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Uniqueness of positive solutions for a boundary blow-up problem

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ABSTRACT

In this paper we prove the uniqueness of the positive solution for the boundary blow-up problem

$$\begin{cases} \Delta u = f(u) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases}$$

where Ω is a C^2 bounded domain in $\mathbb{R}^N$, under the hypotheses that $f(t)$ is nondecreasing in $t > 0$ and $f(t)/t^p$ is increasing for large t and some $p > 1$. We also consider the uniqueness of a related problem when the equation includes a nonnegative weight $a(x)$.

1. Introduction

In this work we deal with positive solutions to the boundary blow-up problem

$$\begin{cases} \Delta u = f(u) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, f a continuous function and by a solution we mean a function $u \in H^1_{loc}(\Omega) \cap C(\Omega)$ verifying the equation in (1.1) in the weak sense, and the boundary condition in the sense $u(x) \to \infty$ as $d(x) := \operatorname{dist}(x, \partial\Omega) \to 0$ (of course it is a consequence of standard regularity theory that $u \in C^{1,\alpha}(\Omega)$ for every $\alpha \in (0,1)$, but we are not needing this extra amount of regularity in what follows).

Problems like (1.1) have been largely dealt with in the recent years. They are usually known as boundary blow-up problems, and their solutions are sometimes termed as *large* solutions. We refer the interested reader to the by now classical papers by Bieberbach [2], Rademacher [23], Keller [16] and Osserman [22], and to the recent survey [24].

One of the earliest known features of problem (1.1) is that a necessary and sufficient condition for existence of solutions when f is increasing is the so-called Keller–Osserman condition:

$$\int_{t_0}^{\infty} \frac{d\tau}{\sqrt{F(\tau)}} < \infty \tag{1.2}$$

for some $t_0 > 0$, where $F(t) = \int_0^t f(\tau)\,d\tau$. See [16], [22] and also [10] where f is not necessarily increasing.

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However, the question of *uniqueness* is not completely understood. While it seems clear that the monotonicity of f is necessary (as the one-dimensional examples show), it is not even known if it is also sufficient for uniqueness to hold. At the best of our knowledge, some additional restrictions are needed.

As a matter of fact, the first important restriction deals with the boundary behavior of all possible solutions to (1.1), a question which is intimately connected to that of uniqueness. It is known in general that such boundary behavior is implicitly given by

$$\lim_{d(x)\to 0} \frac{\psi_0(u(x))}{d(x)} = 1,$$

for any positive solution u to (1.1), where the function ψ_0 is defined by

$$\psi_0(t) = \int_t^\infty \frac{d\tau}{\sqrt{2F(\tau)}}, \quad t > 0$$

(see Theorem 1.6 in [10]). But to obtain an explicit characterization for the solution u itself, some further restrictions are needed. For instance, a condition like

$$\liminf_{t\to\infty} \frac{\psi_0(\kappa t)}{\psi_0(t)} > 1 \quad \text{for every } \kappa \in (0,1), \tag{1.3}$$

implies

$$\lim_{d(x)\to 0} \frac{u(x)}{\phi_0(d(x))} = 1, \tag{1.4}$$

where ϕ_0 is the inverse function of ψ_0, which solves the one-dimensional problem $\phi_0'' = f(\phi_0)$ in $x > 0$ with $\phi_0(0) = \infty$. One simple condition on f to guarantee (1.3) is that $f(t)/t^p$ be increasing for some $p > 1$ and all large t. Under this condition it follows from (1.4) that for any two solutions u, v to (1.1) we have $u/v \to 1$ as $d(x) \to 0$. Then an additional monotonicity condition like

$$\frac{f(t)}{t} \text{ is increasing for } t > 0 \tag{1.5}$$

implies uniqueness (see [1]). Let us also mention that the monotonicity of $f(t)/t^p$ for large t and some $p > 1$ implies condition (1.2).

