J. Math. Anal. Appl. 355 (2009) 12–21

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Bounds for the L^q-spectra of a self-similar multifractal not satisfying the Open Set Condition

L. Olsen

Department of Mathematics, University of St. Andrews, St. Andrews, Fife KY16 9SS, Scotland, United Kingdom

ARTICLE INFO

Article history:
Received 13 June 2008
Available online 24 January 2009
Submitted by B.S. Thomson

Keywords:
Multifractals
Self-similar measure
L^q-spectra

ABSTRACT

By now the L^q-spectra of self-similar measures satisfying the so-called Open Set Condition is well understood. However, if the Open Set Condition is not satisfied, then almost nothing is known. In this paper we provide non-trivial bounds for the L^q-spectra of an arbitrary self-similar measure. We emphasize that we are considering arbitrary self-similar measures (and sets) which are not assumed to satisfy the Open Set Condition or similar separation conditions.

By now the L^q-spectra of self-similar measures satisfying the so-called Open Set Condition is well understood. However, if the Open Set Condition is not satisfied, then almost nothing is known. In this paper we provide non-trivial bounds for the L^q-spectra of an arbitrary self-similar measure, see Theorem 1.1 and its corollaries, and non-trivial bounds for the mixed L^q-spectra of finite families of arbitrary self-similar measures, see Theorem 2.1. We emphasize that we are considering arbitrary self-similar measures and sets which are not assumed to satisfy any separation conditions; in particular, we are not assuming that the Open Set Condition is satisfied. As an application of our results we obtain bounds for the L^q-spectra of the $(2,3)$-Bernoulli convolution, see Section 3.

1. Bounds for L^q-spectra

1.1. Self-similar measures

Let $S_i : \mathbb{R}^d \to \mathbb{R}^d$ for $i = 1, \ldots, N$ be contracting similarities and let $(p_1, \ldots, p_N)$ be a probability vector. For each i, we denote the Lipschitz constant of S_i by $r_i \in (0, 1)$. Let K and μ be the self-similar set and the self-similar measure associated with the list $(S_1, \ldots, S_N, p_1, \ldots, p_N)$, i.e. K is the unique non-empty compact subset of $\mathbb{R}^d$ such that

$$K = \bigcup_i S_i(K), \tag{1.1}$$

and μ is the unique Borel probability measure on $\mathbb{R}^d$ such that

$$\mu = \sum_i p_i \mu \circ S_i^{-1}, \tag{1.2}$$

cf. [6]. It is well known that the support of μ equals K. We will say that the list $(S_1, \ldots, S_N)$ satisfies the Open Set Condition (OSC) if there exists a non-empty, bounded and open set U such that $S_i(U) \subseteq U$ for all i and $S_i(U) \cap S_j(U) = \emptyset$ for all $i \neq j$.

E-mail address: lo@st-and.ac.uk.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.01.037

1.2. L^q-spectra of self-similar measures

During the past 15 years the multifractal structure of μ has received much attention. Multifractal analysis refers (among other things) to the study of the L^q-spectra of μ. For $q \in \mathbb{R}$, we define the lower L^q-spectrum $\underline{\tau}_\mu(q)$ and the upper L^q-spectrum $\overline{\tau}_\mu(q)$ of μ as follows. For $r > 0$ write

$$I_\mu(r;q) = \int_K \mu\big(B(x,r)\big)^{q-1}\, d\mu(x). \tag{1.3}$$

The lower and upper L^q-spectra of μ are now defined by

$$\begin{aligned}\underline{\tau}_\mu(q) &= \liminf_{r\searrow 0} \frac{\log I_\mu(r;q)}{-\log r},\\ \overline{\tau}_\mu(q) &= \limsup_{r\searrow 0} \frac{\log I_\mu(r;q)}{-\log r}.\end{aligned} \tag{1.4}$$

The main significance of the L^q-spectra is their relationship with multifractal analysis, cf. [3] and references therein. Indeed, during the 1990's there has been an enormous interest in computing the L^q-spectra in the mathematical literature, and within the last 15 years the L^q-spectra of various classes of measures in Euclidean space $\mathbb{R}^d$ exhibiting some degree of self-similarity have been computed rigorously, cf. [3].

In particular, in the 1990's Arbeiter and Patzschke [1] succeeded in computing the L^q-spectra of self-similar measures satisfying the OSC; their results are summarized in Theorem A below. However, before we can state Theorem A we introduce the following definition. Namely, define the function $\beta : \mathbb{R} \to \mathbb{R}$ by

$$\sum_{i=1}^N p_i^q r_i^{\beta(q)} = 1. \tag{1.5}$$

We can now state Theorem A from [1].

Theorem A. *(See [1].) Assume that the OSC is satisfied. Then the L^q-spectra $\underline{\tau}_\mu(q)$ and $\overline{\tau}_\mu(q)$ are given by*

$$\underline{\tau}_\mu(q) = \overline{\tau}_\mu(q) = \beta(q)$$

for all $q \in \mathbb{R}$.

The reader is referred to the textbook [3] for a more detailed discussion of multifractal analysis.

1.3. Main results – Part 1: Bounds for the L^q-spectra

Unfortunately, except for a few special classes of measures, almost nothing is known about the L^q-spectra of μ if the OSC is not satisfied. For example, Feng et al. [4,5] have recently proved that all self-similar measures (and, in particular, self-similar measures not satisfying the OSC) satisfies some version of the multifractal formalism, and Lau et al. [10–12] and Testud [18] have investigated multifractal properties of various special classes of self-similar measures not satisfying the OSC.

The main purpose of this paper is to provide non-trivial bounds for the L^q-spectra of an arbitrary self-similar measure. We emphasize that we are not imposing any condition on the size of the overlaps $S_iK \cap S_jK$. Theorem 1.1 below is our first main result. However, before we can state Theorem 1.1 we need to introduce the following notation. Let

$$\begin{aligned}\Sigma^n &= \{1,\dots,N\}^n,\\ \Sigma^{\mathbb{N}} &= \{1,\dots,N\}^{\mathbb{N}},\end{aligned} \tag{1.6}$$

i.e. Σ^n is the family of all finite strings $\mathbf{i} = i_1 \dots i_n$ of length n with entries $i_j \in \{1,\dots,N\}$ and $\Sigma^{\mathbb{N}}$ denotes the family of all infinite strings $\mathbf{i} = i_1 i_2 \dots$ with entries $i_j \in \{1,\dots,N\}$. For $\mathbf{i} = i_1 i_2 \dots \in \Sigma^{\mathbb{N}}$ and a positive integer n, let $\mathbf{i}\,|\,n = i_1 \dots i_n$ denote the truncation of $\mathbf{i}$ to the nth place. Furthermore, for $\mathbf{i} = i_1 \dots i_n \in \Sigma^n$, we write $S_{\mathbf{i}} = S_{i_1} \circ \cdots \circ S_{i_n}$ and

$$K_{\mathbf{i}} = S_{\mathbf{i}} K.$$

Also, write

$$p_{\mathbf{i}} = p_{i_1} \cdots p_{i_n}, \qquad r_{\mathbf{i}} = r_{i_1} \cdots r_{i_n}$$

for $\mathbf{i} = i_1 \dots i_n \in \Sigma^n$. We can now state our first main result providing non-trivial bounds for the L^q-spectra of an arbitrary self-similar measure.

