Sobolev orthogonal polynomials has attracted much attention. Let us mention some results related with special cases of (3) dealing with general classes of measures.

For $N=0$, in [4] the authors study the limit of the sequence $\{Q_n/P_n\}$, $n\in\mathbb{N}$, where $\{P_n\}$, $n\in\mathbb{N}$, is the sequence of monic orthogonal polynomials with respect to μ_0 and $\mu_0'>0$ on its support consisting of a bounded interval of $\mathbb{R}$. In [1], the authors consider a similar problem for general measures μ_0 in the Szegő class supported on an arc or a closed rectifiable Jordan curve in $\mathbb{C}$.

When $N>0$ and $\mathcal{A}\equiv\mathbf{0}$ (known as the continuous case) the strong asymptotic of Sobolev orthogonal polynomials and their first derivative ($N=1$) was studied in [5] assuming that μ_0 and μ_1 belong to the Szegő class. A natural extension when $N>1$ was given in [6].

In this work, we extend the results on strong asymptotics contained in the papers mentioned above. To this aim, we compare the norms and the monic orthogonal polynomials Q_n with respect to the general inner product (3) with the norms and the monic orthogonal polynomials with respect to (2).

Let μ be a finite positive Borel measure supported on Γ and $\mu=\rho(\xi)|d\xi|+\mu_s$ its Lebesgue decomposition on Γ with respect to $|d\xi|$. We say that μ satisfies the Szegő condition on Γ, and we write $\mu\in S(\Gamma)$, if

$$\int_\Gamma\left(\log\rho(\xi)\right)\left|\Phi'(\xi)\right||d\xi|>-\infty, \tag{5}$$

where Φ is the conformal mapping of Ω onto the exterior of the unit circle such that

$$\Phi(\infty)=\infty, \qquad \Phi'(\infty)=\lim_{z\to\infty}\frac{\Phi(z)}{z}=\frac{1}{C(\Gamma)}>0.$$

$C(\Gamma)$ is the logarithmic capacity of Γ. In particular, (5) implies that $\rho>0$ almost everywhere on Γ. If μ is absolutely continuous with respect to $|d\xi|$ and verifies (5), we write $\rho\in S(\Gamma)$, and use ρ in place of μ in the notation of norms and polynomials.

Set

$$\tau_n=\|L_n\|_N^2=\inf_{R_n(z)=z^n+\cdots}\|R_n\|_N^2, \tag{6}$$

where $L_n(z)=z^n+\cdots$. Our first result establishes the asymptotic behavior of the Sobolev norms (4).

Theorem 1. *Let $\{\mu_k\}_{k=0}^{N-1}$ be a set of N finite positive Borel measures supported on $\Gamma\in C^{2+}$, $d\mu_N(\xi)=\rho_N(\xi)|d\xi|$, with $\rho_N\in S(\Gamma)$, and $\{z_1,\dots,z_m\}\subset\Omega$ a fixed set of points. Then*

$$\lim_{n\to\infty}\frac{\kappa_n}{n^{2N}\tau_{n-N}}=1. \tag{7}$$

Consider the extremal problem

$$\nu^*(\rho_N)=\min_{f\in\widetilde{H}}\|f\|^2_{H^2(\Omega,\rho_N)}=\|\mathcal{F}^*\|^2_{H^2(\Omega,\rho_N)}, \tag{8}$$

where $\widetilde{H}=\{f\in H^2(\Omega,\rho_N)\colon f(\infty)=1,\ f^{(j)}(z_i)=0,\ j=0,\dots,d_i,\ i=1,\dots,m\}$ (see Section 2 for the definition of $H^2(\Omega,\rho_N)$). Standard arguments show that a minimizing function $\mathcal{F}^*\in\widetilde{H}$ exists and is unique. Notice that

$$(\mathcal{F}^*)^{(j)}(z_i)=0, \quad j=0,\dots,d_i,\ i=1,\dots,m. \tag{9}$$

The function $\mathcal{F}^*$ allows to describe the asymptotic behavior of the Nth derivative of the polynomials Q_n.

By $\lim_{n\to\infty}\varphi_n=\varphi$ in Ω, we denote local uniform convergence in the region Ω of the sequence of functions $\{\varphi_n\}_{n\in\mathbb{N}}$ as $n\to\infty$ (i.e. uniform convergence on each compact subset of Ω).

Theorem 2. *Under the assumptions of Theorem* 1, *we have*

$$Q_n^{(N)}(z)=n^N\big[C(\Gamma)\Phi(z)\big]^{n-N}\mathcal{F}^*(z)\big(1+\varepsilon_n(z)\big), \tag{10}$$

where $\lim_{n\to\infty}\varepsilon_n=0$ *in Ω. Equivalently,*

$$\lim_{n\to\infty}\frac{Q_n^{(N)}(z)}{n^N L_{n-N}(z)}=1 \quad \text{in } \Omega. \tag{11}$$

Notice that the asymptotic behavior of $\{Q_n^{(N)}\}$ only depends on the weight ρ_N and the interaction matrix $\mathcal{A}$. Some of the measures μ_k, $k=0,\dots,N-1$, may even be zero as long as $\langle\cdot,\cdot\rangle_S$ is an inner product in the space of polynomials. If we assume additionally that the measures $\{\mu_k\}_{k=0}^{N-1}$ satisfy the Szegő condition on Γ, the asymptotic behavior of the polynomials Q_n and all its derivatives can be described as follows:

Theorem 3. *If $\mu_j\in S(\Gamma)$, $j=0,\dots,N$, and μ_N is absolutely continuous, then for $k\in\mathbb{Z}_+$*

$$Q_n^{(k)}(z)=n^k\frac{C(\Gamma)^{n-N}\Phi(z)^{n-k}}{(\Phi'(z))^{N-k}}\mathcal{F}^*(z)\big(1+\varepsilon_n(z)\big), \tag{12}$$

where $\lim_{n\to\infty}\varepsilon_n=0$ *in Ω. Equivalently, for $k\in\mathbb{Z}_+$*

$$\lim_{n\to\infty}\frac{Q_n^{(k)}(z)}{n^k L_{n-k}(z)}=\frac{1}{([C(\Gamma)\Phi]'(z))^{N-k}} \quad \text{in } \Omega. \tag{13}$$

From (9), (12), and Hurwitz' theorem, the asymptotic behavior of the zeros of the polynomials Q_n is derived. Let $G = \mathbb{C} \setminus \overline{\Omega}$; therefore, G is an open connected set if Γ is a closed curve, and it is empty if Γ is an arc.

Corollary 1. *Under the assumptions of Theorem* 3, *for n sufficiently large, each point z_i, $i = 1, \dots, m$, attracts exactly $d_i + 1$ zeros of Q_n. The rest of the zeros accumulate on $\Gamma \cup G$.*

The paper is organized as follows. The next section, includes the definitions and auxiliary results needed in the sequel. Section 3 is dedicated to the proof of Theorem 1. Finally, Section 4 contains the proof of Theorems 2 and 3.

2. Definitions and auxiliary results

In order to give a unified treatment to the cases of an arc and a closed curve, we will consider a rectifiable Jordan arc as a two sided cut of the complex plane. If Γ is an arc and g is an analytic function in Ω with boundary values almost everywhere on Γ (with respect to the Lebesgue measure) when we approach the arc from each side, we denote by $g_+(\xi)$ and $g_-(\xi)$ the two boundary values of g at a given point $\xi \in \Gamma$.

