Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 340 (2008) 606–620
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Solving the optimal control problem of the parabolic PDEs in exploitation of oil

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Received 6 March 2007
Available online 30 August 2007
Submitted by Goong Chen

Abstract

In this paper, the optimal control problem is governed by weak coupled parabolic PDEs and involves pointwise state and control constraints. We use measure theory method for solving this problem. In order to use the weak solution of problem, first problem has been transformed into measure form. This problem is reduced to a linear programming problem. Then we obtain an optimal measure which is approximated by a finite combination of atomic measures. We find piecewise-constant optimal control functions which are an approximate control for the original optimal control problem.

Keywords: Weak coupled parabolic PDEs; Optimal control; Measure theory; Atomic measure; Linear programming; Approximation

1. Introduction

In this paper we use the notation of [11] and [3]. Let $\Omega \subset R^3$ denote the domain of reservoir which is bounded with smooth boundary $\partial\Omega$. Given $0 < T < \infty$, we set $Q = \Omega \times (0, T)$, $\Sigma = \partial\Omega \times (0, T)$; and suppose that there exist N_o producing wells and N_w water injection wells in the oilfield. Let $u(x, t)$, $v(x, t)$ denote the fluid pressures of system one and system two, respectively, in a point $x \in \Omega$ and a time $t \in (0, T)$, then $u(x, t)$, $v(x, t)$ satisfy the following PDEs:

$$\nabla \cdot (\alpha_1 \nabla u) - \beta_1 \frac{\partial u}{\partial t} + \gamma(v - u) = (1 - \lambda)\left(\sum_{i=1}^{N_w} m_i(t)\delta\big(x - P_i^w\big) - \sum_{j=1}^{N_o} n_j(t)\delta\big(x - P_j^o\big)\right), \quad (x, t) \in Q, \tag{1}$$

$$\nabla \cdot (\alpha_2 \nabla v) - \beta_2 \frac{\partial v}{\partial t} + \gamma(u - v) = \lambda\left(\sum_{i=1}^{N_w} m_i(t)\delta\big(x - P_i^w\big) - \sum_{j=1}^{N_o} n_j(t)\delta\big(x - P_j^o\big)\right), \quad (x, t) \in Q, \tag{2}$$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.08.037

$$\frac{\partial u}{\partial n} = 0, \qquad \frac{\partial v}{\partial n} = 0, \quad (x,t) \in \Sigma, \tag{3}$$

$$u(x,0) = \sigma(x), \qquad v(x,0) = \tau(x), \quad x \in \Omega, \tag{4}$$

where $\alpha_l = \alpha_l(x) = \frac{hK_l}{\mu}$, $\beta_l = C_l \phi_l h$ $(l = 1,2)$; K_1, K_2 are mean permeability of system one and system two, respectively, $K_1 \gg K_2 > 0$; C_1, C_2 are the synthetical compressibility and ϕ_1, ϕ_2 are porosities of system one and system two, respectively, n is the unit outward normal on $\partial\Omega$, h is thickness of the reservoir bed; μ is the viscosity of the mixture of oil and water in the global control time interval, γ is the penetration coefficient, $C_l, \phi_l, h, \gamma, \mu, \lambda$ $(l = 1,2)$ are positive constants here and $0 \leqslant \lambda \ll 1$; $\delta(x - p_i^w), \delta(x - p_j^o)$ are the Dirac Functions at points $p_i^w \in \Omega$, $p_j^o \in \Omega$ which are the locations of the injection water entrance and the extraction oil exit of wells, respectively; $m_i(t)$ $(i = 1,2,\ldots,N_w)$ is the discharge of the ith water injection well at $t \in [0,T]$ and p_i^w, $n_j(t)$ $(j = 1,2,\ldots,N_o)$ is the output of the jth producing well at $t \in [0,T]$ and p_j^o, and finally the functions $\sigma(x)$ and $\tau(x)$ are bounded and measurable in Ω.

According to the practical requirement of oil exploitation, we suppose that the pressures of the reservoir must satisfy the following condition:

$$G(u,v) = \int_\Omega g\big(x,t,u(x,t),v(x,t)\big)\,dx \leqslant \nu, \quad t \in [0,T], \tag{5}$$

where $\nu > 0$ is a constant; $g(x,t,\cdot,\cdot) : \Omega \times [0,T] \to R$ is a Caratheodory function, that is, measurable with respect to x in Ω for every (x,t) in $\Omega \times [0,T]$, and continuous with respect to t in $[0,T]$ for almost every x in Ω and $g(\cdot,\cdot,u,v) \in C^1$.

We denote

$$\eta_1 = (m_1, m_1, \ldots, m_{N_w}, n_1, n_1, \ldots, n_{N_o}) \in R^{N_w+N_o},$$
$$\eta_2 = \big(\sigma(x), \tau(x)\big) \in R^2.$$

Let f_i be the cost function of the ith water injection well and g_j be the cost function of producing well, both of them are the functions of $m_i(t)$ $(i = 1,2,\ldots,N_w)$ and $n_j(t)$ $(j = 1,2,\ldots,N_o)$, respectively. In the meantime, we assume $f_i, g_j \in C^2[0,\infty)$ are convex functions. Therefore, the global cost of oil extraction in a time interval $[0,T]$ is given by

$$J(\eta_1,\eta_2) = \int_0^T f_0\big(\eta_1(t)\big)\,dt, \tag{6}$$

where

$$f_0\big(\eta_1(t)\big) = \sum_{i=1}^{N_w} f_i\big(m_i(t)\big) + \sum_{j=1}^{N_o} g_j\big(n_j(t)\big).$$

Suppose that $m_i(t)$, $i = 1,2,\ldots,N_w$, and $n_j(t)$, $j = 1,2,\ldots,N_o$, are bounded and measurable functions in $[0,T]$. Let

$$U_{\mathrm{ad}} = \left\{ (\eta_1,\eta_2) \in \big(L^\infty(0,T)\big)^{N_w+N_o} \times \big(L^\infty(\Omega)\big)^2 \;\middle|\; \begin{array}{l} 0 < \underline{n}_j \leqslant n_j(t) \leqslant \bar{n}_j \\ 0 < \underline{m}_i \leqslant m_i(t) \leqslant \bar{m}_i \\ 0 < \underline{\sigma} \leqslant \sigma(x) \leqslant \bar{\sigma} \\ 0 < \underline{\tau} \leqslant \tau(x) \leqslant \bar{\tau} \\ i = 1,2,\ldots,N_w \\ j = 1,2,\ldots,N_o \end{array} \right\},$$

where $\underline{n}_j, \bar{n}_j, \underline{m}_i, \bar{m}_i, \underline{\sigma}, \bar{\sigma}, \underline{\tau}, \bar{\tau}$ are positive constants. For given $(\eta_1,\eta_2) \in U_{\mathrm{ad}}$, we denote $w(x,t;\eta_1,\eta_2) = (u(x,t;\eta_1,\eta_2), v(x,t;\eta_1,\eta_2))$ which is the solution of the system (1)–(4) corresponding to $(\eta_1,\eta_2) \in U_{\mathrm{ad}}$, and denote it as $w(\eta_1,\eta_2) = (u(\eta_1,\eta_2), v(\eta_1,\eta_2))$ for simplicity. Now we define the following optimal control problem.

Problem **P**: The problem is

$$\inf_{(\eta_1,\eta_2)\in U_{\mathrm{ad}}} J(\eta_1,\eta_2),$$

where $w(\eta_1,\eta_2)=(u(\eta_1,\eta_2),v(\eta_1,\eta_2))$ satisfies the system (1)–(4) and the condition (5).