Theorem 1.1. *For a positive integer n, let*

$$\mathcal{I}_n = \left\{ I \subseteq \Sigma^n \,\middle|\, \bigcap_{\mathbf{i}\in I} K_{\mathbf{i}} \neq \emptyset \right\}$$

(*observe that $\mathcal{I}_n$ is non-empty since $\{\mathbf{i}\} \in \mathcal{I}_n$ for all $\mathbf{i} \in \Sigma^n$*). *There exists a unique $s_n \in \mathbb{R}$ such that*

$$1 = \max_{I\in\mathcal{I}_n} \sum_{\mathbf{i}\in I} p_{\mathbf{i}} r_{\mathbf{i}}^{-s_n}.$$

Let $s = \sup_n s_n$. For all $q \in \mathbb{R}$ with $q \geqslant 1$, we have

$$\beta(q) \leqslant \underline{\tau}_\mu(q) \leqslant \overline{\tau}_\mu(q) \leqslant s(1-q).$$

Theorem 1.1 follows from a more general result in Section 2, namely Theorem 2.1, providing non-trivial bounds for the mixed L^q-spectra of a finite family of arbitrary self-similar measures. If K is not a singleton, then the next result shows that $s > 0$. In particular, this implies that $s(1-q) < 0$ for all $q > 1$, and since it is not difficult to see that $\overline{\tau}_\mu(q) \leqslant 0$ for all $q \geqslant 1$, we therefore conclude that the upper bound for $\overline{\tau}_\mu(q)$ provided by Theorem 1.1 is non-trivial.

Proposition 1.2. *Let s be as in Theorem* 1.1. *If K is not a singleton, then $s > 0$. In particular, if K is not a singleton, then*

$$\overline{\tau}_\mu(q) \leqslant s(1-q) < 0$$

for all $q > 1$.

Proposition 1.2 also follows from a more general result in Section 2, namely Proposition 2.2, providing bounds for the mixed L^q-spectra of a finite family of self-similar measures.

If all the contraction ratios $r_1, \ldots, r_N$ coincide, then the results in Theorem 1.1 can be simplified. Indeed, if $r_1 = \cdots = r_N = r$ for some $r \in (0,1)$, then $s_n = \frac{1}{n\log r}\log(\max_{I\in\mathcal{I}_n}\sum_{\mathbf{i}\in I} p_{\mathbf{i}}^q)$, and we therefore obtain the following corollary from Theorem 1.1.

Corollary 1.3. *Assume that $r_1 = \cdots = r_N = r$. Put*

$$s = \sup_n \frac{1}{n\log r} \log\left(\max_{I\in\mathcal{I}_n} \sum_{\mathbf{i}\in I} p_{\mathbf{i}}^q \right).$$

For all $q \geqslant 1$, we have

$$\beta(q) \leqslant \underline{\tau}_\mu(q) \leqslant \overline{\tau}_\mu(q) \leqslant s(1-q).$$

2. Bounds for the mixed L^q-spectra

Recently mixed (or simultaneous) multifractal L^q-spectra have generated an enormous interest in the mathematical literature, cf. [2,7,14–16] and references therein. Previously, only the scaling behaviour of a single measure μ has been investigated, see [3]. However, mixed multifractal analysis investigates the *simultaneous* scaling behaviour of finitely many measures $\mu_1, \ldots, \mu_k$. Mixed multifractal analysis thus combines local characteristics which depend simultaneously on various different aspects of the underlying dynamical system, and provides the basis for a significantly better understanding of the underlying dynamics.

We will now generalize Theorem 1.1 to the mixed multifractal setting. We therefore fix a positive integer k, and let $\mathbf{p}_j = (p_{j,i})_{i=1,\ldots,N}$ be probability vectors for $j = 1, \ldots, k$ with $p_{j,i} > 0$ for all j, i. Next, let μ_j denote the self-similar measure associated with the list $(S_1, \ldots, S_N, p_{j,1}, \ldots, p_{j,N})$, i.e. μ_j is the unique Borel probability measure such that

$$\mu_j = \sum_i p_{j,i} \mu_j \circ S_i^{-1}.$$

As in the previous section, K denotes the common support of the measures $\mu_1, \ldots, \mu_k$, i.e. K is the unique compact set satisfying (1.1). Finally, we define $\beta : \mathbb{R}^k \to \mathbb{R}$ by

$$\sum_i p_{1,i}^{q_1} \cdots p_{k,i}^{q_k} r_i^{\beta(\mathbf{q})} = 1 \tag{2.1}$$

for $\mathbf{q} = (q_1, \ldots, q_k) \in \mathbb{R}^k$, and observe that this definition reduces to (1.5) for $k = 1$.

2.1. Mixed L^q-spectra of finite families of self-similar measures

Mixed L^q-spectra of the list $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$ are defined as follows. Let $\mathbb{D}_k$ denote the diagonal ray in $\mathbb{R}^k$, i.e.

$$\mathbb{D}_k = \{(x, \ldots, x) \in \mathbb{R}^k \mid x \in \mathbb{R}\}.$$

If E is a subset of $\mathbb{R}^k$ and $r > 0$, we will write $B(E, r)$ for the r neighbourhood of E, i.e. $B(E, r) = \{x \in \mathbb{R}^d \mid \operatorname{dist}(x, E) < r\}$. For $\mathbf{q} = (q_1, \ldots, q_k) \in \mathbb{R}^k$ and $r > 0$, we define the mixed integral moment scaling function of the list $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$ by

$$I_{\boldsymbol{\mu}}(r; \mathbf{q}) = \int\limits_{K^k \cap B(\mathbb{D}_k, r)} \mu_1\big(B(x_1, r)\big)^{q_1 - 1} \cdots \mu_k\big(B(x_k, r)\big)^{q_k - 1}\, d(\mu_1 \times \cdots \times \mu_k)(x_1, \ldots, x_k). \tag{2.2}$$

The lower and upper mixed L^q-spectra, denoted $\underline{\tau}_{\boldsymbol{\mu}}(\mathbf{q})$ and $\overline{\tau}_{\boldsymbol{\mu}}(\mathbf{q})$, of $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$ are now defined by

$$\begin{aligned} \underline{\tau}_{\boldsymbol{\mu}}(\mathbf{q}) &= \liminf_{r \searrow 0} \frac{\log I_{\boldsymbol{\mu}}(r; \mathbf{q})}{-\log r}, \\ \overline{\tau}_{\boldsymbol{\mu}}(\mathbf{q}) &= \limsup_{r \searrow 0} \frac{\log I_{\boldsymbol{\mu}}(r; \mathbf{q})}{-\log r}. \end{aligned} \tag{2.3}$$

For $k = 1$, the above definitions coincide with (1.3) and (1.4). Assuming the OSC Moran [14] and Olsen [16] computed the mixed L^q-spectra $\underline{\tau}_{\boldsymbol{\mu}}(\mathbf{q})$ and $\overline{\tau}_{\boldsymbol{\mu}}(\mathbf{q})$; this result is summarized in Theorem B.

