J. Math. Anal. Appl. 368 (2010) 498–507

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The Black–Scholes equation in stochastic volatility models

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ARTICLE INFO

Article history:
Received 3 August 2009
Available online 9 April 2010
Submitted by R. Kiesel

Keywords:
Parabolic equations
Feynman–Kac theorems
Option pricing
Stochastic volatility
Boundary conditions

ABSTRACT

We study the Black–Scholes equation in stochastic volatility models. In particular, we show that the option price is the unique classical solution to a parabolic differential equation with a certain boundary behaviour for vanishing values of the volatility. If the boundary is attainable, then this boundary behaviour serves as a boundary condition and guarantees uniqueness in appropriate function spaces. On the other hand, if the boundary is non-attainable, then the boundary behaviour is not needed to guarantee uniqueness, but is nevertheless very useful for instance from a numerical perspective.

1. Introduction

In financial mathematics there are two main approaches to the calculation of option prices. Either the price of an option is viewed as a risk-neutral expected value, or it is obtained by solving the Black–Scholes partial differential equation. The connection between these approaches is furnished by the classical Feynman–Kac theorem, which states that a classical solution to a linear parabolic PDE has a stochastic representation in terms of an expected value. In the standard Black–Scholes model, a standard logarithmic change of variables transforms the Black–Scholes equation into an equation with constant coefficients. Since such an equation is covered by standard PDE theory, the existence of a unique classical solution is guaranteed. Consequently, the option price given by the risk-neutral expected value is the unique classical solution to the Black–Scholes equation. However, in many situations outside the standard Black–Scholes setting, the pricing equation has degenerate, or too fast growing, coefficients and standard PDE theory does not apply. Examples include

i) local volatility models with an unbounded volatility for small stock values such as the CEV-model,
ii) one-factor models for the short rate where the volatility is non-Lipschitz at 0 such as the CIR-model,
iii) models for bubbles such as the CEV-model with unbounded volatility at infinity, and
iv) stochastic volatility models such as the Heston model.

In these cases, the existence of solutions to the pricing PDE does not follow from classical theory. Instead, additional analysis is needed if one wants to prove the correspondence between the stochastic representation and the pricing PDE. A treatment of these problems when the boundary of the state space is not hit can be found in [12]. In the setting of that article boundary conditions are mathematically redundant, and are therefore not discussed. However, the knowledge of the boundary behaviour is crucial when using numerical methods to calculate option prices even if these conditions are redundant from a strict mathematical point of view. Indeed, in [5], boundary conditions for several pricing PDEs are discussed. That refer-

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[1] Support from the Swedish Research Council is gratefully acknowledged.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.04.014

ence even has a section entitled "The thorny issue of boundary conditions" for equations of the type under consideration in this article. Furthermore, in many cases of practical importance, the boundary of the state space is reached with positive probability, and boundary conditions are then both theoretically needed and numerically useful. To simplify the terminology, we will speak of boundary *conditions* rather than boundary behaviour, even when these are redundant from a mathematical viewpoint.

Existing literature has treated boundary conditions for the pricing equations in cases i)–iii) above. Indeed, in [15], the connection between the option price given as a risk-neutral expected value and the corresponding Black–Scholes equation is made precise for options on several underlying assets in a local volatility model with absorbing boundary conditions. In particular, it is shown that the option price is the unique classical solution to the Black–Scholes equation with boundary conditions given by a certain iterative procedure. A similar study is performed for the term structure equation in [7] and [6]. It is shown that the option price in a one-factor model for the short rate r is the unique classical solution to the corresponding pricing equation with boundary conditions given by formally plugging $r=0$ into the equation. Recently, in [8], the authors studied the one-dimensional Black–Scholes equation in local volatility models with unbounded volatilities at infinity, i.e the kind of models that have been suggested for the modeling of bubbles, see [4,14,16,17]. In that case, uniqueness of solutions is lost for general contracts, but it is shown that the stochastic representation formula is the unique solution to the Black–Scholes PDE in the class of contracts of strictly sublinear growth.

The purpose of the present paper is to provide the precise connection between the risk-neutral expected value and the pricing PDE with appropriate boundary conditions for stochastic volatility models, thus extending the one-dimensional results in [7] to a setting with two spatial dimensions. Mathematically, the main difficulty is to study the behaviour of the option price for vanishing values of the volatility parameter, and to show that the option price given by the stochastic representation indeed satisfies the stated boundary conditions. Our treatment does not distinguish between models where the boundary of the state space is hit and models where the boundary is not hit. There is therefore no single change of variables that transforms the equation to a parabolic equation in standard form. Using our approach, we obtain boundary conditions that are applicable for all models.

