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Vector measure as controls for explicit nonlinear impulsive system of fed-batch culture

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ABSTRACT

In this paper, we consider an optimal control problem of microbial fermentation process in which glycerol is converted to 1,3-propanediol by *Klebsiella pneumoniae* in fed-batch culture. During the period of reaction, the variation of pH value is monitored to determine glycerol replenishment quantity, guaranteeing that microorganism can always keep growing fast under enough nutrition. Every time pH value is lower than seven, the quantity of glycerol added is such that pH value returns seven again. Glycerol is poured into reactor at discrete time instant and the quantity is controllable. The problem is to determine for each discrete time instant the glycerol quantity to add and maximize the final concentration of 1,3-propanediol. We present a controlled explicit nonlinear impulsive dynamical system of fed-batch culture with state independent vector measures as controls and study the existence, uniqueness, boundedness, continuous dependence and Gâteaux differentiability of its solution with respect to controls. We then propose a multiple objective programming model and demonstrate the regularity of cost functionals and weak compactness of admissible control set. Finally we discuss the existence of optimal control and implement a hybrid particle swarm optimization algorithm to solve the model optimally. Computational results are presented on a numerical example.

1. Introduction

In recent years, there has been growing interest in microbial production of 1,3-propanediol (1,3-PD) throughout the world because of its lower cost, higher production and no pollution [1]. Among various microbial production of 1,3-PD, the dissimilation of glycerol to 1,3-PD by *Klebsiella pneumoniae* has been widely investigated since 1980s [2]. The experimental investigations showed that the fermentation of glycerol by *K. pneumoniae* is a complex bio-process in that the microbial growth is subjected to multiple inhibitions of substrate and products [3]. At present, the researches about the quantitative description of the cell growth kinetics of multiple-inhibitions and the metabolic overflow kinetics of substrate consumption and product formation have particularly been attractive in the fermentation of glycerol by *K. pneumoniae* [4].

Only over the past several years has great progress been made in studying the nonlinear dynamical system of continuous culture. Model analysis and simulations for the dynamical system of continuous culture were made [5]. In [6], a parameter identification model of continuous culture was established to obtain the optimal parameters by constructing an optimization algorithm. The stability analysis and optimal control problem for the dynamical system of continuous culture were studied together with optimality conditions [7,8]. In contrast to continuous culture, the dynamics of microbial bio-conversion in fed-batch cultures just started. Gang Wang proposed the explicit nonlinear impulsive dynamical system of fed-batch culture

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driven by measure and developed a hybrid particle swarm optimization algorithm to carry on parameter identification for the explicit impulsive system [9]. However, the optimal control problem for the explicit impulsive dynamical system has not yet been investigated. Our main contribution is to present the controlled explicit impulsive dynamical system of fed-batch culture with state independent vector measures as controls and discuss the existence, uniqueness, boundedness and regularities of solutions with respect to controllable variables. And then we study properties of cost functionals and admissible control set and existence of optimal controls. Finally, we offer a numerical example and secure an optimal control by using the hybrid particle swarm optimization [9].

The rest of the paper is organized as follows. In Section 2, some basic notations and terminologies are introduced. In Section 3, we present a controlled explicit nonlinear impulsive system. Section 4 is devoted to the existence, uniqueness, boundedness and regularities of solutions for the controlled explicit nonlinear impulsive system with respect to controls. In Section 5, we develop the optimal control problem for controlled impulsive system of fed-batch culture and demonstrate the existence of optimal control. In Section 6, a numerical example is given to justify the optimal control problem we addressed. Section 7 concludes.

2. Some notations and terminologies

Let $\Lambda_n \triangleq \{1, 2, \ldots, n\}$, $I_1 \triangleq [0, t_1)$ be the time interval of batch culture, $I_2 \triangleq [t_1, T]$ be the time interval of fed-batch culture, $I \triangleq I_1 \cup I_2$, $D \triangleq \{t_1, t_2, \ldots, t_n\} \subset [t_1, T)$, where t_i are the impulsive moments of adding glycerol and alkali and $0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = T$. $T \in (0, +\infty)$ denotes the stopping time of fed-batch process. Let $PWC_r(I, R^5)$ denote the space of piecewise right continuous bounded functions on I with values in R^5 having left-hand limits. Furnished with the sup norm topology

$$\|z\|_{\mathrm{pwc}} = \sup\bigl\{\bigl\|z(t)\bigr\|:\ t \in I\bigr\}, \quad \forall z \in PWC_r\bigl(I, R^5\bigr),$$

it is a Banach space.