If there exists $(\eta_1^\star,\eta_2^\star)\in U_{\mathrm{ad}}$, such that

$$J(\eta_1^\star,\eta_2^\star)=\inf_{(\eta_1,\eta_2)\in U_{\mathrm{ad}}} J(\eta_1,\eta_2) \tag{7}$$

and the solution $w(\eta_1^\star,\eta_2^\star)=(u(\eta_1^\star,\eta_2^\star),v(\eta_1^\star,\eta_2^\star))$ of the system (1)–(4) corresponding to $(\eta_1^\star,\eta_2^\star)\in U_{\mathrm{ad}}$ satisfies the constraint condition (5), then $(\eta_1^\star,\eta_2^\star)\in U_{\mathrm{ad}}$, is called the solution of problem **P**, $w(\eta_1^\star,\eta_2^\star)=(u(\eta_1^\star,\eta_2^\star),v(\eta_1^\star,\eta_2^\star))$ is called the optimal state and $(w(\eta_1^\star,\eta_2^\star),\eta_1^\star,\eta_2^\star)$ the optimal triples.

We consider the following assumptions on α_l $(l=1,2)$ and $\partial\Omega$ which have been introduced in [3].

(A$_1$) $\Omega\subset R^3$ is bounded domain with sufficiently smooth boundary $\partial\Omega$.
(A$_2$) $\alpha_l(x)\in L^2(\Omega)$, $0<\underline{\alpha}_l\leqslant\alpha_l(x)\leqslant\bar{\alpha}_l$ for all $x\in\Omega$, where $\underline{\alpha}_l,\bar{\alpha}_l$ $(l=1,2)$ are constants.

Base on these assumptions the weak solution of the problem (1)–(4) can be defined as follows (for details see [3]).

Definition 1.1. For arbitrary pair $(\eta_1,\eta_2)\in U_{\mathrm{ad}}$, $w(\eta_1,\eta_2)=(u(\eta_1,\eta_2),v(\eta_1,\eta_2))\in[L^1((0,T);\ W^{1,1}(\Omega))]^2$ is called the weak solution of (1)–(4) if for arbitrary function $\theta(x,t)=(\theta_1,\theta_2)\in[C^\infty(\overline{Q})]^2$ with $\theta(x,T)=0$, the following identical equation will be valid:

$$\begin{aligned}
&\int_\Omega\left(\nabla\cdot(\alpha_1\nabla\theta_1)-\beta_1\frac{\partial\theta_1}{\partial t}+\gamma(\theta_2-\theta_1)\right)u(x,t)+\left(\nabla\cdot(\alpha_2\nabla\theta_2)-\beta_2\frac{\partial\theta_2}{\partial t}+\gamma(\theta_1-\theta_2)\right)v(x,t)\,dx\,dt\\
&\quad=\lambda\int_0^T\left(\sum_{i=1}^{N_w}m_i(t)\theta_2\big(P_i^w,t\big)-\sum_{j=1}^{N_o}n_j(t)\theta_2\big(P_j^o,t\big)\right)dt\\
&\qquad+(1-\lambda)\int_0^T\left(\sum_{i=1}^{N_w}m_i(t)\theta_1\big(P_i^w,t\big)-\sum_{j=1}^{N_o}n_j(t)\theta_1\big(P_j^o,t\big)\right)dt\\
&\qquad-\int_\Omega\big(\beta_1\sigma(x)\theta_1(x,0)+\beta_2\tau(x)\theta_2(x,0)\big)\,dx.
\end{aligned} \tag{8}$$

Definition 1.2. A triple (w,η_1,η_2) of the trajectory function w and two vector control functions η_1 and η_2 is said to be admissible if:

1. The trajectory function $w(\eta_1,\eta_2)\in[L^1((0,T);\ W^{1,1}(\Omega))]^2\cap[L^2((0,T);\ L^2(\Omega))]^2$ satisfies the problem (1)–(4) and the constraint condition (5).
2. The vector control functions η_1 and η_2 are in $(L^\infty(0,T))^{N_w+N_o}$ and $(L^\infty(\Omega))^2$, respectively.

Let $\digamma$ be the nonempty set of admissible triples. We intend to find a triple in $\digamma$, so that minimize the functional (6).

2. Change of the space

In the given classical control problem, it is not possible to find a triple in $\digamma$ so that minimize the functional (6) in general. So we may extend the problem to measure space which is larger space than the classical space of controls, then we obtain a solution in the new space for the problem and finally we obtain an approximate solution for the original problem in the classical space.

First consider the following theorem of [3].

Theorem 2.1. *Assume* (A$_1$), (A$_2$) *are valid. Then for given* $(\eta_1,\eta_2)\in U_{\mathrm{ad}}$, *problem* (1)–(4) *has a unique weak solution* $w(\eta_1,\eta_2)\in[L^1((0,T);W^{1,1}(\Omega))]^2\cap[L^2((0,T);L^2(\Omega))]^2$ *which satisfies*

$$\begin{aligned}
&\int_\Omega\left(\nabla\cdot(\alpha_1\nabla\theta_1)-\beta_1\frac{\partial\theta_1}{\partial t}+\gamma(\theta_2-\theta_1)\right)u(x,t)+\left(\nabla\cdot(\alpha_2\nabla\theta_2)-\beta_2\frac{\partial\theta_2}{\partial t}+\gamma(\theta_1-\theta_2)\right)v(x,t)\,dx\,dt\\
&\quad=\lambda\int_0^T\left(\sum_{i=1}^{N_w}m_i(t)\theta_2\big(P_i^w,t\big)-\sum_{j=1}^{N_o}n_j(t)\theta_2\big(P_j^o,t\big)\right)dt\\
&\qquad+(1-\lambda)\int_0^T\left(\sum_{i=1}^{N_w}m_i(t)\theta_1\big(P_i^w,t\big)-\sum_{j=1}^{N_o}n_j(t)\theta_1\big(P_j^o,t\big)\right)dt\\
&\qquad-\int_\Omega\beta_1\sigma(x)\theta_1(x,0)+\beta_2\tau(x)\theta_2(x,0)\,dx,
\end{aligned}\tag{9}$$

where $\theta(x,t)=(\theta_1(x,t),\theta_2(x,t))\in[C^\infty(\overline{Q})]^2$. *Moreover, there exist nonnegative constants* N_k, $k=1,2,\dots,6$, *such that the following inequalities hold*:

$$\begin{aligned}
\|w\|_{[L^1((0,T),W^{1,1}(\Omega))]^2}&\leqslant N_1\left(\sum_{i=1}^{N_w}\big\|m_i(t)\big\|_{L^\infty(0,T)}+\sum_{j=1}^{N_o}\big\|n_j(t)\big\|_{L^\infty(0,T)}\right)\\
&\quad+\beta_1N_2\big\|\sigma(x)\big\|_{L^\infty(\Omega)}+\beta_2N_3\big\|\tau(x)\big\|_{L^\infty(\Omega)},
\end{aligned}\tag{10}$$

$$\begin{aligned}
\|w\|_{[L^2((0,T),L^1(\Omega))]^2}&\leqslant N_4\left(\sum_{i=1}^{N_w}\big\|m_i(t)\big\|_{L^\infty(0,T)}+\sum_{j=1}^{N_o}\big\|n_j(t)\big\|_{L^\infty(0,T)}\right)\\
&\quad+\beta_1N_5\big\|\sigma(x)\big\|_{L^\infty(\Omega)}+\beta_2N_6\big\|\tau(x)\big\|_{L^\infty(\Omega)}.
\end{aligned}\tag{11}$$

From (10) and (11) and with respect to bounded control functions m_i and n_j we can find the compact set $A\subset R^2$ such that

$$w(x,t)\in A,\quad\forall(x,t)\in\overline{Q}.\tag{12}$$

The following lemma is useful to use constraint (5) by measure theory method.

Lemma 2.2. *If* T_d *is an arbitrary countable discrete subset of points in* $[0,T]$ *which is dense in* $[0,T]$, *then the constraint condition* (5) *at* $t\in[0,T]$ *is equivalent to the following constraints*:

$$\int_\Omega g\big(x,t_n,u(x,t_n),v(x,t_n)\big)\,dx\leqslant\nu,\quad n=1,2,\dots,\tag{13}$$

where t_n *is a sequence from point in* T_d *and convergent to* t *as* $n\to\infty$.

