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Exponential energy decay of solutions for an integro-differential equation with strong damping

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ABSTRACT

The initial boundary value problem for an integro-differential equation with strong damping in $\Omega \times (0, \infty)$:

$$u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + \int_0^t g(t-s)\Delta u(s)\,ds - \Delta u_t = f(u),$$

is considered, where Ω is a bounded domain with a smooth boundary $\partial\Omega$. The asymptotic behavior of solutions is discussed under some conditions on g. Decay estimates of the energy function of solutions are also given, via an integral inequality introduced by Komornik (1994) [11].

1. Introduction

In this paper we consider the initial boundary value problem for the following nonlinear integro-differential equation:

$$u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + \int_0^t g(t-s)\Delta u(s)\,ds - \Delta u_t = f(u), \quad \text{in } \Omega \times (0, \infty), \tag{1.1}$$

with initial conditions

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

and boundary condition

$$u(x,t) = 0, \quad x \in \partial\Omega,\ t \geqslant 0, \tag{1.3}$$

where $\Omega \subset R^N$, $N \geqslant 1$, is a bounded domain with a smooth boundary $\partial\Omega$ so that Divergence theorem can be applied. Here, M is a positive C^1-function and g represents the kernel of the memory term they will be specified later (see assumptions (A1), (A2)). f is a nonlinear function like $f(u) = |u|^{p-2}u$, $p > 2$.

Before going further, Eq. (1.1) without the viscoelastic term, that is $g \equiv 0$, for the case that $M \equiv 1$, Eq. (1.1) becomes a nonlinear wave equation which has been extensively studied and several results concerning existence and nonexistence have been established [1,3,4,8,10,12,14]. When M is not a constant function, a special case of Eq. (1.1) is Kirchhoff equation which has been introduced in order to describe the nonlinear vibrations of an elastic string. Kirchhoff [9] was the first one

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to study the oscillations of stretched strings and plates. In this case the existence and nonexistence of solutions have been discussed by many authors and the references cited therein [5,6,16–20].

For Eq. (1.1) with $g \neq 0$, in the case that $M \equiv 1$, Eq. (1.1) becomes a semilinear viscoelastic equation. Cavalcanti et al. [2] treated (1.1) with damping term $a(x)u_t$, here $a(x)$ may be null on a part of the boundary. By assuming the kernel g in the memory term decays exponentially, they obtained an exponentially decay rate of the energy. On the other hand, Jiang and Rivera [7] proved, in the framework of nonlinear viscoelasticity, the exponential decay of the energy provided that the kernel g decays exponentially without imposing damping term. In the case that M is not a constant function, Eq. (1.1) is a model to describe the motion of deformable solids as hereditary effect is incorporated. This equation was first studied by Torrejon and Young [23] who proved the existence of weakly asymptotic stable solution for large analytical datum. Later, Rivera [15] showed the existence of global solutions for small datum and the total energy decays to zero exponentially under some restrictions. Recently, Wu and Tsai [22] discuss the global as well as energy decay of Eq. (1.1) for f is a power like function. In that paper, the following assumption on the nonnegative kernel $g'(t) \leqslant -rg(t)$, $\forall t \geqslant 0$ for some $r > 0$ was assumed. This motivates us to consider the problem of how to obtain the energy decay of the solution when we replace the above assumption by $g'(t) \leqslant 0$, for $t \geqslant 0$.

In this paper we show that under some conditions on g the solution is global in time and the energy decays exponentially. In this way, we can extend the result of [22] to a weaker condition on g and the result of [24] to nonconstant $M(s)$. The content of this paper is organized as follows. In Section 2, we give some lemmas and assumptions which will be used later, and then state the local existence Theorem 2.3. In Section 3, we first define an energy function $E(t)$ in (3.4) and show that it is a nonincreasing function of t. Then, the results of global existence and decay properties of the solutions of (1.1)–(1.3) are given in Theorem 3.5.