Let $\mathcal{M}_c(I, R^n)$ denote the space of bounded countably additive vector measures on the sigma algebra $\mathcal{B}$ of subsets of the set I with values in R^n. We assume that $\mu(\{0\}) = 0$ for each $\mu \in \mathcal{M}_c(I, R^n)$. We equip this vector space with the total variation norm

$$\|\mu\|_{\mathrm{var}} \triangleq |\mu|(I) \triangleq \sup_{\pi}\left\{\sum_{A\in\pi} \bigl\|\mu(A)\bigr\|\right\},$$

where the supremum is taken over all partition π of the internal I into a finite number of disjoint members of $\mathcal{B}$ given by

$$\pi \triangleq \bigl\{0 = t_0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n \leqslant t_{n+1} = T,\ A_i = (t_i, t_{i+1}],\ n \in \mathbb{N}\bigr\}.$$

With respect to this norm topology, $\mathcal{M}_c(I, R^n)$ is a Banach space. For any $A \in \mathcal{B}$, define the variation of μ on A by

$$V(\mu)(A) \triangleq V(\mu, A) \triangleq |\mu|(A).$$

Clearly this induces a countably additive bounded positive scalar valued measure $V(\mu)$ on $\mathcal{B}$ now that μ is countably additive and bounded. Here, we represent the space of real valued countably additive bounded signed measures by $\mathcal{M}_c(I)$.

3. Controlled explicit nonlinear impulsive system

In this section, we will develop the controlled explicit nonlinear impulsive system of fed-batch culture based on explicit nonlinear impulsive system of microbial bioconversion in fed-batch culture given in the literature [9]. Since alkali is added into the reactor to decrease the influence of the inhibition of substrates and multi-products while adding glycerol and to neutralize the product acetate, which has little effect on the productivity of 1,3-PD, we ignore its effect in the dynamical system. Consequently, we make the following assumptions.

- (H1) The glycerol concentration is uniform in reactor while adding glycerol to the reactor, time delay and nonuniform space distribution are ignored.
- (H2) The feed rate of glycerol can be infinitely large. Consequently, it is possible to add the substrate instantaneously to the reactor at various discrete time instants.

The fed-batch culture of glycerol bioconversion to 1,3-PD includes batch culture in the early stage and later fed-batch culture. Let $W \triangleq \{x \in R^5 \mid x_1 \in [0.001, x_1^*],\ x_2 \in [100, x_2^*],\ x_i \in [0, x_i^*],\ i = 3, 4, 5\}$. Mass balances of biomass, substrate and products in fed-batch cultures are written as follows (see [9]):

$$dx(t) = f\bigl(x(t)\bigr)\,dt + g\bigl(x(t-)\bigr)\nu_\alpha(dt), \quad t \in I, \qquad x(0) = x_0, \tag{3.1}$$

where $\nu_\alpha \in \mathcal{M}_c(I)$ and $\nu_\alpha(dt) = \sum_{t_i \leqslant t} \alpha_i \delta_{t_i}(dt)$, where δ_{t_i} denotes the Dirac measure concentrated at t_i and α_i is dilution rate at time $t_i \in D$. $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t))^T \in W$ is the state variable and $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$, $x_5(t)$ are the concentrations of biomass, glycerol, 1,3-PD, acetic acid and ethanol at time t in reactor, respectively. $x_0 \in W$ is the initial state.

$$f\big(x(t)\big) = \big(\sigma x_1(t), -q_2 x_1(t), q_3 x_1(t), q_4 x_1(t), q_5 x_1(t)\big)^T, \tag{3.2}$$

$$g\big(x(t-)\big) = \big(-x_1(t-), -x_2(t-) + c, -x_3(t-), -x_4(t-), -x_5(t-)\big)^T. \tag{3.3}$$