Proof. Suppose (13) is valid and let $g_n(x):=g(x,t_n,u(x,t_n),v(x,t_n))$. From property of g, $\lim_{n\to\infty}g_n(x)=g(x)$, so by Fatou's lemma

$$\int_\Omega g\big(x,t,u(x,t),v(x,t)\big)\,dx=\int_\Omega\lim_{n\to\infty}\inf g_n\,dx\leqslant\lim_{n\to\infty}\inf\int_\Omega g_n\,dx\leqslant\nu.$$

The converse of lemma is trivial. □

By the continuity of the cost function, let M_d and σ_τ be the compact sets, then $\eta_1(t)\in M_d$ and $\eta_2(x)\in\sigma_\tau$, for all $t\in[0,T]$ and $x\in\Omega$. If $D_1=A\times Q$, $D_2=M_d\times[0,T]$ and $D_3=\sigma_\tau\times\Omega$, then for $F\in C(D_1)$, $G\in C(D_2)$ and $H\in C(D_3)$, define $\Lambda_1:C(D_1)\mapsto\mathcal{C}$, $\Lambda_2:C(D_2)\mapsto\mathcal{C}$, $\Lambda_3:C(D_1)\mapsto\mathcal{C}$ by

$$\Lambda_1(F)=\int_Q F(w,x,t)\,dx\,dt,$$

$$\Lambda_2(G)=\int_0^T G\big(\eta_1(t),t\big)\,dt$$

and

$$\Lambda_3(H)=\int_\Omega H\big(\eta_2(x),x\big)\,dx.$$

One can see that Λ_1, Λ_2 and Λ_3 are positive bounded linear functionals on $C(D_1), C(D_2)$ and $C(D_3)$, respectively. By the Riesz representation theorem [9], there exist positive Radon measures μ_1, μ_2 and μ_3 such that

$$\Lambda_1(F)=\int_Q F(w,x,t)\,dx\,dt=\int_{D_1} F\,d\mu_1,$$

$$\Lambda_2(G)=\int_{[0,T]} G\big(\eta_1(t),t\big)\,dt=\int_{D_2} G\,d\mu_2$$

and

$$\Lambda_3(H)=\int_\Omega H\big(\eta_2(x),x\big)\,dx=\int_{D_3} H\,d\mu_3.$$

Now (9) changes to the following form:

$$\int_{D_1} F_\theta\,d\mu_1=\int_{D_2} G_\theta\,d\mu_2+\int_{D_3} H_\theta\,d\mu_3, \tag{14}$$

where

$$\begin{aligned}
F_\theta(w,x,t)&=\left(\nabla\cdot(\alpha_1\nabla\theta_1)-\beta_1\frac{\partial\theta_1}{\partial t}+\gamma(\theta_2-\theta_1)\right)u(x,t)+\left(\nabla\cdot(\alpha_2\nabla\theta_2)-\beta_2\frac{\partial\theta_2}{\partial t}+\gamma(\theta_1-\theta_2)\right)v(x,t),\\
G_\theta\big(\eta_1(t),t\big)&=(1-\lambda)\left(\sum_{i=1}^{N_w} m_i(t)\theta_1\big(P_i^w,t\big)-\sum_{j=1}^{N_o} n_j(t)\theta_1\big(P_j^o,t\big)\right)\\
&\quad+\lambda\left(\sum_{i=1}^{N_w} m_i(t)\theta_2\big(P_i^w,t\big)-\sum_{j=1}^{N_o} n_j(t)\theta_2\big(P_j^o,t\big)\right),\\
H_\theta\big(\eta_2(x),x\big)&=-\beta_1\sigma(x)\theta_1(x,0)+\beta_2\tau(x)\theta_2(x,0)
\end{aligned} \tag{15}$$

and (13) changes to

$$\frac{1}{T}\mu_1\big(g\big(x,t_n,u(x,t_n),v(x,t_n)\big)\big)\leqslant\nu,\quad n=1,2,\ldots, \tag{16}$$

where F_θ, G_θ and H_θ are belong to $C(D_1), C(D_2)$ and $C(D_3)$, respectively. Using these concepts we can put our nonclassical problem (14) with functional (6) in its definitive form. Thus, we seek measures μ_1, μ_2 and μ_3 which minimize the functional

$$I(\mu_1,\mu_2,\mu_3)=\mu_2(f_0)=\int_{D_2} f_0\,d\mu_2 \tag{17}$$

subject to (by (14))

$$\mu_1(F_\theta) - \mu_2(G_\theta) - \mu_3(H_\theta) = 0, \quad \forall \theta \in \big(C^\infty(\overline{Q})\big)^2,$$
$$\frac{1}{T}\mu_1\big(g\big(x, t_n, u(x, t_n), v(x, t_n)\big)\big) \leqslant \nu, \quad n = 1, 2, \ldots. \tag{18}$$

So the problem of minimizing (6) on $\mathcal{F}$ will be converted to the problem of minimizing (17) by the triples (μ_1, μ_2, μ_3) so that these triples are satisfied in (18). We call the set of all positive Radon measures on D_1, D_2 and D_3 by $M^+(D_1)$, $M^+(D_2)$, and $M^+(D_3)$, respectively. We choose (μ_1, μ_2, μ_3) from $M^+(D_1) \times M^+(D_2) \times M^+(D_3)$. Now consider the functions $\zeta : D_1 \to R$ and $\xi : D_2 \to R$ and $\varsigma : D_3 \to R$ so that these functions depend only on $(x, t) \in Q$ and $t \in [0, T]$ and $x \in \Omega$, respectively. We have

$$\mu_1(\zeta) = \int_{D_1} \zeta(x, t)\, dx\, dt = a_\zeta, \tag{19}$$

$$\mu_2(\xi) = \int_{D_2} \xi(t)\, dt = b_\xi \tag{20}$$

and

$$\mu_3(\varsigma) = \int_{D_3} \varsigma(x)\, dx = c_\varsigma. \tag{21}$$

Note that a_ζ, b_ξ and c_ς are the Lebesgue integrals of the functions ζ, ξ and ς on D_1, D_2 and D_3, respectively. Thus if 1_{D_1}, 1_{D_2} and 1_{D_3} are characteristic functions of D_1, D_2 and D_3 and L is the Lebesgue measure of Ω, respectively, then

$$\mu_1(1_{D_1}) = TL,$$
$$\mu_2(1_{D_2}) = T,$$
$$\mu_3(1_{D_3}) = L. \tag{22}$$

3. The existence of approximate optimal measure

Let $\mathcal{P}$ be the subset of measures in $M^+(D_1) \times M^+(D_2) \times M^+(D_3)$ which satisfies (18)–(22). We intend to show there exists an optimal triple measure (μ_1, μ_2, μ_3) in $\mathcal{P}$ such that minimizes the functional (17).

To find an optimal triple measure we have to use a convenient topology for $M^+(D_1) \times M^+(D_2) \times M^+(D_3)$ so that $\mathcal{P}$ be a compact subset of this space. If we topologize the space $M^+(D_1) \times M^+(D_2) \times M^+(D_3)$ by the product topology of the weak* topology, we can say $\mathcal{P}$ is compact and so by Theorem II.1 in [7] any continuous function gets its minimum on a compact subset of a Hausdorff space.

Proposition 3.1. *The set $\mathcal{P}$ of all triple measures in $M^+(D_1) \times M^+(D_2) \times M^+(D_3)$ that satisfy*

$$\mu_1(F_\theta) - \mu_2(G_\theta) - \mu_3(H_\theta) = 0, \quad \forall \theta \in \big(C^\infty(\overline{Q})\big)^2,$$
$$\frac{1}{T}\mu_1\big(g\big(x, t_n, u(x, t_n), v(x, t_n)\big)\big) \leqslant \nu, \quad n = 1, 2, \ldots,$$
$$\mu_1(\zeta) = a_\zeta,$$
$$\mu_2(\xi) = b_\xi,$$
$$\mu_3(\varsigma) = c_\varsigma \tag{23}$$

for all ζ's, ξ's and ς's, is compact in respect to weak topology on $M^+(D_1) \times M^+(D_2) \times M^+(D_3)$.*

Proof. The proof is similar to the proof of Proposition 4.4 in [2]. □

The functional $(\mu_1, \mu_2, \mu_3) \to \mu_2(f_0)$ is continuous (see [2]) and thus we have the following proposition.