Although there is an extensive literature on equations with degenerating coefficients, compare the classical reference [20], not even C^1 regularity of the pricing function is available in the generality that is needed here. In fact, in an example in [14] a term structure equation with degenerating coefficients is considered, similar to the equations studied here. Two different solutions to the equation are provided, one of which is bounded and continuous but fails to be continuously differentiable up to the boundary, whereas the second solution satisfies the appropriate boundary conditions. The authors consider these different solutions as alternative prices. We, however, take the point of view that only the solution corresponding to the stochastic representation is the actual price and our boundary conditions single out the stochastic representation among all possible solutions to the pricing PDE.

In addition to the complications at the boundary at which the volatility parameter vanishes, the pricing equation in stochastic volatility models falls outside the standard classes of parabolic equations since the volatility of the stock price is unbounded. Indeed, the instantaneous variance is now an underlying state variable, so the diffusion coefficients grow faster than quadratically at infinity. This situation resembles the one with bubbles treated in [8], and uniqueness of solutions may be lost for general contracts. We specify classes of functions for which uniqueness of solutions are proven using maximum principle arguments.

The remainder of the paper is organised as follows. In Section 2 we introduce the model and present our main theorem that states that the option price, given as the risk-neutral expected value of the terminal pay-off, is a classical solution to the corresponding pricing PDE. This result is proved in detail in Sections 3–5 using a mix of analytic and probabilistic arguments. Finally, in Section 6, uniqueness results for the pricing PDE are provided.

2. Stochastic volatility models and the main result

We assume that the stock price process X is specified under the pricing measure as

$$dX_t = \sqrt{Y_t} X_t \, dW_1, \tag{1}$$

where the variance process Y satisfies

$$dY_t = \beta(Y_t)\, dt + \sigma(Y_t)\, dW_2. \tag{2}$$

Here W_1 and W_2 are two standard Brownian motions with constant correlation $\rho \in (-1, 1)$, and β and σ are given functions specified so that Y remains non-negative at all times, compare Hypothesis 2.1 below. Given a pay-off function $g : [0, \infty) \to \mathbb{R}$, the price at time t of a European option that at time T pays $g(X_T)$ is $u(X_t, Y_t, t)$, where

$$u(x, y, t) = E_{x,y,t}\big[g(X_T)\big]. \tag{3}$$

In (3), the subindices indicate that the processes X and Y should be started at time t at the values x and y, respectively. The corresponding Black–Scholes equation is given by

$$u_t + \mathcal{L}u = 0 \tag{4}$$

where

$$\mathcal{L}=\frac{1}{2}yx^2\frac{\partial^2}{\partial x^2}+\rho\sigma(y)\sqrt{y}x\frac{\partial^2}{\partial x\partial y}+\frac{\sigma^2(y)}{2}\frac{\partial^2}{\partial y^2}+\beta(y)\frac{\partial}{\partial y},$$

with terminal condition $u(x,y,T)=g(x)$. Throughout the article, the following hypothesis is assumed to hold unless otherwise stated.

Hypothesis 2.1. *The drift satisfies $\beta\in C^1([0,\infty))$ with a Hölder (α) continuous derivative for some α, and $\beta(0)\geqslant 0$. The volatility $\sigma:[0,\infty)\to[0,\infty)$ satisfies $\sigma(0)=0$ and $\sigma(y)>0$ for all $y>0$, and the function $\sigma^2(y)$ is continuously differentiable on $[0,\infty)$ with a Hölder (α) continuous derivative. The growth condition*

$$\left|\beta(y)\right|+\sigma(y)\leqslant C(1+y) \tag{5}$$

holds for all $y\geqslant 0$, where C is a constant. The pay-off function g is bounded and is twice continuously differentiable on $[0,\infty)$. Moreover, $xg'(x)$ and $x^2g''(x)$ are bounded.

Remark. Note that the conditions on β and σ guarantee a unique strong solution to (2), see Section IX.3 in [21]. This solution stays non-negative automatically, i.e. there is no need to specify any boundary behaviour of Y at the boundary $y=0$. Moreover, the stock price process in (1) has the explicit solution

$$X_T=x\exp\left\{-\frac{1}{2}\int_t^T Y_s\,ds+\int_t^T\sqrt{Y_s}\,dW_1\right\}. \tag{6}$$

Thus

$$X_T=xH_T, \tag{7}$$

where $H_T:=\exp\{-\frac{1}{2}\int_t^T Y_s\,ds+\int_t^T\sqrt{Y_s}\,dW_1\}$ is independent of the initial state x.

Remark. For ease of exposition, the model is specified with a zero interest rate and with time-homogeneous coefficients β and σ. Generalisations to a deterministic interest rate and time-dependent coefficients are straightforward.