If g_+ and g_- are integrable on Γ with respect to the weight ρ, we define

$$\oint_\Gamma g(\xi)\rho(\xi)|d\xi| = \int_\Gamma g_+(\xi)\rho(\xi)|d\xi| + \int_\Gamma g_-(\xi)\rho(\xi)|d\xi|.$$

When Γ is a closed curve and g is analytic on Ω with boundary values $g(\xi)$ for almost all $\xi \in \Gamma$, we define

$$\oint_\Gamma g(\xi)\rho(\xi)|d\xi| = \int_\Gamma g(\xi)\rho(\xi)|d\xi|.$$

Assume that μ is a finite positive Borel measure supported on $\Gamma \in C^{2+}$ and $d\mu = \rho(\xi)|d\xi|$, $\rho \in S(\Gamma)$. Let $P_n(\rho; z)$ denote the nth monic orthogonal polynomial with respect to ρ, and

$$\lambda_n(\rho) = \int_\Gamma \left|P_n(\rho;\xi)\right|^2 \rho(\xi)|d\xi| = \inf_{p(z)=z^n+\cdots} \int_\Gamma \left|p(\xi)\right|^2 \rho(\xi)|d\xi|. \tag{14}$$

Set $\Gamma_r = \Phi^{-1}(\{w \in \mathbb{C}\colon |w| = r\})$, $r > 1$. Since $\Gamma \in C^{2+}$, by Carathéodory's theorem (see [7, Chapter 2]), Φ and Φ^{-1} have one-to-one continuous extensions to the boundary of their respective domains of definition (considering in the case that Γ is an arc that it has two sides) and $|\Phi(\xi)| = 1$ ($|\Phi_\pm(\xi)| = 1$ if Γ is an arc) for all $\xi \in \Gamma$.

Since $\rho \in S(\Gamma)$, there exists a unique function $D(\rho; \cdot)$ analytic on Ω satisfying:

(1) for all $z \in \Omega$, $D(\rho; z) \neq 0$,
(2) $D(\rho; \infty) > 0$,
(3) for almost all $\xi \in \Gamma$, $D(\rho; \cdot)$ has non-tangential limit $D(\rho;\xi)$ ($D_+(\rho;\xi)$, $D_-(\rho;\xi)$ if Γ is an arc) and $|D(\rho;\xi)| = \rho(\xi)$ ($|D_\pm(\rho;\xi)| = \rho(\xi)$).

If $g(z;\infty) = \log|\Phi(z)|$ denotes the Green function of Ω with singularity at infinity, and $\partial/\partial\eta$ denotes the exterior normal derivative to Γ, then

$$D(\rho;\infty) = \exp\left\{\frac{1}{2\pi}\oint_\Gamma \log\rho(\xi)\frac{\partial g(\xi;\infty)}{\partial\eta}|d\xi|\right\}. \tag{15}$$

An analytic function f on Ω is said to belong to $E^1(\Omega)$, if

$$\sup_{r>1}\oint_{\Gamma_r}\left|f(z)\right||dz| < \infty.$$

By $H^2(\Omega, \rho)$ we denote the space of analytic functions f on Ω such that $f^2 D(\rho, \cdot) \in E^1(\Omega)$.

Each $f \in H^2(\Omega, \rho)$ has a non-tangential limit $f(\xi)$ ($f_+(\xi)$, $f_-(\xi)$ if Γ is an arc) at almost all $\xi \in \Gamma$. Moreover, $f(\xi) \in L^2(\rho)$, ($f_+(\xi)$, $f_-(\xi) \in L^2(\rho)$). The inner product

$$\langle f, g\rangle_{H^2(\Omega,\rho)} = \oint_\Gamma f(\xi)\overline{g(\xi)}\rho(\xi)|d\xi|, \tag{16}$$

makes $H^2(\Omega, \rho)$ a Hilbert space. By the Riesz representation theorem it follows that there exists (see [8, §7]) a function $K(t, z)$, called *Szegő reproducing kernel*, such that for any $f \in H^2(\Omega, \rho)$

$$f(z) = \langle f, K(., z)\rangle_{H^2(\Omega,\rho)}, \quad z \in \Omega. \tag{17}$$

Let $\|\cdot\|_{H^2(\Omega,\rho)}$ be the norm induced by (16). The extremal property (14) of the polynomials $P_n(\rho;\cdot)$ motivates the following extremal problem:

$$\nu(\rho) = \min\bigl\{\|f\|^2_{H^2(\Omega,\rho)}\colon f \in H^2(\Omega, \rho),\ f(\infty) = 1\bigr\}. \tag{18}$$

It has a unique minimizing function, which we denote by $\mathcal{F}$, satisfying additionally (see [8, §6 and §7])

$$\mathcal{F}^2(z) = \Phi'(z)\frac{D(\rho;\infty)}{D(\rho;z)}, \quad z \in \Omega,$$

$$\nu(\rho) = 2\pi D(\rho;\infty)C(\Gamma), \tag{19}$$

$$\mathcal{F}(z) = \nu(\rho)K(z,\infty), \quad z \in \Omega. \tag{20}$$

The function $\mathcal{F}$ and the extremal constant $\nu(\rho)$ are intimately connected with the asymptotic properties of the orthogonal polynomials $P_n(\rho;\cdot)$. In fact,

$$\lim_{n\to\infty}\frac{\lambda_n(\rho)}{C(\Gamma)^{2n}} = \nu(\rho), \tag{21}$$

$$P_n(\rho;z) = \bigl[C(\Gamma)\Phi(z)\bigr]^n\mathcal{F}(z)\bigl(1+\varepsilon_n(z)\bigr), \tag{22}$$

where $\lim_{n\to\infty}\varepsilon_n = 0$ in Ω.

In [1], the following extremal problem is considered

$$\nu^*(\rho) = \min_{f\in\widetilde{H}}\|f\|^2_{H^2(\Omega,\rho)}, \tag{23}$$

where $\widetilde{H} = \{f \in H^2(\Omega,\rho)\colon f(\infty) = 1,\ f^{(j)}(z_i) = 0,\ j = 0,\dots,d_i,\ i = 1,\dots,m\}$. Let $\mathcal{F}^*$ be the extremal function for the problem (23). There exists a close connection between the extremal constants and functions of problems (18) and (23) (see [2,3]). In fact,

$$\mathcal{F}^*(z) = B(z)\mathcal{F}(z), \tag{24}$$

$$\nu^*(\rho) = \nu(\rho)\prod_{i=1}^m\bigl|\Phi(z_i)\bigr|^{2(d_i+1)}, \tag{25}$$

where

$$B(z) = \prod_{i=1}^m\left(\frac{\Phi(z)-\Phi(z_i)}{\Phi(z)\overline{\Phi(z_i)}-1}\,\frac{|\Phi(z_i)|^2}{\Phi(z_i)}\right)^{d_i+1}$$

has the following properties:

(1) $B \in H^2(\Omega,\rho)$, $B(\infty) = 1$,
(2) $B^{(j)}(z_i) = 0$, $j = 0,\dots,d_i$, $i = 1,\dots,m$,
(3) $|B(\xi)| = \prod_{i=1}^m|\Phi(z_i)|^{d_i+1}$, $\xi \in \Gamma$ ($|B_\pm(\xi)| = \prod_{i=1}^m|\Phi(z_i)|^{d_i+1}$, if Γ is an arc).

