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Uniqueness and value-sharing of differential polynomials of meromorphic functions

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ABSTRACT

In this paper, we investigate uniqueness problems of meromorphic functions concerning differential polynomials sharing non-zero finite value and give some results. As particular cases of our results we deduce some significant results which improve several earlier results of Fang and Hong (2001) [2], Yang and Yi (2003) [7], Lin and Yi (2004) [5] and others.

1. Introduction and main results

Let $f(z)$ be a non-constant meromorphic function in the whole complex plane. We shall use the following standard notations of the value distribution theory:

$$T(r,f),\quad m(r,f),\quad N(r,f),\quad \overline{N}(r,f),\quad \ldots.$$

See Hayman [4], Yang [8] and Yi and Yang [7]. We denote by $S(r,f)$ any quantity satisfying

$$S(r,f)=o\big(T(r,f)\big),$$

as $r\to+\infty$, possibly outside of a set with finite measure. For any constant 'a' we define

$$\Theta(a,f)=1-\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,\frac{1}{f-a})}{T(r,f)},$$

where $\overline{N}(r,\frac{1}{f-a})$ is the counting function which counts zeros of $f-a$ in $|z|\leqslant r$, counted only once.

Let $g(z)$ be a meromorphic function. If $f(z)-a$ and $g(z)-a$ assume the same zeros with the same multiplicities then we say that $f(z)$ and $g(z)$ share the value 'a', CM, where 'a' is any constant.

It is assumed that the reader is familiar with the notations of the Nevanlinna theory that can be found in [4,8,7].

In 1996, Fang and Hua [1] obtained the following theorem.

Theorem A. *Let f and g be two transcendental entire functions, $n\geqslant 6$ an integer. If f^nf' and g^ng' share the value 1 CM, then either $f=dg$ for some $(n+1)$-th root of unity d or $f^nf'g^ng'=1$.*

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In 1997, Yang and Hua [6] proved the following result.

Theorem B. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. $n \geqslant 11$ an integer and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share the value a CM, then either $f = dg$ for some $(n+1)$-th root of unity d or $g(z) = c_1 e^{ez}$ and $f(z) = c_2 e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1 c_2)^{n+1} c^2 = -a^2$.*

In 2001, Fang and Hong [2] proved the following result.

Theorem C. *Let $f(z)$ and $g(z)$ be two transcendental entire functions, $n \geqslant 11$ be a positive integer. If $f^n(z)(f(z)-1)f'(z)$ and $g^n(z)(g(z)-1)g'(z)$ share 1 CM, then $f(z) \equiv g(z)$.*

In 2004, Lin and Yi [5] proved the following three theorems.

Theorem D. *Let f and g be two transcendental entire functions, $n \geqslant 7$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $f(z) \equiv g(z)$.*

Theorem E. *Let f and g be two non-constant meromorphic functions satisfying $\Theta(\infty, f) > \frac{2}{n+1}$, $n \geqslant 12$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $f \equiv g$.*

Theorem F. *Let f and g be two non-constant meromorphic functions, $n \geqslant 13$ an integer. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share the value 1 CM, then $f(z) \equiv g(z)$.*

In this paper, by introducing the notion of multiplicity, we improve Theorems A, B, C, D, E, F by obtaining the following results.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let $n \geqslant 2$ be an integer satisfying $(n+1)s \geqslant 12$. If $f^n f'$ and $g^n g'$ share the value 1 CM, then either $f = dg$, for some $(n+1)$-th root of unity d or $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$.*

Remark 1.1. If $s = 1$ in Theorem 1.1, then Theorem 1.1 reduces to Theorem B.

Theorem 1.2. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer and $\Theta(\infty, f) > \frac{2}{n+1}$. Let $n \geqslant 4$ be an integer satisfying $(n+1)s \geqslant 12$. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $f(z) \equiv g(z)$.*

Remark 1.2. If $s = 1$ in Theorem 1.2, then Theorem 1.2 improves Theorem E: the condition $n \geqslant 12$ is replaced by $n \geqslant 11$ in Theorem E, then the conclusion remains the same.