Theorem B. *(See [14,16].) Assume that the OSC is satisfied. Then the mixed L^q-spectra $\underline{\tau}_{\boldsymbol{\mu}}(\mathbf{q})$ and $\overline{\tau}_{\boldsymbol{\mu}}(\mathbf{q})$ are given by*

$$\underline{\tau}_{\boldsymbol{\mu}}(\mathbf{q}) = \overline{\tau}_{\boldsymbol{\mu}}(\mathbf{q}) = \beta(\mathbf{q})$$

for all $\mathbf{q} \in \mathbb{R}^k$.

2.2. Main results – Part 2: Bounds for the mixed L^q-spectra

Unfortunately, nothing is known about the mixed L^q-spectra $\underline{\tau}_{\boldsymbol{\mu}}(\mathbf{q})$ and $\overline{\tau}_{\boldsymbol{\mu}}(\mathbf{q})$ if the OSC is not satisfied, and the second main purpose of this the paper is to provide non-trivial bounds for the mixed L^q-spectra of a list of arbitrary self-similar measures. We emphasize that we are not imposing any condition on the size of the overlaps $S_i K \cap S_j K$. Theorem 2.1 below is the main result. In Theorem 2.1 and Proposition 2.2 we write $\mathbf{x} \geqslant \mathbf{y}$ for $\mathbf{x} = (x_1, \ldots, x_k)$, $\mathbf{y} = (y_1, \ldots, y_k) \in \mathbb{R}^k$ if $x_i \geqslant y_i$ for all i and we write $\mathbf{x} > \mathbf{y}$ for $\mathbf{x} = (x_1, \ldots, x_k)$, $\mathbf{y} = (y_1, \ldots, y_k) \in \mathbb{R}^k$ if $x_i > y_i$ for all i. We also write

$$p_{i,\mathbf{i}} = p_{i,i_1} \cdots p_{i,i_n}$$

for $\mathbf{i} = i_1 \ldots i_n \in \Sigma^n$, and put $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^k$ and $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^k$. Finally, we let $\langle \cdot \mid \cdot \rangle$ denote the usual inner product in $\mathbb{R}^k$. We can now state Theorem 2.1.

Theorem 2.1. *For a positive integer n, let*

$$\mathcal{I}_n = \left\{ I \subseteq \Sigma^n \;\middle|\; \bigcap_{\mathbf{i} \in I} K_{\mathbf{i}} \neq \emptyset \right\}$$

(*observe that $\mathcal{I}_n$ is non-empty since $\{\mathbf{i}\} \in \mathcal{I}_n$ for all $\mathbf{i} \in \Sigma^n$). There exists a unique $s_n \in \mathbb{R}$ such that*

$$1 = \max_{\substack{i \\ I \in \mathcal{I}_n}} \sum_{\mathbf{i} \in I} p_{i,\mathbf{i}} r_{\mathbf{i}}^{-s_n}.$$

Let $s = \sup_n s_n$ and $\mathbf{s} = (s, \ldots, s) \in \mathbb{R}^k$. For all $\mathbf{q} \in \mathbb{R}^k$ with $\mathbf{q} \geqslant \mathbf{1}$, we have

$$\beta(\mathbf{q}) \leqslant \underline{\tau}_{\boldsymbol{\mu}}(\mathbf{q}) \leqslant \overline{\tau}_{\boldsymbol{\mu}}(\mathbf{q}) \leqslant \langle \mathbf{s} \mid \mathbf{1} - \mathbf{q} \rangle.$$

It is clear that Theorem 1.1 follows from Theorem 2.1 by setting $k = 1$. Also, in order to prove Theorem 2.1 we must show that if $\mathbf{q} \geqslant \mathbf{1}$, then

$$\overline{\tau}_{\boldsymbol{\mu}}(\mathbf{q}) \leqslant \langle \mathbf{s} \mid \mathbf{1} - \mathbf{q} \rangle \tag{2.4}$$

and

$$\beta(\mathbf{q}) \leqslant \underline{\tau}_{\boldsymbol{\mu}}(\mathbf{q}). \tag{2.5}$$

The proof of (2.4) is given in Section 4 and the proof of (2.5) is given in Section 5.

Analogous to Section 1, if K is not a singleton, then we prove below that $\mathbf{s} > \mathbf{0}$. In particular, this implies that $\langle \mathbf{s} \mid \mathbf{1} - \mathbf{q} \rangle < 0$ for all $\mathbf{q} > \mathbf{1}$, and since it is not difficult to see that $\overline{\tau}_{\boldsymbol{\mu}}(\mathbf{q}) \leqslant 0$ for all $\mathbf{q} > \mathbf{1}$, we therefore conclude the upper bound for $\overline{\tau}_{\boldsymbol{\mu}}(\mathbf{q})$ provided by Theorem 2.1 is non-trivial.

Proposition 2.2. *Let s be as in Theorem* 2.1. *If K is not a singleton, then $\mathbf{s} > \mathbf{0}$. In particular, if K is not a singleton, then*

$$\overline{\tau}_{\boldsymbol{\mu}}(\mathbf{q}) \leqslant \langle \mathbf{s} \mid \mathbf{1} - \mathbf{q} \rangle < 0$$

for all $\mathbf{q} > \mathbf{1}$.

Proof. For $\mathbf{i} = i_1 i_2 \ldots \in \Sigma^{\mathbb{N}}$ and a positive integer n, we write $\mathbf{i} \mid n = i_1 \ldots i_n$. Also, define $\pi : \Sigma^{\mathbb{N}} \to \mathbb{R}^d$ by $\{\pi(\mathbf{i})\} = \bigcap_n K_{\mathbf{i}|n}$. It is well known that $K = \pi(\Sigma^{\mathbb{N}})$. If K is not a singleton, then we can find $x, y \in K$ with $x \neq y$. Let $x = \pi(\mathbf{i})$ and $y = \pi(\mathbf{j})$ where $\mathbf{i}, \mathbf{j} \in \Sigma^{\mathbb{N}}$. Since $\lim_n \operatorname{diam} K_{\mathbf{i}|n} = \lim_n \operatorname{diam} K_{\mathbf{j}|n} = 0$ and $\bigcap_n K_{\mathbf{i}|n} = \{\pi(\mathbf{i})\} = \{x\}$ and $\bigcap_n K_{\mathbf{j}|n} = \{\pi(\mathbf{j})\} = \{y\}$, we conclude from the fact that $x \neq y$, that there exists a positive integer n such that $K_{\mathbf{i}|n} \cap K_{\mathbf{j}|n} = \emptyset$. This implies that, if $I \in \mathcal{I}_n$, then I cannot contain both $\mathbf{i} \mid n$ and $\mathbf{j} \mid n$, whence I is a proper subset of Σ^n. We infer from this that $\sum_{\mathbf{k} \in I} p_{i,\mathbf{k}} r_{\mathbf{k}}^{-s_n} < \sum_{\mathbf{k} \in \Sigma^n} p_{i,\mathbf{k}} r_{\mathbf{k}}^{-s_n}$ for all i and all $I \in \mathcal{I}_n$. Hence

$$1 = \max_{\substack{i \\ I \in \mathcal{I}_n}} \sum_{\mathbf{k} \in I} p_{i,\mathbf{k}} r_{\mathbf{k}}^{-s_n} < \max_i \sum_{\mathbf{k} \in \Sigma^n} p_{i,\mathbf{k}} r_{\mathbf{k}}^{-s_n} = \max_i \left(\sum_j p_{i,j} r_j^{-s_n} \right)^n.$$

It follows from this that $0 < s_n$. $\square$

It is clear that Theorem 1.2 follows from Theorem 2.2 by setting $k = 1$.