The authors of [1] prove that

$$\lim_{n\to\infty}\frac{\lambda_n(\rho)}{\tau_n} = \prod_{i=1}^m\bigl|\Phi(z_i)\bigr|^{-2(d_i+1)}, \tag{26}$$

and the relation of the extremal function $\mathcal{F}^*$ and constant $\nu^*(\rho)$ with the asymptotic behavior of the orthogonal polynomials L_n with respect to (3) with $N = 0$ is

$$\lim_{n\to\infty} \frac{\tau_n}{C(\Gamma)^{2n}} = \nu^*(\rho), \tag{27}$$

$$L_n(z) = \big[C(\Gamma)\Phi(z)\big]^n \mathcal{F}^*(z)\big(1+\varepsilon_n(z)\big), \tag{28}$$

where $\lim_{n\to\infty} \varepsilon_n = 0$ in Ω.

Notice that the asymptotic formulas (27) and (28) coincide in form with those given in (21) and (22) for $\lambda_n(\rho)$ and $P_n(\rho,\cdot)$, respectively, only the extremal constant and function change.

The following result shows that for functions in $H^2(\Omega,\rho)$, estimates of Γ in the L^2 norm imply estimates on $K \subset \Omega$ in the uniform norm (see [8, Corollary 7.4]).

Lemma 1. *Let $K \subset \Omega$ be a compact set and $\rho \in S(\Gamma)$. Then, there exists a constant $A(K)$, which only depends on K, such that for all $f \in H^2(\Omega,\rho)$*

$$\max_{z\in K} \big|f(z)\big|^2 \leqslant A(K)\|f\|^2_{H^2(\Omega,\rho)}.$$

3. Asymptotics of the Sobolev norms

We begin establishing two auxiliary results needed for the proof of (7) in Theorem 1.

Let $\mathbb{P}^*$ be the family of monic polynomials in $\mathbb{P}$. Fix a point $\xi_0 \in \Gamma$ and consider the mapping $\Pi : \mathbb{P}^* \to \mathbb{P}^*$ given by

$$R(z) = z^n + \cdots \quad \Rightarrow \quad \Pi(R)(z) := (n+1)\int_{\xi_0}^{z} R(\xi)\,d\xi,$$

that is, $\Pi(R)$ is the monic primitive of the monic polynomial R that vanishes at ξ_0. Now, define $\Pi_0 = I$ (the identity operator on $\mathbb{P}^*$), $\Pi_k = \Pi \circ \Pi_{k-1}$, $k \in \mathbb{N}$. If $R(z) = z^n + \cdots$, then $\deg(\Pi_k(R)) = n+k$ and for each $j \in \{0,\dots,k\}$

$$\Pi_k^{(j)}(R)(z) = \frac{(n+k)!}{(n+k-j)!}\Pi_{k-j}(R)(z). \tag{29}$$

Lemma 2. *Let μ be a finite positive Borel measure supported on Γ, $d\mu(\xi) = \rho(\xi)|d\xi|$, ω a monic polynomial whose zeros lie in $\mathbb{C}\setminus\Gamma$, $\deg\omega = M$, $W = |\omega|^2$ and $P_n = P_n(W\rho;\cdot)$ the nth monic orthogonal polynomial with respect to the measure $W\rho|d\xi|$. If $1/\rho \in L^1(\Gamma)$, then for each $k \in \mathbb{N}$, the sequence of polynomials*

$$\alpha_{n,k}(z) := \frac{n!}{(n+k)!}\frac{\Pi_k(\omega P_{n-M})(z)}{\sqrt{\lambda_{n-M}(W\rho)}}, \quad n \in \mathbb{N},\ n > M, \tag{30}$$

is uniformly bounded on Γ and tends to zero for all $z \in \Gamma$ as $n \to \infty$.

Proof. Let l be the length of Γ and $\xi = \xi(t)$, $t \in [0,l]$, $\xi(0) = \xi(l) = \xi_0$, $\xi_0 \in \Gamma$, be the parametrization of Γ with respect to the arc length. If $z \in \Gamma$,

$$\alpha_{n,1}(z) := \frac{1}{n+1}\frac{\Pi_1(\omega P_{n-M})(z)}{\sqrt{\lambda_{n-M}(W\rho)}} = \int_{\Lambda(z)} \omega(\xi)p_{n-M}(\xi)\,d\xi = \int_0^s \omega\big(\xi(t)\big)p_{n-M}\big(\xi(t)\big)\xi'(t)\,dt,$$

where $\Lambda(z) = \xi[0,s]$ denotes the arc along Γ from ξ_0 to z, following the orientation given by the parametrization, and $p_n = p_n(W\rho;\cdot)$ is the nth orthonormal polynomial with respect to the measure $W\rho|d\xi|$.

If $\mathbf{1}_{\Lambda(z)}$ is the characteristic function of $\Lambda(z)$, then

$$\begin{aligned}\alpha_{n,1}(z) &= \int_{[0,l]} \mathbf{1}_{\Lambda(z)}\big(\xi(t)\big)\frac{W(\xi(t))\rho(\xi(t))}{\overline{\omega(\xi(t))}\rho(\xi(t))}p_{n-M}\big(\xi(t)\big)\xi'(t)\,dt = \int_\Gamma \overline{f(z;\xi)}p_{n-M}(\xi)W(\xi)\rho(\xi)|d\xi| \\ &= \overline{\big\langle f(z;\cdot), p_{n-M}\big\rangle}_{L^2(W\rho)},\end{aligned} \tag{31}$$

where

$$f(z;\xi)=\frac{\mathbf{1}_{A(z)}(\xi)\overline{\xi'(t)}}{\omega(\xi)\,\rho(\xi)},\quad \xi\in\Gamma.$$

Thus, $\alpha_{n,1}(z)$ is the conjugate of the $(n-M)$th Fourier coefficient of $f(z;\cdot)$ with respect to the orthonormal system $\{p_n\}$ in $L^2(W\rho)$. Since $1/\rho\in L^1(\Gamma)$, then for all $z\in\Gamma$,

$$\|f(z;\cdot)\|^2_{L^2(W\rho)}=\int_\Gamma\left|f(z;\xi)\right|^2W(\xi)\rho(\xi)|d\xi|\leqslant\left\|\frac{1}{\rho}\right\|_{L^1(\Gamma)}. \tag{32}$$

Using (31), (32), and the Bessel inequality, we obtain

$$\left|\alpha_{n,1}(z)\right|\leqslant\left\|f(z;\cdot)\right\|_{L^2(W\rho)}\leqslant\left\|\frac{1}{\rho}\right\|^{1/2}_{L^1(\Gamma)},\quad z\in\Gamma,\ n>M,$$

$$\lim_{n\to\infty}\alpha_{n,1}(z)=0,\quad z\in\Gamma,$$

which establishes Lemma 2 for $k=1$. For the case of $k\geqslant 2$ one can proceed by induction making use of the identity

$$\alpha_{n,k+1}(z)=\int_{\xi_0}^{z}\alpha_{n,k}(x)\,dx,\quad n,k\in\mathbb{N},\ n>M,\ z\in\mathbb{C},$$

and the Lebesgue dominated convergence theorem. □

For what follows, we fix the notation

$$\omega(z)=\prod_{i=1}^{m}(z-z_i)^{d_i+1},\quad M=\deg\omega=m+\sum_{i=1}^{m}d_i,\quad W=|\omega|^2. \tag{33}$$

Lemma 3. *Assume that the assumptions of Theorem* 1 *are satisfied and let $P_n=P_n(W\rho_N;\cdot)$ be the nth monic orthogonal polynomials with respect to the measure $W\rho_N|d\xi|$, with W given in (33). Then*

$$\limsup_{n\to\infty}\frac{\kappa_n}{n^{2N}\lambda_{n-N-M}(W\rho_N)}\leqslant 1, \tag{34}$$

where $\kappa_n=\|Q_n\|_S^2$ and $\lambda_n(W\rho_N)=\|P_n\|^2_{L^2(W\rho_N|d\xi|)}$.