Definition 2.2. A continuous function $v:[0,\infty)^2\times[0,T]\to\mathbb{R}$ is a classical solution to the pricing equation if $v\in C^{2,2,1}((0,\infty)^2\times[0,T))\cap C^{0,1,1}((0,\infty)\times[0,\infty)\times[0,T))$, with

$$\begin{cases} v_t(x,y,t)+\mathcal{L}v(x,y,t)=0 & \text{if } (x,y,t)\in(0,\infty)^2\times[0,T),\\ v(0,y,t)=g(0) & \text{for } (y,t)\in[0,\infty)\times[0,T],\\ v_t(x,0,t)+\beta(0)v_y(x,0,t)=0 & \text{for } (x,t)\in(0,\infty)\times[0,T),\\ v(x,y,T)=g(x) & \text{for } (x,y)\in(0,\infty)^2. \end{cases}$$

Remark. In accordance with the discussion in the introduction, boundary conditions are included in the notion of a classical solution regardless if the boundary can be hit or not. The boundary condition at $x=0$ corresponds to the fact that if X_t is small, then also X_T is likely to be small, compare (7). The boundary condition at $y=0$ is obtained by formally plugging in $y=0$ into (4). This boundary condition is also specified for the particular case of the Heston model in [13] and Chapter 22.4 in [5], but without further discussion under what conditions it holds.

We next present our main result concerning existence of solutions to the pricing equation.

Theorem 2.3. *Assume that Hypothesis* 2.1 *holds. Then the option price u defined in* (3) *above is a classical solution to the pricing equation.*

Remark. Theorem 2.3 is formulated for a rather general setting covering most models for the volatility process Y that are used in practice. Examples of commonly used models covered by Hypothesis 2.1 include the Heston model, in which the variance process satisfies

$$dY_t=(b-aY_t)\,dt+\sigma\sqrt{Y_t}\,dW_2$$

(see [13]), and the 'Garch diffusion model' (see [18])

$$dY_t=(b-aY_t)\,dt+\sigma Y_t\,dW_2,$$

where a, b and σ are positive constants. Clearly, Hypothesis 2.1 also covers a whole range of other interesting models that may lack the analytical tractability of the Heston model, but that can be treated numerically thanks to Theorem 2.3.

The regularity condition imposed on the pay-off function g may seem restrictive. Note, however, that if Eq. (4) has a regularising effect, then the methods below together with the Markov property would yield an extension of Theorem 2.3 to irregular pay-off functions. We believe that (4) should have a sufficient regularising effect if $\beta(0) > 0$. We also conjecture that Theorem 2.3 holds for contracts of linear growth. However, the extension to irregular contracts and to contracts of linear growth seem to entail many technical difficulties. Since the price of a linear contract can be approximated arbitrarily well on compact sets with prices of bounded and regular contracts, this investigation is not carried out.

The proof of Theorem 2.3 is carried out in several steps in Sections 3–5 below.

3. Continuity and interior regularity

Proposition 3.1. *The option price $u(x,y,t)$ is continuous on $[0,\infty)^2 \times [0,T]$.*

Proof. Note that by time-homogeneity, the dependence in u on time is only through the time $T-t$ left to maturity. Let (x_n, y_n, T_n) be a sequence of points converging to (x,y,T) where the last coordinate denotes time left to maturity, and let X^n and Y^n be defined by

$$\begin{cases} dX_t^n = \sqrt{Y_t^n} X_t^n \, dW_1, \\ X_0^n = x_n \end{cases}$$

and

$$\begin{cases} dY_t^n = \beta(Y_t^n)\, dt + \sigma(Y_t^n)\, dW_2, \\ Y_0^n = y_n, \end{cases}$$

respectively. Moreover, let X and Y be defined accordingly with $X_0 = x$ and $Y_0 = y$. It follows from Theorem 2.4 in [3] that

$$E\Big[\sup_{0\leqslant t\leqslant T+\delta} \big|Y_t^n - Y_t\big|^2\Big] \to 0$$

as $n\to\infty$ (here $\delta > 0$ is such that $T_n \leqslant T+\delta$). Consequently,

$$E\bigg[\bigg(\int_0^{T_n} Y_t^n\, dt - \int_0^T Y_t\, dt\bigg)^2\bigg] \leqslant 2E\bigg[\bigg(\int_0^{T_n} Y_t^n - Y_t\, dt\bigg)^2\bigg] + 2E\bigg[\bigg(\int_{T_n}^T Y_t\, dt\bigg)^2\bigg] \to 0$$

as $n\to\infty$. Similarly,

$$\begin{aligned} E\bigg[\bigg(\int_0^{T_n} \sqrt{Y_t^n}\, dW_t - \int_0^T \sqrt{Y_t}\, dW_t\bigg)^2\bigg] &\leqslant 2E\bigg[\bigg(\int_0^{T_n} \sqrt{Y_t^n} - \sqrt{Y_t}\, dW_t\bigg)^2\bigg] + 2E\bigg[\bigg(\int_{T_n}^T \sqrt{Y_t}\, dW_t\bigg)^2\bigg] \\ &\leqslant 2\int_0^{T_n} E\big[\big|Y_t^n - Y_t\big|\big]\, dt + 2\int_{T_n}^T E[Y_t]\, dt \to 0 \end{aligned}$$

as $n\to\infty$. In view of (6) above, it follows that $X_{T_n}^n$ converges to X_T in probability. Thus $E_{x_n,y_n,0}[g(X_{T_n}^n)] \to E_{x,y,0}[g(X_T)]$ as $n\to\infty$ since g is bounded, so u is continuous. □