2. Preliminary results

In this section, we shall give some lemmas and assumptions which will be used throughout this work. Let $W^{m,p}(\Omega)$ be the usual Sobolev space. Specially, $W^{m,2}(\Omega)$ and $W^{0,p}(\Omega)$ will be marked by $H^m(\Omega)$ and $L^p(\Omega)$, respectively. And we denote $\|\cdot\|_p$ to be L^p-norm for $1 \leqslant p \leqslant \infty$. $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{H_0^1} = \|\nabla u\|_2$.

Lemma 2.1. *(See Sobolev–Poincaré inequality [13].) If $2 \leqslant p \leqslant \frac{2N}{N-2}$, then*

$$\|u\|_p \leqslant c_s \|\nabla u\|_2,$$

for $u \in H_0^1(\Omega)$ holds with some constant c_s.

Lemma 2.2. *(See [11].) Let $h : [0,\infty) \to [0,\infty)$ be a nonincreasing function and assume that there exists a constant $r > 0$ such that*

$$\int_t^\infty h(s)\,ds \leqslant rh(t), \quad \forall t \in [0,\infty).$$

Then we have

$$h(t) \leqslant h(0)\exp\left(1 - \frac{t}{r}\right), \quad \forall t \geqslant r.$$

We state the general hypotheses on M, g and f:

(A1) $M(s)$ is a positive C^1-function for $s \geqslant 0$ satisfying $M(s) = m_0 + bs^\gamma$, $m_0 > 0$, $b \geqslant 0$, $\gamma \geqslant 1$ and $s \geqslant 0$.
(A2) $g \in C^1([0,\infty))$ is a nonnegative and nonincreasing function satisfying

$$m_0 - \int_0^\infty g(s)\,ds = l > 0. \tag{2.1}$$

(A3) $f(0) = 0$ and there is a positive constant k_1 such that

$$\left|f(u) - f(v)\right| \leqslant k_1|u - v|\left(|u|^{p-2} + |v|^{p-2}\right),$$

for $u, v \in R$ and $2 < p \leqslant \frac{2(N-1)}{N-2}$ (∞, if $N \leqslant 2$).

Now, we are ready to state the local existence of problem (1.1)–(1.3), whose proof can be found in [21,22].

Theorem 2.3. *Suppose that* (A1), (A2) *and* (A3) *hold, and that $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in L^2(\Omega)$, then there exists a unique solution u of* (1.1)–(1.3) *satisfying*

$$u \in C\left([0,T]; H_0^1(\Omega) \cap H^2(\Omega)\right) \quad \textit{and} \quad u_t \in C\left([0,T]; L^2(\Omega)\right) \cap L^2\left([0,T]; H_0^1(\Omega)\right).$$

Moreover, at least one of the following statements holds true:

(i) $T = \infty$,
(ii) $\|u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2 \to \infty$ *as* $t \to T^-$.

3. Global existence and energy decay

In this section, we consider the global existence and energy decay of solutions for a kind of problem (1.1) with initial and boundary conditions (1.2) and (1.3), respectively:

$$u_{tt} - M\big(\|\nabla u\|_2^2\big)\Delta u + \int_0^t g(t-s)\Delta u(s)\,ds - \Delta u_t = |u|^{p-2}u, \tag{3.1}$$

where $2 < p \leqslant \frac{2(N-1)}{N-2}$ and $M(s) = m_0 + bs^\gamma$, $m_0 > 0$, $b \geqslant 0$, $\gamma \geqslant 1$ and $s \geqslant 0$. In order to state our results, we define

$$I(t) \equiv I\big(u(t)\big) = \left(m_0 - \int_0^t g(s)\,ds\right)\big\|\nabla u(t)\big\|_2^2 + (g \diamond \nabla u)(t) - \big\|u(t)\big\|_p^p, \tag{3.2}$$

$$\begin{aligned} J(t) \equiv J\big(u(t)\big) = {} & \frac{1}{2}\left(m_0 - \int_0^t g(s)\,ds\right)\big\|\nabla u(t)\big\|_2^2 + \frac{1}{2}(g \diamond \nabla u)(t) \\ & + \frac{b}{2(\gamma+1)}\big\|\nabla u(t)\big\|_2^{2(\gamma+1)} - \frac{1}{p}\big\|u(t)\big\|_p^p, \end{aligned} \tag{3.3}$$

and the energy function by

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + J(t), \tag{3.4}$$

where

$$(g \diamond \nabla u)(t) = \int_0^t g(t-s)\int_\Omega \big|\nabla u(s) - \nabla u(t)\big|^2\,dx\,ds,$$

for $u(t) \in H_0^1(\Omega)$, $t \geqslant 0$.