3. An example: The (2, 3)-Bernoulli convolution

In this section we illustrate Theorem 1.1 and Theorem 2.1 by analyzing the L^q spectra of the so-called (2, 3)-Bernoulli convolution. We note that Feng and Olivier [5] have very recently proved that the (2, 3)-Bernoulli convolution satisfies the multifractal formalism (see Section 1.1 for the definition of the multifractal formalism). Unfortunately, Feng and Olivier's result does not easily provide explicit values for the L^q multifractal spectra. We will now use the results in this paper to obtain explicit and non-trivial bounds for the L^q multifractal spectra of the (2, 3)-Bernoulli convolution. The (2, 3)-Bernoulli convolution is defined as follows. Define $S_1, S_2, S_3 : \mathbb{R} \to \mathbb{R}$ by $S_i(x) = \frac{1}{2}x + \frac{i-1}{4}$ and let (p_1, p_2, p_3) denote the uniform probability vector, i.e. $p_1 = p_2 = p_3 = \frac{1}{3}$. The (2, 3)-Bernoulli convolution is by definition the self-similar measure μ associated with the list $(S_1, S_2, S_3, p_1, p_2, p_3)$, i.e. μ is the unique probability measure such that

$$\mu = \frac{1}{3}\mu \circ S_1^{-1} + \frac{1}{3}\mu \circ S_2^{-1} + \frac{1}{3}\mu \circ S_3^{-1}.$$

As in Example 1, the main difficulty in analyzing the multifractal spectrum of μ is due to the fact that the OSC is not satisfied. It is clear that in this case $N = 3$ and $r_1 = r_2 = r_3 = \frac{1}{2}$, whence

$$s = \sup_n s_n = \sup_n \frac{1}{n \log 2} \log\left(\max_{I \in \mathcal{I}_n} \sum_{\mathbf{i} \in I} p_{\mathbf{i}} \right) = \sup_n \frac{1}{n \log 2} \log\left(\max_{I \in \mathcal{I}_n} |I| 3^{-n} \right) = \frac{\log 3}{\log 2} - \frac{1}{\log 2} \inf_n \frac{1}{n} \log \max_{I \in \mathcal{I}_n} |I|.$$

Since clearly $\beta(q) = \frac{\log 3}{\log 2}(1 - q)$, we conclude from Theorem 1.1 that if $q \geqslant 1$, then

$$\frac{\log 3}{\log 2}(1 - q) \leqslant \underline{\tau}_\mu(q) \leqslant \overline{\tau}_\mu(q) \leqslant \left(\frac{\log 3}{\log 2} - \frac{1}{\log 2} \inf_n \frac{1}{n} \log \max_{I \in \mathcal{I}_n} |I| \right)(1 - q). \tag{3.1}$$

It is not difficult to see that, for example, $\max_{I \in \mathcal{I}_2} |I| = 5$, and it therefore follows from (3.1) that if $q \geqslant 1$, then

$$\frac{\log 3}{\log 2}(1 - q) \leqslant \underline{\tau}_\mu(q) \leqslant \overline{\tau}_\mu(q) = \left(\frac{\log 3}{\log 2} - \frac{\log 5}{\log 4} \right)(1 - q). \tag{3.2}$$

4. Proof of inequality (2.4) in Theorem 2.1

The purpose of this section is to prove inequality (2.4). For a positive integer n, we define $\rho_n : K \to \mathbb{R}$ by

$$\rho_n(x) = \sup_{t \in K} \min_{\substack{\mathbf{i} \in \Sigma^n \\ t \notin K_{\mathbf{i}}}} \operatorname{dist}(x, K_{\mathbf{i}}),$$

and write

$$\delta_n = \inf_{x \in K} \rho_n(x).$$

Lemma 4.1. *Fix a positive integer n. We have* $\delta_n > 0$.

Proof. It is not difficult to see that $\rho_n(x) > 0$ for all $x \in K$. Also, it not difficult to see that ρ_n is continuous (in fact, one can easily prove that $|\rho_n(x) - \rho_n(y)| \leqslant |x - y|$ for all $x, y \in K$), and the compactness of K therefore implies the existence of a point $x_0 \in K$ such that $\inf_{x \in K} \rho_n(x) = \rho_n(x_0)$. Hence $\delta_n = \inf_{x \in K} \rho_n(x) = \rho_n(x_0) > 0$. □

Define $M : (0, \infty) \to \mathbb{R}$ by

$$M(r) = \sup_{\substack{i \\ x \in K}} \mu_i\big(B(x, r)\big).$$

Theorem 4.2. *Fix a positive integer n. We have*

$$M(r) \leqslant \max_{\substack{i \\ I \in \mathcal{I}_n}} \sum_{\mathbf{i} \in I} p_{i,\mathbf{i}} M\left(\frac{r}{r_{\mathbf{i}}}\right)$$

for all $0 < r < \delta_n$.

Proof. Fix $0 < r < \delta_n$ and $x \in K$.

Observe that

$$S_{\mathbf{i}}^{-1}\big(B(x, r)\big) \subseteq B\left(S_{\mathbf{i}}^{-1}x, \frac{r}{r_{\mathbf{i}}}\right) \quad \text{for all } \mathbf{i} \in \Sigma^n. \tag{4.1}$$

Next, we prove that there exists $t \in K$ such that

$$\big\{\mathbf{i} \in \Sigma^n \,\big|\, S_{\mathbf{i}}^{-1}\big(B(x, r)\big) \cap K \neq \emptyset\big\} \subseteq \big\{\mathbf{i} \in \Sigma^n \,\big|\, t \in K_{\mathbf{i}}\big\}. \tag{4.2}$$

Indeed, since $r < \delta_n \leqslant \rho_n(x) = \sup_{t \in K} \min_{\mathbf{i} \in \Sigma^n,\, t \notin K_{\mathbf{i}}} \operatorname{dist}(x, K_{\mathbf{i}})$, there exists $t \in K$ such that

$$r < \min_{\substack{\mathbf{i} \in \Sigma^n \\ t \notin K_{\mathbf{i}}}} \operatorname{dist}(x, K_{\mathbf{i}}). \tag{4.3}$$

We now claim that $\{\mathbf{i} \in \Sigma^n \mid S_{\mathbf{i}}^{-1}(B(x, r)) \cap K \neq \emptyset\} \subseteq \{\mathbf{i} \in \Sigma^n \mid t \in K_{\mathbf{i}}\}$. We will now prove this. Let $\mathbf{i} \in \Sigma^n$ with $S_{\mathbf{i}}^{-1}(B(x, r)) \cap K \neq \emptyset$. We can thus choose $u \in S_{\mathbf{i}}^{-1}(B(x, r)) \cap K$. Hence $\operatorname{dist}(x, K_{\mathbf{i}}) = \operatorname{dist}(x, S_{\mathbf{i}}K) \leqslant |x - S_{\mathbf{i}}u| \leqslant r$, and we therefore conclude from (4.3) that $t \in K_{\mathbf{i}}$. This proves (4.2).