Proposition 3.2. *The option price u is in $C^{2,2,1}((0,\infty)^2 \times [0,T))$, and it satisfies $u_t + \mathcal{L}u = 0$ at all points $(x,y,t) \in (0,\infty)^2 \times [0,T)$.*

Proof. The result follows from the continuity of u by a standard argument as shown below. Let $(x,y,t) \in (0,\infty)^2 \times [0,T)$, and let $\mathcal{R} = (x_1,x_2) \times (y_1,y_2) \times (0,T)$ be a rectangle which contains (x,y,t). Since u is continuous by Proposition 3.1, it follows from parabolic theory (see Theorem 6.3.6 on page 138 in [10]) that there is a unique solution $f(x,y,t)$ to the boundary value problem

$$\begin{cases} f_t + \mathcal{L}f = 0 & \text{for } (x,y,t) \in \mathcal{R}, \\ f = u & \text{for } (x,y,t) \in \partial_0\mathcal{R}, \end{cases}$$

where $\partial_0\mathcal{R} = \partial\mathcal{R} \setminus (x_1,x_2) \times (y_1,y_2) \times \{0\}$ is the parabolic boundary of $\mathcal{R}$. The Ito formula shows that the process $Z_s = f(X_s, Y_s, s)$ is a martingale on $[t, \tau_{\mathcal{R}})$, where

$$\tau_{\mathcal{R}} = \inf\big\{s \geqslant t\colon (X_s, Y_s, s) \notin \mathcal{R}\big\}.$$

Thus

$$f(x,y,t)=Z_t=E_{x,y,t}Z_{T\wedge\tau_{\mathcal{R}}}=E_{x,y,t}u(X_{T\wedge\tau_{\mathcal{R}}},Y_{T\wedge\tau_{\mathcal{R}}},T\wedge\tau_{\mathcal{R}})=u(x,y,t)$$

by the strong Markov property. Consequently, $u=f$ on $\mathcal{R}$, so u is in $C^{2,2,1}(\mathcal{R})$ and satisfies $u_t+\mathcal{L}u=0$ in $\mathcal{R}$. Since the initial point (x,y,t) was arbitrary, the result follows. $\square$

The boundary conditions at $t=T$ and $x=0$ follow from the continuity of u. It therefore only remains to study the boundary behaviour at $y=0$. To do this, we first derive extra regularity of the value function at this boundary.

4. Regularity at the boundary $y=0$

In this section we use an approximation procedure to show that the derivative u_y is continuous up to the boundary $y=0$. Note that the coefficients of X and Y are not differentiable with respect to the y-variable, so the theory for differentiating stochastic flows is not directly applicable, see for example §8, Chapter 2, Part 2 in [11]. Instead, we take the approach of first formally differentiating the pricing equation, which in fact has differentiable coefficients, and then we consider the stochastic representation of the differentiated equation. This stochastic representation is finally shown to agree with the derivative u_y, see Proposition 4.2 below.

If one formally differentiates the pricing Eq. (4) with respect to y one obtains the equation

$$v_t+\hat{\mathcal{L}}v+\frac{1}{2}x^2u_{xx}=0, \tag{8}$$

where

$$\hat{\mathcal{L}}=\frac{1}{2}yx^2\frac{\partial^2}{\partial x^2}+\rho\sigma(y)\sqrt{y}x\frac{\partial^2}{\partial x\partial y}+\frac{\sigma^2(y)}{2}\frac{\partial^2}{\partial y^2}+\rho\hat{a}(y)x\frac{\partial}{\partial x}+\hat{\beta}(y)\frac{\partial}{\partial y}+\beta'(y), \tag{9}$$

$$\hat{a}(y)=\frac{\partial(\sigma(y)\sqrt{y})}{\partial y} \tag{10}$$

and

$$\hat{\beta}(y)=\beta(y)+\frac{1}{2}\frac{\partial\sigma^2}{\partial y}(y). \tag{11}$$

Moreover, since the terminal condition for u does not depend on y, the terminal condition of the derivative should be 0. We next define a function v to be the stochastic representation of Eq. (8) with terminal condition 0. More precisely, let $\hat{X}$ and $\hat{Y}$ satisfy

$$d\hat{X}_t=\rho\hat{a}(\hat{Y}_t)\hat{X}_t\,dt+\sqrt{\hat{Y}_t}\hat{X}_t\,dW_1$$

and

$$d\hat{Y}_t=\hat{\beta}(\hat{Y}_t)\,dt+\sigma(\hat{Y}_t)\,dW_2, \tag{12}$$

respectively, and define the function $v:[0,\infty)^2\times[0,T]$ by

$$v(x,y,t)=\frac{1}{2}E_{x,y,t}\left[\int_t^T e^{\int_t^s\beta'(\hat{Y}_u)\,du}\hat{X}_s^2u_{xx}(\hat{X}_s,\hat{Y}_s,s)\,ds\right]. \tag{13}$$

Here the indices indicate that $\hat{X}_t=x$ and $\hat{Y}_t=y$.