Here the specific growth rate of cells σ, specific consumption rate of substrate q_2 and specific formation rate of product q_i are expressed by Eqs. (3.4)–(3.7), respectively:

$$\sigma = \sigma_m \left(\frac{x_2(t)}{x_2(t) + k_s}\right)^{n_1} \prod_{i=2}^{5} \left(1 - \frac{x_i(t)}{x_i^*}\right)^{n_i}, \tag{3.4}$$

$$q_2 = m_2 + \frac{\sigma}{Y_2} + \Delta_2 \frac{x_2(t)}{x_2(t) + k_2}, \tag{3.5}$$

$$q_i = m_i + \sigma Y_i + \Delta_i \frac{x_2(t)}{x_2(t) + k_i}, \quad i = 3, 4, \tag{3.6}$$

$$q_5 = q_2 \left(\frac{b_1}{c_1 + \sigma x_2(t)} + \frac{b_2}{c_2 + \sigma x_2(t)}\right) \tag{3.7}$$

where $c = k_i F_i C_{s_0}/(1 + k_i)F_i$, where C_{s_0} is the concentration of glycerol added and $k_i F_i$ and F_i are the volumes of added glycerol and alkali at time $t_i \in D$, $i \in \Lambda_n$, $n \triangleq (n_1, n_2, n_3, n_4, n_5)^T \in N_0 = (1, 2, 3, 4, 5)^5$. Under anaerobic conditions at 37 °C and pH 7.0, the maximum specific growth rate of cells σ_m is 0.67 h^{-1}, and Monod saturation constant k_s is 0.28 mmol/L. The critical concentrations of biomass, glycerol, 1,3-PD, acetate and ethanol for cell growth are $x_1^* = 10$ g/L, $x_2^* = 2039$ mmol/L, $x_3^* = 939.5$ mmol/L, $x_4^* = 1026$ mmol/L and $x_5^* = 360.9$ mmol/L, respectively. b_1, b_2, c_1, c_2, m_i, Y_i, Δ_i, k_i, $i = 2, 3, 4$, are parameters given in the previous report [4].

Next, we formulate the controlled explicit nonlinear impulsive system of fed-batch culture. Since the quantity of added glycerol and alkali can be changeable relatively in a prescribed range, the volume of adding glycerol is a controllable variable each time. Define the range set of glycerol volume $\mathscr{V} \triangleq \prod_{i=1}^{n} [0.2 \cdot v_i^0, 1.8 \cdot v_i^0]$, where $v_0 \triangleq \{v_1^0, v_2^0, \ldots, v_n^0\}$ is a fed-batch flow strategy of glycerol provided by the experiment. Hence we may define the admissible control set

$$\mathscr{U}_{\mathrm{ad}} \triangleq \left\{ u \in \mathcal{M}_c\big(I, R^5\big) \;\middle|\; u : \mathscr{B} \to R^5,\ u(dt) = \sum_{i=1}^{n} u_i \delta_{t_i}(dt),\ u_i = (0, \nu, 0, 0, 0)^T,\ \nu \in \mathcal{M}_c(I) \right\}, \tag{3.8}$$

where $\nu(\{t_i\}) = [(1 + k(t_i))\omega(\{t_i\})]/[\sum_{t \leqslant t_i} (1 + k(t_i))\omega(\{t_i\}) + V_0]$. Here $k(t)$ is a continuous bounded function on I which satisfies $k(0) = 0$ and $k(t) = k_i$ when $t \in D$, where k_i is the ratio of the volume glycerol to that alkali added at t_i, nonnegative and bounded. And $\omega \in \mathcal{M}_c(I)$ and $\omega(dt) = \sum_{t \leqslant t_i} V_i \delta_{t_i}(dt)$, where $V_i \in \mathscr{V}$ denotes the volume of adding glycerol, $i \in \Lambda_n$. Thus, the controlled explicit nonlinear impulsive system of fed-batch can be formulated below:

$$dx(t) = f\big(x(t)\big)\,dt + h\big(x(t-)\big)\nu(dt) + c(t)u(dt), \quad t \in I, \qquad x(0) = x_0, \tag{3.9}$$

where

$$h\big(x(t-)\big) = \big(-x_1(t-), -x_2(t-), -x_3(t-), -x_4(t-), -x_5(t-)\big)^T, \tag{3.10}$$

$$c(t) = 1/\big[1 + k(t)\big]. \tag{3.11}$$

Here the function $f(\cdot)$ is identical with function $f(\cdot)$ of (3.2), $\nu \in \mathcal{M}_c(I)$ and $u \in \mathscr{U}_{\mathrm{ad}}$.

4. Existence, uniqueness and regularity

For the study of optimal control problem, it is necessary to guarantee the existence, uniqueness and regularity of solutions to the controlled explicit nonlinear impulsive system with respect to the control laws.

4.1. Existence and uniqueness

First let us give the definition of an integral solution to the controlled impulsive system (3.9).