Proof. Let us assume initially that $\frac{1}{\rho_N}\in L^1(\Gamma)$. By the extremal property of Q_n, we have

$$\begin{aligned}\kappa_n=\|Q_n\|_S^2&\leqslant\sum_{k=0}^{N-1}\left\|\left(\Pi_N(\omega P_{n-N-M})\right)^{(k)}\right\|_k^2+\left\|\left(\Pi_N(\omega P_{n-N-M})\right)^{(N)}\right\|_N^2\\&\leqslant\sum_{k=0}^{N-1}\left[\frac{n!}{(n-k)!}\right]^2\left\|\Pi_{N-k}(\omega P_{n-N-M})\right\|_k^2+\left[\frac{n!}{(n-N)!}\right]^2\|\omega P_{n-N-M}\|_N^2.\end{aligned}$$

Hence

$$\left[\frac{(n-N)!}{n!}\right]^2\frac{\kappa_n}{\lambda_{n-N-M}(W\rho_N)}\leqslant\sum_{k=0}^{N-1}\left[\frac{(n-N)!}{(n-k)!}\right]^2\frac{\|\Pi_{N-k}(\omega P_{n-N-M})\|_k^2}{\lambda_{n-N-M}(W\rho_N)}+1,$$

and

$$\frac{\kappa_n}{n^{2N}\lambda_{n-N-M}(W\rho_N)}\leqslant\sum_{k=0}^{N-1}\|\alpha_{n-N,N-k}\|_k^2+1.$$

From Lemma 2 and the Lebesgue dominated convergence theorem, it follows that

$$\lim_{n\to\infty} \|\alpha_{n-N,N-k}\|_k^2 = 0, \quad k = 0,\dots,N-1.$$

Consequently, we have (34) when $1/\rho_N \in L^1(\Gamma)$.

Now, assume that the weight $\rho_N \in S(\Gamma)$ is arbitrary. Take a constant $\delta > 0$ (to be fixed later) and define $\tilde\rho_N(\xi) = \rho_N(\xi) + \delta$. Let $\widetilde{P}_n = \widetilde{P}_n(W\tilde\rho_N;\cdot)$ be the nth monic orthogonal polynomial with respect to the measure $d\tilde\mu(\xi) = W(\xi)\tilde\rho_N(\xi)|d\xi|$, and let $\tilde\alpha_{n,k}$ be the sequence defined in (30) corresponding to the measure $\tilde\mu$. Then

$$\frac{\kappa_n}{n^{2N}\lambda_{n-N-M}(W\rho_N)} \leqslant \left[\sum_{k=0}^{N-1} \|\tilde\alpha_{n-N,N-k}\|_k^2 + 1\right] \frac{\lambda_{n-N-M}(W\tilde\rho_N)}{\lambda_{n-N-M}(W\rho_N)}. \tag{35}$$

But $1/\tilde\rho_N \in L^1(\Gamma)$; therefore, from Lemma 2 and (35), we have that

$$\limsup_{n\to\infty} \frac{\kappa_n}{n^{2N}\lambda_{n-N-M}(W\rho_N)} \leqslant \lim_{n\to\infty} \frac{\lambda_{n-N-M}(W\tilde\rho_N)}{\lambda_{n-N-M}(W\rho_N)}. \tag{36}$$

Since $\rho_N \in S(\Gamma)$, then $W\rho_N, W\tilde\rho_N \in S(\Gamma)$. Therefore, from (21), (18) and (15), we obtain

$$\lim_{n\to\infty} \frac{\lambda_{n-N-M}(W\tilde\rho_N)}{\lambda_{n-N-M}(W\rho_N)} = \frac{\nu(W\tilde\rho_N)}{\nu(W\rho_N)} = \exp\left\{\frac{1}{2\pi}\int_\Gamma \left[\log(\rho_N+\delta) - \log\rho_N\right](\xi)\frac{\partial g(\xi;\infty)}{\partial\eta}|d\xi|\right\}. \tag{37}$$

It remains to use the continuity of the extremal constant $\nu(\rho)$, $\rho \in S(\Gamma)$, in the metric

$$\operatorname{dist}(\rho,\sigma) = \frac{1}{\pi}\int_\Gamma |\log\rho - \log\sigma|(\xi)\frac{\partial g(\xi;\infty)}{\partial\eta}|d\xi|, \quad \rho,\sigma \in S(\Gamma).$$

In fact, by the Lebesgue dominated convergence theorem

$$\operatorname{dist}(\tilde\rho_N,\rho_N) \to 0, \qquad \delta \to 0.$$

Then, for an arbitrary fixed $\varepsilon > 0$, we can take $\delta > 0$ in (37) so that

$$\lim_{n\to\infty} \frac{\lambda_{n-N-M}(W\tilde\rho_N)}{\lambda_{n-N-M}(W\rho_N)} \leqslant 1 + \varepsilon. \tag{38}$$

Now, (34) follows from (36), (38), and the arbitrariness of $\varepsilon > 0$. □

Proof of Theorem 1. By the extremal property (6), for all $n \in \mathbb{Z}_+$, $n > N$, we have

$$\kappa_n = \sum_{k=0}^{N-1} \left\| Q_n^{(k)} \right\|_k^2 + \left\| Q_n^{(N)} \right\|_N^2 \geqslant \left\| Q_n^{(N)} \right\|_N^2 \geqslant \left(\frac{n!}{(n-N)!}\right)^2 \tau_{n-N}.$$

Therefore,

$$\liminf_{n\to\infty} \frac{\kappa_n}{n^{2N}\tau_{n-N}} \geqslant 1. \tag{39}$$

On the other hand,

$$\frac{\kappa_n}{n^{2N}\tau_{n-N}} = \frac{\kappa_n}{n^{2N}\lambda_{n-N-M}(W\rho_N)} \frac{\lambda_{n-N-M}(W\rho_N)}{\lambda_{n-N}(\rho_N)} \frac{\lambda_{n-N}(\rho_N)}{\tau_{n-N}}.$$

According to (21) and (19), we have

$$\lim_{n\to\infty} \frac{\lambda_{n-N-M}(W\rho_N)}{\lambda_{n-N}(\rho_N)} = \frac{D(W\rho_N;\infty)}{D(\rho_N;\infty)} \frac{1}{C(\Gamma)^{2M}} = \prod_{i=1}^m \left|\Phi(z_i)\right|^{2(d_i+1)},$$

since due to (15),

$$\frac{D(W\rho_N;\infty)}{D(\rho_N;\infty)} = C(\Gamma)^{2M} \prod_{i=1}^m \left|\Phi(z_i)\right|^{2(d_i+1)}.$$

Using (26) and (34), we conclude that

$$\limsup_{n\to\infty} \frac{\kappa_n}{n^{2N}\tau_{n-N}} \leqslant 1. \tag{40}$$

Now, (7) follows from (39) and (40). □

4. Asymptotics of Sobolev polynomials

The following two lemmas pave the way for the proof of Theorem 2.

