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Lattice points in rational ellipsoids

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We combine exponential sums, character sums and Fourier coefficients of automorphic forms to improve the best known upper bound for the lattice error term associated to rational ellipsoids.

1. Introduction

The classical lattice point problem in dimension three consists of computing the error exponent

$$\theta_3 = \inf\{\alpha:\ E(R) = O(R^\alpha)\}$$

where $E(R)$ is the lattice error term associated to a fixed smooth and strictly convex body $\mathcal{B} \subset \mathbb{R}^3$

$$E(R) = \#\{\vec{n} \in \mathbb{Z}^3:\ \vec{n}/R \in \mathcal{B}\} - \mathrm{Vol}(\mathcal{B})R^3.$$

It is known that $\theta_3 \geqslant 1$, even in a sharper form [18]. The conjecture is $\theta_3 = 1$ but the best known general upper bound is $\theta_3 \leqslant 63/43$ [19], improved to $\theta_3 \leqslant 4/3$ for rational ellipsoids (see [4, §6]) and to $\theta_3 \leqslant 21/16$ for the sphere [14]. The disparity between the first and the second bound lies in the possibility of glueing variables twice when we have three rational axes (the argument was introduced in [6] and [23]). The third result is rather different because it uses the classical and deep result due to Gauss [11] that relates the number of lattice points on the sphere to the class number of imaginary quadratic fields (see [12]). This interpretation, via class number formula, allows to complement the exponential sums approach with character sums estimates.

The number of lattice points on a rational ellipsoid can be understood as the coefficient of a modular form that decomposes into a non-cuspidal part (a linear combination of Eisenstein series) and a cuspidal part. The former encodes local information and gives the main contribution in dimension greater than three, on the other hand W. Duke and R. Schulze-Pillot have shown in several works a different and more intricate truth in dimension 3 (see [1,10,13]).

In our approach we start using exponential sums techniques (as in [5]) to analyze a regularization of the error term. The loss in the regularization can be reduced controlling the number of lattice points in thin layers which is expressed as a short sum of coefficients of a modular form decomposed as before into a cuspidal and a non-cuspidal form. The Siegel mass formula [22] allows to express the non-cuspidal contribution as a sum of a product of local factors that mimics a short

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sum of L-functions and we succeed adapting the method of [14] to bound this kind of sums. The cuspidal coefficients cannot be controlled individually in a satisfactory way but using the structure of Shimura lift, Duke–Iwaniec bound for half-integral weight [9,16] and Ramanujan–Petersson conjecture for weight 2 (due to Eichler and Shimura), they can be manipulated to prove that in the cuspidal contribution we can save a small power over the admitted error term in the non-cuspidal part. (This saving would not occur with any nontrivial bound.)

In few words, the key idea in [5] and [14] is a decomposition of the lattice error term for the sphere using additive characters (exponential sums) and multiplicative characters (character sums) and we prove that for rational ellipsoids a modular part must be added to the decomposition (it does not appear in the sphere problem because the form $x_1^2+x_2^2+x_3^2$ is the only form in the genus). To our knowledge this is the first time that the properties of modular forms appear intrinsically in the classical lattice point problem (as described above). Although a sufficiently large amount of saving is needed in the modular part, the bounds are not critical and recent improvements [2] do not affect the result because with our estimates the additive and the multiplicative parts dominate. In this way we reach for rational ellipsoids the best known result for the sphere [14]:

Theorem 1. *Let $\mathcal{B}=\{\vec{x}\in\mathbb{R}^3:\ Q(\vec{x})\leqslant 1\}$ where Q is a positive definite ternary quadratic form with rational coefficients, then the corresponding error exponent satisfies $\theta_3\leqslant 21/16$.*

2. Decomposition of the error term

From now on we shall assume that Q has integral coefficients because θ_3 is invariant by dilations of $\mathcal{B}$. Then we have a symmetric matrix $\mathbf{A}\in\mathrm{GL}_3(\mathbb{Z})$ with even diagonal entries such that

$$Q(\vec{x})=\frac{1}{2}\vec{x}^t\mathbf{A}\vec{x}.$$

We denote by D the determinant of $\mathbf{A}$.