The first objective of the present work is to show that condition (1.5) is not necessary for uniqueness to hold. That is, it suffices to assume that f is increasing and there exists $p > 1$ such that $f(t)/t^p$ is increasing for large t. Our proof is inspired in [10], where the radial case was considered under the assumption that either $f(t)$ is convex or $f(t)/t$ is increasing for large t (although it can be shown that the former condition implies the latter at least for smooth functions, as long as the Keller–Osserman condition holds).

Our second aim is to extend the previously mentioned uniqueness result to slightly more general problems than (1.1), involving weights, namely

$$\begin{cases} \Delta u = a(x) f(u) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $a \in C(\overline{\Omega})$ is a nonnegative function, possibly vanishing on $\partial\Omega$. If this is the case then the profile of solutions near $\partial\Omega$ is affected, depending on the vanishing rate of a (see [4–7,9,13,14,18,19,21,25,26]).

However, the most interesting point regarding problem (1.6) in the present paper is that to prove that the quotient of any two solutions tends to one at the boundary we do not obtain the boundary behavior of each single solution, but proceed directly. This entails that no precise asymptotic profile needs to be assumed for the weight near $\partial\Omega$, only a growth condition, which will take the form

$$C_1 d(x)^\gamma \leqslant a(x) \leqslant C_2 d(x)^\gamma \quad \text{near } \partial\Omega,$$

with $\gamma \geqslant 0$, and C_1, C_2 positive constants. The price to pay is imposing an additional growth condition on f, relating f' and the primitive of $1/f$. This condition already appears in [12]. Precisely

$$\sigma := \lim_{t\to\infty} f'(t) \int_t^\infty \frac{d\tau}{f(\tau)} > 1. \tag{1.7}$$

It is fulfilled by a function f if for instance f' is asymptotic to a positive power at infinity, or more generally if f' is of regular variation at infinity with index $\rho \neq 0$ (see [5]). We remark that if the limit in (1.7) exists, it is always greater than or equal to one.

It is also important to mention that the correct growth of the solutions is determined in terms of the solution ϕ of the one-dimensional first-order equation $\phi' = -f(\phi)$ in $x > 0$ with $\phi(0) = \infty$, which is the inverse function of

$$\psi(t) = \int_t^\infty \frac{d\tau}{f(\tau)}.$$

In terms of this function, the growth of the solutions turns out to be essentially $\phi(d^{\gamma+2})$. This device goes back to [12].

Let us finally mention that our proof of the boundary behavior is a refinement of an iterative technique attributed to Safonov that we learned from [17], and has been further adapted and used in [3,11] or [8]. Also, some other related uniqueness results, whose main interest is put on weakening the smoothness of the boundary $\partial\Omega$, have been obtained in [20], by means of a different iterative technique.

We come now to the precise statement of our results. We begin with those pertinent to problem (1.1).

Theorem 1. *Assume f is continuous and nondecreasing in $\mathbb{R}^+$ with $f(0) = 0$. If there exist $p > 1$ and $t_0 > 0$ such that $f(t)/t^p$ is increasing for $t \geqslant t_0$, then problem* (1.1) *admits a unique positive solution, which in addition verifies*

$$\lim_{d(x)\to 0} \frac{u(x)}{\phi_0(d(x))} = 1.$$

As for problem (1.6) we only mention that condition (1.7) implies in particular that $f(t)/t^q$ is increasing for large t if $1 < q < \sigma/(\sigma - 1)$, and thus also condition (1.2) holds (see the details in Section 2, right after the statement of Theorem 3).

Theorem 2. *Assume f is continuous and nondecreasing in $\mathbb{R}^+$ with $f(0) = 0$. Assume moreover that f is differentiable for large t and* (1.7) *holds. If $a \in C(\overline{\Omega})$ is a nonnegative function which verifies*

$$C_1 d(x)^\gamma \leqslant a(x) \leqslant C_2 d(x)^\gamma \quad \text{in } \Omega_\eta,$$

for some constants $C_1, C_2 > 0$, $\gamma \geqslant 0$ and some small $\eta > 0$, where $\Omega_\eta := \{x \in \Omega\colon\ d(x) < \eta\}$, then problem (1.6) *admits a unique positive solution.*

One final word on the proofs: the existence of a positive solution to both problems (1.1) and (1.6) (more precisely, a minimal positive solution) is well known, see [16] and [12], respectively. Hence we are only showing uniqueness.