Finally, observe that iteration of the self-similar identity (1.3) shows that

$$\mu_i(E) = \sum_{\mathbf{i} \in \Sigma^n} p_{i,\mathbf{i}} \mu_i\big(S_{\mathbf{i}}^{-1}E\big) \tag{4.4}$$

for all $i = 1, \ldots, k$ and all $E \subseteq \mathbb{R}^d$.

Combining (4.1), (4.2) and (4.4) we now obtain

$$\begin{aligned} \mu_i\big(B(x, r)\big) &= \sum_{\mathbf{i} \in \Sigma^n} p_{i,\mathbf{i}} \mu_i\big(S_{\mathbf{i}}^{-1}B(x, r)\big) = \sum_{\substack{\mathbf{i} \in \Sigma^n \\ S_{\mathbf{i}}^{-1}(B(x,r)) \cap K \neq \emptyset}} p_{i,\mathbf{i}} \mu_i\big(S_{\mathbf{i}}^{-1}B(x, r)\big) \leqslant \sum_{\substack{\mathbf{i} \in \Sigma^n \\ t \in K_{\mathbf{i}}}} p_{i,\mathbf{i}} \mu_i\left(B\left(S_{\mathbf{i}}^{-1}x, \frac{r}{r_{\mathbf{i}}}\right)\right) \\ &\leqslant \sum_{\substack{\mathbf{i} \in \Sigma^n \\ t \in K_{\mathbf{i}}}} p_{i,\mathbf{i}} M\left(\frac{r}{r_{\mathbf{i}}}\right). \end{aligned} \tag{4.5}$$

However, since clearly $\{\mathbf{i} \in \Sigma^n \mid t \in K_{\mathbf{i}}\} \in \mathcal{I}_n$, this implies that

$$\mu_i\big(B(x, r)\big) \leqslant \max_{I \in \mathcal{I}_n} \sum_{\mathbf{i} \in I} p_{i,\mathbf{i}} M\left(\frac{r}{r_{\mathbf{i}}}\right).$$

Since $i = 1, \ldots, k$ and $x \in K$ were arbitrary, this completes the proof. □

Proposition 4.3. *Fix a positive integer n. There exists a constant* $c > 0$ *such that*

$$M(r) \leqslant c r^{s_n}$$

for all $r > 0$.

Proof. Define $W:(0,\infty)\to[0,\infty)$ by $W(r)=r^{-s_n}M(r)$. It follows from Theorem 4.2 that

$$W(r)\leqslant \max_{\substack{i\\ I\in\mathcal{I}_n}}\sum_{\mathbf{i}\in I}p_{i,\mathbf{i}}r_{\mathbf{i}}^{-s_n}\left(\frac{r}{r_{\mathbf{i}}}\right)^{-s_n}M\left(\frac{r}{r_{\mathbf{i}}}\right)=\max_{\substack{i\\ I\in\mathcal{I}_n}}\sum_{\mathbf{i}\in I}p_{i,\mathbf{i}}r_{\mathbf{i}}^{-s_n}W\left(\frac{r}{r_{\mathbf{i}}}\right) \tag{4.6}$$

for all $0<r<\delta_n$. Next, write $\lambda=(\min_i r_i)^n$ and $\Lambda=(\max_i r_i)^n$. Observe that if a is a positive real number with $0<a<\frac{a}{\Lambda}<\delta_n$, then (4.6) and the definition of s_n imply that

$$\sup_{a\leqslant r<\delta_n}W(r)\leqslant \max_{\substack{i\\ I\in\mathcal{I}_n}}\sum_{\mathbf{i}\in I}p_{i,\mathbf{i}}r_{\mathbf{i}}^{-s_n}\sup_{a\leqslant r<\delta_n}W\left(\frac{r}{r_{\mathbf{i}}}\right)\leqslant \max_{\substack{i\\ I\in\mathcal{I}_n}}\sum_{\mathbf{i}\in I}p_{i,\mathbf{i}}r_{\mathbf{i}}^{-s_n}\sup_{\frac{a}{\Lambda}\leqslant r<\frac{\delta_n}{\lambda}}W(r)=\sup_{\frac{a}{\Lambda}\leqslant r<\frac{\delta_n}{\lambda}}W(r),$$

and so

$$\sup_{a\leqslant r}W(r)=\max\Bigl(\sup_{a\leqslant r<\delta_n}W(r),\sup_{\delta_n\leqslant r}W(r)\Bigr)\leqslant\max\Bigl(\sup_{\frac{a}{\Lambda}\leqslant r<\frac{\delta_n}{\lambda}}W(r),\sup_{\delta_n\leqslant r}W(r)\Bigr)=\sup_{\frac{a}{\Lambda}\leqslant r}W(r).$$

Next, let k_n be the unique integer with $\Lambda^{k_n}<\delta_n\leqslant\Lambda^{k_n-1}$. Repeated application of the previous inequality now yields

$$\sup_{0<r}W(r)=\sup_{k>k_n}\sup_{\Lambda^k\leqslant r}W(r)\leqslant\sup_{k>k_n}\sup_{\Lambda^{k-1}\leqslant r}W(r)\leqslant\cdots\leqslant\sup_{k>k_n}\sup_{\Lambda^{k_n}\leqslant r}W(r)=\sup_{\Lambda^{k_n}\leqslant r}W(r)=\sup_{\Lambda^{k_n}\leqslant r}r^{-s_n}M(r)\leqslant\sup_{\Lambda^{k_n}\leqslant r}r^{-s_n}.$$

However, since $\max_{i,I\in\mathcal{I}_n}\sum_{\mathbf{i}\in I}p_{i,\mathbf{i}}\leqslant\max_i\sum_{\mathbf{i}\in\Sigma^n}p_{i,\mathbf{i}}=1=\max_{i,I\in\mathcal{I}_n}\sum_{i,\mathbf{i}\in I}p_{i,\mathbf{i}}^{q}r_{\mathbf{i}}^{-s_n}$, we conclude that $s_n\geqslant 0$. It therefore follows from the above inequality that

$$\sup_{0<r}W(r)\leqslant\sup_{\Lambda^{k_n}\leqslant r}r^{-s_n}\leqslant\Lambda^{-k_ns_n}=c.$$