If fact perhaps changing $\mathbf{A}$ by $2\mathbf{A}$ we can assume also that $D\mathbf{A}^{-1}$ has even diagonal entries and $4\mid D$. Hence the *dual form* $Q^*(\vec{x})=\frac{1}{2}\vec{x}^tD\mathbf{A}^{-1}\vec{x}$ is also an integral quadratic form.

It is well known (see [21, p. 456], [15, Proposition 10.6]) that the theta function $\theta_Q(z)=\sum r_Q(n)e(nz)$ is a modular form in $\mathfrak{M}_{3/2}(\Gamma_0(D),\chi)$ for a certain quadratic character χ (we abbreviate $e^{2\pi ix}$ by $e(x)$ and $r_Q(n)$ indicates the number of representations of n by Q). Obviously it is not a cusp form, in fact its projection onto the linear space generated by Eisenstein series is the genus θ-function $\theta_{\mathrm{gen}}(z)=\sum r(n,\mathrm{gen}\,Q)e(nz)$ whose coefficients are given by the Siegel mass formula [22]

$$r(n,\mathrm{gen}\,Q)=\frac{4\pi\sqrt{2n}}{\sqrt{D}}L(n)\quad\text{with } L(n)=\prod_p\delta_p(n)$$

where $\delta_p(n)$ are the p-adic densities of the solutions of $Q(\vec{x})=n$. In fact for α large enough

$$\delta_p(n)=p^{-2\alpha}N_{p^\alpha}(n)$$

where $N_q(n)$ is the number of the solutions of $Q(\vec{x})=n$ in $(\mathbb{Z}/q\mathbb{Z})^3$.

We write

$$a_n=r_Q(n)-r(n,\mathrm{gen}\,Q).$$

This is the nth coefficient of a form in $S_{3/2}(\Gamma_0(D),\chi)$, the linear space of cusp forms of weight 3/2 with the θ-multiplier system.

Before stating the decomposition of the error term we introduce a smoothing function for technical purposes. To facilitate references we follow the choice of [5] for $0<H<1$, the function

$$f(x)=\begin{cases}x & \text{if } 0\leqslant x\leqslant R,\\ R(R+H-x)/H & \text{if } R\leqslant x\leqslant R+H,\\ 0 & \text{otherwise.}\end{cases}$$

This election gives an explicit though slow-decaying Fourier sine transform

$$\widetilde{f}(\xi):=2\int_0^\infty f(t)\sin(2\pi\xi t)\,dt=\frac{\sin(2\pi R\xi)}{2\pi^2\xi^2}-\frac{R}{H}\frac{\sin(\pi H\xi)}{\pi^2\xi^2}\cos\bigl(\pi(2R+H)\xi\bigr).$$

After these definitions, we decompose the lattice error term in three parts $\mathcal{T}$, $\mathcal{C}$ and $\mathcal{M}$ that will be treated by methods of trigonometric sums, character sums and modular functions, respectively, in Sections 3–5. The parameter H is adjustable. In Section 6 we shall take $H=R^{-5/8}$ that corresponds to the optimal value for the sphere problem in [14].

Proposition 2. *With the previous definitions, we have for $R > 1$ and $0 < H < 1$,*

$$E(R) = \mathcal{T} + \mathcal{C} + \mathcal{M}$$

where

$$\mathcal{T} = \frac{8\pi\sqrt{2}}{\sqrt{D}} \int_R^{R+H} r f(r)\,dr - 1 + \sqrt{2}\sum_{n=1}^{\infty} r_{Q^*}(n)\frac{\widetilde{f}(\sqrt{4n/D})}{\sqrt{n}},$$

$$\mathcal{C} = -\frac{4\pi\sqrt{2}}{\sqrt{D}} \sum_{R<\sqrt{n}<R+H} L(n) f(\sqrt{n}) \quad \text{and} \quad \mathcal{M} = -\sum_{R<\sqrt{n}<R+H} a_n \frac{f(\sqrt{n})}{\sqrt{n}}.$$

In the next section (see the proof of Proposition 4) we shall check that the series in the definition of $\mathcal{T}$ converges, indeed it does uniformly for R in a bounded interval. Considering this, the decomposition of $E(R)$ is a consequence of the following summation formula (cf. [5]).