Remark. The pathwise uniqueness of solutions to (12) follows from [2] and Corollary IX.3.4 in [21].

Proposition 4.1. *The function v defined in* (13) *above is continuous on* $[0,\infty)^2\times[0,T]$.

Proof. Using the notation of Eq. (7), we have

$$x^2u_{xx}(x,y,t)=x^2E\big[H_T^2g''(xH_T)\big]=E\big[X_T^2g''(X_T)\big].$$

Since the function $x^2g''(x)$ is bounded by assumption, it follows that also $x^2u_{xx}(x,y,t)$ is bounded, and it is continuous by a similar argument as in the proof of Proposition 3.1. As a consequence, the function

$$(x,y,t)\mapsto E_{x,y,t}\big[e^{\int_t^s\beta'(\hat{Y}_u)\,du}\hat{X}_s^2u_{xx}(\hat{X}_s,\hat{Y}_s,s)\big]$$

is of a similar type as the option price u in (3), so continuity again follows by the same argument as before. $\square$

Proposition 4.2. *We have $u_y = v$. Consequently, u_y is continuous on $(0,\infty)\times[0,\infty)\times[0,T)$.*

Proof. First let $f_n \in C^\infty([0,\infty))$ be a smooth approximation of the function $f(y)=\sqrt{y}$ such that $1/n \leqslant f_n(y)\leqslant n$, $f_n(y)$ is increasing in y, f_n^2 is Lipschitz continuous uniformly in n, and such that $f_n \to f$ uniformly on compacts as $n\to\infty$. Moreover, let $\sigma_n \in C^\infty([0,\infty))$ be a smooth approximation of the function $\sigma(y)$ such that $\sigma_n(y)\leqslant y$ for $y\in[0,1/n)$ and $\sigma_n\to\sigma$ uniformly on compacts. Define Y^n, X^n and u^n by

$$dY_t^n = \beta\big(Y_t^n\big)\,dt + \sigma_n\big(Y_t^n\big)\,dW_2,$$

$$dX_t^n = f_n\big(Y_t^n\big)X_t^n\,dW_1$$

and

$$u^n(x,y,t) = E_{x,y,t}\big[g\big(X_T^n\big)\big], \tag{14}$$

respectively. Now consider the parabolic equation

$$\begin{cases} \dfrac{\partial u^n}{\partial t} + \dfrac{1}{2}f_n^2x^2\dfrac{\partial^2 u^n}{\partial x^2} + \rho\sigma_n f_n x\dfrac{\partial^2 u^n}{\partial x\partial y} + \dfrac{\sigma_n^2}{2}\dfrac{\partial^2 u^n}{\partial y^2} + \beta\dfrac{\partial u^n}{\partial y} = 0, \\ u^n(x,y,T) = g(x). \end{cases} \tag{15}$$

If one rewrites this equation in the logarithmic variables $\ln x$ and $\ln y$, then one obtains a uniformly parabolic equation with smooth and bounded diffusion and drift coefficients. From standard parabolic theory it is known that such an equation has a unique bounded classical solution (see Theorem 1.12 on page 25 and Theorem 1.16 on page 29 in [9]; boundedness of the solution follows from maximum principle arguments), and it follows from the Feynman–Kac theorem that the unique solution is given by u^n defined in (14). In addition, it follows from standard parabolic theory that $u^n \in C^\infty((0,\infty)^2\times[0,T))\cap C^{2,2,1}((0,\infty)^2\times[0,T])$.

Moreover,

$$E_{x,y,t}\Big[\sup_{s\in[t,T]}\big|Y_s^n - Y_s\big|^2\Big]\to 0$$

as $n\to\infty$ by Theorem 2.5 in [3], so

$$E_{x,y,t}\left[\int_t^T\big|f_n^2\big(Y_s^n\big)-Y_s\big|\,ds\right] \leqslant E_{x,y,t}\left[\int_t^T\big|f_n^2\big(Y_s^n\big)-f_n^2(Y_s)\big|\,ds\right] + E_{x,y,t}\left[\int_t^T\big|f_n^2(Y_s)-Y_s\big|\,ds\right]\to 0$$

as $n\to\infty$ since f_n^2 is Lipschitz continuous uniformly in n, and similarly

$$E_{x,y,t}\left[\left|\int_t^T\big(f_n\big(Y_s^n\big)-\sqrt{Y_s}\big)\,dW_1\right|^2\right] = E_{x,y,t}\left[\int_t^T\big|f_n\big(Y_s^n\big)-\sqrt{Y_s}\big|^2\,ds\right]\to 0.$$

Consequently, $X_T^n \to X_T$ in probability, so $u^n(x,y,t)\to u(x,y,t)$ at all points $(x,y,t)\in(0,\infty)^2\times[0,T]$. Therefore, by interior Schauder estimates, $u_y^n(x,y,t)\to u_y(x,y,t)$ at all interior points.