Lemma 4. *Assume that the assumptions of Theorem* 1 *are satisfied and let*

$$\Psi_{n-N}(z) = \eta_{n-N} \frac{Q_n^{(N)}}{\Phi^{n-N}}(z), \quad \text{where } \eta_{n-N} = \frac{(n-N)!}{n!C(\Gamma)^{n-N}},\ n \in \mathbb{N},\ n > N. \tag{41}$$

Then $\{\Psi_{n-N}\}$ *is uniformly bounded on compact subsets of* Ω *and*

$$\lim_{n\to\infty} \Psi_{n-N}^{(j)}(z_i) = 0, \quad j = 0, \dots, d_i,\ i = 1, \dots, m. \tag{42}$$

Proof. From the definition of κ_n, we have

$$\eta_{n-N}^2 \kappa_n = \sum_{k=0}^{N-1} \eta_{n-N}^2 \left\| Q_n^{(k)} \right\|_k^2 + \left\| \eta_{n-N} Q_n^{(N)} \right\|_{L^2(\rho_N)}^2 + \eta_{n-N}^2 Q_n^{(N)}(Z)\mathcal{A}Q_n^{(N)}(Z)^*. \tag{43}$$

Using (7) and (27), it follows that

$$\lim_{n\to\infty} \eta_{n-N}^2 \kappa_n = \nu^*(\rho_N). \tag{44}$$

On the other hand, (43) and (44) imply

$$\limsup_{n\to\infty} \|\Psi_{n-N}\|_{H^2(\Omega,\rho_N)}^2 \leqslant \nu^*(\rho_N).$$

Therefore, the sequence $\{\Psi_{n-N}\}$ is bounded in $H^2(\Omega, \rho_N)$ and Cauchy's integral formula renders that $\{\Psi_{n-N}\}$ is uniformly bounded on compact subsets of Ω.

For $\mathcal{A}$ hermitian and positive definite,

$$\min_{x\in\mathbb{C}^M,\, x\neq 0} \frac{x^*\mathcal{A}x}{x^*x} = \lambda_1 > 0,$$

where $\lambda_1 > 0$ is the smallest eigenvalue of $\mathcal{A}$. In particular, for every $x \in \mathbb{C}^M$,

$$x^*x \leqslant \frac{x^*\mathcal{A}x}{\lambda_1}. \tag{45}$$

From (43) and (44) we have that the sequence $\{\eta_{n-N}^2 Q_n^{(N)}(Z)\mathcal{A}Q_n^{(N)}(Z)^*\}_{n\in\mathbb{N}}$ is bounded; that is, there exists a constant $C > 0$ such that

$$\left[\eta_{n-N}\, Q_n^{(N)}(Z)\right]\mathcal{A}\left[\eta_{n-N}\, Q_n^{(N)}(Z)^*\right] \leqslant C, \quad \forall n \in \mathbb{N}. \tag{46}$$

Hence, by (45), sequence

$$\left\{\eta_{n-N}\, Q_n^{(N)}(Z)^*\right\}_{n\in\mathbb{N}} \subset \mathbb{C}^M, \tag{47}$$

is uniformly bounded with respect to the Euclidean norm. Since in a finite-dimensional space all norms are equivalent, we conclude that sequence (47) is bounded with respect to the uniform norm; that is,

$$\left\{\eta_{n-N}\, Q_n^{(N+j)}(z_i)\right\}_{n\in\mathbb{N}}, \quad j = 0, \dots, d_i,\ i = 1, \dots, m,$$

are uniformly bounded.

We have

$$\left|\eta_{n-N} Q_n^{(N+j)}(z_i)\right| = \left(\frac{(n-N)!}{n!\,C(\Gamma)^j}\left|\Phi(z_i)\right|^{n-N-j}\right)\left|\frac{Q_n^{(N+j)}}{[C(\Gamma)\Phi]^{n-N-j}}(z_i)\right|,$$

and

$$\lim_{n\to\infty}\frac{(n-N)!}{n!}\left|\Phi(z_i)\right|^{n-N-j}=\infty.$$

Therefore,

$$\lim_{n\to\infty}\frac{Q_n^{(N+j)}}{[C(\Gamma)\Phi]^{n-N-j}}(z_i)=0, \quad j=0,\dots,d_i,\ i=1,\dots,m, \tag{48}$$

and convergence takes place with geometric rate. Let us see that

$$\lim_{n\to\infty}\left(\frac{Q_n^{(N)}}{[C(\Gamma)\Phi]^{n-N}}\right)^{(j)}(z_i)=0, \quad j=0,\dots,d_i,\ i=1,\dots,m, \tag{49}$$

also converges with geometric rate.

According to the Leibnitz formula,

$$\left(\frac{Q_n^{(N)}}{[C(\Gamma)\Phi]^{n-N}}\right)^{(j)}(z_i)=\sum_{k=0}^{j}\binom{j}{k}Q_n^{(N+k)}(z_i)\left(\frac{1}{[C(\Gamma)\Phi]^{n-N}}\right)^{(j-k)}(z_i). \tag{50}$$

The derivative of order l, $l\in\mathbb{N}$, of the function $1/[C(\Gamma)\Phi]^{n-N}$, is of the form

$$\sum_{t=1}^{l}(-1)^t\prod_{r=1}^{t}(n-N+r-1)\frac{C_t}{[C(\Gamma)\Phi]^{n-N+t}}(z), \tag{51}$$

where $C_t(z)$, $t=1,\dots,l$, do not depend on n, but only on $C(\Gamma)$ and on some of the values $\Phi'(z),\dots,\Phi^{(l)}(z)$. From (50) and (51), we have

$$\left(\frac{Q_n^{(N)}}{[C(\Gamma)\Phi]^{n-N}}\right)^{(j)}(z_i)=\sum_{k=0}^{j}\binom{j}{k}\sum_{t=1}^{j-k}(-1)^t\prod_{r=1}^{t}(n-N+r-1)\frac{Q_n^{(N+k)}}{[C(\Gamma)\Phi]^{n-N-k}}(z_i)\frac{C_t}{[C(\Gamma)\Phi]^{k+t}}(z_i). \tag{52}$$

Now, using (52) and (48), we obtain (49). Finally, (42) follows from (41) and (49). $\square$

Associated with the extremal function $\mathcal{F}^*$ and the conformal mapping Φ, we consider the sequence of functions $\{H_n^*\}$, defined almost everywhere on Γ by

$$H_n^*(\xi)=\begin{cases}\mathcal{F}^*(\xi)\Phi^n(\xi), & \text{if } \Gamma \text{ is a closed curve},\\ \mathcal{F}_+^*(\xi)\Phi_+^n(\xi)+\mathcal{F}_-^*(\xi)\Phi_-^n(\xi), & \text{if } \Gamma \text{ is an arc}.\end{cases}$$

Lemma 5. *Under the assumptions of Theorem* 1*, we have*

$$\lim_{n\to\infty}\left\|H_{n-N}^*-\frac{Q_n^{(N)}}{n^N\,C(\Gamma)^{n-N}}\right\|_{L^2(\rho_N)}=0. \tag{53}$$