Lemma 3. *Let $g \in C_0^\infty([0, +\infty))$ with $g(0) = g''(0) = 0$ then*

$$\sum_{n=1}^{\infty} r_Q(n)\frac{g(\sqrt{n})}{\sqrt{n}} = \frac{8\pi\sqrt{2}}{\sqrt{D}} \int_0^{\infty} r g(r)\,dr - g'(0) + \sqrt{2}\sum_{n=1}^{\infty} r_{Q^*}(n)\frac{\widetilde{g}(\sqrt{4n/D})}{\sqrt{n}}.$$

Proof. Let $G(\vec{x})$ be the $C_0^2(\mathbb{R}^3)$ extension of $g(\sqrt{Q(\vec{x})})/\sqrt{Q(\vec{x})}$. By Poisson summation formula

$$g'(0) + \sum_{n=1}^{\infty} r_Q(n)\frac{g(\sqrt{n})}{\sqrt{n}} = \sum_{\vec{n}} \widehat{G}(\vec{n}).$$

Using spherical coordinates we know (cf. [5, Lemma 2.1]) that the Fourier transform of $g(\|\vec{x}\|)/\|\vec{x}\|$ is $\widetilde{g}(\|\vec{x}\|)/\|\vec{x}\|$. With the change of variables $\vec{x} \mapsto \frac{1}{\sqrt{2}}\mathbf{C}^{-1}\vec{x}$ where $\mathbf{C}^t\mathbf{A}\mathbf{C} = \mathbf{I}$, or equivalently $\mathbf{A}^{-1} = \mathbf{C}\mathbf{C}^t$, we have

$$\widehat{G}(\vec{n}) = \frac{\sqrt{2}\widetilde{g}(\sqrt{4Q^*(\vec{n})/D})}{\sqrt{Q^*(\vec{n})}} \quad \text{for } \vec{n} \neq \vec{0} \quad \text{and} \quad \widehat{G}(\vec{0}) = \frac{8\pi\sqrt{2}}{\sqrt{D}} \int_0^{\infty} r g(r)\,dr.$$

Substituting we get the result. □

Proof of Proposition 2. Write

$$E(R) = \sum_{1 \leqslant n \leqslant R^2} r_Q(n) - \mathrm{Vol}(\mathcal{B})R^3 = \sum_{n=1}^{\infty} r_Q(n)\frac{f(\sqrt{n})}{\sqrt{n}} - \sum_{R<\sqrt{n}<R+H} r_Q(n)\frac{f(\sqrt{n})}{\sqrt{n}} - \frac{8\pi\sqrt{2}}{\sqrt{D}} \int_0^R r f(r)\,dr.$$

The first and the third terms give $\mathcal{T}$ by Lemma 3. After the definition of a_n and the Siegel mass formula, the second term gives $\mathcal{C}$ and $\mathcal{M}$. □

3. The trigonometric sum

The exponential sum is analog to the one treated in [5]. We just outline the argument.

Proposition 4. *For any $1 < H^{-1} < R < H^{-2}$ and $\epsilon > 0$ we have*

$$\mathcal{T} = \frac{4\sqrt{2}\pi}{\sqrt{D}} H R^2 + O_\epsilon\big(\big(RH^{-1/2} + R^{21/16} + R^{11/8}H^{1/8}\big)R^\epsilon\big).$$

Remark. In a recent work F. Chamizo and E. Cristóbal have proved that the terms $R^{21/16}$ and $R^{11/8}H^{1/8}$ can be diminished. This work is not published yet and we prefer to appeal to the arguments of [5]. We take this opportunity to correct a misprint in the last but one formula of p. 427 [5], firstly pointed out by G. Kuba and not affecting the result of [5].