Next, since u^n is a smooth solution to (15), straightforward differentiation and standard estimates show that

$$v^n(x,y,t) := \frac{\partial u^n}{\partial y}(x,y,t)$$

is a bounded solution to

$$\begin{cases} v_t^n + \hat{\mathcal{L}}^n v^n + \dfrac{1}{2}\gamma_n(y)x^2u_{xx}^n = 0, \\ v^n(x,y,T) = 0, \end{cases} \tag{16}$$

where

$$\hat{\mathcal{L}}^n = \frac{1}{2}f_n^2(y)x^2\frac{\partial^2}{\partial x^2} + \rho\sigma_n(y)f_n(y)x\frac{\partial^2}{\partial x\partial y} + \frac{\sigma_n^2(y)}{2}\frac{\partial^2}{\partial y^2} + \rho\hat{a}_n(y)x\frac{\partial}{\partial x} + \hat{\beta}_n(y)\frac{\partial}{\partial y} + \beta'(y),$$

$$\hat{a}_n(y) = \frac{\partial(\sigma_n(y)f_n(y))}{\partial y},$$

$$\hat{\beta}_n(y) = \beta(y) + \frac{1}{2}\frac{\partial\sigma_n^2(y)}{\partial y}$$

and

$$\gamma_n(y) = \frac{\partial f_n^2(y)}{\partial y}.$$

Eq. (16) is, after a logarithmic change of variables, covered by existing parabolic theory (see [9]) and has a unique bounded classical solution. Thus the Feynman–Kac theorem applies, so v^n has a stochastic representation

$$v^n(x,y,t) = \frac{1}{2}E_{x,y,t}\left[\int_t^T e^{\int_t^s \beta'(\hat{Y}_u^n)\,du}\gamma_n(\hat{Y}_s^n)(\hat{X}_s^n)^2 u_{xx}^n(\hat{X}_s^n,\hat{Y}_s^n,s)\,ds\right]$$

for some processes $\hat{X}^n$ and $\hat{Y}^n$. Using the stability results of [3] and the same reasoning as in the proofs of Propositions 3.1 and 4.1, it follows that $v_n \to v$ as $n \to \infty$ for all points $(x,y,t) \in [0,\infty)^2 \times [0,T]$. Consequently, $u_y = v$ at all interior points. Since v is continuous up to the boundary $y = 0$, it is easy to check that so is u_y. □

5. Estimates of the second spatial derivatives

Since the function v defined in (13) is continuous by Proposition 4.1, it follows by the methods used in the proof of Proposition 3.2 that v indeed solves the differentiated equation

$$v_t + \hat{\mathcal{L}}v + \frac{1}{2}x^2 u_{xx} = 0$$

on $(0,\infty)\times(0,\infty)\times[0,T)$, where $\hat{\mathcal{L}}$ is as in (9). In this section we use a scaling argument to show that $\sigma^2(y)v_y(x,y,t) \to 0$ and $\sigma\sqrt{y}v_x(x,y,t) \to 0$ as $y \to 0$.

Proposition 5.1. *The function $v = u_y$ satisfies*

$$\lim_{(x,y,t)\to(x_0,0,t)} \sigma^2 v_y(x,y,t) = 0 \tag{17}$$

and

$$\lim_{(x,y,t)\to(x_0,0,t_0)} \sigma\sqrt{y}v_x(x,y,t) = 0 \tag{18}$$

for any $t_0 \in [0,T)$ and any positive x_0. Consequently, Proposition 4.2 gives that

$$\lim_{(x,y,t)\to(x_0,0,t_0)} \sigma^2 u_{yy}(x,y,t) = 0 \quad \text{and} \quad \lim_{(x,y,t)\to(x_0,0,t_0)} \sigma\sqrt{y}u_{xy}(x,y,t) = 0.$$

Remark. Note that Proposition 5.1 shows that the two terms $\frac{1}{2}\sigma^2 u_{yy}$ and $\rho\sigma\sqrt{y}xu_{xy}$ in the Black–Scholes equation (4) approach zero close to the boundary $y = 0$. In addition to this, recall from the proof of Proposition 4.1 that the function $x^2u_{xx}(x,y,t)$ is bounded, so also the term $\frac{1}{2}yx^2u_{xx}$ vanishes near $y = 0$. Consequently, it follows that

$$\lim_{(x,y,t)\to(x_0,0,t_0)} \big(u_t(x,y,t) + \beta(x,y,t)u_y(x,y,t)\big) = 0. \tag{19}$$