Proof. Obviously,

$$\left\|H_{n-N}^*-\eta_{n-N}Q_n^{(N)}\right\|_{L^2(\rho_N)}^2=\left\|H_{n-N}^*\right\|_{L^2(\rho_N)}^2+\left\|\eta_{n-N}Q_n^{(N)}\right\|_{L^2(\rho_N)}^2-2\operatorname{Re}\left\langle H_{n-N}^*,\eta_{n-N}Q_n^{(N)}\right\rangle_{L^2(\rho_N)}. \tag{54}$$

In connection with the first term of (54), if Γ is a closed curve, then

$$\begin{aligned}\|H_{n-N}^*\|_{L^2(\rho_N)}^2 &= \int_\Gamma |\mathcal{F}^*(\xi)|^2 |\Phi(\xi)|^{2(n-N)} \rho_N(\xi)|d\xi| = \oint_\Gamma |\mathcal{F}^*(\xi)|^2 \rho_N(\xi)|d\xi| \\ &= \|\mathcal{F}^*\|_{H^2(\Omega,\rho_N)}^2 = \nu^*(\rho_N),\end{aligned} \tag{55}$$

since $\mathcal{F}^*$ is the solution of the extremal problem (23) and $|\Phi(\xi)| = 1$ for $\xi \in \Gamma$. If Γ is an arc, we have

$$\begin{aligned}\|H_{n-N}^*\|_{L^2(\rho_N)}^2 &= \langle \mathcal{F}_+^* \Phi_+^{n-N} + \mathcal{F}_-^* \Phi_-^{n-N}, \mathcal{F}_+^* \Phi_+^{n-N} + \mathcal{F}_-^* \Phi_-^{n-N} \rangle_{L^2(\rho_N)} \\ &= \int_\Gamma |\mathcal{F}_+^*(\xi)|^2 |\Phi_+(\xi)|^{2(n-N)} \rho_N(\xi)|d\xi| + \int_\Gamma |\mathcal{F}_-^*(\xi)|^2 |\Phi_-(\xi)|^{2(n-N)} \rho_N(\xi)|d\xi| \\ &\quad + 2\,\mathrm{Re}\langle \mathcal{F}_+^* \Phi_+^{n-N}, \mathcal{F}_-^* \Phi_-^{n-N} \rangle_{L^2(\rho_N)} \\ &= \oint_\Gamma |\mathcal{F}^*(\xi)|^2 \rho_N(\xi)|d\xi| + 2\,\mathrm{Re} \int_\Gamma \Phi_+^{n-N}(\xi) \overline{\Phi_-^{n-N}(\xi)} \mathcal{F}_+^*(\xi) \overline{\mathcal{F}_-^*(\xi)} \rho_N(\xi)|d\xi| \\ &= \|\mathcal{F}^*\|_{H^2(\Omega,\rho_N)}^2 + 2\,\mathrm{Re} \int_\Gamma \Phi_+^{n-N}(\xi) \Phi_-^{-(n-N)}(\xi) \mathcal{F}_+^*(\xi) \overline{\mathcal{F}_-^*(\xi)} \rho_N(\xi)|d\xi|,\end{aligned} \tag{56}$$

and the second term tends to zero when $n \to \infty$ (see [8, Lemma 12.1]). Therefore, if Γ is an arc or a closed curve,

$$\lim_{n\to\infty} \|H_{n-N}^*\|_{L^2(\rho_N)}^2 = \nu^*(\rho_N). \tag{57}$$

Let us consider the second term in the right-hand side of (54). We have

$$\|Q_n^{(N)}\|_{L^2(\rho_N)}^2 \leqslant \|Q_n^{(N)}\|_N^2 \leqslant \|Q_n\|_S^2 = \kappa_n.$$

Hence,

$$\eta_{n-N}^2 \|Q_n^{(N)}\|_{L^2(\rho_N)}^2 \leqslant \left(\frac{(n-N)!}{n!}\right)^2 \frac{\kappa_n}{\tau_{n-N}} \frac{\tau_{n-N}}{C(\Gamma)^{2(n-N)}}. \tag{58}$$

From (27), (7) and (58),

$$\limsup_{n\to\infty} \|\eta_{n-N} Q_n^{(N)}\|_{L^2(\rho_N)}^2 \leqslant \nu^*(\rho_N). \tag{59}$$

Consider the third term in (54). If Γ is a closed curve,

$$\langle Q_n^{(N)}, H_{n-N}^* \rangle_{L^2(\rho_N)} = \int_\Gamma Q_n^{(N)}(\xi) \overline{\mathcal{F}^*(\xi)\Phi^{n-N}(\xi)} \rho_N(\xi)|d\xi| = \left\langle \frac{Q_n^{(N)}}{\Phi^{n-N}}, \mathcal{F}^* \right\rangle_{H^2(\Omega,\rho_N)}.$$

If Γ is an arc,

$$\begin{aligned}\langle Q_n^{(N)}, H_{n-N}^* \rangle_{L^2(\rho_N)} &= \int_\Gamma Q_n^{(N)}(\xi) \overline{H_{n-N}^*(\xi)} \rho_N(\xi)|d\xi| \\ &= \int_\Gamma \left(Q_n^{(N)}(\xi) \overline{\mathcal{F}_+^*(\xi)\Phi_+^{n-N}(\xi)} + Q_n^{(N)}(\xi) \overline{\mathcal{F}_-^*(\xi)\Phi_-^{n-N}(\xi)}\right) \rho_N(\xi)|d\xi| \\ &= \int_\Gamma \left(\frac{Q_n^{(N)}(\xi)}{\Phi_+^{n-N}(\xi)} \overline{\mathcal{F}_+^*(\xi)} + \frac{Q_n^{(N)}(\xi)}{\Phi_-^{n-N}(\xi)} \overline{\mathcal{F}_-^*(\xi)}\right) \rho_N(\xi)|d\xi| \\ &= \oint_\Gamma \frac{Q_n^{(N)}(\xi)}{\Phi^{n-N}(\xi)} \overline{\mathcal{F}^*(\xi)} \rho_N(\xi)|d\xi| = \left\langle \frac{Q_n^{(N)}}{\Phi^{n-N}}, \mathcal{F}^* \right\rangle_{H^2(\Omega,\rho_N)},\end{aligned}$$

where we used again that $|\Phi| = 1$ ($|\Phi_\pm| = 1$) on Γ. Therefore, if Γ is an arc or a closed curve, from the definition of Ψ_{n-N}, (24), (25), (20), and the properties of $B(z)$, it follows that

$$\begin{aligned}\big\langle \eta_{n-N} Q_n^{(N)}, H_{n-N}^*\big\rangle_{L^2(\rho_N)} &= \langle \Psi_{n-N}, \mathcal{F}^*\rangle_{H^2(\Omega,\rho_N)} = \nu(\rho_N)\big\langle \Psi_{n-N}, BK(\cdot,\infty)\big\rangle_{H^2(\Omega,\rho_N)}\\ &= \nu(\rho_N)\left\langle \frac{\Psi_{n-N}}{B}|B|^2, K(\cdot,\infty)\right\rangle_{H^2(\Omega,\rho_N)}\\ &= \nu^*(\rho_N)\left\langle \frac{\Psi_{n-N}}{B}, K(\cdot,\infty)\right\rangle_{H^2(\Omega,\rho_N)}. \end{aligned} \tag{60}$$