Proof of Proposition 4. A calculation proves

$$\frac{8\pi\sqrt{2}}{\sqrt{D}}\int_R^{R+H} rf(r)\,dr = \frac{4\pi\sqrt{2}}{\sqrt{D}}HR^2 + \frac{4\pi\sqrt{2}}{3\sqrt{D}}H^2R \tag{1}$$

where f is the smoothing function that we have shown before. The task is now to estimate

$$S_{\widetilde{f}}(R) = \sqrt{2}\sum_{n=1}^{\infty} r_{Q^*}(n)\frac{\widetilde{f}(\sqrt{4n/D})}{\sqrt{n}}. \tag{2}$$

The application of Lemma 3 is justified because we shall deduce the bound (3) for each dyadic block

$$V_N(R) = \sum_{N\leqslant n<2N} r_{Q^*}(n)e(R\sqrt{4n/D}),$$

which assures the uniform convergence on compacta.

Diagonalizing Q^* and clearing denominators, we can write

$$V_N(R) = \sum_{\vec{n}} e\left(\frac{R}{d}\sqrt{an_1^2+bn_2^2+cn_3^2}\right)$$

where $a, b, c, d \in \mathbb{Z}^+$ only depend on $\mathbf{A}$, and $\vec{n}$ runs over a subset of $\mathbb{Z}^3$ included in $\|\vec{n}\| \ll N^{1/2}$.

By Lemma 3.1 of [5] with minor modifications, we have

$$V_N(R) \ll N^{5/4+\epsilon} + N^{\epsilon}\min\bigl(R^{3/8}N^{15/16} + R^{1/8}N^{17/16}, R^{7/24}N^{49/48} + R^{5/24}N^{53/48}\bigr). \tag{3}$$

This gives

$$N^{-1}V_N(R) \ll R^{5/16+\epsilon} \quad \text{for } N \leqslant R,$$
$$N^{-1}V_N(R) \ll \bigl(H^{-1/2} + R^{5/16}\bigr)H^{-\epsilon} \quad \text{for } R \leqslant N \leqslant H^{-2},$$
$$N^{-3/2}V_N(R) \ll \bigl(H^{1/2} + R^{3/8}H^{9/8}\bigr)H^{-\epsilon} \quad \text{for } H^{-2} \leqslant N.$$

Note the misprint in p. 427 of [5] and that in (5.3) $R^{9/8}H^{-1/8}$ should be replaced by $R^{11/8}H^{1/8}$. Now, employing the last inequalities to bound $S_{\widetilde{f}}(R)$ in the same way as in the proof of Theorem 1.1 of [5] we obtain,

$$S_{\widetilde{f}}(R) \ll \bigl(RH^{-1/2} + R^{21/16} + R^{11/8}H^{1/8}\bigr)H^{-\epsilon}$$

and combining this bound with (1) the proof is completed. □

4. The character sum

The aim of this section is to estimate $\mathcal{C}$. The main point is to translate $\mathcal{C}$ into a character sum proving that $L(n)$ is a sum over square divisors of n of L-functions up to some local factors. The periodicity of these factors is important to disregard their influence, so we begin by studying it.

Recall that $N_q(n)$ is defined to be the number of the solutions of $Q(\vec{x}) = n$ in $(\mathbb{Z}/q\mathbb{Z})^3$. We define $N_q^*(n)$ in the same way but considering only primitive solutions (those with at least one coordinate coprime to q). Note that both functions are multiplicative in q.

Lemma 5. *Let p be a prime with $p^r \parallel D$ and α a positive integer. We have that*

$$\delta_p^*(n) = p^{-2\alpha}N_{p^\alpha}^*(n)$$

is well defined, i.e. it does not depend on the choice of α provided $\alpha \geqslant r+1$ if $p \neq 2$ and $\alpha \geqslant r+3$ if $p = 2$. Furthermore, by definition, it follows that $\delta_p^*(n)$ is p^{r+1}-periodic in n if $p \neq 2$ and 2^{r+3}-periodic if $p = 2$.