Remark 3.1. From the definition of $E(t)$ by (3.4), $m_0 - \int_0^t g(s)\,ds \geqslant m_0 - \int_0^\infty g(s)\,ds = l$ by (A2) and by Lemma 2.1, we have

$$\begin{aligned} E(t) & \geqslant \frac{1}{2}\left(m_0 - \int_0^t g(s)\,ds\right)\big\|\nabla u(t)\big\|_2^2 + \frac{1}{2}(g \diamond \nabla u)(t) - \frac{1}{p}\|u\|_p^p \\ & \geqslant \frac{1}{2}\big(l\big\|\nabla u(t)\big\|_2^2 + (g \diamond \nabla u)(t)\big) - \frac{B_1^p l^{\frac{p}{2}}}{p}\|\nabla u\|_2^p \\ & \geqslant G\Big[\big(l\big\|\nabla u(t)\big\|_2^2 + (g \diamond \nabla u)(t)\big)^{\frac{1}{2}}\Big], \end{aligned} \tag{3.5}$$

$t \geqslant 0$, where

$$G(\lambda) = \frac{1}{2}\lambda^2 - \frac{B_1^p}{p}\lambda^p \quad \text{and} \quad B_1 = \frac{c_s}{\sqrt{l}}.$$

It is easy to verify that $G(\lambda)$ has a maximum at $\lambda_1 = B_1^{-\frac{p}{p-2}}$ and the maximum value is

$$E_1 = \frac{p-2}{2p}B_1^{-\frac{2p}{p-2}}.$$

Before we prove our main result, we need the following lemmas.

Lemma 3.2. *(See [22].) $E(t)$ is a nonincreasing function on $[0,T)$ and*

$$E'(t) = -\|\nabla u_t\|_2^2 + \frac{1}{2}\big(g' \diamond \nabla u\big)(t) - \frac{1}{2}g(t)\big\|\nabla u(t)\big\|_2^2. \tag{3.6}$$

Lemma 3.3. *Let u be the solution of* (3.1), (1.2) *and* (1.3) *with initial data satisfy $E(0) < E_1$ and $l^{\frac{1}{2}}\|\nabla u_0\|_2 < \lambda_1$, then*

$$\big(l\big\|\nabla u(t)\big\|_2^2 + (g \diamond \nabla u)(t)\big)^{\frac{1}{2}} < \lambda_1, \tag{3.7}$$

for $t \in [0,T)$.

Proof. From the definition of $G(\lambda)$, we see that $G(\lambda)$ is increasing in $(0,\lambda_1)$ and decreasing in (λ_1,∞), and $G(\lambda) \to -\infty$, as $\lambda \to \infty$. We establish (3.7) by contradiction. Suppose (3.7) does not hold, then, it follows from the continuity of $u(t)$ that there exists $t_0 \in (0,T)$ such that

$$\big(l\big\|\nabla u(t_0)\big\|_2^2 + (g \diamond \nabla u)(t_0)\big)^{\frac{1}{2}} = \lambda_1.$$