The paper is organized as follows: in Section 2 we prove that the quotients of two arbitrary positive solutions to (1.6) tends to one as the boundary is approached, while Section 3 is devoted to prove the uniqueness claimed in Theorems 1 and 2.

2. Asymptotic behavior of solutions

The aim of the present section is proving that any two positive solutions to (1.6) "agree" on the boundary of Ω. As has been already said in the introduction, this fact is nowadays well known for problem (1.1), and it usually follows because in that case it is possible to ascertain the boundary behavior of a single solution. However, the procedure we are following here will consist in directly comparing two solutions.

Let us precisely state the result we are going to prove.

Theorem 3. *Assume $f(t)$ is differentiable for large t and* (1.7) *holds. Then if u, v are positive solutions to* (1.6)*, we have*

$$\lim_{d(x)\to 0} \frac{u(x)}{v(x)} = 1.$$

Before coming to the proof of Theorem 3, let us quote two simple consequences of hypothesis (1.7). For fixed $t_1 > 0$ and $t > t_1$, we have, thanks to an integration by parts:

$$\int_{t_1}^t f'(s) \int_s^\infty \frac{d\tau}{f(\tau)} = f(t) \int_t^\infty \frac{d\tau}{f(\tau)} - f(t_1) \int_{t_1}^\infty \frac{d\tau}{f(\tau)} + t - t_1.$$

We divide by $t - t_1$ and let $t \to \infty$ to obtain

$$\lim_{t\to\infty} \frac{f(t)}{t} \int_t^\infty \frac{d\tau}{f(\tau)} = \sigma - 1 > 0 \tag{2.1}$$

(notice in passing that this also shows that in general $\sigma \geqslant 1$). Finally, we observe that (1.7) also implies that $f(t)/t^q$ is increasing for large t if $1 < q < \sigma/(\sigma-1)$. Indeed, notice that

$$\left(f'(t) - q\frac{f(t)}{t}\right)\psi(t) \to \sigma - q(\sigma-1) > 0 \quad \text{as } t \to \infty.$$

Hence $(f(t)/t^q)' = t^{-q}(f'(t) - qf(t)/t) > 0$ for large t, as claimed.

Let us return now to Theorem 3. The first step in the proof consists in obtaining bounds with the right growth near the boundary for the solutions.

Lemma 4. *Assume $f(t)$ is differentiable for large t and* (1.7) *holds. Let u be a positive solution to* (1.6)*. Then there exist positive numbers η, ε_0 and K_0 such that*

$$\frac{1}{2}\phi\big(Kd(x)^{\gamma+2}\big) \leqslant u(x) \leqslant 2\phi\big(\varepsilon d(x)^{\gamma+2}\big) \quad \text{in } \Omega_\eta, \tag{2.2}$$

whenever $0 < \varepsilon < \varepsilon_0$, $K > K_0$.

Proof. Since $\partial\Omega$ is of class C^2, it is well known that the distance function $d(x)$ is C^2 in Ω_η for some small positive η, where it verifies $|\nabla d| = 1$ (cf. for instance [15]). For small $\delta > 0$, define

$$\bar{u}_\delta = \phi\big(\varepsilon(d-\delta)^{\gamma+2}\big), \quad x \in \Omega_\eta^\delta,$$

where $\varepsilon > 0$ is to be chosen small and $\Omega_\eta^\delta = \{x \in \Omega\colon\ \delta < d(x) < \eta\}$. We claim that $\bar{u}_\delta$ is a supersolution in Ω_η^δ for small enough ε, independent of δ. Indeed, a calculation shows that