Definition 4.1. Suppose $x_0 \in W$ and $u \in \mathscr{U}_{\mathrm{ad}}$, $x(t) \in PWC_r(I, R^5)$ is said to be an integral solution of the controlled system (3.9) provided that it satisfies the following integral equation:

$$x(t) = x_0 + \int_0^t f\big(x(s)\big)\,ds + \int_0^t h\big(x(s-)\big)\nu(ds) + \int_0^t c(s)u(ds), \quad t \in I. \tag{4.1}$$

Lemma 4.1. *Suppose $\tau \in \mathcal{M}_+(I)$ with $\tau(\{0\}) = 0$ and $a(t)$, $\psi(t)$, $t \geqslant 0$ are nonnegative bounded measurable functions on I, $K(\cdot)$ is positive integrable function on I and satisfies the following inequality* (*see* [10]):

$$\psi(t) \leqslant a(t) + \int_0^t K(s)\psi(s)\tau(ds), \quad t \in I.$$

Then

$$\psi(t) \leqslant a(t) + \int_0^t \exp\left\{\int_0^t K(\theta)\mu(d\theta)\right\} a(s)K(s)\tau(ds), \quad t \in I.$$

Next we discuss some properties of the solutions of the impulsive system (3.9) and the functions $f(\cdot)$, $g(\cdot)$ and $c(\cdot)$.

Theorem 4.1. *There exists a finite positive number r such that $x(t) \in B_r$ for any $t \in I$, where $B_r \triangleq \{x \in W \mid \|x\|_{\mathrm{pwc}} \leqslant r\}$.*

Proof. From the formula (4.1), we have

$$\|x(t)\| \leqslant \|x_0\| + K_r \int_0^t \big(1 + \|x(s)\|\big)\,ds + L\int_0^t \big(1 + \|x(s)\|\big)V(\nu, ds) + \int_0^t V(u, ds).$$

Define $\beta(A) \triangleq K_r\lambda(A) + LV(\nu, A)$, $A \in \mathcal{B}$. Hence it follows from the preceding inequality that

$$\|x(t)\| \leqslant \big(\|x_0\| + K_r\lambda(I) + LV(\nu, I) + V(u, I)\big) + \int_0^t \|x(t)\|\beta(ds).$$

By Lemma 4.1, we can conclude that

$$\|x(t)\| \leqslant \big(\|x_0\| + K_r\lambda(I) + LV(\nu, I) + V(u, I)\big)\exp\big\{\beta(I)\big\}.$$

Set $r = (\|x_0\| + K_r\lambda(I) + LV(\nu, I) + V(u, I))\exp\{\beta(I)\}$ and we have $\|x\|_{\mathrm{pwc}} \leqslant r$, which completes the proof. □

Theorem 4.2. *The function $f(\cdot)$ defined in* (3.9) *satisfies that*

(1) *$f(\cdot)$ is Borel measurable in t on I.*
(2) *$f(\cdot)$ is locally Lipschitz continuous having at most linear growth on $PWC_r(I, R^5)$, that is, there exists a constant $K_r > 0$, for each $r > 0$, such that*

$$\|f(x) - f(y)\| \leqslant K_r\|x - y\|_{\mathrm{pwc}}, \quad \forall x, y \in B_r,$$

and

$$\|f(x)\| \leqslant K(t)\big(1 + \|x\|_{\mathrm{pwc}}\big), \quad \forall x \in W$$

where B_r is defined in Theorem 4.1, $K \in L_1^+(I, \lambda)$, $L_1^+(I, \lambda)$ denotes the set of bounded nonnegative Lebesgue integrable functions on I, λ denotes the Lebesgue measure.
(3) *$f(\cdot)$ is Gâteaux differentiable on R^5 with the derivatives bounded and uniformly measurable.*

Proof. The proofs of (1) and (2) are given in Theorem 1 [10]. We now turn to the proof of (3). From Eq. (3.2), we can have that $\nabla f_1(x), \nabla f_2(x), \ldots, \nabla f_5(x)$ all exist and are continuous with respect to the state x. Thus, the Jacobian matrix of the function $f(\cdot)$ can be written as $Jf(x) = (\nabla f_1(x), \nabla f_2(x), \ldots, \nabla f_5(x))^T$. Therefore, the function $f(\cdot)$ is Gâteaux differentiable on R^5. Again because $x \in W$ and the continuity of the Jacobian of the function $f(\cdot)$, we can conclude that the derivative of $f(\cdot)$ is bounded and uniformly measurable, which completes our proof. □