By (3.5), we observe that

$$\begin{aligned} E(t_0) &\geqslant G\big[\big(l\big\|\nabla u(t_0)\big\|_2^2 + (g \diamond \nabla u)(t_0)\big)^{\frac{1}{2}}\big] \\ &= G(\lambda_1) \\ &= E_1, \end{aligned}$$

which contradicts $E(t) \leqslant E(0) < E_1$, $t \geqslant 0$. Hence, we get

$$\big(l\big\|\nabla u(t)\big\|_2^2 + (g \diamond \nabla u)(t)\big)^{\frac{1}{2}} < \lambda_1,$$

for $t \in [0,T)$. $\square$

Remark 3.4. From (3.7) and $\lambda_1 = B_1^{-\frac{p}{p-2}} = l^{\frac{p}{2(p-2)}} c_s^{\frac{-p}{p-2}}$, we have $l\|\nabla u(t)\|_2^2 \leqslant l\|\nabla u(t)\|_2^2 + (g \diamond \nabla u)(t) < \lambda_1^2 = l^{\frac{p}{p-2}} c_s^{\frac{-2p}{p-2}}$, which implies that

$$\begin{aligned} I(t) &= \left(m_0 - \int_0^t g(s)\,ds\right)\big\|\nabla u(t)\big\|_2^2 + (g \diamond \nabla u)(t) - \big\|u(t)\big\|_p^p \\ &\geqslant \left(m_0 - \int_0^\infty g(s)\,ds\right)\big\|\nabla u(t)\big\|_2^2 + (g \diamond \nabla u)(t) - \big\|u(t)\big\|_p^p \\ &\geqslant l\big\|\nabla u(t)\big\|_2^2 - c_s^p\big\|\nabla u(t)\big\|_2^p \\ &\geqslant 0. \end{aligned}$$

Further, by (3.2) and (3.3), we obtain

$$\begin{aligned} J(t) &\geqslant \frac{1}{2}\left(m_0 - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(g \diamond \nabla u)(t) - \frac{1}{p}\|u\|_p^p \\ &= \frac{p-2}{2p}\left[\left(m_0 - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + (g \diamond \nabla u)(t)\right] + \frac{1}{p}I(t), \end{aligned}$$

from which, the assumption (A2), the definition of $E(t)$ by (3.4) and $E(t) \leqslant E(0)$ by Lemma 3.2, we deduce that

$$l\|\nabla u\|_2^2 \leqslant \left(m_0 - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 \leqslant \frac{2p}{p-2}J(t) \leqslant \frac{2p}{p-2}E(t) \leqslant \frac{2p}{p-2}E(0). \tag{3.8}$$

In addition, it follows from Lemma 2.1 and (3.8) that

$$\|u\|_p^p \leqslant c_s^p\|\nabla u\|_2^p \leqslant \frac{c_s^p}{l}\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{p-2}{2}} l\|\nabla u\|_2^2 = \alpha l\|\nabla u\|_2^2 \leqslant \frac{2p\alpha}{p-2}E(t), \quad \text{for } t \in [0,T), \tag{3.9}$$

where $\alpha = \frac{c_s^p}{l}(\frac{2p}{l(p-2)}E(0))^{\frac{p-2}{2}}$. Here, we also note that $E(0) < E_1$ if and only if

$$\alpha = \frac{c_s^p}{l}\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{p-2}{2}} < 1. \tag{3.10}$$

Theorem 3.5 *(Global existence and energy decay). Suppose that* (A2), $E(0) < E_1$ *and* $l^{\frac{1}{2}}\|\nabla u_0\|_2 < \lambda_1$ *hold, then problem* (3.1), (1.2) *and* (1.3) *admits a global solution u if $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and $u_1 \in L^2(\Omega)$. Furthermore, if $E(0)$ is small enough and*

$$m_0 > \left(1 + \frac{5p}{2(p-2)}\right)\int_0^\infty g(s)\,ds, \tag{3.11}$$

then we have the following decay estimates:

$$E(t) \leqslant E(0)e^{1-\tau_1 t} \quad \text{on } [0,\infty),$$

where τ_1 is given in (3.29).