Proof. The proof falls naturally into two cases, p odd and $p = 2$. We first prove the lemma when p is an odd prime.

We can write $\mathbf{A} = \mathbf{C}^t\mathbf{T}\mathbf{C}$ where $\mathbf{T}$ is a diagonal matrix with integral coefficients, $\mathbf{C}$ has coefficients in $\mathbb{Z}_p$ and $|\mathbf{C}| \in \mathbb{Z}_p^\times$ [17, Theorem 32]. Hence, without loss of generality, we can assume that $Q(\vec{x})$ is a diagonal form, so we can write

$$Q(\vec{x}) = t_1x_1^2 + t_2x_2^2 + t_3x_3^2,$$

and suppose that $p^{a_j} \parallel t_j$ for each $j \in \{1,2,3\}$ and a_j decreasing. For any $\alpha > a_1$ we define $\mathcal{N}_{p^\alpha}^*(n)$ as the number of primitive solutions of $Q(\vec{x}) \equiv n \pmod{p^\alpha}$ with $1 \leqslant x_j \leqslant p^{\alpha-a_j}$ for each $j \in \{1,2,3\}$.

We can write each component of a solution counted by $N^*_{p^\alpha}(n)$ as $w_j + p^{\alpha-a_j}h_j$ with $1 \leqslant w_j \leqslant p^{\alpha-a_j}$ and $0 \leqslant h_j < p^{a_j}$. So we have that

$$N^*_{p^\alpha}(n) = p^{a_1+a_2+a_3}\mathcal{N}^*_{p^\alpha}(n).$$

We consider $\vec{w}$, a primitive solution of $Q(\vec{x}) \equiv n \pmod{p^\alpha}$ with $1 \leqslant w_j \leqslant p^{\alpha-a_j}$. Let $\vec{x} = \vec{w} + (p^{\alpha-a_1}m_1, p^{\alpha-a_2}m_2, p^{\alpha-a_3}m_3)$ with $0 \leqslant m_j \leqslant p-1$; it is easy to check that $Q(\vec{x}) \equiv n \pmod{p^{\alpha+1}}$ is equivalent to

$$Aw_1m_1 + Bw_2m_2 + Cw_3m_3 \equiv \frac{n - Q(\vec{w})}{2p^\alpha} \pmod{p} \quad \text{with } p \nmid A, B, C.$$

Given $\vec{w}$ as before this equation has p^2 solutions (m_1, m_2, m_3). Therefore $\mathcal{N}^*_{p^{\alpha+1}}(n) = p^2\mathcal{N}^*_{p^\alpha}(n)$, consequently $N^*_{p^{\alpha+1}}(n) = p^2N^*_{p^\alpha}(n)$ and a trivial verification shows that $p^{-2\alpha}N^*_{p^\alpha}(n)$ is constant for $\alpha \geqslant r+1 > a_1$.

We now turn to the case $p = 2$. In this case we employ that every quadratic form is equivalent over $\mathbb{Z}_2$ to $s_1x_1^2 + s_2x_2^2 + s_3x_3^2$, $s_1x_1^2 + s_2x_2x_3$ or $s_1x_1^2 + s_2(x_2^2 + x_2x_3 + x_3^2)$ [17, pp. 84–85]. The three cases are alike and we restrict ourselves to $Q(\vec{x}) = s_1x_1^2 + s_2(x_2^2 + x_2x_3 + x_3^2)$.