Proposition 3.2. *There exists an optimal triple measure* $(\mu_1^*, \mu_2^*, \mu_3^*)$ *in* $\mathcal{P}$ *so that for any triple,* (μ_1, μ_2, μ_3) *in* $\mathcal{P}$,

$$I(\mu_1^*, \mu_2^*, \mu_3^*) = \mu_2^*(f_0) \leqslant \mu_2(f_0) = I(\mu_1, \mu_2, \mu_3)$$

thus the functional $I = \int_{D_2} f_0 \, d\mu_2$ *achieves a minimum on* $\mathcal{P}$.

The problem (17)–(18) is an infinite dimensional linear programming, the underlying space $M^+(D_1) \times M^+(D_2) \times M^+(D_3)$ is not finite dimensional and the number of equations in (18) are not finite. In following we intend to find a way for converting this problem to a finite dimensional linear programming problem.

Proposition 3.3. *Let* $\mathcal{P}^\diamond \subset \mathcal{P}$ *be the set of triple measures* (μ_1, μ_2, μ_3) *in* $\mathcal{P}$ *corresponding to triples* (w, η_1, η_2) *of piecewise constant functions on* Q, $[0, T]$ *and* Ω *that satisfy* (23), *then* $\mathcal{P}^\diamond$ *is weak* dense in* $\mathcal{P}$.

Proof. The proof is like the proposition in Appendix of [4]. □

Definition 3.4. We call the functions $(\theta_{1_i}, \theta_{2_i}) \in [C^\infty(\overline{Q})]^2, i = 1, 2, \ldots,$ total if for each $(\theta_1, \theta_2) \in [C^\infty(\overline{Q})]^2$ and for given $\epsilon > 0$, there exist a positive integer N and real numbers as ρ_i, $i = 1, 2, \ldots, N$ such that

$$\begin{aligned}
&\max_{\overline{Q}} \left| \theta_1 - \sum_{i=1}^N \rho_i \theta_{1_i} \right| < \epsilon, \qquad \max_{\overline{Q}} \left| \theta_2 - \sum_{i=1}^N \rho_i \theta_{2_i} \right| < \epsilon, \\
&\max_{\overline{Q}} \left| \frac{\partial}{\partial t}\theta_1 - \sum_{i=1}^N \rho_i \frac{\partial}{\partial t}\theta_{1_i} \right| < \epsilon, \qquad \max_{\overline{Q}} \left| \frac{\partial}{\partial t}\theta_2 - \sum_{i=1}^N \rho_i \frac{\partial}{\partial t}\theta_{2_i} \right| < \epsilon, \\
&\max_{\overline{Q}} \left\| \nabla\theta_1 - \sum_{i=1}^N \rho_i \nabla\theta_{1_i} \right\|_E < \epsilon, \qquad \max_{\overline{Q}} \left\| \nabla\theta_2 - \sum_{i=1}^N \rho_i \nabla\theta_{2_i} \right\|_E < \epsilon.
\end{aligned} \tag{$\star$}$$

Now by (15) we define

$$F_i := F_{\theta_i}, \qquad G_i := G_{\theta_i}, \qquad H_i := H_{\theta_i}, \quad i = 1, 2, 3, \ldots,$$

furthermore we consider a different form of total functions in $C(Q)$, $C([0, T])$ and $C(\Omega)$, respectively, corresponding to the functions in (19)–(21) as follows:

$$\{\zeta_i,\ i = 1, 2, \ldots\}, \qquad \{\xi_i,\ i = 1, 2, \ldots\}, \qquad \{\varsigma_i,\ i = 1, 2, \ldots\},$$

respectively, such that Lebesgue integral of them on D_1, D_2 and D_3 are a_j, b_k and c_l for a_{ζ_j}, b_{ξ_k} and c_{ς_l}.

Now we consider the following proposition that its proof is similar to Theorem 3 of [8].

Proposition 3.5. *Let* M_1, M_2, M_3, M_4 *and* M_5 *be positive integers. We consider the problem of minimizing the functional as*

$$(\mu_1, \mu_2, \mu_3) \to \mu_2(f_0) \tag{24}$$

on the set $P(M_1, M_2, M_3, M_4, M_5) \subset \mathcal{P}$ *of measures in* $M^+(D_1) \times M^+(D_2) \times M^+(D_3)$ *that satisfy*

$$\begin{aligned}
&\mu_1(F_i) - \mu_2(G_i) - \mu_3(H_i) = 0, \quad i = 1, 2, \ldots, M_1, \\
&\frac{1}{T}\mu_1\big(g\big(x, t_n, u(x, t_n), v(x, t_n)\big)\big) \leqslant \nu, \quad n = 1, 2, \ldots, M_2, \\
&\mu_1(\zeta_j) = a_j, \quad j = 1, 2, \ldots, M_3, \\
&\mu_2(\xi_k) = b_k, \quad k = 1, 2, \ldots, M_4, \\
&\mu_3(\varsigma_l) = c_l, \quad l = 1, 2, \ldots, M_5,
\end{aligned} \tag{25}$$

then as $M_1, M_2, M_3, M_4, M_5 \to \infty$,

$$\inf_{P(M_1, M_2, M_3, M_4, M_5)} \mu_2(f_0) \to \inf_{\mathcal{P}} \mu_2(f_0).$$

One can show that (see [7])

$$\inf_{\mathcal{P}} I \leqslant \inf_{F} J.$$

Now we can proceed the construction of suboptimal triples of trajectory and controls for functional (6). In the first step we obtain the optimal triples $(\mu_1^*, \mu_2^*, \mu_3^*)$ in $\mathcal{P}$ corresponding to triples (w, η_1, η_2) of piecewise constant functions on $Q, [0, T]$ and Ω that satisfy (23) which we called the set of all these triples $\mathcal{P}^\diamond$. By Proposition 3.3, $\mathcal{P}^\diamond$ is dense in $\mathcal{P}$, thus we apply proposition (3.5) for member of $\mathcal{P}^\diamond \cap P(M_1, M_2, M_3, M_4, M_5)$. Now by optimal measure obtained from (24)–(25), we find a triple (w, η_1, η_2) of piecewise constant functions. The vector function η_2 belongs to $[L^2(\Omega)]^2$, because Ω is bounded and η_2 is piecewise constant. By a similar reason the piecewise constant vector function η_2 belongs $[L^2((0, T))]^{N_w+N_o}$. We call function w corresponding to η_1 and η_2 in any triple by $w_{\eta_1}^{\eta_2}$. Now by the weak solution of (1)–(4) and definition (1.1), the function $w_{\eta_1}^{\eta_2}$ which is belong to $[L^1((0, T); W^{1,1}(\Omega))]^2$ is a weak solution of (1)–(4) as well. The existence of this weak solution is shown in Theorem 2.1. One can see a framework in [5,6,8] and [1], by borrowing a term from [10], we call the triple $(w_{\eta_1}^{\eta_2}, \eta_1, \eta_2)$ of trajectory and control functions *asymptotically admissible* if:

(i) The vector control functions $\eta_1 \in [L^2((0, T))]^2$, $\eta_1(t) \in M_d$ and $\eta_2 \in [L^2(\Omega)]^2$, $\eta_2 \in \sigma_\tau$.
(ii) Trajectory function $(w_{\eta_1}^{\eta_2}, \eta_1, \eta_2)$ is the weak solution of (1)–(4) corresponding to the control functions $\eta_1(\cdot)$ and $\eta_2(\cdot)$ and satisfy (9) and (5).