Proof. First, we show that $T = \infty$. Multiplying (3.1) by $-2\Delta u$, and integrating it over Ω, we get

$$\frac{d}{dt}\left\{\|\Delta u\|_2^2 - 2\int_\Omega u_t \Delta u\,dx\right\} + 2M\big(\|\nabla u\|_2^2\big)\|\Delta u\|_2^2$$
$$\leqslant 2\|\nabla u_t\|_2^2 - 2\int_\Omega |u|^{p-2}u\Delta u\,dx + 2\int_0^t g(t-\tau)\int_\Omega \Delta u(\tau)\Delta u(t)\,dx\,d\tau.$$

Since

$$\int_0^t g(t-\tau)\int_\Omega \Delta u(\tau)\Delta u(t)\,dx\,d\tau \leqslant 2\eta\big\|\Delta u(t)\big\|_2^2 + \frac{\|g\|_{L^1}}{2\eta}\int_0^t g(t-\tau)\big\|\Delta u(\tau)\big\|_2^2\,d\tau,$$

we obtain

$$\frac{d}{dt}\left\{\|\Delta u\|_2^2 - 2\int_\Omega u_t \Delta u\,dx\right\} + \big(2M\big(\|\nabla u\|_2^2\big) - 2\eta\big)\|\Delta u\|_2^2 \leqslant 2\|\nabla u_t\|_2^2 + \frac{\|g\|_{L^1}}{2\eta}\int_0^t g(t-\tau)\big\|\Delta u(\tau)\big\|_2^2\,d\tau$$
$$- 2\int_\Omega |u|^{p-2}u\Delta u\,dx, \tag{3.12}$$

where $0 < \eta \leqslant \frac{\|g\|_{L^1}}{2}$. Multiplying (3.12) by ε, $0 < \varepsilon \leqslant 1$, and multiplying (3.6) by 2, and then adding them together, we obtain

$$\frac{d}{dt}E^*(t) + 2(1-\varepsilon)\|\nabla u_t\|_2^2 + 2\varepsilon\big(M\big(\|\nabla u\|_2^2\big) - \eta\big)\|\Delta u\|_2^2$$
$$\leqslant -2\varepsilon\int_\Omega |u|^{p-2}u\Delta u\,dx + \varepsilon\frac{\|g\|_{L^1}}{2\eta}\int_0^t g(t-\tau)\big\|\Delta u(\tau)\big\|_2^2\,d\tau, \tag{3.13}$$

where

$$E^*(t) = 2E(t) - 2\varepsilon\int_\Omega u_t\Delta u\,dx + \varepsilon\|\Delta u\|_2^2.$$

By Young's inequality, we get

$$\left|2\varepsilon\int_\Omega u_t\Delta u\,dx\right| \leqslant 2\varepsilon\|u_t\|_2^2 + \frac{\varepsilon}{2}\|\Delta u\|_2^2.$$

Noting that $J(t) \geqslant 0$ by Remark 3.4, then, from the definition of $E(t)$, we have

$$\|u_t\|_2^2 \leqslant 2E(t). \tag{3.14}$$

Hence, choosing $\varepsilon = \frac{2}{5}$, we see that

$$E^*(t) \geqslant \frac{1}{5}\big(\|u_t\|_2^2 + \|\Delta u\|_2^2\big). \tag{3.15}$$

Moreover, we note that

$$2\left|\int_\Omega |u|^{p-2} u \Delta u\, dx\right| = 2(p-1)\int_\Omega |u|^{p-2}|\nabla u|^2\, dx \leqslant 2(p-1)\|u\|_{(p-2)\theta_1}^{p-2}\|\nabla u\|_{2\theta_2}^2,$$

where $\frac{1}{\theta_1} + \frac{1}{\theta_2} = 1$, so that, we put $\theta_1 = 1$ and $\theta_2 = \infty$, if $N = 1$; $\theta_1 = 1 + \varepsilon_1$ (for arbitrary small $\varepsilon_1 > 0$), if $N = 2$; and $\theta_1 = \frac{N}{2}$, $\theta_2 = \frac{N}{N-2}$, if $N \geqslant 3$. Then, by Poincaré inequality, $l\|\nabla u\|_2^2 \leqslant \frac{2p}{p-2}E(0)$ by (3.8) and by (3.15), we have