Notice that $(\Psi_{n-N}/B)(\infty) = 1$, and $\Psi_{n-N}/B \notin H^2(\Omega,\rho_N)$, since the points z_i, $i = 1,\dots,m$, are poles of this function. For all $n \in \mathbb{N}$,

$$\frac{\Psi_{n-N}(z)}{B(z)} = \sum_{i=1}^{m}\sum_{j=0}^{d_i} \frac{a_{n-N}^{(i,j)}}{(z-z_i)^j} + g_n(z), \quad g_n \in H^2(\Omega,\rho_N),\ g_n(\infty) = 1. \tag{61}$$

Substituting (61) in (60) and using property (17) of the reproducing kernel, we obtain

$$\big\langle \eta_{n-N} Q_n^{(N)}, H_{n-N}^*\big\rangle_{L^2(\rho_N)} = \nu^*(\rho_N)\left(1 + \sum_{i=1}^{m}\sum_{j=0}^{d_i} a_{n-N}^{(i,j)}\left\langle \frac{1}{(z-z_i)^j}, K(\cdot,\infty)\right\rangle_{H^2(\Omega,\rho_N)}\right). \tag{62}$$

Now, suppose that

$$\liminf_{n\to\infty} \operatorname{Re}\big\langle \eta_{n-N} Q_n^{(N)}, H_{n-N}^*\big\rangle_{L^2(\rho_N)} < \nu^*(\rho_N). \tag{63}$$

Let $\Lambda \subset \mathbb{N}$ be such that

$$\lim_{n\in\Lambda} \operatorname{Re}\big\langle \eta_{n-N} Q_n^{(N)}, H_{n-N}^*\big\rangle_{L^2(\rho_N)} = \liminf_{n\to\infty} \operatorname{Re}\big\langle \eta_{n-N} Q_n^{(N)}, H_{n-N}^*\big\rangle_{L^2(\rho_N)}. \tag{64}$$

Since $\{\Psi_{n-N}\}$ is uniformly bounded on compact subsets of Ω, there exists $\Lambda' \subset \Lambda$ and $\Psi_{\Lambda'}$, holomorphic in Ω, such that for $j \in \mathbb{Z}_+$,

$$\lim_{n\to\infty} \Psi_{n-N}^{(j)} = \Psi_{\Lambda'}^{(j)} \quad \text{in } \Omega \text{ where } n \in \Lambda'. \tag{65}$$

From (42) and (65), we have that

$$\Psi_{\Lambda'}^{(j)}(z_i) = 0, \quad j = 0,\dots,d_i,\ i = 1,\dots,m. \tag{66}$$

Using the definition of $B(z)$, (65) and (66), we have that $\Psi_{\Lambda'}/B \in H^2(\Omega,\rho_N)$ and for $n \in \Lambda'$,

$$\lim_{n\to\infty} \frac{\Psi_{n-N}}{B} = \frac{\Psi_{\Lambda'}}{B} \quad \text{in } \Omega \setminus \{z_1,\dots,z_m\} \text{ where } n \in \Lambda'. \tag{67}$$

On account of (61) and (67), we conclude that $a_{n-N}^{(i,j)} \to 0$, $j = 0,\dots,d_i$, $i = 1,\dots,m$, for $n \in \Lambda'$, and from (62), we obtain

$$\lim_{n\in\Lambda'} \operatorname{Re}\big\langle \eta_{n-N} Q_n^{(N)}, H_{n-N}^*\big\rangle_{L^2(\rho_N)} = \nu^*(\rho_N). \tag{68}$$

Since $\Lambda' \subset \Lambda$, (64) and (68) contradict the assumption (63). Therefore,

$$\liminf_{n\to\infty} \operatorname{Re}\big\langle H_{n-N}^*, \eta_{n-N} Q_n^{(N)}\big\rangle_{L^2(\rho_N)} \geqslant \nu^*(\rho_N). \tag{69}$$

Taking upper limit on both sides of (54), (53) follows from (57), (59) and (69). □

Proof of Theorem 2. If Γ is a closed curve,

$$\left\| \mathcal{F}^* - \frac{(n-N)!}{n!}\frac{Q_n^{(N)}}{[C(\Gamma)\Phi]^{n-N}} \right\|_{H^2(\Omega,\rho_N)} = \left\| H_{n-N}^* - \frac{(n-N)!}{n!}\frac{Q_n^{(N)}}{C(\Gamma)^{n-N}} \right\|_{L^2(\rho_N)}.$$

Lemma 1 and (53) imply

$$\lim_{n\to\infty} \frac{Q_n^{(N)}}{n^N[C(\Gamma)\Phi]^{n-N}}(z) = \mathcal{F}^*(z) \quad \text{in } \Omega,$$

which is (10). Due to (28), (10) is equivalent to (11).

If Γ is an arc, using (17), we have

$$\begin{aligned}
\frac{(n-N)!}{n!}\frac{Q_n^{(N)}}{[C(\Gamma)\Phi]^{n-N}}(z) &= \left\langle \frac{(n-N)!}{n!}\frac{Q_n^{(N)}}{[C(\Gamma)\Phi]^{n-N}}, K(\cdot,z)\right\rangle_{H^2(\Omega,\rho_N)} \\
&= \oint_\Gamma \frac{(n-N)!}{n!}\frac{Q_n^{(N)}}{[C(\Gamma)\Phi]^{n-N}}(\xi)\overline{K(\xi,z)}\rho_N(\xi)|d\xi| \\
&= \oint_\Gamma \frac{(n-N)!}{n!}\frac{Q_n^{(N)}(\xi)}{C(\Gamma)^{n-N}}\overline{\Phi^{n-N}(\xi)K(\xi,z)}\rho_N(\xi)|d\xi| \\
&= \int_\Gamma \frac{(n-N)!}{n!}\frac{Q_n^{(N)}(\xi)}{C(\Gamma)^{n-N}}\big(\overline{\Phi_+^{n-N}(\xi)K_+(\xi,z)+\Phi_-^{n-N}(\xi)K_-(\xi,z)}\big)\rho_N(\xi)|d\xi| \\
&= \big\langle H_{n-N}^*, \Phi_+^{n-N}K_+(\cdot,z)+\Phi_-^{n-N}K_-(\cdot,z)\big\rangle_{L^2(\rho_N)} \\
&\quad - \left\langle H_{n-N}^* - \frac{(n-N)!}{n!}\frac{Q_n^{(N)}}{C(\Gamma)^{n-N}}, \Phi_+^{n-N}K_+(\cdot,z)+\Phi_-^{n-N}K_-(\cdot,z)\right\rangle_{L^2(\rho_N)}.
\end{aligned}$$

Using the Cauchy–Schwartz inequality and (53), it is easy to see that the second term in the last equality tends to zero as $n\to\infty$. Additionally,

$$\begin{aligned}
&\big\langle H_{n-N}^*, \Phi_+^{n-N}K_+(\cdot,z)+\Phi_-^{n-N}K_-(\cdot,z)\big\rangle_{L^2(\rho_N)} \\
&\quad = \big\langle \mathcal{F}^*, K(.,z)\big\rangle_{H^2(\Omega,\rho_N)} \\
&\qquad + \int_\Gamma \big\{\mathcal{F}_+^*(\xi)\Phi_+^{n-N}(\xi)\Phi_-^{-(n-N)}(\xi)\overline{K_-(\xi,z)} + \mathcal{F}_-^*(\xi)\Phi_+^{-(n-N)}(\xi)\Phi_-^{n-N}(\xi)\overline{K_+(\xi,z)}\big\}\rho_N(\xi)|d\xi| \\
&\quad = \mathcal{F}^*(z) + \int_\Gamma \big\{\mathcal{F}_+^*(\xi)\Phi_+^{n-N}(\xi)\Phi_-^{-(n-N)}(\xi)\overline{K_-(\xi,z)} + \mathcal{F}_-^*(\xi)\Phi_+^{-(n-N)}(\xi)\Phi_-^{n-N}(\xi)\overline{K_+(\xi,z)}\big\}\rho_N(\xi)|d\xi|.
\end{aligned}$$

In the last equality, the second term tends to zero since it is the Fourier coefficient of an integrable function. Therefore, we also have (10) when Γ is an arc and we conclude. □

In proving Theorem 3, we use the following auxiliary result.