We conclude from this that $r^{-s_n}M(r)=W(r)\leqslant c$ for all $0<r$. This completes the proof. $\square$

Proof of inequality (2.4) in Theorem 2.1. Let $\mathbf{q}=(q_1,\dots,q_k)\in\mathbb{R}^k$ with $\mathbf{q}\geqslant\mathbf{1}$. Fix a positive integer n and let c be the constant in Proposition 4.3. For $r>0$, we have

$$\begin{aligned}I_{\boldsymbol\mu}(r;\mathbf{q})&=\int_{K^k\cap B(\mathbb{D}_k,r)}\mu_1\bigl(B(x_1,r)\bigr)^{q_1-1}\cdots\mu_k\bigl(B(x_k,r)\bigr)^{q_k-1}\,d(\mu_1\times\cdots\times\mu_k)(x_1,\dots,x_k)\\&\leqslant\int_{K^k\cap B(\mathbb{D}_k,r)}M(r)^{q_1-1+\cdots+q_k-1}\,d(\mu_1\times\cdots\times\mu_k)(x_1,\dots,x_k)\\&\leqslant M(r)^{q_1-1+\cdots+q_k-1}\\&\leqslant\bigl(cr^{s_n}\bigr)^{q_1-1+\cdots+q_k-1}.\end{aligned}$$

This clearly implies that $\overline{\tau}_{\boldsymbol\mu}(\mathbf{q})\leqslant s_n(1-q_1+\cdots+1-q_k)$. Since this is true for all n, we now conclude that $\overline{\tau}_{\boldsymbol\mu}(\mathbf{q})\leqslant s(1-q_1+\cdots+1-q_k)=\langle\mathbf{s}\mid\mathbf{1}-\mathbf{q}\rangle$. This completes the proof. $\square$

5. Proof of inequality (2.5) in Theorem 2.1

The purpose of this section is to prove inequality (2.5).

Theorem 5.1. *Let $\mathbf{q}\in\mathbb{R}^k$ with $\mathbf{q}\geqslant\mathbf{1}$. Then*

$$I_{\boldsymbol\mu}(r;\mathbf{q})\geqslant\sum_i p_{1,i}^{q_1}\cdots p_{k,i}^{q_k}I_{\boldsymbol\mu}\left(\frac{r}{r_i};\mathbf{q}\right)$$

for all $r>0$.

Proof. For a set X, we let 1_X denote the indicator function on X. We now have

$$\begin{aligned}I_{\boldsymbol\mu}(r;\mathbf{q})&=\int 1_{K^k\cap B(\mathbb{D}_k,r)}(x_1,\dots,x_k)\prod_l\mu_l\bigl(B(x_l,r)\bigr)^{q_l-1}\,d(\mu_1\times\cdots\times\mu_k)(x_1,\dots,x_k)\\&=\int\cdots\int 1_{K^k\cap B(\mathbb{D}_k,r)}(x_1,\dots,x_k)\prod_l\mu_l\bigl(B(x_l,r)\bigr)^{q_l-1}\,d\mu_1(x_1)\cdots d\mu_k(x_k)\end{aligned}$$

$$= \int \cdots \int \sum_{i_1} \cdots \sum_{i_k} p_{1,i_1} \cdots p_{k,i_k} 1_{K^k \cap B(\mathbb{D}_k, r)}(x_1, \ldots, x_k) \prod_l \mu_l\big(B(x_l, r)\big)^{q_l - 1} d(\mu_1 \circ S_{i_1})(x_1) \cdots d(\mu_k \circ S_{i_k})(x_k)$$

$$= \int \cdots \int \sum_{i_1} \cdots \sum_{i_k} p_{1,i_1} \cdots p_{k,i_k} 1_{K^k \cap B(\mathbb{D}_k, r)}(S_{i_1} x_1, \ldots, S_{i_k} x_k) \prod_l \mu_l\big(B(S_{i_l} x_l, r)\big)^{q_l - 1} d\mu_1(x_1) \cdots d\mu_k(x_k) \tag{5.1}$$

for all $r > 0$.

Next, notice that if $x \in K$ and $l = 1, \ldots, k$, then we have

$$\begin{aligned} \mu_l\big(B(S_{i_l} x, r)\big)^{q_l - 1} &= \left(\sum_j p_{l,j} \mu_l\big(S_j^{-1} B(S_{i_l} x, r)\big)\right)^{q_l - 1} = \left(\sum_j p_{l,j} \mu_l\left(B\left(S_j^{-1} S_{i_l} x, \frac{r}{r_j}\right)\right)\right)^{q_l - 1} \\ &\geqslant \left(p_{l,i_l} \mu_l\left(B\left(S_{i_l}^{-1} S_{i_l} x, \frac{r}{r_{i_l}}\right)\right)\right)^{q_l - 1} = p_{l,i_l}^{q_l - 1} \mu_l\left(B\left(x, \frac{r}{r_{i_l}}\right)\right)^{q_l - 1} \end{aligned} \tag{5.2}$$

for all $r > 0$.

Combining (5.1) and (5.2) gives

$$\begin{aligned} I_{\boldsymbol{\mu}}(r; \mathbf{q}) &\geqslant \int \cdots \int \sum_{i_1} \cdots \sum_{i_k} p_{1,i_1} \cdots p_{k,i_k} 1_{K^k \cap B(\mathbb{D}_k, r)}(S_{i_1} x_1, \ldots, S_{i_k} x_k) \prod_l p_{l,i_l}^{q_l - 1} \mu_l\left(B\left(x, \frac{r}{r_{i_l}}\right)\right)^{q_l - 1} d\mu_1(x_1) \cdots d\mu_k(x_k) \\ &= \int \cdots \int \sum_{i_1} \cdots \sum_{i_k} p_{1,i_1}^{q_1} \cdots p_{k,i_k}^{q_k} 1_{K^k \cap B(\mathbb{D}_k, r)}(S_{i_1} x_1, \ldots, S_{i_k} x_k) \prod_l \mu_l\left(B\left(x, \frac{r}{r_{i_l}}\right)\right)^{q_l - 1} d\mu_1(x_1) \cdots d\mu_k(x_k) \\ &\geqslant \int \cdots \int \sum_i p_{1,i}^{q_1} \cdots p_{k,i}^{q_k} 1_{K^k \cap B(\mathbb{D}_k, r)}(S_i x_1, \ldots, S_i x_k) \prod_l \mu_l\left(B\left(x, \frac{r}{r_i}\right)\right)^{q_l - 1} d\mu_1(x_1) \cdots d\mu_k(x_k). \end{aligned} \tag{5.3}$$

We now observe that if $(x_1, \ldots, x_k) \in K^d$ and $i = 1, \ldots, k$, then we have

$$1_{K^k \cap B(\mathbb{D}_k, r)}(S_i x_1, \ldots, S_i x_k) \geqslant 1_{K^k \cap B(\mathbb{D}_k, \frac{r}{r_i})}(x_1, \ldots, x_k) \tag{5.4}$$

for all $r > 0$.