$$2\left|\int_\Omega |u|^{p-2} u \Delta u\, dx\right| \leqslant 2c_s^p(p-1)\|\nabla u\|_2^{p-2}\|\Delta u\|_2^2 \leqslant c_1 E^*(t), \tag{3.16}$$

where $c_1 = 10c_s^p(p-1)(\frac{2p}{l(p-2)}E(0))^{\frac{p-2}{2}}$. Substituting (3.16) into (3.13), and then integrating it over $(0,t)$, we obtain

$$E^*(t) + \frac{4}{5}\left(m_0 - \eta - \frac{\|g\|_{L^1}^2}{4\eta}\right)\int_0^t \big\|\Delta u(s)\big\|_2^2\, ds \leqslant E^*(0) + \int_0^t c_1 E^*(s)\, ds. \tag{3.17}$$

Taking $\eta = \frac{\|g\|_{L^1}}{2}$ in (3.17), and then by Gronwall's Lemma, we deduce

$$E^*(t) \leqslant E^*(0)\exp(c_1 t),$$

for any $t \geqslant 0$. Therefore by (3.15) and Theorem 2.3, we have $T = \infty$.

Next, we want to derive the decay rate of energy function for problem (3.1). Multiplying (3.1) by u and integrating it over $\Omega \times [t_1, t_2]$, we have

$$\begin{aligned}\int_{t_1}^{t_2}\big[M\big(\|\nabla u\|_2^2\big)\|\nabla u\|_2^2 - \|u\|_p^p\big]\, dt = &-\int_{t_1}^{t_2}\int_\Omega u_{tt} u\, dx\, dt - \int_{t_1}^{t_2}\int_\Omega \nabla u_t \cdot \nabla u\, dx\, dt\\ &- \int_{t_1}^{t_2}\int_\Omega\int_0^t g(t-s)u(t)\Delta u(s)\, ds\, dx\, dt.\end{aligned}$$

Then, through integrating by parts, we obtain

$$\begin{aligned}\int_{t_1}^{t_2}\big[M\big(\|\nabla u\|_2^2\big)\|\nabla u\|_2^2 - \|u\|_p^p\big]\, dt = &-\int_\Omega u_t u\, dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\|u_t\|_2^2\, dt - \int_{t_1}^{t_2}\int_\Omega \nabla u_t \cdot \nabla u\, dx\, dt\\ &- \int_{t_1}^{t_2}\int_\Omega\int_0^t g(t-s)u(t)\Delta u(s)\, ds\, dx\, dt.\end{aligned} \tag{3.18}$$

It follows from (3.4) that

$$\begin{aligned}2\int_{t_1}^{t_2} E(t)\, dt + \frac{2-p}{p}\int_{t_1}^{t_2}\|u\|_p^p\, dt \leqslant &-\int_\Omega u_t u\, dx\Big|_{t_1}^{t_2} + 2\int_{t_1}^{t_2}\|u_t\|_2^2\, dt - \int_{t_1}^{t_2}\int_\Omega \nabla u_t \cdot \nabla u\, dx\, dt - \int_{t_1}^{t_2}\int_0^t g(s)\, ds\big\|\nabla u(t)\big\|_2^2\, dt\\ &+ \int_{t_1}^{t_2}(g \diamond \nabla u)(t)\, dt - \int_{t_1}^{t_2}\int_\Omega\int_0^t g(t-s)u(t)\Delta u(s)\, ds\, dx\, dt.\end{aligned} \tag{3.19}$$

Since

$$-\int_\Omega \int_0^t g(t-s)u(t)\Delta u(s)\,ds\,dx = \frac{1}{2}\left[\int_0^t g(t-s)\left(\left\|\nabla u(t)\right\|_2^2 + \left\|\nabla u(s)\right\|_2^2\right)ds - \int_0^t g(t-s)\left\|\nabla u(s)-\nabla u(t)\right\|_2^2 ds\right]$$
$$= \frac{1}{2}\left[\int_0^t g(t-s)\left\|\nabla u(t)\right\|_2^2 ds + \int_0^t g(t-s)\left\|\nabla u(s)\right\|_2^2 ds - (g\diamond\nabla u)(t)\right],$$