Let $2^{a_i} \parallel s_i$ and define $\mathcal{N}^*_{2^\alpha}(n)$ for $\alpha > \max(a_1+2, a_2)$ as the number of primitive solutions of $Q(\vec{w}) \equiv n \pmod{2^\alpha}$ satisfying $1 \leqslant w_1 \leqslant 2^{\alpha-a_1-1}$ and $1 \leqslant w_2, w_3 \leqslant 2^{\alpha-a_2}$. Note that changing x_1 into $x_1 + 2^{\alpha-a_1-1}m_1$ or x_j into $x_j + 2^{\alpha-a_2}m_j$, $j = 2, 3$, leaves invariant the congruence $Q(\vec{x}) \equiv n \pmod{2^\alpha}$ hence we have as before $N^*_{2^\alpha}(n) = 2^{a_1+1+2a_2}\mathcal{N}^*_{2^\alpha}(n)$ and a solution counted by $\mathcal{N}^*_{2^{\alpha+1}}(n)$ is of the form $\vec{w} + (2^{\alpha-a_1-1}m_1, 2^{\alpha-a_2}m_2, 2^{\alpha-a_2}m_3)$ for some $m_i \in \{0, 1\}$. Substituting a solution $\vec{x}$ in $Q(\vec{x}) \equiv n \pmod{2^{\alpha+1}}$ we obtain

$$s_12^{\alpha-a_1}w_1m_1 + s_22^{\alpha-a_2}(w_2m_3 + w_3m_2) \equiv n - Q(\vec{w}) \quad \bigl(\text{mod } 2^{\alpha+1}\bigr).$$

Given $\vec{w}$ there are 2^2 possibilities for (m_1, m_2, m_3) (note that $s_12^{-a_1}$ and $s_22^{-a_2}$ are both odd integers) then $\mathcal{N}^*_{2^{\alpha+1}}(n) = 2^2\mathcal{N}^*_{2^\alpha}(n)$ holds leading to $N^*_{2^{\alpha+1}}(n) = 2^2N^*_{2^\alpha}(n)$ and $2^{-2\alpha}N^*_{2^\alpha}(n)$ is constant for $\alpha \geqslant r+3 > \max(a_1+2, a_2)$. $\square$

We can now state our result for $L(n)$.

Lemma 6. *There exists an $8D^2$-periodic function B such that*

$$L(n) = \sum_{d^2|n} d^{-1}B\bigl(n/d^2\bigr)L(1, \chi_{Dn/d^2}) \quad \textit{where } \chi_m = \left(\frac{m}{\cdot}\right).$$

Proof. Our proof starts with the observation that $N^*_{p^\alpha}(n)$ and $N_{p^\alpha}(n)$ are related by

$$N_{p^\alpha}(n) = \sum_{p^{2\gamma}|(n,p^\alpha)} N^*_{p^{\alpha-\gamma}}\left(\frac{n}{p^{2\gamma}}\right)p^\gamma.$$

From the definitions of $\delta_p(n)$ and $\delta^*_p(n)$ we have $p^{2\alpha}\delta_p(n) = N_{p^\alpha}(n)$ and $p^{2\alpha}\delta^*_p(n) = N^*_{p^\alpha}(n)$ for α large enough, so that

$$\delta_p(n) = \sum_{p^{2\gamma}|n} \frac{1}{p^\gamma}\delta^*_p\left(\frac{n}{p^{2\gamma}}\right). \tag{4}$$

For $p \nmid D$ Hilfssatz 16 of [22] assures

$$\delta_p(n) = \sum_{p^{2\gamma}|n} \frac{1}{p^\gamma}\frac{1 - p^{-2}}{1 - \chi_{Dn/p^{2\gamma}}(p)p^{-1}}$$

or equivalently, by (4), $\delta^*_p(n) = (1 - p^{-2})/(1 - \chi_{Dn}(p)p^{-1})$. Hence

$$L^*(n) := \prod_p \delta^*_p(n) = B(n)L(1, \chi_{Dn}),$$

where $B(n) = C_D\prod_{p|D}\delta^*_p(n)$ and $C_D = \prod_{p\nmid D}(1 - p^{-2})$. Note that B is $8D^2$-periodic by Lemma 5.