Theorem 1.3. *Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let $n \geqslant 3$ be an integer satisfying $(n+1)s \geqslant 12$. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share the value 1 CM, then $f(z) \equiv g(z)$.*

Remark 1.3. If $s = 1$ in Theorem 1.3, then Theorem 1.3 improves Theorem F: the condition $n \geqslant 13$ is replaced by $n \geqslant 11$ in Theorem F, then the conclusion remains the same.

Theorem 1.4. *Let $f(z)$ and $g(z)$ be two transcendental entire functions, whose zeros are of multiplicities at least s, where s is a positive integer. Let $n \geqslant 1$ be an integer satisfying $(n+1)s \geqslant 5$. If $f^n f'$ and $g^n g'$ share the value 1 CM, then either $f = dg$, for some $(n+1)$-th root of unity d or $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$.*

Remark 1.4. If $s = 1$ in Theorem 1.4, then Theorem 1.4 improves Theorem A, that is, the condition $n \geqslant 6$ in Theorem A is replaced by $n \geqslant 4$, then the conclusion remains the same.

Theorem 1.5. *Let f and g be two transcendental entire functions, whose zeros are of multiplicities at least s, where s is a positive integer. Let $n \geqslant 1$ be an integer satisfying $(n+1)s \geqslant 5$. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $f(z) \equiv g(z)$.*

Remark 1.5. If $s = 1$ in Theorem 1.5, then Theorem 1.5 improves Theorem D, that is, the condition $n \geqslant 7$ is replaced by $n \geqslant 4$ in Theorem D, then the conclusion remains the same.

Remark 1.6. Giving specific values for s, we get the following interesting cases for meromorphic functions.

(i) If $s=1$, then $n \geqslant 11$.
(ii) If $s=2$, then $n \geqslant 5$.
(iii) If $s=3$, then $n \geqslant 3$.
(iv) If $s=4$ or 5, then $n \geqslant 2$.
(v) If $s \geqslant 6$, then $n \geqslant 1$.

We can conclude that if f and g have zeros and poles of higher order multiplicity, then we can reduce the value of n.

2. Some lemmas

Lemma 2.1. *(See [8,7].) Let $f(z)$ be a non-constant meromorphic function, k a positive integer and let c be a non-zero finite complex number. Then*

$$\begin{aligned} T(r,f) &\leqslant \overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}-c}\right) - N_0\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f) \\ &\leqslant \overline{N}(r,f) + (k+1)\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}-c}\right) - N_0\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f), \end{aligned} \tag{2.1}$$

here $N_0(r,\frac{1}{f^{(k+1)}})$ is the counting function which only counts those points such that $f^{(k+1)}=0$ but $f(f^{(k)}-c)\neq 0$.

In order to prove our theorems we shall first prove the following lemmas:

Lemma 2.2. *Let $f(z)$ and $g(z)$ be two non-constant transcendental meromorphic functions, k be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share the value 1 CM and if*

$$\Delta = (k+2)\Theta(\infty,f) + 2\Theta(\infty,g) + (k+2)\big[\Theta(0,f)+\Theta(0,g)\big] > 3k+7,$$

then either $f^{(k)}g^{(k)} \equiv 1$ or $f \equiv g$.

Proof. Let

$$\Phi(z) = \frac{f^{(k+2)}}{f^{(k+1)}} - 2\frac{f^{(k+1)}}{f^{(k)}-1} - \frac{g^{(k+2)}}{g^{(k+1)}} + 2\frac{g^{(k+1)}}{g^{(k)}-1}. \tag{2.2}$$

Clearly $m(r,\Phi) = S(r,f)+S(r,g)$. We consider the cases $\Phi(z)\not\equiv 0$ and $\Phi(z)\equiv 0$.

Let $\Phi(z)\not\equiv 0$. Then if z_0 is a common simple 1-point of $f^{(k)}$ and $g^{(k)}$, substituting their Taylor series at z_0 into (2.2), we see that z_0 is a zero of $\Phi(z)$. Thus, we have

$$\begin{aligned} N_{1)}\left(r,\frac{1}{f^{(k)}-1}\right) &= N_{1)}\left(r,\frac{1}{g^{(k)}-1}\right) \\ &\leqslant \overline{N}\left(r,\frac{1}{\Phi}\right) \leqslant T(r,\Phi)+O(1) \leqslant N(r,\Phi)+S(r,f)+S(r,g), \end{aligned} \tag{2.3}$$

here $N_{1)}(r,\frac{1}{f^{(k)}-1})$ is the counting function which only counts those points such that $f^{(k)}-1=0$ but $f^{(k+1)}\neq 0$.