Proof. Let $\{(x_n,y_n,t_n)\}_{n=1}^\infty \subseteq (0,\infty)\times(0,\infty)\times[0,T)$ be a sequence of points converging to $(x_0,0,t_0)$, where $t_0 \in [0,T)$ and $x_0 > 0$. Define new coordinates $(\bar{x},\bar{y},s)$ by letting $\bar{x} = k(x - x_0)$, $\bar{y} = my$ and $s = m(t - t_0)$, where k and m are specified more precisely below. Then the function w defined by

$$w(\bar{x},\bar{y},s) = v(x,y,t)$$

satisfies

$$\begin{aligned} &w_s + \frac{1}{2}\bar{y}(x_0+\bar{x}/k)^2\frac{k^2}{m^2}w_{\bar{x}\bar{x}} + \rho\sigma\left(\frac{\bar{y}}{m}\right)\sqrt{\bar{y}}(x_0+\bar{x}/k)\frac{k}{\sqrt{m}}w_{\bar{x}\bar{y}} + \frac{1}{2}\sigma^2\left(\frac{\bar{y}}{m}\right)mw_{\bar{y}\bar{y}} + \rho\hat{a}(\bar{y}/m)(x_0+\bar{x}/k)\frac{k}{m}w_{\bar{x}} \\ &\quad + \hat{\beta}(\bar{y}/m)w_{\bar{y}} + \beta'(\bar{y}/m)\frac{1}{m}w = 0, \end{aligned} \tag{20}$$

where $\hat{a}$ and $\hat{\beta}$ are as in (10) and (11), respectively. Now consider a box $\mathcal{R} = \mathcal{R}^n$ which contains the point (x_n,y_n,t_n), and is such that m can be chosen so that

$$1 \leqslant \sigma^2(y)m \leqslant 2$$

in $\mathcal{R}$. Since $\sigma^2(y)$ by assumption is continuously differentiable up to the boundary, the region $\mathcal{R}$ in $(\bar{x}, \bar{y}, s)$-coordinates does not collapse as $n \to \infty$, but it can rather be chosen to consist of a box of fixed size (the location of the box is not necessarily fixed though). In this box, the coefficients of Eq. (20) satisfy

$$1 \leqslant \sigma^2\left(\frac{\bar{y}}{m}\right)m \leqslant 2, \tag{21}$$

and

$$1 \leqslant \bar{y}x_0^2\frac{k^2}{m^2} \leqslant 2 \tag{22}$$

if k is chosen appropriately. Since $w(\bar{x}, \bar{y}, s) = v(x, y, t)$ we have that w converges to the constant $v(x_0, 0, t_0) = u_y(x_0, 0, t_0)$ uniformly on $\mathcal{R}$ as $n \to \infty$. Therefore, by interior Schauder estimates, $w_{\bar{y}}$ tends to 0 as $n \to \infty$. Since

$$\sigma^2(y)v_y(x, y, t) = \sigma^2\left(\frac{\bar{y}}{m}\right)mw_{\bar{y}}(\bar{x}, \bar{y}, s),$$

and since $\sigma^2(\frac{\bar{y}}{m})m$ is bounded on $\mathcal{R}$, Eq. (17) follows. Similarly, to prove (18), interior estimates show that $w_{\bar{x}}$ tends to 0 as $n \to \infty$, so

$$\sigma(y)\sqrt{y}v_x(x, y, t) = \sigma\left(\frac{\bar{y}}{m}\right)\sqrt{\bar{y}}\frac{k}{\sqrt{m}}w_{\bar{x}}(\bar{x}, \bar{y}, s) \to 0$$

by (21) and (22). $\square$

To show that the boundary condition $u_t + \beta u_y = 0$ also holds *at* the boundary, i.e. not merely in the limit as in (19), we need to show that $u_t(x, 0, t)$ exists and that $u_t(x, 0, t) = -\beta(0)u_y(x, 0, t)$. Since u_y is continuous up to the boundary, we have

$$\begin{aligned} u_t(x, 0, t) &= \lim_{h \to 0} \frac{u(x, 0, t+h) - u(x, 0, t)}{h} \\ &= \lim_{h \to 0} \frac{u(x, h^2, t+h) - u(x, h^2, t) + \mathcal{O}(h^2)}{h} \\ &= \lim_{h \to 0} u_t\bigl(x, h^2, t+\xi\bigr) + \mathcal{O}(h), \end{aligned}$$

for some $\xi \in (0, h)$ by the mean value theorem. Since u_t approaches $-\beta u_y$ at the boundary, this finishes the proof of Theorem 2.3.

6. Uniqueness results

In this section we discuss uniqueness results for the pricing equation in the case of pay-off functions of at most linear growth. In fact, higher moments do not exist in general for these models, see [1], so superlinear contracts may have infinite prices. One should note that some care needs to be taken since uniqueness does not hold even for general linear contracts, compare [18] and the example below.