$$\begin{aligned}\Delta\bar{u}_\delta &= \varepsilon^2 f'\big(\phi\big(\varepsilon(d-\delta)^{\gamma+2}\big)\big) f\big(\phi\big(\varepsilon(d-\delta)^{\gamma+2}\big)\big)(\gamma+2)^2(d-\delta)^{2\gamma+2} \\ &\quad - \varepsilon f\big(\phi\big(\varepsilon(d-\delta)^{\gamma+2}\big)\big)(\gamma+2)(\gamma+1)(d-\delta)^{\gamma} - \varepsilon f\big(\phi\big(\varepsilon(d-\delta)^{\gamma+2}\big)\big)(\gamma+2)(d-\delta)^{\gamma+1}\Delta d.\end{aligned}$$

Thus, taking into account that $a(x) \geqslant C_1 d(x)^\gamma \geqslant C_1(d(x)-\delta)^\gamma$ in Ω_δ^η, $\bar{u}_\delta$ will be a supersolution provided that

$$\varepsilon^2 f'\big(\phi\big(\varepsilon(d-\delta)^{\gamma+2}\big)\big)(\gamma+2)^2(d-\delta)^{\gamma+2} - \varepsilon(\gamma+2)(\gamma+1) + \varepsilon(\gamma+2)(d-\delta)\sup|\Delta d| \leqslant C$$

where C is a positive constant, whose exact value from now on will be unimportant. It can be easily seen that this inequality holds for small ε and all δ whenever

$$f'\big(\phi\big(\varepsilon(d-\delta)^{\gamma+2}\big)\big)\varepsilon(d-\delta)^{\gamma+2} \leqslant C,$$

which is a direct consequence of

$$\limsup_{t\to 0} f'\big(\phi(t)\big)t < \infty. \tag{2.3}$$

Finally observe that (2.3) is implied by hypothesis (1.7) on f. Thus $\bar{u}_\delta$ is a supersolution in Ω_η^δ if ε is small enough. It is also clear that $\bar{u}_\delta = \infty$ when $d = \delta$. If z denotes the unique solution to $-\Delta z = 1$ in Ω with $z = 0$ on $\partial\Omega$, we have that the function $\bar{u}_\delta + Mz$ is also a supersolution in Ω_η^δ for $M > 0$, which blows up on $d = \delta$.

Next let u be a positive solution to (1.6). We may select a large M so that $\bar{u}_\delta + Mz \geqslant u$ on $d = \eta$. Since $u < \infty$ on $d = \delta$, it follows by comparison that $u \leqslant \bar{u}_\delta + Mz$ in Ω_η^δ. Letting $\delta \to 0$, we arrive at

$$u(x) \leqslant \phi\big(\varepsilon d(x)^{\gamma+2}\big) + Mz(x) \quad \text{in } \Omega_\eta,$$

and the upper inequality in (2.2) follows since $Mz \leqslant \phi(\varepsilon d(x)^{\gamma+2})$, taking ε possibly smaller.

To prove the lower inequality we similarly check that $\underline{u}_\delta = \phi(K(d+\delta)^{\gamma+2})$ is a subsolution in Ω_η. The reasoning in this case is not completely symmetric. Indeed, the condition $\underline{u}_\delta$ needs to verify to be a subsolution is

$$K\big(f'\big(\phi\big(K(d+\delta)^{\gamma+2}\big)\big)(\gamma+2)^2K(d+\delta)^{\gamma+2} - (\gamma+2)(\gamma+1) - (\gamma+2)(d+\delta)\sup|\Delta d|\big) \geqslant C. \tag{2.4}$$