Finally we will show if the numbers M_1, M_2, M_3, M_4, and M_5, that are introduced in proposition (3.5), are sufficiently large and the approximate optimal triple measure, that is obtained by above manner, be sufficiently good, then the value of $J(w_{\eta_1}^{\eta_2}, \eta_1, \eta_2)$, the value of functional in (6) by $(w_{\eta_1}^{\eta_2}, \eta_1, \eta_2)$, is close to $\inf_{\mathcal{P}} I$.

Note that we do not need to obtain the trajectory function which is made by the control functions $\eta_1(\cdot)$ and $\eta_2(\cdot)$.

Theorem 3.6. *Let $(w_{\eta_1}^{\eta_2}, \eta_1, \eta_2)$ be the triple of controls and trajectory that is obtained by the mentioned discussion, then*

(i) *The triple is asymptotically admissible.*
(ii) *As $M_1, M_2, M_3, M_4, M_5 \to \infty$, then*

$$J\big(w_{\eta_1}^{\eta_2}, \eta_1, \eta_2\big) \to \inf_{\mathcal{P}} I.$$

Proof. We assumed that $\{\theta_i\}_{i\in\mathbb{N}}$ is a total sequence of functions, so for given ϵ and $\theta = (\theta_1, \theta_2) \in [C^\infty(\overline{Q})]^2$, there is M_1 and ρ_i, $i = 1, 2, \ldots, M_1$, such that $(\star)$ holds, for M_1 instead of N. Fix the values of M_2, M_3, M_4 and M_5. Let $(\mu_1^*, \mu_2^*, \mu_3^*)$ be the minimizer for the functional (24) over the set of $P(M_1, M_2, M_3, M_4, M_5)$. Because of density of $\mathcal{P}^\diamond$ in $\mathcal{P}$, Proposition 3.3, we can find a triple of piecewise constant trajectory-control functions $(w_{\eta_1}^{\eta_2}, \eta_1, \eta_2)$ on $\mathcal{P}^\diamond$ so that

$$\begin{aligned}
&\big|\mu_2^{\eta_1}(f_0) - \mu_2^*(f_0)\big| < \epsilon, \\
&\big|\mu_1^{w}(F_i) - \mu_2^{\eta_1}(G_i) - \mu_3^{\eta_2}(H_i)\big| < \epsilon, \quad i = 1, 2, \ldots, M_1, \\
&\frac{1}{T}\mu_1^{w}\big(g_n\big(x, t_n, w(x, t_n)\big)\big) \leqslant \nu, \quad n = 1, 2, \ldots, M_2.
\end{aligned} \tag{26}$$

In the above inequalities, we have used the triple $(\mu_1^w, \mu_2^{\eta_1}, \mu_3^{\eta_2})$ to denote measures in $M^+(D_1) \times M^+(D_2) \times M^+(D_3)$ generated by the triple trajectory-control $(w_{\eta_1}^{\eta_2}, \eta_1, \eta_2)$.

Now we prove that the triple $(w_{\eta_1}^{\eta_2}, \eta_1, \eta_2)$ obtained as described above is asymptotically admissible.

(a) We use $\eta = (\eta_1, \eta_2)$ for obtaining $w(\eta)$ as above. Now, we must prove that $w(\eta)$ for the above given $\theta \in [C^\infty(\overline{Q})]^2$ satisfies

$$\bigg|\int_Q -\beta_1 \frac{\partial \theta_1}{\partial t} + \alpha_1 \nabla\theta_1 \cdot \nabla u + \gamma(u - v)\theta_1\, dx\, dt + \int_Q -\beta_2 \frac{\partial \theta_2}{\partial t} + \alpha_2 \nabla\theta_2 \cdot \nabla v + \gamma(v - u)\theta_2\, dx\, dt$$

$$-(1-\lambda)\int_0^T\left(\sum_{j=1}^{N_o} n_j(t)\theta_1\big(P_j^o,t\big)-\sum_{i=1}^{N_w} m_j(t)\theta_1\big(P_i^w,t\big)\right)dt$$

$$-\lambda\int_0^T\left(\sum_{j=1}^{N_o} n_j(t)\theta_2\big(P_j^o,t\big)-\sum_{i=1}^{N_w} m_j(t)\theta_2\big(P_i^w,t\big)\right)dt-\int_\Omega\big(\beta_1\sigma(x)\theta_1(x,0)-\beta_2\tau(x)\theta_2(x,0)\big)\,dx\Bigg|<\epsilon.$$

We know that for $i=1,2,\ldots,M_1$,

$$\Bigg|\int_Q\left(-\beta_1\frac{\partial\theta_{1_i}}{\partial t}+\alpha_1\nabla\theta_{1_i}\cdot\nabla u+\gamma(u-v)\theta_{1_i}\right)dx\,dt+\int_Q\left(-\beta_2\frac{\partial\theta_{2_i}}{\partial t}+\alpha_2\nabla\theta_{2_i}\cdot\nabla v+\gamma(v-u)\theta_{2_i}\right)dx\,dt$$

$$-(1-\lambda)\int_0^T\left(\sum_{j=1}^{N_o} n_j(t)\theta_1\big(P_j^o,t\big)-\sum_{i=1}^{N_w} m_j(t)\theta_1\big(P_i^w,t\big)\right)dt$$

$$-\lambda\int_0^T\left(\sum_{j=1}^{N_o} n_j(t)\theta_{2_i}\big(P_j^o,t\big)-\sum_{i=1}^{N_w} m_j(t)\theta_{2_i}\big(P_i^w,t\big)\right)dt-\int_\Omega\big(\beta_1\sigma(x)\theta_1(x,0)-\beta_2\tau(x)\theta_{2_i}(x,0)\big)\,dx|<\epsilon.$$

Let $K_1=\int_Q dx\,dt$, $K_u=\int_Q|\nabla u|\,dx\,dt$, $K_v=\int_Q|\nabla v|\,dx\,dt$, $K_2=\int_Q|u-v|\,dx\,dt$, $K_3=\int_0^T|\sum_{j=1}^{N_o}n_j(t)|\,dt$, $K_4=\int_0^T|\sum_{i=1}^{N_w}m_j(t)|\,dt$, $K_5=\int_\Omega|\sigma(x)|\,dx$, $K_6=\int_\Omega|\tau(x)|\,dx$, therefore

$$\Bigg|\int_Q\left(-\beta_1\frac{\partial\theta_1}{\partial t}+\alpha_1\nabla\theta_1\cdot\nabla u+\gamma(u-v)\theta_1\right)dx\,dt+\int_Q\left(-\beta_2\frac{\partial\theta_2}{\partial t}+\alpha_2\nabla\theta_2\cdot\nabla v+\gamma(v-u)\theta_2\right)dx\,dt$$

$$-(1-\lambda)\int_0^T\left(\sum_{j=1}^{N_o} n_j(t)\theta_1\big(P_j^o,t\big)-\sum_{i=1}^{N_w} m_j(t)\theta_1\big(P_i^w,t\big)\right)dt$$

$$-\lambda\int_0^T\left(\sum_{j=1}^{N_o} n_j(t)\theta_2\big(P_j^o,t\big)-\sum_{i=1}^{N_w} m_j(t)\theta_2\big(P_i^w,t\big)\right)dt-\int_\Omega\big(\beta_1\sigma(x)\theta_1(x,0)-\beta_2\tau(x)\theta_2(x,0)\big)\,dx\Bigg|$$

$$=\Bigg|\int_Q\left(-\beta_1\left[\frac{\partial\theta_1}{\partial t}-\sum_{i=1}^{M_1}\rho_i\frac{\partial\theta_{1_i}}{\partial t}\right]+\alpha_1\left(\nabla\theta_1-\sum_{i=1}^{M_1}\rho_i\nabla\theta_{1_i}\right)\cdot\nabla u+\gamma(u-v)\left(\theta_1-\sum_{i=1}^{M_1}\rho_i\theta_{1_i}\right)\right)dx\,dt$$

$$+\int_Q\left(\sum_{i=1}^{M_1}\rho_i\left(-\beta_1\frac{\partial\theta_{1_i}}{\partial t}\right)+\sum_{i=1}^{M_1}\rho_i(\alpha_1\nabla\theta_{1_i}\cdot\nabla u)+\sum_{i=1}^{M_1}\rho_i\big(\gamma(u-v)\theta_{1_i}\big)\right)dx\,dt$$