Example. Assume that the variance process is given by

$$dY_t = \sigma Y_t\, dW_2,$$

and that the correlation ρ is positive. Then the stock price X is a strict local martingale, see [22] or [19], so the price of the stock option, i.e. the option to buy the stock itself, satisfies

$$u(x, y, t) = E_{x,y,t}[X_T] < x. \tag{23}$$

On the other hand, the function $u \equiv x$ is also a classical solution to the corresponding Black–Scholes equation, so there are multiple solutions. (To be precise, the function u in (23) is not formally covered by Theorem 2.3 since the pay-off function $g(x) = x$ is unbounded. Nevertheless, this example strongly indicates that uniqueness is lost for linear contracts.)

Theorem 6.1. *There is at most one classical solution to the pricing equation which is of strictly sublinear growth in x and polynomial in y.*

Remark. More explicitly, the growth assumption in Theorem 6.1 means that the solution v satisfies

$$\bigl|v(x, y, t)\bigr| \leqslant C\bigl(1 + x^{1-\epsilon} + y^m\bigr) \tag{24}$$

for some constants $C, m \geqslant 0$ and $\epsilon > 0$. The boundedness of g and its derivatives assumed in Hypothesis 2.1 is not needed for Theorem 6.1, but we rather assume the pay-off function to be continuous and of strictly sublinear growth.

Proof. Assume that a function v is a classical solution to the pricing equation with $g = 0$, and that v is of strictly sublinear growth in x and polynomial growth in y. For simplicity, we reverse the time by a change of variable $t \to T - t$, so $v_t = \mathcal{L}v$ with an initial condition $v(x, y, 0) = 0$.

Let $h(x, y) = (1 + x + y^{m+1})$, where m is as in (24). In view of the growth condition (5), we can find a large constant M so that

$$Mh > \frac{m(m+1)}{2}\sigma^2(y)y^{m-1} + m\beta(y)y^m = \mathcal{L}h \tag{25}$$

on $(0,\infty)^2 \times [0, T]$. For $\epsilon > 0$, define the function $v^\epsilon : [0,\infty)^2 \times [0, T] \to \mathbb{R}$ by

$$v^\epsilon(x, y, t) = v(x, y, t) + \epsilon e^{Mt} h(x, y).$$

Then

$$v^\epsilon_t - \mathcal{L}v^\epsilon = \epsilon e^{Mt}(Mh - \mathcal{L}h) > 0$$

at all interior points. Let

$$\Gamma := \{(x, y, t)\colon\ v^\epsilon(x, y, t) < 0\},$$

and note that Γ is a bounded set. Assume that $\Gamma \neq \emptyset$, and define

$$t_0 = \inf\{t \geqslant 0\colon\ (x, y, t) \in \overline{\Gamma} \text{ for some } (x, y) \in [0,\infty)^2\}.$$

Since $\overline{\Gamma}$ is compact, there exists a point (x_0, y_0, t_0) with $(x_0, y_0, t_0) \in \overline{\Gamma}$, and by continuity it follows that $v^\epsilon = 0$ at that point. Therefore, $t_0 > 0$ and $x_0 > 0$. First assume that $y_0 = 0$. Then, by the definition of t_0, we have

$$v^\epsilon_t(x_0, y_0, t_0) \leqslant 0$$

and

$$v^\epsilon_y(x_0, y_0, t_0) \geqslant 0.$$

Since $\beta(0)$ is non-negative, it follows that

$$0 \geqslant v^\epsilon_t(x_0, y_0, t_0) - \beta(0)v^\epsilon_y(x_0, y_0, t_0) = \epsilon e^{Mt_0}(M + Mx_0) > 0.$$

This contradiction shows that $y_0 > 0$, so (x_0, y_0, t_0) has to be an interior point. However, at such a point we have $v^\epsilon_t \leqslant 0$, $v^\epsilon_y = 0$, $v^\epsilon_{xx} \geqslant 0$ and $v^\epsilon_{yy} \geqslant 0$ since (x_0, y_0, t_0) is a local minimum. Consequently,

$$0 \geqslant v^\epsilon_t - \mathcal{L}v^\epsilon = \epsilon e^{Mt_0}(Mh - \mathcal{L}h) > 0$$

by (25). This contradiction yields that $\Gamma = \emptyset$. Since $\epsilon > 0$ is arbitrary, it follows that $v \geqslant 0$. The same argument applied to $-v$ shows that $v = 0$, which finishes the proof. $\square$

Theorem 6.2. *Assume that $\rho \leqslant 0$ or that $\sigma(y) \leqslant C(1 + y^\gamma)$ for all y and some constants $C, \gamma \leqslant 1/2$. Then there is at most one classical solution to the pricing equation in the class of functions that are at most linear in x and polynomial in y.*

Proof. Assume that v is a classical solution to the pricing equation with terminal condition $g = 0$. Let $h(x, y) = (1 + x\ln x + xy + y^{m+1})$, where m is large so that

$$v(x, y, t) \leqslant D(1 + x + y^m)$$

for some D. It is straightforward to check that one can find a constant M large so that

$$Mh > \mathcal{L}h$$

at all interior points and

$$Mh > \beta h_y$$

for $y = 0$. The proof can then be finished along the same lines as the proof above. $\square$

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