We now notice that from hypothesis (1.7) it immediately follows that

$$\sigma = \liminf_{t\to 0} f'\big(\phi(t)\big)t > 1$$

so that for $\tilde{\sigma}$ verifying $1 < \tilde{\sigma} < \sigma$, there exists t_0 such that $f'(\phi(t))t > \tilde{\sigma}$ if $t \leqslant t_0$. Hence (2.4) will hold provided we have $K(d+\delta)^{\gamma+2} \leqslant t_0$ and

$$K\big(\tilde{\sigma}(\gamma+2)^2 - (\gamma+2)(\gamma+1) - (\gamma+2)(d+\delta)\sup|\Delta d|\big) \geqslant C. \tag{2.5}$$

Thus we may choose and fix a large value of K, say K_0, and small η and δ so that (2.5) holds and then diminish η and δ if necessary so that $K_0(d+\delta)^{\gamma+2} \leqslant t_0$ if $d < \eta$.

Hence $\underline{u}_\delta$ is a subsolution in Ω_η, and $\underline{u}_\delta - Mz$ also is for $M > 0$. A comparison as in the first part of the proof shows that if u is an arbitrary solution to (1.6) then $u \geqslant \underline{u}_\delta - Mz$ in Ω_η if M is large enough. Letting $\delta \to 0$, the proof of the lower inequality in (2.2) concludes as before. □

Our next task will be to show that with no further conditions on f, the bounds given in (2.2) suffice to guarantee that the quotient of any two solutions to (1.6) is bounded.

Lemma 5. *Assume $f(t)$ is differentiable for large t and* (1.7) *holds, and let u, v be positive solutions to* (1.6)*. Then the quotient $\frac{u}{v}$ is bounded in Ω.*

Proof. Thanks to Lemma 4 it suffices to prove that

$$\frac{\phi(Bt)}{\phi(At)} \leqslant C \tag{2.6}$$

for sufficiently small t if $0 < B < A$. Notice that for $M > 1$ and sufficiently large s we have, for every q such that $1 < q < \sigma/(\sigma-1)$:

$$\psi(Ms) = \int_{Ms}^{\infty} \frac{d\tau}{f(\tau)} = M \int_{s}^{\infty} \frac{d\tau}{f(M\tau)} \leqslant M^{-(q-1)} \int_{s}^{\infty} \frac{d\tau}{f(\tau)} = M^{-(q-1)}\psi(s) \tag{2.7}$$

since $f(t)/t^q$ is increasing for large t. Thus we can find M such that for large s,

$$\frac{\psi(Ms)}{\psi(s)} \leqslant \frac{B}{A}.$$

This implies $\phi(Bt/A) \leqslant M\phi(t)$, and substituting t by At we obtain (2.6) for small positive t. This concludes the proof. □

Remark 1. We mention for later use that (2.7) implies in particular $\phi(\lambda t) \leqslant \lambda^{-\frac{1}{q-1}}\phi(t)$ for all $\lambda \in (0,1)$ and sufficiently small t.

We finally proceed to the actual proof of Theorem 3. It will be by contradiction, and is inspired in an argument in [17] (see also [8]).

Proof of Theorem 3. Let u, v arbitrary positive solutions to (1.6). According to Lemma 5,

$$\theta := \limsup_{d(x)\to 0} \frac{u(x)}{v(x)}$$

is finite. To prove the theorem it suffices to show that $\theta \leqslant 1$, since reversing the roles of u and v we would then get that the lim inf is greater than or equal to one.

Thus assume $\theta > 1$. Given a small $\varepsilon > 0$ so that $\theta - \varepsilon > 1$, there exists $\delta > 0$ such that

$$\frac{u(x)}{v(x)} \leqslant (\theta+\varepsilon) \quad \text{if } d(x) < \delta, \tag{2.8}$$

and x_0 with $d(x_0) < 2\delta/3$ verifying $u(x_0) > (\theta-\varepsilon)v(x_0)$. By diminishing δ if necessary, we may assume $v \geqslant t_0$ in Ω_δ, where t_0 is such that $f(t)/t^q$ is increasing for $t \geqslant t_0$ and some $q \in (1, \sigma/(\sigma-1))$. Define