$$+\int_Q\left(-\beta_2\left[\frac{\partial\theta_2}{\partial t}-\sum_{i=1}^{M_1}\rho_i\frac{\partial\theta_{2_i}}{\partial t}\right]+\alpha_2\left(\nabla\theta_1-\sum_{i=1}^{M_1}\rho_i\nabla\theta_{2_i}\right)\cdot\nabla v+\gamma(u-v)\left(\theta_2-\sum_{i=1}^{M_1}\rho_i\theta_{2_i}\right)\right)dx\,dt$$

$$+\int_Q\left(\sum_{i=1}^{M_1}\rho_i\left(-\beta_2\frac{\partial\theta_{2_i}}{\partial t}\right)+\sum_{i=1}^{M_1}\rho_i(\alpha_2\nabla\theta_{2_i}\cdot\nabla v)+\sum_{i=1}^{M_1}\rho_i\big(\gamma(v-u)\theta_{2_i}\big)\right)dx\,dt$$

$$-(1-\lambda)\int_0^T\left(\sum_{j=1}^{N_o} n_j(t)\left(\theta_1-\sum_{i=1}^{M_1}\rho_i\theta_{1_i}\right)\big(P_j^o,t\big)-\sum_{i=1}^{N_w} m_j(t)\left(\theta_1-\sum_{i=1}^{M_1}\rho_i\theta_{1_i}\right)\big(P_i^w,t\big)\right)dt$$

$$-\lambda\int_0^T\left(\sum_{j=1}^{N_o}n_j(t)\left(\theta_2-\sum_{i=1}^{M_1}\rho_i\theta_{2_i}\right)\left(P_j^o,t\right)-\sum_{i=1}^{N_w}m_j(t)\left(\theta_2-\sum_{i=1}^{M_1}\rho_i\theta_{2_i}\right)\left(P_i^w,t\right)\right)dt$$

$$-\int_\Omega\left(\beta_1\sigma(x)\left(\theta_1-\sum_{i=1}^{M_1}\rho_i\theta_{1_i}\right)(x,0)-\beta_2\tau(x)\left(\theta_2-\sum_{i=1}^{M_1}\rho_i\theta_{2_i}\right)(x,0)\right)dx$$

$$-(1-\lambda)\int_0^T\sum_{i=1}^{M_1}\rho_i\left(\sum_{j=1}^{N_o}n_j(t)\theta_{1_i}\left(P_j^o,t\right)-\sum_{i=1}^{N_w}m_j(t)\theta_{1_i}\left(P_i^w,t\right)\right)dt$$

$$-\lambda\int_0^T\sum_{i=1}^{M_1}\rho_i\left(\sum_{j=1}^{N_o}n_j(t)\theta_{2_i}\left(P_j^o,t\right)-\sum_{i=1}^{N_w}m_j(t)\theta_{2_i}\left(P_i^w,t\right)\right)dt$$

$$-\int_\Omega\sum_{i=1}^{M_1}\rho_i\left(\beta_1\sigma(x)\theta_{1_i}(x,0)-\beta_2\tau(x)\theta_{2_i}(x,0)\right)dx\Bigg|$$

$$\leqslant\left((\beta_1+\beta_2)K_1+\alpha_1K_u+2\gamma K_2+\alpha_2K_v+(K_3+K_4)+\beta_1K_5+\beta_2K_6\right)\epsilon$$

$$+\sum_{i=1}^{M_1}|\rho_i|\left|\int_Q-\beta_1\frac{\partial\theta_{1_i}}{\partial t}+\alpha_1\nabla\theta_{1_i}\cdot\nabla u+\gamma(u-v)\theta_{1_i}\,dx\,dt\right.$$

$$+\int_Q-\beta_2\frac{\partial\theta_{2_i}}{\partial t}+\alpha_2\nabla\theta_{2_i}\cdot\nabla v+\gamma(v-u)\theta_{2_i}\,dx\,dt$$

$$-(1-\lambda)\int_0^T\left(\sum_{j=1}^{N_o}n_j(t)\theta_1\left(P_j^o,t\right)-\sum_{i=1}^{N_w}m_j(t)\theta_1\left(P_i^w,t\right)dt\right.$$

$$\left.-\lambda\int_0^T\sum_{j=1}^{N_o}n_j(t)\theta_{2_i}\left(P_j^o,t\right)-\sum_{i=1}^{N_w}m_j(t)\theta_{2_i}\left(P_i^w,t\right)\right)dt-\int_\Omega\left(\beta_1\sigma(x)\theta_1(x,0)-\beta_2\tau(x)\theta_{2_i}(x,0)\right)dx\Bigg|$$

$$<\epsilon C+\sum_{i=1}^{M_1}|\rho_i|\epsilon<\epsilon_1,$$

where C is the constant appeared in the inequality. By Theorem 2.1, we have

$$\int_\Omega\left(\nabla\cdot(\alpha_1\nabla\theta_1)-\beta_1\frac{\partial\theta_1}{\partial t}+\gamma(\theta_2-\theta_1)\right)u(x,t)\,dx\,dt+\left(\nabla\cdot(\alpha_2\nabla\theta_2)-\beta_2\frac{\partial\theta_2}{\partial t}+\gamma(\theta_1-\theta_2)\right)v(x,t)\,dx\,dt$$

$$=\lambda\int_0^T\left(\sum_{i=1}^{N_w}m_i(t)\theta_2\left(P_i^w,t\right)-\sum_{j=1}^{N_o}n_j(t)\theta_2\left(P_j^o,t\right)\right)dt$$

$$+(1-\lambda)\int_0^T\left(\sum_{i=1}^{N_w}m_i(t)\theta_1\left(P_i^w,t\right)-\sum_{j=1}^{N_o}n_j(t)\theta_1\left(P_j^o,t\right)\right)dt$$

$$-\int_\Omega\beta_1\sigma(x)\theta_1(x,0)+\beta_2\tau(x)\theta_2(x,0)\,dx. \tag{27}$$

Therefore $w(\eta,\eta)$ is asymptotically admissible.

(b) Since

$$\left|J(\eta) - \inf_{D_2} I(\mu_1, \mu_2, \mu_3)\right| = \left|\mu_2^{\eta_1}(f_0) - \mu_2^*(f_0)\right|,$$

by using (26), we have

$$\left|J(\eta) - \inf_{D_2} I(\mu_1, \mu_2, \mu_3)\right| < \epsilon. \qquad \square$$

4. The approximate optimal triple measure

Let $(\mu_1^*, \mu_2^*, \mu_3^*)$ be the optimal triple measure that is obtained by solving the linear programming (24)–(25). Now by unitary atomic measure we can write μ_1^*, μ_2^*, and μ_3^* as a finite linear combination of unitary atomic measure as follows:

$$\mu_1^* = \sum_{m=1}^{M} \alpha_m^* \delta(z_m^*),$$
$$\mu_2^* = \sum_{n=1}^{p} \beta_n^* \delta(\hat{z}_n^*),$$
$$\mu_3^* = \sum_{r=1}^{R} \gamma_r^* \delta(\hat{\hat{z}}_r^*),$$

where $\alpha_m^* \geqslant 0$, $m = 1, 2, \ldots, M$, $\beta_n^* \geqslant 0$, $n = 1, 2, \ldots, p$, and $\gamma_r^* \geqslant 0$, $r = 1, 2, \ldots, R$, $z_m^* \in D_1$, $\hat{z}_n^* \in D_2$, and $\hat{\hat{z}}_r^* \in D_3$ for any m, n, and r and $\delta(z_m)$, $\delta(\hat{z}_n)$ and $\delta(\hat{\hat{z}}_r)$ are unitary atomic measures, respectively, supported by $z, \hat{z}$, and $\hat{\hat{z}}$.