Finally, using that $\delta^*_p(n) = \delta^*_p(m^2n)$ for $p \nmid m$, we obtain

$$L(n) = \sum_{d^2|n} d^{-1}L^*\bigl(n/d^2\bigr)$$

that gives the expected formula. $\square$

Now we employ [14] to estimate $\mathcal{C}$.

Proposition 7. *For any $R \geqslant 1 \geqslant H > 0$ and $\epsilon > 0$ there exists a constant $c = c(Q)$ such that*

$$\mathcal{C} = cR^2H + O_\epsilon\big(R^{1+\epsilon}\big(R^{5/6}H^{5/6} + R^{4/15} + R^{1/6}H^{-1/6}\big)\big).$$

Proof. By Lemma 6 we can rewrite $\mathcal{C}$ as

$$\mathcal{C} = -\frac{4\pi\sqrt{2}}{\sqrt{D}}\sum_{\nu=1}^{8D^2} B(\nu)\sum_{d<R+H} d^{-1}\sum_{\substack{R/d<\sqrt{m}<(R+H)/d\\ m\equiv\nu \pmod{8D^2}}} L(1,\chi_{Dm})f(d\sqrt{m}).$$

By partial summation, Lemmas 1 and 2 of [14] (they also work with a congruence condition) the assertion of the proposition is proved (see also Corollary 4.2 and the proof of Theorem 1.1 in [5]). Note that the actual value of $B(\nu)$ is not required and this avoids the computation of the densities when $p \mid D$. $\square$

5. The modular sum

In this section we bound the modular contribution, $\mathcal{M}$.

Proposition 8. *For any $R \geqslant 1 \geqslant H > 0$ and $\epsilon > 0$ we have*

$$\mathcal{M} = O_\epsilon\big(R^{27/14+\epsilon}H + R\big).$$

This result is linked to the estimation of the Fourier coefficients of modular forms of half-integral weight treated in [16] and continued in [9] and [10]. In this last paper (see also [1,20]) it is discussed the relation between the number of representations by forms in the same genus and in the same spinor genus. Manipulating these results it is possible to derive Proposition 8 but we prefer to employ the following bound for cuspidal form of weight 3/2 that extends Lemma 2 of [10].

Lemma 9. *Let $f(z) = \sum a_n e(nz) \in S_{3/2}(\varGamma_0(D),\chi)$, $4 \mid D$, such that its Shimura lift is a cusp form, then $a_n \ll_{f,\epsilon} n^{13/28+\epsilon}$ for every $\epsilon > 0$.*

Remark. In Appendix 2 of the recent work [2] Duke–Iwaniec bound [9,16] is improved. Using Theorem 6 of [2] with $\theta = 7/64$ (see Hypothesis H_θ) in the proof of this lemma one can replace $13/28 = 0.4642\ldots$ by $231/512 = 0.4517\ldots$ but it does not affect our final result. In principle even better bounds could be obtained if [3] or [8] are extended to modular forms of weight 2 and arbitrary level D.

Proof of Lemma 9. We can find a basis of $S_{3/2}(\varGamma_0(D),\chi)$ such that their elements are eigenfunctions of the Hecke operators T_{p^2} for every $p \nmid D$ (they commute and are normal) and we can assume that f is one of these basis elements because we can always express f as a linear combination of them. Let $V \subset S_{3/2}(\varGamma_0(D),\chi)$ be the subspace of forms with the same eigenvalues as f for every $p \nmid D$ (in particular the Shimura lift of each form in V is also a cusp form). As $0 < \dim V < \infty$ the elements of V are determined by their first N Fourier coefficients for a suitable N and we have an isomorphism $i: V \to S$ for some subspace $S \subset \mathbb{C}^N$.