$$D = \{x \in \Omega\colon\ u(x) > (\theta-\varepsilon)v(x)\} \cap B_r(x_0),$$

where $r = d(x_0)/2$. In the set D we have

$$\begin{aligned}\Delta\big(u-(\theta-\varepsilon)v\big) &= a(x)\big(f(u)-(\theta-\varepsilon)f(v)\big) \geqslant a(x)\big(f\big((\theta-\varepsilon)v\big)-(\theta-\varepsilon)f(v)\big)\\ &\geqslant a(x)\big((\theta-\varepsilon)^q-(\theta-\varepsilon)\big)f(v) \geqslant Cd^\gamma(\theta-\varepsilon)f(v)\end{aligned}$$

where C is a positive constant which can be taken independently of ε. Now thanks to (2.2), and noticing that $r \leqslant d \leqslant 3r$ in D, we have $v \geqslant \frac{1}{2}\phi(Kd^{\gamma+2}) \geqslant \frac{1}{2}\phi(Cr^{\gamma+2})$ in D, so that

$$\Delta\big(u-(\theta-\varepsilon)v\big) \geqslant C(\theta-\varepsilon)f\left(\frac{1}{2}\phi\big(Cr^{\gamma+2}\big)\right)r^\gamma \quad \text{in } D. \tag{2.9}$$

Now let z_r be the unique solution to $-\Delta z = 1$, in $B_r(x_0)$, with $z = 0$ on $\partial B_r(x_0)$. It is worth noticing that $z_r(x) = C(r^2 - |x - x_0|^2)$, for a constant which only depends on N. According to (2.9), we have $\Delta(u - (\theta - \varepsilon)v + \kappa z_r) \geqslant 0$ in D, where $\kappa = C(\theta - \varepsilon) f(\frac{1}{2}\phi(Cr^{\gamma+2}))r^{\gamma}$. Then, the maximum principle implies the existence of $y_0 \in \partial D$ such that

$$u(x_0) - (\theta - \varepsilon)v(x_0) + \kappa z_r(x_0) \leqslant u(y_0) - (\theta - \varepsilon)v(y_0) + \kappa z_r(y_0). \tag{2.10}$$

If we have $y_0 \in B_r(x_0)$, then $u(y_0) = (\theta - \varepsilon)v(y_0)$, and (2.10) gives the contradiction $z_r(x_0) < z_r(y_0)$. Hence $y_0 \in \partial B_r(x_0)$, and in particular

$$C(\theta - \varepsilon) f\left(\frac{1}{2}\phi\left(Cr^{\gamma+2}\right)\right) r^{\gamma+2} \leqslant u(y_0) - (\theta - \varepsilon)v(y_0). \tag{2.11}$$

We now notice that condition (2.1) implies that

$$f\left(\frac{1}{2}\phi\left(Cr^{\gamma+2}\right)\right) r^{\gamma+2} \geqslant C\phi\left(Cr^{\gamma+2}\right),$$

and since $d(y_0) \geqslant r$, using again (2.2) we derive from (2.11) the inequality $u(y_0) - (\theta - \varepsilon)v(y_0) \geqslant C(\theta - \varepsilon)\phi(Cd(y_0)^{\gamma+2})$, which thanks to Remark 1 implies

$$u(y_0) \geqslant (1 + C)(\theta - \varepsilon)v(y_0). \tag{2.12}$$

Finally, observing that $d(y_0) \leqslant 3d(x_0)/2 < \delta$, we obtain from (2.12) and (2.8) the inequality $(\theta + \varepsilon) \geqslant (1 + C)(\theta - \varepsilon)$. After letting $\varepsilon \to 0$, we arrive at a clear contradiction.

Thus, our initial assumption $\theta > 1$ is incorrect, and we have $\theta \leqslant 1$. This concludes the proof. □

3. Uniqueness

In this section we prove our main results. We notice that, once we have shown that the quotient of two positive solutions approaches one on the boundary, no differences arise between the proof of both theorems. Thus we only consider that of Theorem 2. As we already said in the introduction, we focus on uniqueness.