Finally, combining (5.3) and (5.4) gives

$$\begin{aligned} I_{\boldsymbol{\mu}}(r; \mathbf{q}) &\geqslant \int \cdots \int \sum_i p_{1,i}^{q_1} \cdots p_{k,i}^{q_k} 1_{K^k \cap B(\mathbb{D}_k, \frac{r}{r_i})}(x_1, \ldots, x_k) \prod_l \mu_l\left(B\left(x, \frac{r}{r_i}\right)\right)^{q_l - 1} d\mu_1(x_1) \cdots d\mu_k(x_k) \\ &\geqslant \int \sum_i p_{1,i}^{q_1} \cdots p_{k,i}^{q_k} 1_{K^k \cap B(\mathbb{D}_k, \frac{r}{r_i})}(x_1, \ldots, x_k) \prod_l \mu_l\left(B\left(x, \frac{r}{r_i}\right)\right)^{q_l - 1} d(\mu_1 \times \cdots \times \mu_k)(x_1, \ldots, x_k) \\ &= \sum_i p_{1,i}^{q_1} \cdots p_{k,i}^{q_k} I_{\boldsymbol{\mu}}\left(\frac{r}{r_i}; \mathbf{q}\right) \end{aligned}$$

for all $r > 0$. This completes the proof. □

Proposition 5.2. *Let $\mathbf{q} \in \mathbb{R}^k$ with $\mathbf{q} \geqslant \mathbf{1}$. There exists a constant $c > 0$ such that*

$$I_{\boldsymbol{\mu}}(r; \mathbf{q}) \geqslant c r^{-\beta(\mathbf{q})}$$

for all $r > 0$.

Proof. Define $V : (0, \infty) \to [0, \infty)$ by $V(r) = r^{\beta(\mathbf{q})} I_{\boldsymbol{\mu}}(r; \mathbf{q})$. It follows from Theorem 5.1 that

$$V(r) \geqslant \sum_i p_{1,i}^{q_1} \cdots p_{k,i}^{q_k} r_i^{\beta(\mathbf{q})} \left(\frac{r}{r_i}\right)^{\beta(\mathbf{q})} I_{\boldsymbol{\mu}}\left(\frac{r}{r_i}; \mathbf{q}\right) = \sum_i p_{1,i}^{q_1} \cdots p_{k,i}^{q_k} r_i^{\beta(\mathbf{q})} V\left(\frac{r}{r_i}\right) \tag{5.5}$$

for all $r > 0$. Next, write $\lambda = \max_i r_i$. Observe that if a is a positive real number, then (5.5) and the definition of $\beta(\mathbf{q})$ imply that

$$\inf_{a \leqslant r} V(r) \geqslant \sum_i p_{1,i}^{q_1} \cdots p_{k,i}^{q_k} r_i^{\beta(\mathbf{q})} \inf_{a \leqslant r} V\left(\frac{r}{r_i}\right) \geqslant \sum_i p_{1,i}^{q_1} \cdots p_{k,i}^{q_k} r_i^{\beta(\mathbf{q})} \inf_{\frac{a}{\lambda} \leqslant r} V(r) = \inf_{\frac{a}{\lambda} \leqslant r} V(r).$$

Repeated application of the previous inequality now yields

$$\inf_{0<r} V(r) = \inf_{k\geqslant 0} \inf_{\lambda^k \leqslant r} V(r) \geqslant \inf_{k\geqslant 0} \inf_{\lambda^{k-1} \leqslant r} V(r) \geqslant \cdots \geqslant \inf_{k\geqslant 0} \inf_{\lambda^0 \leqslant r} V(r) = \inf_{\lambda^0 \leqslant r} V(r) = \inf_{1\leqslant r} V(r) = \inf_{1\leqslant r} r^{\beta(\mathbf{q})} I_{\boldsymbol{\mu}}(r;\mathbf{q})$$
$$\geqslant \inf_{1\leqslant r} r^{\beta(\mathbf{q})} I_{\boldsymbol{\mu}}(1;\mathbf{q}).$$

Now write $\mathbf{q} = (q_1,\ldots,q_k)$ and note that $q_i \geqslant 1$ for all i. Since $\sum_i p_{1,i}^{q_1}\cdots p_{k,i}^{q_k} \leqslant \sum_i p_{1,i}\cdots p_{k,i} \leqslant (\sum_i p_{1,i})\cdots(\sum_i p_{k,i}) = 1 = \sum_i p_{1,i}^{q_1}\cdots p_{k,i}^{q_k} r_i^{\beta(\mathbf{q})}$, we conclude that $\beta(\mathbf{q}) \leqslant 0$. It therefore follows from the above inequality that $\inf_{0<r} V(r) \geqslant \inf_{1\leqslant r} r^{\beta(\mathbf{q})} I_{\boldsymbol{\mu}}(1;\mathbf{q}) \geqslant I_{\boldsymbol{\mu}}(1;\mathbf{q}) = c$. We conclude from this that $r^{\beta(\mathbf{q})} I_{\boldsymbol{\mu}}(r;\mathbf{q}) = V(r) \geqslant c$ for all $0 < r$. This completes the proof. □

Proof of inequality (2.5) in Theorem 2.1. Inequality (2.5) follows immediately from Proposition 5.2. □

6. Further remarks

6.1. Connections with renewal theory

It is clear that the two key results used for proving (2.4) and (2.5), namely Theorems 4.2 and 5.1, have a renewal theoretical flavour. Indeed, the inequalities in Theorems 4.2 and 5.1 are very closely related to the so-called renewal equation, cf. [3, Chapter 7] or [9]. Renewal theoretical techniques have recently been used very successfully in fractal geometry, see, for example [3,8,9,13,17]. In particular, we note that Lau and Wang [13] and Olsen [17] have used renewal theory to study the L^q-spectra of self-similar measures satisfying the OSC. This suggests that a more careful analysis based on ideas from renewal theory might give estimates for the L^q-spectra that are more precise than those presented in this paper.