Now by using Proposition III.3 of [7], by considering dense sets as $\widehat{D_1} \subset D_1$, $\widehat{D_2} \subset D_2$, and $\widehat{D_3} \subset D_3$, and by choosing $z_m^* \in \widehat{D_1}$, $m = 1, 2, \ldots, M$, $\hat{z}_n^* \in \widehat{D_2}$, $n = 1, 2, \ldots, p$, and $\hat{\hat{z}}_r^* \in \widehat{D_3}$, $r = 1, 2, \ldots, R$, the optimal triple measure $(\mu_1^*, \mu_2^*, \mu_3^*)$ that is obtained from problem of (24)–(25) can be approximated by triple measure (μ_1, μ_2, μ_3), where

$$\mu_1 = \sum_{m=1}^{M} \alpha_m \delta(z_m), \qquad \mu_2 = \sum_{n=1}^{p} \beta_n \delta(\hat{z}_n), \qquad \mu_3 = \sum_{r=1}^{R} \gamma_r \delta(\hat{\hat{z}}_r) \tag{28}$$

and $\alpha_m \geqslant 0$, $m = 1, 2, \ldots, M$, $\beta_n \geqslant 0$, $n = 1, 2, \ldots, p$, and $\gamma_r \geqslant 0$, $r = 1, 2, 3, \ldots, R$, will be obtained by solving a linear programming problem as follows:

$$\text{Minimize } \sum_{n=1}^{p} \beta_n f_0(\hat{z}_n)$$

subject to

$$\sum_{m=1}^{M} \alpha_m F_i(z_m) - \sum_{n=1}^{p} \beta_n G_i(\hat{z}_n) - \sum_{r=1}^{R} \gamma_r H_i(\hat{\hat{z}}_r) = 0, \quad i = 1, 2, \ldots, M_1,$$
$$\frac{1}{T} \sum_{m=1}^{M} \alpha_m g_s(z_m) \leqslant \nu, \quad s = 1, 2, \ldots, M_2,$$
$$\sum_{m=1}^{M} \alpha_m \zeta_j(z_m) = a_j, \quad j = 1, 2, \ldots, M_3,$$
$$\sum_{n=1}^{p} \beta_n \xi_k(\hat{z}_n) = b_k, \quad k = 1, 2, \ldots, M_4,$$

$$\sum_{r=1}^{R} \gamma_r \varsigma_l(\hat{\hat{z}}_r) = c_l, \quad l = 1, 2, \ldots, M_5,$$
$$\sum_{m=1}^{M} \alpha_m = 1_{D_1},$$
$$\sum_{n=1}^{p} \beta_n = 1_{D_2},$$
$$\sum_{r=1}^{R} \gamma_r = 1_{D_3},$$
$$\alpha_m \geqslant 0, \quad m = 1, 2, \ldots, M,$$
$$\beta_n \geqslant 0, \quad n = 1, 2, \ldots, p,$$
$$\gamma_r \geqslant 0, \quad r = 1, 2, \ldots, R, \tag{29}$$

where $1_{D_1} = TL$, $1_{D_2} = T$ and $1_{D_3} = L$ (L is Lebesgue measure of Ω). For obtaining z_m's, $\hat{z}_n$'s and $\hat{\hat{z}}_r$'s that are dense in D_1, D_2 and D_3 we divided the sets of B, M_d, σ_τ, A and $[0, T]$, respectively, to r_1, r_2, r_3, r_4, and r_5 subrectangulars, where $r_1 = d_x\, d_y\, d_z$, $r_2 = d_{m_1}\, d_{m_2} \cdots d_{m_{N_w}}\, d_{n_1}\, d_{n_2} \cdots d_{n_{N_o}}$, $r_3 = d_\sigma\, d_\tau$ and $r_4 = d_u\, d_v$, so that we have $M = r_1 r_2 r_3 r_4 r_5$, $p = r_5$ and $R = r_1$ subrectangulars of D_1, D_2, and D_3, respectively, as D_1^m, $m = 1, 2, \ldots, M$, D_2^n, $n = 1, 2, \ldots, p$, D_3^r, $r = 1, 2, \ldots, R$. We choose from each D_1^m, D_2^n and D_3^r a point as $z_m = (t_m, x_m, y_m, z_m, u_m, v_m)$, $\hat{z}_n = (t_n, \eta_{1_n})$ and $\hat{\hat{z}}_r = (x_r, y_r, z_r, \sigma_r, \tau_r)$, respectively.

5. Numerical example

Example 1. In this example, we apply the mentioned method for finding control functions $m_1(\cdot), n_1(\cdot), \sigma(\cdot)$ and $\tau(\cdot)$ for an optimal control problem that is to minimize a certain given functional of $m_1(\cdot)$ and $n_1(\cdot)$ at time interval $[0, 1]$ as

$$J(\eta) = \int_0^1 \big(m_1(t)\big)^2\, dt + \int_0^1 \big(n_1(t)\big)^3\, dt \tag{30}$$

on a system governed by following parabolic equations:

$$(0.1)\nabla^2 u - (0.1)\frac{\partial u}{\partial t} + 0.01(v - u) = (0.99)\big(m_1(t)\delta(x - 0.2, y - 0.2, z - 0.2) - n_1(t)\delta(x - 0.5, y - 0.5, z - 0.5)\big), \tag{31}$$

$$(0.01)\nabla^2 v - (0.01)\frac{\partial v}{\partial t} + 0.01(u - v) = (0.01)\big(m_1(t)\delta(x - 0.2, y - 0.2, z - 0.2) - n_1(t)\delta(x - 0.5, y - 0.5, z - 0.5)\big), \tag{32}$$

where initial and boundary conditions are as follows:

$$\frac{\partial u}{\partial n} = 0, \qquad \frac{\partial u}{\partial n} = 0, \quad (x, y, z, t) \in \Sigma, \tag{33}$$
$$u(x, y, z, 0) = \sigma(x, y, z), \qquad v(x, y, z, 0) = \tau(x, y, z), \quad (x, y, z) \in \Omega, \tag{34}$$

and according to the practical requirement of oil exploitation, we suppose that the pressures of the reservoir must satisfy the following condition:

$$G(u, v) = \int_\Omega \big(x^2 + y^2 + z^2 + t^2 + u^2(x, t) + v^2(x, t)\big)\, dx\, dy\, dz \leqslant 2, \quad t \in [0, 1], \tag{35}$$

where $\Omega = (0, 1) \times (0, 1) \times (0, 1)$ and $Q = \Omega \times (0, 1)$. For our numerical example we consider $T = 1$, $A = (0, 1) \times (0, 1)$, $M_d = [0, 1] \times [0, 1]$ and $\sigma_\tau = [0, 1] \times [0, 1]$. We divide the sells $(0, T) = (0, 1)$, A, $\Omega = (0, 1) \times (0, 1) \times (0, 1)$,

M_d and σ_τ to $5, 6\times 5, 4\times 4\times 4, 4\times 4$ and 5×5, respectively, thus we have $M=9600$, $p=80$ and $R=1600$. We define the functions θ which are defined in Section 2 as

$$\theta(x,y,z,t)=(T-t)^{p_1}\big(\sin(p_2\pi x)\sin(p_3\pi y)\sin(p_4\pi z),\cos(p_2\pi x)\cos(p_3\pi y)\cos(p_4\pi z)\big).$$