Theorem 4.3. *The function $h(\cdot)$ given by Eq.* (3.10) *satisfies that*

(1) *$h(\cdot)$ is locally Lipschitz having at most linear growth on $PWC_r(I, R^5)$ with respect to the measure $V(\nu)$, that is, there exists a positive constant L such that for any $x, y \in PWC_r(I, R^5)$,*

$$\|h(x) - h(y)\| \leqslant L\|x - y\|_{\mathrm{pwc}}, \qquad \|h(x)\| \leqslant L\big(1 + \|x\|_{\mathrm{pwc}}\big). \tag{4.2}$$

(2) *$h(\cdot)$ is Gâteaux differentiable on R^5 with the derivatives bounded and uniformly measurable.*

Proof. (1) From Eq. (3.10), we can easily obtain that Eq. (4.2) holds.

(2) Since the Jacobian matrix Jh of the function $h(\cdot)$ is an identity matrix, we conclude that the function $h(\cdot)$ is Gâteaux differentiable on R^5. It is clear that its derivatives are bounded and uniformly measurable. □

Theorem 4.4. *The function $c(t)$ defined by Eq.* (3.11) *is continuous on I and locally integral with respect to the measure $V(u)$ induced by vector measure u.*

Proof. It is easy to verify that $c(t)$ is continuous on I from the continuity of the function $k(t)$. On the other hand, the local integrability of the function $c(t)$ is easy to conclude with respect to the measure $V(u)$ because $c(t)$ is continuous and $k(t)$ is bounded on I. □

Theorem 4.5. *For each $x_0 \in W$ and $u \in \mathscr{U}_{\mathrm{ad}}$, the controlled system* (3.9) *has unique integral solution in $PWC_r(I, R^5)$.*

Proof. See [10, Theorem 1]. □

4.2. Continuous dependence and Gâteaux differentiability of solutions on controls

Here we discuss the regularity properties of the piecewise continuous solutions of (3.9) with respect to the controlled variables. Later we denote the solutions of (3.9) with respect to the controls by $x(\cdot\,; u)$.

Theorem 4.6. *For a given initial value $x_0 \in W$, there exists $M > 0$ such that the piecewise solution of the controlled system* (3.9) *satisfies that*

$$\big\|x(t; u_1) - x(t; u_2)\big\| \leqslant M\|u_1 - u_2\|_{\mathrm{var}}, \quad \forall u_1, u_2 \in \mathscr{U}_{\mathrm{ad}},\ t \in I.$$

Proof. Define $M = \sup\{\|c(t)\|\colon\ t \in I\}$. Then let $u_1, u_2 \in \mathscr{U}_{\mathrm{ad}}$ and $x_{u_1}, x_{u_2} \in PWC_r(I, R^5)$ the corresponding integral solutions. In terms of Eq. (4.1), Theorems 4.1–4.3, we can have that

$$\big\|x(t; u_1) - x(t; u_2)\big\| \leqslant M\|u_1 - u_2\|_{\mathrm{var}} + \int_0^t \big\|x(s; u_1) - x(s; u_2)\big\|\alpha(ds), \quad t \in I,$$

where $\alpha(A) \triangleq K_r\lambda(A) + LV(\nu, A)$, $\forall A \in \mathcal{B}$. Now using Lemma 4.1, it follows from the preceding inequality that

$$\big\|x(t; u_1) - x(t; u_2)\big\| \leqslant M\|u_1 - u_2\|_{\mathrm{var}} \exp\left\{\int_0^t \alpha(ds)\right\}, \quad \forall t \in I.$$

Set $K = M\exp\{\alpha(I)\}$, and we can obtain

$$\big\|x(t; u_1) - x(t; u_2)\big\| \leqslant K\|u_1 - u_2\|_{\mathrm{var}}. \qquad \Box$$

Theorem 4.7. *For a given initial value $x_0 \in W$, the piecewise solution of the controlled system* (3.9) *is continuously Gâteaux differentiable with respect to the control $u \in \mathscr{U}_{\mathrm{ad}}$.*

Proof. See [11, Corollary 3.2]. □

5. Optimal control problem

The optimal control problem using the productivity of 1,3-PD and the total volume of consumed glycerol as cost functionals, based on the controlled explicit nonlinear impulsive system (3.9), can be formulated as follows.