Lemma 6. *Under the assumptions of Theorem* 3*, we have for $k=0,\dots,i-1$ and $i=1,\dots,N$,*

$$\lim_{n\to\infty}\frac{Q_n^{(k)}}{n^i[C(\Gamma)\Phi]^{n-k}}(z)=0 \quad \text{in } \Omega.$$

Proof. Let $\rho_k(\xi)|d\xi|$, $k=0,\dots,N$, be the absolutely continuous component of $d\mu_k(\xi)$. We have

$$\begin{aligned}
\frac{\kappa_n}{n^{2N}C(\Gamma)^{2(n-N)}} &= \sum_{k=0}^{N-1}\left\|\frac{Q_n^{(k)}}{n^N C(\Gamma)^{n-N}}\right\|_k^2 + \left\|\frac{Q_n^{(N)}}{n^N C(\Gamma)^{n-N}}\right\|_N^2 \\
&\geqslant \sum_{k=0}^{N-1}\left\|\frac{Q_n^{(k)}}{n^N C(\Gamma)^{n-N}}\right\|_{L^2(\rho_k)}^2 + \left\|\frac{Q_n^{(N)}}{n^N C(\Gamma)^{n-N}}\right\|_{L^2(\rho_N)}^2.
\end{aligned}$$

Since (see (44))

$$\lim_{n\to\infty}\frac{\kappa_n}{n^{2N}C(\Gamma)^{2(n-N)}} = \nu^*(\rho_N),$$

and, according to (53) and (57),

$$\lim_{n\to\infty}\left\|\frac{Q_n^{(N)}}{n^N C(\Gamma)^{n-N}}\right\|^2_{L^2(\rho_N)}=\lim_{n\to\infty}\left\|H^*_{n-N}\right\|^2_{L^2(\rho_N)}=\|\mathcal{F}^*\|^2_{H^2(\Omega,\rho_N)}=\nu^*(\rho_N),$$

it follows that

$$\lim_{n\to\infty}\left\|\frac{Q_n^{(k)}}{n^N [C(\Gamma)\Phi]^{n-k}}\right\|^2_{L^2(\rho_k)}=0,\quad k=0,\dots,N-1.$$

Since $\mu_k\in S(\Gamma)$, $k=0,\dots,N$, using Lemma 1, we obtain for $k=0,\dots,N-1$,

$$\lim_{n\to\infty}\frac{Q_n^{(k)}}{n^N [C(\Gamma)\Phi]^{n-k}}(z)=0\quad\text{in }\Omega. \tag{70}$$

Therefore, the lemma is true when $i=N$.

By the Weierstrass theorem, for $k=0,\dots,N-1$,

$$\lim_{n\to\infty}\left[\frac{Q_n^{(k)}}{n^N [C(\Gamma)\Phi]^{n-k}}\right]'(z)=0\quad\text{in }\Omega. \tag{71}$$

It is easy to verify that

$$\frac{(n-k)\Phi'(z)}{n\Phi(z)}\frac{Q_n^{(k)}}{n^{N-1}[C(\Gamma)\Phi]^{n-k}}(z)=\frac{1}{C(\Gamma)\Phi(z)}\frac{Q_n^{(k+1)}}{n^N[C(\Gamma)\Phi]^{n-(k+1)}}(z)-\left[\frac{Q_n^{(k)}}{n^N[C(\Gamma)\Phi]^{n-k}}\right]'(z).$$

Therefore, using (70) and (71),

$$\lim_{n\to\infty}\frac{Q_n^{(k)}(z)}{n^{N-1}[C(\Gamma)\Phi]^{n-k}(z)}=0\quad\text{in }\Omega$$

for $k=0,\dots,N-2$. Repeating the arguments, we conclude the proof. □

Proof of Theorem 3. For a given $k\in\mathbb{Z}_+$, (12) and (13) are equivalent due to (28). On the other hand, using the Weierstrass theorem it is easy to prove that if (12) is true for a certain k then it holds for all $i\geqslant k$. For $k=N$, (12) coincides with (10). Therefore, (12) takes place for all $k\geqslant N$. To conclude, we will assume that (12) is satisfied for some k, $1\leqslant k\leqslant N$, and prove that it also holds for $k-1$.

We have

$$\frac{(n+1-k)\Phi'(z)}{n\Phi(z)}\frac{Q_n^{(k-1)}}{n^{k-1}[C(\Gamma)\Phi]^{n-(k-1)}}(z)=\frac{1}{C(\Gamma)\Phi(z)}\frac{Q_n^{(k)}}{n^k[C(\Gamma)\Phi]^{n-k}}(z)-\left[\frac{Q_n^{(k-1)}}{n^k[C(\Gamma)\Phi]^{n-(k-1)}}\right]'(z).$$

From Lemma 6 and the Weierstrass theorem, the second term on the right-hand side of this equality tends to zero uniformly on compact subsets of Ω. By assumption,

$$\lim_{n\to\infty}\frac{1}{C(\Gamma)\Phi(z)}\frac{Q_n^{(k)}(z)}{n^k[C(\Gamma)\Phi]^{n-k}(z)}=\frac{1}{C(\Gamma)\Phi(z)}\frac{\mathcal{F}^*(z)}{([C(\Gamma)\Phi]'(z))^{N-k}}\quad\text{in }\Omega.$$

Taking limit on both sides of the equality above, (12) immediately follows for $k-1$ and we are done. □

Remark. The following questions arise naturally. First of all, can the assumption that μ_N is absolutely continuous be suppressed? Moreover, consider a Sobolev inner product of the form

$$\langle p,q\rangle_{\widetilde{S}}=\sum_{k=0}^N\int_\Gamma p^{(k)}(\xi)\overline{q^{(k)}(\xi)}\,d\mu_k(\xi)+p(Z)\mathcal{A}q(Z)^*,$$

where

$$p(Z)=\big(p(z_1),\dots,p^{(d_1)}(z_1),p(z_2),\dots,p^{(d_2)}(z_2),\dots,p(z_m),\dots,p^{(d_m)}(z_m)\big),$$

$z_i\in\Omega$, $i=1,\dots,m$, and $\mathcal{A}$ is a hermitian semi positive definite matrix of order $M=m+\sum_{i=1}^m d_i$. Under what assumptions can we obtain theorems similar to those above for the corresponding sequence of Sobolev orthogonal polynomials? How does the appearance of derivatives of order $<N$ in the discrete part affect the results?

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