hence, (3.19) becomes

$$\begin{aligned}2\int_{t_1}^{t_2} E(t)\,dt + \frac{2-p}{p}\int_{t_1}^{t_2}\|u\|_p^p\,dt &\leqslant -\int_\Omega u_t u\,dx\Big|_{t_1}^{t_2} + 2\int_{t_1}^{t_2}\|u_t\|_2^2\,dt - \int_{t_1}^{t_2}\int_\Omega \nabla u_t\cdot\nabla u\,dx\,dt \\ &\quad - \frac{1}{2}\int_{t_1}^{t_2}\int_0^t g(s)\,ds\left\|\nabla u(t)\right\|_2^2 dt + \frac{1}{2}\int_{t_1}^{t_2}(g\diamond\nabla u)(t)\,dt \\ &\quad + \frac{1}{2}\int_{t_1}^{t_2}\int_0^t g(t-s)\left\|\nabla u(s)\right\|_2^2 ds\,dt \\ &\leqslant -\int_\Omega u_t u\,dx\Big|_{t_1}^{t_2} + 2\int_{t_1}^{t_2}\|u_t\|_2^2\,dt - \int_{t_1}^{t_2}\int_\Omega \nabla u_t\cdot\nabla u\,dx\,dt \\ &\quad + \frac{1}{2}\int_{t_1}^{t_2}(g\diamond\nabla u)(t)\,dt + \frac{1}{2}\int_{t_1}^{t_2}\int_0^t g(t-s)\left\|\nabla u(s)\right\|_2^2 ds\,dt. \end{aligned} \tag{3.20}$$

For the left-hand side of (3.20), since $\|u\|_p^p \leqslant \frac{2p\alpha}{p-2}E(t)$ by (3.9), we have

$$2\int_{t_1}^{t_2} E(t)\,dt + \frac{2-p}{p}\int_{t_1}^{t_2}\|u\|_p^p\,dt \geqslant \beta_1\int_{t_1}^{t_2} E(t)\,dt, \tag{3.21}$$

where $\beta_1 = 2-2\alpha > 0$ (by (3.10)). Next, we shall estimate every term of the right-hand side of (3.20). First, by Hölder inequality, Young's inequality and Lemma 2.1, we have

$$\int_\Omega |u_t u|\,dx \leqslant \int_\Omega |u|^2\,dx + \int_\Omega |u_t|^2\,dx \leqslant c_s^2\|\nabla u\|_2^2 + \|u_t\|_2^2.$$

Then, since $l\|\nabla u\|_2^2 \leqslant \frac{2p}{p-2}E(t)$ by (3.8), $\|u_t\|_2^2 \leqslant 2E(t)$ by (3.14) and $E(t)$ is a nonincreasing function by Lemma 3.2, we obtain

$$\int_\Omega u_t u\,dx\Big|_{t_1}^{t_2} \leqslant 2c_2E(t_1) \tag{3.22}$$

and use $\|\nabla u_t\|_2^2 \leqslant -E'(t)$ by (3.6) to get

$$2\int_{t_1}^{t_2}\|u_t\|_2^2\,dt \leqslant 2c_s^2\int_{t_1}^{t_2}\|\nabla u_t\|_2^2\,dt \leqslant 2c_s^2E(t_1), \tag{3.23}$$

where $c_2 = \frac{2c_s^2p}{l(p-2)} + 2$. And for the third term, we have

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega \nabla u_t\cdot\nabla u\,dx\,dt &= \frac{1}{2}\int_{t_1}^{t_2}\frac{d}{dt}\|\nabla u\|_2^2\,dt \\ &= \frac{1}{2}\left[\left\|\nabla u(t_2)\right\|_2^2 - \left\|\nabla u(t_1)\right\|_2^2\right] \\ &\leqslant \frac{2p}{l(p-2)}E(t_1). \end{aligned} \tag{3.24}$$