$$\begin{aligned}
(P_1)\quad \max\quad & J_1(u) \triangleq \phi\big(x(T; u)\big) \triangleq x(T; u)^T Gx(T; u)/T^2,\\
\min\quad & J_2(u) \triangleq \|\omega\|_{\mathrm{var}},\\
\text{s.t.}\quad & dx(t) = f\big(x(t)\big)\,dt + h\big(x(t-)\big)\nu(dt) + c(t)u(dt),\\
& x(0) = x_0,\\
& x(t) \in W,\\
& u \in \mathscr{U}_{\mathrm{ad}},
\end{aligned}$$

where $\omega \in \mathcal{M}_c(I)$ and $\omega(dt) = \sum_{t \leqslant t_i} V_i \delta_{t_i}(dt)$, where $V_i \in \mathscr{V}$ denotes the volume of adding glycerol as explained previously, $i \in \Lambda_n$ and w involves the flow policy of glycerol at the impulsive moments, that is, its value at t_i equals the volume V_i of added glycerol at time t_i. Here we use linear weighting method to transform the multiple objective optimal control problem into simple objective one. Hence (P_1) is equivalent to the following optimization problem:

$$
\begin{aligned}
(P_2)\quad \min\quad & J(\eta) \triangleq a/J_1(u) + bJ_2(u) \\
& \quad = aT^2/x(T;u)^T Gx(T;u) + b\|\omega\|_{\mathrm{var}}, \\
\text{s.t.}\quad & dx(t) = f\big(x(t)\big)\,dt + h\big(x(t-)\big)\nu(dt) + c(t)u(dt), \\
& x(0) = x_0, \\
& x(t) \in W, \\
& u \in U.
\end{aligned}
$$

Here G is a 5×5 diagonal matrix, $a, b \in [0,1]$ and $\eta = (a,b,u)^T \in U \triangleq [0,1] \times [0,1] \times \mathscr{U}_{\mathrm{ad}} \subset R^7$. The properties of cost functionals and admissible control set play the important role in the study of optimal control problem. First we give the weak compactness of the admissible control set.

Definition 5.1. A subset $\mathscr{U}$ of $\mathcal{M}_c(I, R^5)$ is said to be compact if it satisfies

(1) $\mathscr{U}$ is bounded,
(2) there exists a nonnegative countably additive finite scalar valued measure κ on $\mathcal{B}$ such that, for any $A \in \mathcal{B}$, $\lim_{\kappa(A)\to 0} V(u)(A) = 0$ uniformly with respect to $u \in \mathscr{U}$,
(3) for each $A \in \mathcal{B}$, the set $\{u(A),\ u \in \mathscr{U}\}$ is a relatively compact subset of R^5.

Theorem 5.1. *The admissible control set $\mathscr{U}_{\mathrm{ad}}$ defined in* (3.8)*, a subset of $\mathcal{M}_c(I, R^5)$, is compact.*

Proof. First, we show that $\mathscr{U}_{\mathrm{ad}}$ is bounded. By the expression (3.8), for $u \in \mathscr{U}_{\mathrm{ad}}$ and $J \in \mathcal{B}$, we have

$$\|u\|_{\mathrm{var}} = \sup_{\pi}\left\{\sum_{A\in\pi} \big\|u(A)\big\|\right\} = \sup_{\pi}\left\{\sum_{A\in\pi} \big|\nu(A)\big|\right\} = \|\nu\|_{\mathrm{var}}.$$

However, for any $A \in \mathcal{B}$, $V(\nu)(A) < (1+M_1)M_2/V_0$, where $M_1 \triangleq \max_{t\in I}\|k(t)\|$ and $M_2 \triangleq \max_{i\in\Lambda_n} V_i$. Again because $V_i \in \mathscr{V}$ and $\mathscr{V}$ is a bounded closed set, we obtain that

$$\|u\|_{\mathrm{var}} \leqslant (1+M_1)M_2/V_0.$$

Next, we will prove (2). Let $\kappa \triangleq V(\nu)$. By definition of the measure ν, we can easily see that κ is a nonnegative countably additive finite scalar valued measure on $\mathcal{B}$. For any $A \in \mathcal{B}$, it follows from Jordan decomposition of the measure ν that $V(\nu) = \nu^+ + \nu^-$, where ν^+ and ν^- are two mutually singular positive measures. Hence, since $\nu = \nu^+ - \nu^-$, for $A \in \mathcal{B}$, $\nu(A) \to 0$ when $\kappa(A) \to 0$. Thus, by (3.8), one arrives at the following result $V(u)(A) \to 0$ for any $A \in \mathcal{B}$.