Proof of Theorem 2. Let u be the minimal solution to (1.6). We are going to prove that for any other positive solution v to (1.6) we have $u = v$. Observe first that $u/v \to 1$ as $x \to \partial\Omega$, thanks to Theorem 3.

For small $\varepsilon > 0$, let $w = w_\varepsilon = (1 + \varepsilon)u$ and define the (open) set

$$D_\varepsilon = \left\{x \in \Omega \colon\ w(x) < v(x)\right\}.$$

We may assume that D_ε is nonempty for some small enough ε, for otherwise there is nothing to prove. Indeed, notice that D_ε monotonically increases as $\varepsilon \downarrow 0$. Moreover, we may also assume that $D_\varepsilon \to \Omega$ as $\varepsilon \downarrow 0$, for if there exist $x \in \Omega$ and a sequence $\varepsilon_n \to 0$ such that $x \notin D_{\varepsilon_n}$ for all n, we have $(1 + \varepsilon_n)u(x) \geqslant v(x)$, and hence $u(x) = v(x)$. The strong maximum principle then yields $u \equiv v$. Finally, notice that $D_\varepsilon \Subset \Omega$, since the quotient u/v tends to 1 as we approach the boundary.

Now choose $\eta > 0$ so that $u \geqslant t_0$ in Ω_η, where t_0 is such that $f(t)/t$ is increasing for $t \geqslant t_0$ (this condition automatically holds if $f(t)/t^q$ is increasing for some $q > 1$ and large t). Define $D_{\varepsilon,\eta} = D_\varepsilon \cap \Omega_\eta$, and notice that $D_{\varepsilon,\eta}$ is a nonempty open set for small ε. Moreover, in $D_{\varepsilon,\eta}$, we have

$$\Delta(v - w) = a(x)\left(f(v) - (1 + \varepsilon)f(u)\right) \geqslant a(x)\left(f(v) - f(w)\right) \geqslant 0,$$

so that, thanks to the maximum principle

$$v - w \leqslant \max_{\partial D_{\varepsilon,\eta}}(v - w) \quad \text{in } D_{\varepsilon,\eta}.$$

Now notice that $\partial D_{\varepsilon,\eta} = (\partial D_\varepsilon \cap \Omega_\eta) \cup (D_\varepsilon \cap \partial\Omega_\eta)$, and the maximum of $v - w$ cannot be achieved on ∂D_ε, for this would imply $v - w \leqslant 0$ in $D_{\varepsilon,\eta}$, which is impossible by its definition. Thus it is achieved on $D_\varepsilon \cap \partial\Omega_\eta = D_\varepsilon \cap \{x\colon\ d(x) = \eta\}$, since $D_\varepsilon \cap \partial\Omega = \emptyset$. Hence

$$v - w \leqslant \max_{D_\varepsilon \cap \{d=\eta\}}(v - w) \quad \text{in } D_{\varepsilon,\eta}. \tag{3.1}$$

Letting $\varepsilon \to 0$ in (3.1) we arrive at

$$v - u \leqslant \max_{d=\eta}(v - u) := \theta \quad \text{in } \Omega_\eta. \tag{3.2}$$

On the other hand, since u is the minimal solution to (1.6) we have $u \leqslant v$ and because $f(t)$ is increasing in $t > 0$, it follows that $\Delta(v - u) = a(x)(f(v) - f(u)) \geqslant 0$ in $\Omega^\eta := \{x \in \Omega\colon\ d(x) > \eta\}$. The maximum principle implies that $v - u \leqslant \theta$ in Ω^η, and hence $v - u \leqslant \theta$ throughout Ω. But then the strong maximum principle gives $v - u \equiv \theta$. When plugged in the equations satisfied by u and v we obtain that $f(u) = f(u + \theta)$ in Ω, which can only hold if $\theta = 0$. Thus $u \equiv v$, and this shows uniqueness. □

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