Note that $\theta(x,y,z,T)=0$. We consider six various forms of this functions for $p_1=1$, $p_2=1$, $p_3=2$ and $p_4=3$, thus in the problem (29) we have $M_1=6$, $M_2=20$, $M_3=320$, $M_4=5$, also $M_5=64$. Thus we have following linear programming problem:

$$\text{Minimize } \sum_{n=1}^{80}\beta_n\big(m_n^2+n_n^2\big)$$

subject to

$$\begin{aligned}
&\sum_{m=1}^{9600}\alpha_m F_i(z_m)-\sum_{n=1}^{80}\beta_n G_i(\hat z_n)-\sum_{r=1}^{1600}\gamma_r H_i(\hat{\hat z}_r)=0, \quad i=1,2,\ldots,6,\\
&\frac{1}{T}\sum_{m=1}^{9600}\alpha_m g_s(z_m)\leqslant \nu, \quad s=1,2,\ldots,20,\\
&\sum_{m=1}^{9600}\alpha_m\zeta_j(z_m)=a_j, \quad j=1,2,\ldots,320,\\
&\sum_{n=1}^{80}\beta_n\xi_k(\hat z_n)=b_k, \quad k=1,2,\ldots,5,\\
&\sum_{r=1}^{1600}\gamma_r\varsigma_l(\hat{\hat z}_r)=c_l, \quad l=1,2,\ldots,64,\\
&\sum_{m=1}^{9600}\alpha_m=1,\\
&\sum_{n=1}^{80}\beta_n=1,\\
&\sum_{r=1}^{1600}\gamma_r=1,\\
&\alpha_m\geqslant 0, \quad m=1,2,\ldots,9600,\\
&\beta_n\geqslant 0, \quad n=1,2,\ldots,80,\\
&\gamma_r\geqslant 0, \quad r=1,2,\ldots,1600.
\end{aligned}\tag{36}$$

By solving this linear programming problem we construct the optimal controls $m_1(\cdot)$ and $n_1(\cdot)$ by the method that is proposed in [7] and Section 6. The value of cost function is 0.1109 and optimal controls are shown in Figs. 1–4.

Example 2. This example is similar to Example 1 but here we have the following changes, $M_1=6$, $M_2=20$, $M_3=448$, $M_4=7$, $M_5=64$. So we have $M=15680$, $p=1512$, $R=3136$. The related linear programming has 548 constraints and 20328 variables. The cost function at time interval [0, 1] is as

$$J(\eta)=\int_0^1\big(m_1(t)\big)^2\,dt+\int_0^1\big(n_1(t)\big)^3\,dt+\int_0^1\big(n_2(t)\big)^3\,dt.$$

The value of cost function is 0.0139 and optimal controls are shown in Figs. 5–9.

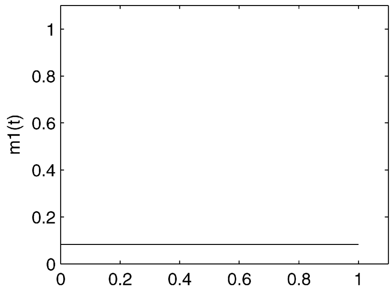

Fig. 1. The piecewise-constant optimal control $m_1(\cdot)$ on $t \in (0, 1]$.

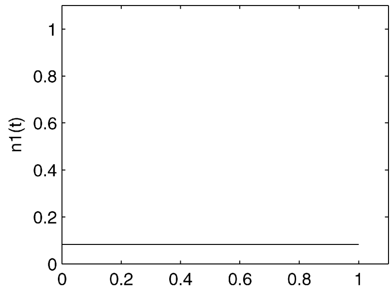

Fig. 2. The piecewise-constant optimal control $n_1(\cdot)$ on $t \in (0, 1]$.

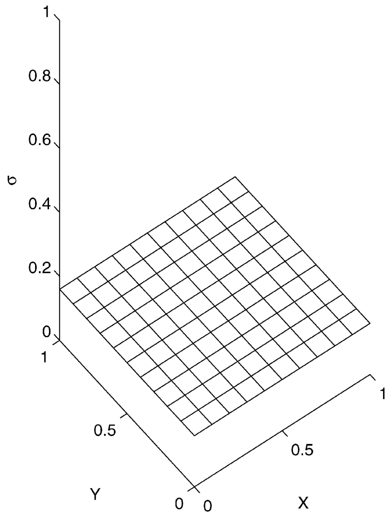

Fig. 3. The piecewise-constant optimal control $\sigma(x, y, 0)$.

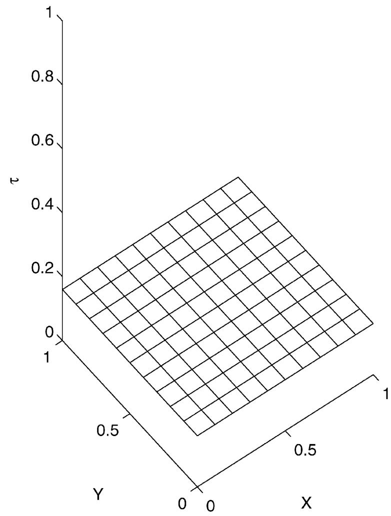

Fig. 4. The piecewise-constant optimal control $\tau(x, y, 0)$.

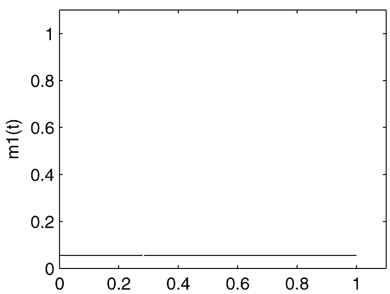

Fig. 5. The piecewise-constant optimal control $m_1(\cdot)$ on $t \in (0, 1]$.

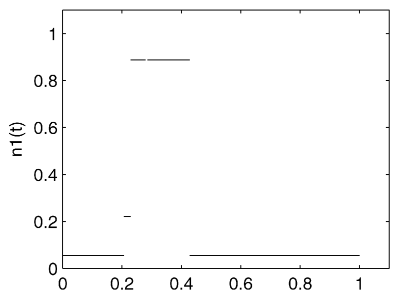

Fig. 6. The piecewise-constant optimal control $n_1(\cdot)$ on $t \in (0, 1]$.

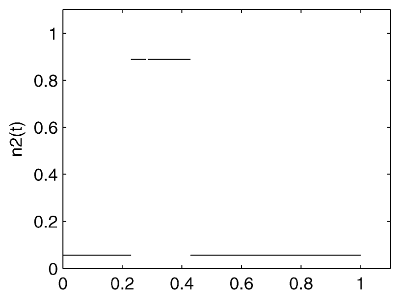

Fig. 7. The piecewise-constant optimal control $n_2(\cdot)$ on $t \in (0, 1]$.

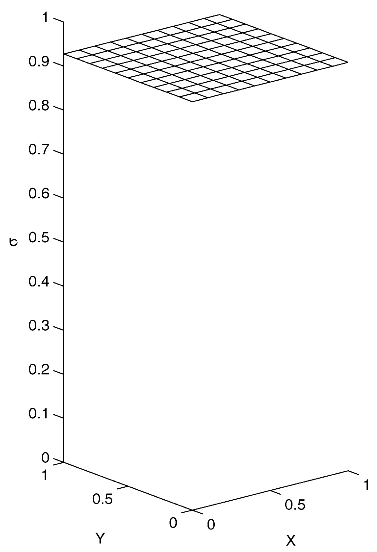

Fig. 8. The piecewise-constant optimal control $\sigma(x, y, 0)$.

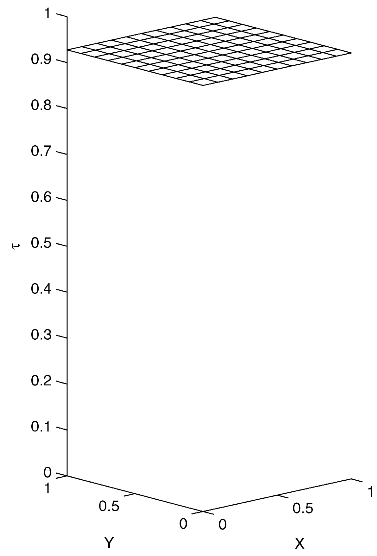

Fig. 9. The piecewise-constant optimal control $\tau(x, y, 0)$.

6. Conclusions

In this paper we introduce a new technique for solving the optimal control problem of the parabolic PDEs in exploitation of oil. Using the weak solution of the problem, the first problem is transformed into an optimization problem in measure space. Then we change this one to a finite dimensional linear programming problem. Finally we obtain piecewise-constant optimal control functions which are an approximate control for the original problem.

Acknowledgments

The authors wish to thank the referees for their kind comments and most valuable remarks.

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