Finally, it is easy to verify that the set $\{u(A),\ u \in \mathscr{U}_{\mathrm{ad}}\}$ is relatively compact in R^5. This completes the proof. □

Theorem 5.2. *The functionals $\phi(\cdot)$ and $J_2(\cdot)$ defined in (P_1) satisfy that*

(1) *$\phi(\cdot)$ is continuous and bounded on bounded sets of R^5 and there exist constants C_1 and C_2 such that for $u \in \mathscr{U}_{\mathrm{ad}}$*

$$C_1\|x\|_{\mathrm{pwc}} \leqslant \phi(x) \leqslant C_2\|x\|_{\mathrm{pwc}}, \quad \forall x \in B_r \subset R^5,$$

where B_r is the same as that in Theorem 4.1.
(2) *$J_2(\cdot)$ is continuous on $\mathcal{M}_c(I, R^5)$.*

Proof. (1) Let C_1 and C_2 denote the minimum and maximum eigenvalues of the diagonal matrix G, respectively, and then, for any $x \in B_r$, we see that $C_1\|x\|_{\mathrm{pwc}} \leqslant \phi(x) \leqslant C_2\|x\|_{\mathrm{pwc}}$ holds.

(2) Let $u_1, u_2 \in \mathscr{U}_{\mathrm{ad}}$. Since $\mathscr{U}_{\mathrm{ad}}$ is compact, there exists a positive constant δ such that $\|u_1(A) - u_2(A)\|_{\mathrm{pwc}} \leqslant \delta$ for any $A \in \mathscr{B}$. Hence we have

$$\|u_1 - u_2\|_{\mathrm{var}} = \sup_{\pi}\left\{\sum_{A\in\pi} \big\|(u_1-u_2)(A)\big\|\right\} = \sup_{\pi}\left\{\sum_{A\in\pi} \big\|u_1(A) - u_2(A)\big\|\right\} \leqslant N\big\|u_1(I) - u_2(I)\big\| \leqslant N\delta,$$

where N is the number of finite members of the partition supremum is reached over, which finishes the proof. □

Next, we will prove the existence of optimal control for (P_1).

Table 1
Optimal flow strategy u^* of glycerol.

Time (h)	5.33–5.83	5.83–6.13	6.13–7.15	7.15–7.83	7.83–8.83
Volume (ml)	1.4227	1.4227	1.9917	2.1561	2.1561

Table 2
Optimal flow strategy u^* of glycerol (continued).

Time (h)	8.33–12.16	12.16–18.53	18.53–18.1	18.1–19.83
Volume (ml)	1.9917	1.7072	1.4134	0.9889

Theorem 5.3. *Consider the system given by* (3.10) *with the cost functionals* (5.1) *and admissible control set* $\mathscr{U}_{\mathrm{ad}}$ *defined in* (3.8)*. Then there is an* $\eta^* = (\lambda_1^*, \lambda_2^*, u^*) \in U$ *minimizing the functional* $J(\cdot)$ *given by* (5.1)*, where* u^* *is the desired optimal control.*

Proof. Since $\mathscr{U}_{\mathrm{ad}}$ is compact, in view of Theorems 5.1 and 5.2, we have

$$\inf\bigl\{J(\eta),\ \eta \in U\bigr\} = m > -\infty.$$

Let $\{\eta_n\}_{n=1}^{\infty}$ be a minimizing sequence such that

$$\lim_{n\to\infty} J(\eta_n) = m.$$

By Theorem 5.2, there exist a subsequence of the sequence $\{u_n\}_{n=1}^{\infty}$, relabeled as $\{u_{n_k}\}_{k=1}^{\infty}$, and an element $u^* \in \mathscr{U}_{\mathrm{ad}}$ so that $u_{n_k} \xrightarrow{\omega} u^*$ as $k \to \infty$. Let $x_n, x^* \in PWC_r(I, R^5)$ denote the corresponding piecewise continuous solutions of the impulsive system (3.9), respectively. Similarly, there are the sequences $\{a_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=1}^{\infty}$ and a^*, b^* in the interval $[0,1]$ satisfying that $a_k \to a^*$ and $b_k \to b^*$ as $k \to \infty$. In fact, we mean that there exists a subsequence $\{\eta_{n_k}\}_{k=1}^{\infty}$ of $\{\eta\}_{n=1}^{\infty}$ such that $\eta_{n_k} \triangleq (a_k, b_k, u_{n_k})^T \xrightarrow{\omega} \eta^* \triangleq (a^*, b^*, u^*)^T \in U$ as $k \to \infty$.