To estimate the last term, use Young's inequality for convolution $\|\phi * \psi\|_q \leqslant \|\phi\|_r \|\psi\|_s$, with $\frac{1}{q} = \frac{1}{r} + \frac{1}{s} - 1$, $1 \leqslant q, r, s \leqslant \infty$, noting that if $q = 1$, then $r = 1$ and $s = 1$, we get

$$\begin{aligned}
\int_{t_1}^{t_2} \int_0^t g(t-s) \left\| \nabla u(s) \right\|_2^2 ds\, dt &\leqslant \int_{t_1}^{t_2} g(t)\, dt \int_{t_1}^{t_2} \left\| \nabla u(t) \right\|_2^2 dt \\
&\leqslant (m_0 - l) \int_{t_1}^{t_2} \left\| \nabla u(t) \right\|_2^2 dt \\
&\leqslant \frac{2(m_0 - l)p}{l(p-2)} \int_{t_1}^{t_2} E(t)\, dt. \qquad (3.25)
\end{aligned}$$

As the fifth term, by (3.25) and using $l\|\nabla u(t)\|_2^2 \leqslant \frac{2p}{p-2} E(t)$ by (3.8) once more, we deduce that

$$\begin{aligned}
\frac{1}{2} \int_{t_1}^{t_2} (g \diamond \nabla u)(t)\, dt &= \frac{1}{2} \int_{t_1}^{t_2} \int_0^t g(t-s) \left\| \nabla u(s) - \nabla u(t) \right\|_2^2 ds\, dt \\
&\leqslant \frac{1}{2} \int_{t_1}^{t_2} \int_0^t g(t-s) \left(\left\| \nabla u(s) \right\|_2 + \left\| \nabla u(t) \right\|_2 \right)^2 ds\, dt \\
&\leqslant \int_{t_1}^{t_2} \int_0^t g(t-s) \left(\left\| \nabla u(s) \right\|_2^2 + \left\| \nabla u(t) \right\|_2^2 \right) ds\, dt \\
&\leqslant \frac{4(m_0 - l)p}{l(p-2)} \int_{t_1}^{t_2} E(t)\, dt. \qquad (3.26)
\end{aligned}$$

Thus, combining (3.21)–(3.26), (3.20) yields

$$\beta_1 \int_{t_1}^{t_2} E(t)\, dt \leqslant c_3 E(t_1) + \frac{5(m_0 - l)p}{l(p-2)} \int_{t_1}^{t_2} E(t)\, dt, \qquad (3.27)$$

where $c_3 = 2c_2 + 2c_s^2 + \frac{2p}{l(p-2)}$. Then, rewriting (3.27), we have

$$\beta_2 \int_{t_1}^{t_2} E(t)\, dt \leqslant c_3 E(t_1), \qquad (3.28)$$

here $\beta_2 = \beta_1 - \frac{5(m_0-l)p}{l(p-2)} = 2 - 2\alpha - \frac{5(m_0-l)p}{l(p-2)}$. Observing that the condition $m_0 > (1 + \frac{5p}{2(p-2)}) \int_0^\infty g(s)\, ds$ given in (3.11) is equivalent to $2 - \frac{5(m_0-l)p}{l(p-2)} > 0$, hence, if $E(0)$ is small enough, then not only the condition $E(0) < E_1$ is satisfied, but also $\beta_2 = 2 - 2\alpha - \frac{5(m_0-l)p}{l(p-2)} = 2 - \frac{5(m_0-l)p}{l(p-2)} - \frac{2c_s^p}{l} (\frac{2p}{l(p-2)} E(0))^{\frac{p-2}{2}} > 0$ is assured. Now, letting $t_2 \to \infty$ in (3.28), we obtain

$$\int_{t_1}^{\infty} E(t)\, dt \leqslant \frac{c_3}{\beta_2} E(t_1), \quad \forall t_1 \geqslant 0.$$

Therefore, by Lemma 2.2, we derive that

$$E(t) \leqslant E(0) \exp\left(1 - \frac{\beta_2 t}{c_3}\right), \qquad (3.29)$$

for $t \geqslant \frac{c_3}{\beta_2}$. $\square$

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