6.2. Modifying the function $M(r)$

Fix $\mathbf{q} = (q_1,\ldots,q_k) \in \mathbb{R}^k$. It is tempting to derive a version of the inequality in Theorem 4.2 for the function $M^{\mathbf{q}} : (0,\infty) \to \mathbb{R}$ defined by

$$M^{\mathbf{q}}(r) = \max_i \int \mu_i\big(B(x,r)\big)^{q_i-1}\, d\mu(x).$$

Such an inequality could subsequently be used for relating the asymptotic behaviour of $M^{\mathbf{q}}(r)$ (as $r \to 0$) to the number s_n defined by

$$1 = \max_{\substack{i\\ I\in\mathcal{I}_n}} \sum_{\mathbf{i}\in I} p_{i,\mathbf{i}}^{q_i} r_{\mathbf{i}}^{-s_n}.$$

This makes s_n (and hence s) a function of $\mathbf{q}$, and could possibly lead to finer upper bounds of the L^q-spectra. Unfortunately, we have not been able to do this. We will now explain this in more detail and discuss the problems that we encountered while attempting to do so. Recall, that we are seeking an inequality of the form

$$M^{\mathbf{q}}(r) \leqslant \max_{\substack{i\\ I\in\mathcal{I}_n}} \sum_{\mathbf{i}\in I} p_{i,\mathbf{i}}^{q_i} M^{\mathbf{q}}\left(\frac{r}{r_{\mathbf{i}}}\right). \tag{6.1}$$

Following the structure of the proof of Theorem 4.2, a proof of (6.1) might proceed as follows. Fix $0 < r < \delta_n$ (recall that δ_n is defined in Section 4) and $x \in K$. As in the proof of Theorem 4.2 we see that there is a point $t_{x,r} \in K$ such that

$$\big\{\mathbf{i} \in \Sigma^n \bigm| S_{\mathbf{i}}^{-1}\big(B(x,r)\big) \cap K \neq \emptyset\big\} \subseteq \big\{\mathbf{i} \in \Sigma^n \bigm| t_{x,r} \in K_{\mathbf{i}}\big\} = I_n(x,r); \tag{6.2}$$

observe that the point $t_{x,r}$ depends on x and r. Assuming that $q_i \geqslant 2$ for all i, and combining (4.1), (4.4) and (6.2), we now obtain

$$\begin{aligned} M^{\mathbf{q}}(r) &= \max_i \int \left(\sum_{\mathbf{i}\in\Sigma^n} p_{i,\mathbf{i}}\mu_i\big(S_{\mathbf{i}}^{-1}B(x,r)\big)\right)^{q_i-1} d\mu(x) \\ &\leqslant \max_i \int \sum_{\mathbf{i}\in\Sigma^n} p_{i,\mathbf{i}}\mu\big(S_{\mathbf{i}}^{-1}B(x,r)\big)^{q_i-1} d\mu(x) \quad \text{[by Jensen's inequality since } q_i \geqslant 2] \\ &= \max_i \int \sum_{\mathbf{i}\in I_n(x,r)} p_{i,\mathbf{i}}\mu_i\left(B\left(S_{\mathbf{i}}^{-1}x, \frac{r}{r_{\mathbf{i}}}\right)\right)^{q_i-1} d\mu(x). \end{aligned} \tag{6.3}$$

Ideally we would now like to interchange the integral sign and the summation sign in (6.3) in order to introduce the term $M^q(\frac{r}{r_{\mathbf{i}}})$. Unfortunately, we cannot do this since $I_n(x,r)$ depends on x.

Even if it was possible to interchange the integral and the summation in (6.3) it is not clear how to proceed. Indeed, assuming that it is possible to interchange sign the integral sign and the summation in (6.3), then we need to show that

$$\int \mu_i\left(B\left(S_{\mathbf{i}}^{-1}x, \frac{r}{r_{\mathbf{i}}}\right)\right)^{q_i-1} d\mu(x) \leqslant p_{i,\mathbf{i}}^{q_i-1} M^{\mathbf{q}}\left(\frac{r}{r_{\mathbf{i}}}\right),$$

or (at the very least) obtain estimates for the difference

$$\left|\int \mu_i\left(B\left(S_{\mathbf{i}}^{-1}x, \frac{r}{r_{\mathbf{i}}}\right)\right)^{q_i-1} d\mu(x) - p_{i,\mathbf{i}}^{q_i-1} M^{\mathbf{q}}\left(\frac{r}{r_{\mathbf{i}}}\right)\right|. \tag{6.4}$$

Unfortunately, this does not appear to be easy. For example, assuming the OSC and using very intricate and delicate arguments, Lalley [8,9], Lau and Wang [13] and Olsen [17] managed to provide estimates for the difference in (6.4). However, because of the complicated nature of these arguments together with the fact that they rely very heavily on the OSC, it is not clear if similar results can be obtained without assuming the OSC.

References

[1] M. Arbeiter, N. Patzschke, Random self-similar multifractals, Math. Nachr. 181 (1996) 5–42.
[2] L. Barreira, B. Saussol, Variational principles and mixed multifractal spectra, Trans. Amer. Math. Soc. 353 (2001) 3919–3944.
[3] K.J. Falconer, Techniques in Fractal Geometry, John Wiley & Sons, Chichester, 1997.
[4] D.-J. Feng, Gibbs properties of self-conformal measures and the multifractal formalism, Ergodic Theory Dynam. Systems 27 (2007) 787–812.
[5] D.-J. Feng, E. Olivier, Multifractal analysis of weak Gibbs measures and phase transition – application to some Bernoulli convolutions, Ergodic Theory Dynam. Systems 23 (2003) 1751–1784.
[6] J. Hutchinson, Fractals and self-similarity, Indiana Univ. Math. J. 30 (1981) 713–747.
[7] M. Kesseböhmer, M. Urbański, Higher-dimensional multifractal value sets for conformal infinite graph directed Markov systems, Nonlinearity 20 (2007) 1969–1985.
[8] S. Lalley, The packing and covering functions of some self-similar fractals, Indiana Univ. Math. J. 37 (1988) 699–710.
[9] S. Lalley, Probabilistic methods in certain counting problems of ergodic theory, in: T. Bedford, M. Keane, C. Series (Eds.), Ergodic Theory, Symbolic Dynamics, and Hyperbolic Spaces, Papers from the Workshop on Hyperbolic Geometry and Ergodic Theory held in Trieste, April 17–28, 1989, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1991, pp. 223–257.
[10] K.-S. Lau, S.-M. Ngai, Multifractal measures and a weak separation condition, Adv. Math. 141 (1999) 45–96.
[11] K.-S. Lau, S.-M. Ngai, L^q-spectrum of Bernoulli convolutions associated with P.V. numbers, Osaka J. Math. 36 (1999) 993–1010.
[12] K.-S. Lau, S.-M. Ngai, H. Rao, Iterated function systems with overlaps and self-similar measures, J. London Math. Soc. 63 (2001) 99–116.
[13] K.-S. Lau, J. Wang, Mean quadratic variations and Fourier asymptotics of self-similar measures, Monatsh. Math. 115 (1993) 99–132.
[14] M. Morán, Multifractal components of multiplicative set functions, Math. Nachr. 229 (2001) 129–160.
[15] L. Olsen, Mixed divergence points for self-similar measures, Indiana Univ. Math. J. 52 (2003) 1343–1372.
[16] L. Olsen, Mixed generalized dimensions of self-similar measures, J. Appl. Math. Anal. Appl. 306 (2005) 516–539.
[17] L. Olsen, Empirical multifractal moment measures and moment scaling functions of self-similar multifractals, Math. Proc. Cambridge Philos. Soc. 133 (2002) 459–485.
[18] B. Testud, Phase transitions for the multifractal analysis of self-similar measures, Nonlinearity 19 (2006) 1201–1207.