Now define

$$e_n(t) = \bigl\|x_n(t) - x^*(t)\bigr\|, \quad t \in I,\ n \in \mathbb{N}.$$

Using Theorem 4.6, one can easily verify that

$$e_n(t) \leqslant M\|u_n - u^*\|_{\mathrm{var}}, \quad \forall t \in I.$$

Thus, we have $\lim_{n\to\infty} e_n(t) = 0$, $\forall t \in I$ and conclude that

$$\liminf_{k\to\infty} J(\eta_{n_k}) \geqslant J\bigl(\eta^*\bigr),$$

which shows that the cost function $J(\cdot)$ is weakly lower semicontinuous on U. Since U is compact, this implies that $J(\cdot)$ attains its minimum on U, that is,

$$m \leqslant J\bigl(\eta^*\bigr) \leqslant \liminf_{k\to\infty} J(\eta_{n_k}) \leqslant \lim_{n\to\infty} J(\eta_n) = m,$$

which finishes our proof. □

6. Numerical example

In this example, the initial value $x_0 = (0.115\ \mathrm{g/L}, 495\ \mathrm{mmol/L}, 0, 0, 0)^T$ and the power $n = (1, 1, 2, 1, 1)^T$. Fed-batch began at $t_1 = 5.33$ h. The impulsive moments have been determined by the experiment. By using the hybrid particle swarm optimization algorithm [9], we obtain an optimal strategy of adding glycerol in the dynamical process of microbial bioconversion to 1,3-PD in fed-batch culture. Tables 1–3 give optimal values of flow volumes of glycerol every 100 seconds during the flow period. Fig. 1 shows the comparison of 1,3-PD concentrations between experimental and computational results, where the points and real line denote the experimental values and the computational curve, respectively. The maximum of 1,3-PD obtained by (P_2) is 895.725 mmol/L, $a^* = b^* = 0.720542$ and $J_2(u^*) = 815.40$ ml, but the maximum concentration of 1,3-PD is 875.7 mmol/L in the experiment.

7. Conclusion

In this paper we have presented a controlled explicit nonlinear impulsive dynamical system of fed-batch culture. We then demonstrated the existence, uniqueness and regularity of solutions to the controlled system in association to the controllable variables. At last optimal control problem of the controlled nonlinear impulsive system is proposed and the

Table 3
Optimal flow strategy u^* of glycerol (continued).

Time (h)	19.83–23.83	23.83–24.16	24.16–25.1	25.1–27.13
Volume (ml)	0.6893	0.3447	0.6892	0.3447

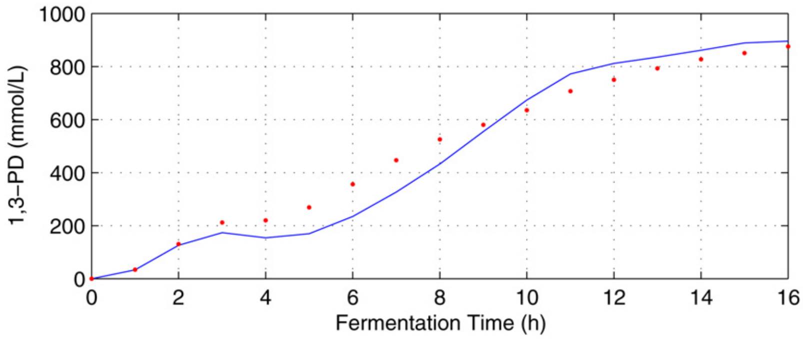

Fig. 1. Comparison of 1,3-PD concentrations between optimal and experimental results, respectively.

questions of the existence of optimal controls are discussed. And we supply a numerical example to justify the existence of optimal control problem for controlled explicit nonlinear impulsive dynamical system of fed-batch culture.

Next, it is intended to delve into the necessary conditions of optimality for optimal control problem of the controlled explicit nonlinear impulsive dynamical system of fed-batch culture and the questions of optimization algorithm for the optimal control problem.

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