Let $g = \sum b_n e(nz) \in V$, by Corollary 1.8 of [21], $(T_{p^2})^k(g) = \sum b_{np^{2k}} e(nz)$ for $p \mid D$ and by Main Theorem in [21] for $n \leqslant N$ fixed $b_{np^{2k}}$ is a sum of the lp^kth Fourier coefficients of a cusp form of weight 2 where l divides the non-squarefree part of n, hence Ramanujan–Petersson conjecture assures $b_{np^{2k}} \ll_{g,N,\epsilon} (p^k)^{1/2+\epsilon}$ or equivalently $\|i \circ (T_{p^2})^k \circ i^{-1}\| \ll (p^k)^{1/2+\epsilon}$ for any $p \mid D$ where $\|\cdot\|$ indicates the operator norm. We conclude

$$\big\|\big(T_{p^2}\big)^k\big|_V\big\| \ll \big(p^k\big)^{1/2+\epsilon} \quad \text{for } p \mid D. \tag{5}$$

Denote by D^∞ an arbitrarily large power of D. Let t squarefree and $m = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r} \mid D^\infty$ and consider the operator $L = (T_{p_1^2})^{\alpha_1}|_V \circ \cdots \circ (T_{p_r^2})^{\alpha_r}|_V$. Fixing an orthonormal basis $\{g_j\}_{j\in J}$ of V we have $f = \sum \lambda_j g_j$ and $L(g_j) = \sum \mu_{jk} g_k$ with $\lambda_j \ll 1$ and $\mu_{jk} \ll \|L\| \ll m^{1/2+\epsilon}$ by (5), with $\ll$-constants not depending on m. By Duke–Iwaniec bound [9,16] the tth Fourier coefficient of g_k is $O(t^{13/28+\epsilon})$ and extracting this coefficient in the identity $L(f) = \sum \lambda_j L(g_j)$ we obtain

$$a_{tm^2} \ll t^{13/28+\epsilon} m^{1/2+\epsilon} \ll \big(tm^2\big)^{13/28+\epsilon} \tag{6}$$

for every t squarefree and $m \mid D^\infty$. Considering the t-Shimura lifting of f with t squarefree we have the identity (see [21] or (2.4) in [1] for a more explicit formulation)

$$\sum_{n=1}^{\infty} \frac{a_{td^2}}{d^s} = \sum_{m\mid D^\infty} \frac{a_{tm^2}}{m^s} \cdot \sum_{(l,D)=1} \frac{\omega_l}{l^s} \cdot L(s,\chi)^{-1}$$

for a certain character χ to the modulus $4tD$ and some $|\omega_l| \ll l^{1/2+\epsilon}$ (this is again Ramanujan–Petersson conjecture for weight 2). Hence if $n = td^2$ with t squarefree, we have

$$a_n \ll \sum_{\substack{ml|d \\ m|D^\infty}} |a_{tm^2}| l^{1/2+\epsilon} \ll m^\epsilon \sum_{m|(d,D^\infty)} |a_{tm^2}| (d/m)^{1/2} \ll n^{13/28+\epsilon}$$

where we have employed (6) for the last inequality. □

Proof of Proposition 8. There exists a linear combination of theta functions $\sum \chi(k) k e(tk^2)$ with $t \mid D$ such that added to $f(z) = \sum a_n e(nz)$ gives $f^*(z) = \sum a_n^* e(nz) \in S_{3/2}(\Gamma_0(D), \chi)$ with cuspidal Shimura lift [7]. Note that in the range $R^2 \leqslant n < (R+H)^2$ we have $a_n = a_n^*$ except for a subset of indexes of bounded cardinality in which $a_n = a_n^* + O(R)$. Hence the result reduces to apply Lemma 9 to f^*. □

6. Proof of the main result

Substituting Propositions 4, 7 and 8 in Proposition 2 and choosing $H = R^{-5/8}$ we obtain

$$E(R) = \left(\frac{4\sqrt{2}\pi}{\sqrt{D}} - c \right) R^{11/8} + O\left(R^{21/16+\epsilon}\right).$$

The bound $\int_0^\infty E(t) e^{-t^2/R^2}\, dt = O(R)$ is an exercise using Poisson summation and implies $c = 4\sqrt{2}\pi/\sqrt{D}$. An alternative approach is to appeal to two-sided Ω-results [18].

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