Our assumptions are that $\Phi(z)$ has poles, all simple only at zeros of $f^{(k+1)}$ and $g^{(k+1)}$ and poles of f and g. Thus, we deduce from (2.2) that

$$N(r,\Phi) \leqslant \overline{N}(r,f) + \overline{N}(r,g) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{g}\right) + N_0\left(r,\frac{1}{f^{(k+1)}}\right) + N_0\left(r,\frac{1}{f^{(k+1)}}\right), \tag{2.4}$$

here $N_0(r,\frac{1}{f^{(k+1)}})$ has the same meaning as in Lemma 2.2. Obviously,

$$\begin{aligned} \overline{N}\left(r,\frac{1}{f^{(k)}-1}\right) + \overline{N}\left(r,\frac{1}{g^{(k)}-1}\right) &= 2\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right) \\ &\leqslant N_{1)}\left(r,\frac{1}{f^{(k)}-1}\right) + N\left(r,\frac{1}{f^{(k)}-1}\right). \end{aligned} \tag{2.5}$$

From Lemma 2.1, we have

$$T(r,f) \leqslant \overline{N}(r,f) + (k+1)\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}-1}\right) - N_0\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f), \tag{2.6}$$

$$T(r,g) \leqslant \overline{N}(r,g) + (k+1)\overline{N}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g^{(k)}-1}\right) - N_0\left(r,\frac{1}{g^{(k+1)}}\right) + S(r,g), \tag{2.7}$$

since

$$N\left(r,\frac{1}{f^{(k)}-1}\right) \leqslant T(r,f) + k\overline{N}(r,f) + S(r,f). \tag{2.8}$$

Thus we deduce from (2.3)–(2.8) that

$$T(r,f) + T(r,g) \leqslant 2\overline{N}(r,f) + 2\overline{N}(r,g) + (k+2)\left[\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{g}\right)\right] + k\overline{N}(r,f) + T(r,f) + S(r,f) + S(r,g).$$

Hence

$$T(r,g) \leqslant (k+2)\overline{N}(r,f) + 2\overline{N}(r,g) + (k+2)\left[\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{g}\right)\right] + S(r,f) + S(r,g).$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f) \leqslant T(r,g)$ for $r \in I$. Hence

$$T(r,g) \leqslant \bigl\{(k+2)\bigl[1-\Theta(\infty,f)\bigr] + 2\bigl[1-\Theta(\infty,g)\bigr] + (k+2)\bigl[2-\bigl(\Theta(0,f)+\Theta(0,g)\bigr)\bigr] + \epsilon\bigr\}T(r,g) + S(r,g)$$

for $r \in I$ and $0 < \epsilon < \Delta - (3k+7)$.

Therefore,

$$T(r,g) \leqslant \bigl\{(3k+8) - \Delta + \epsilon\bigr\}T(r,g) + S(r,g)$$

for $r \in I$. This gives

$$\Delta - (3k+7) \leqslant 0, \quad \text{i.e.,} \quad \Delta \leqslant 3k+7$$

which is a contradiction to our hypothesis $\Delta > 3k+7$. Hence, we get $\Phi(z) \equiv 0$. Therefore by (2.2), we have

$$\frac{f^{(k+2)}}{f^{(k+1)}} - \frac{2f^{(k+1)}}{f^{(k)}-1} \equiv \frac{g^{(k+2)}}{g^{(k+1)}} - \frac{2g^{(k+1)}}{g^{(k)}-1}.$$

By solving this, we obtain

$$\frac{1}{f^{(k)}-1} = \frac{bg^{(k)}+a-b}{g^{(k)}-1}, \tag{2.9}$$

where a and b are two constants and $a \neq 0$. Next, we consider three cases:

Case 1. $a = b$.

(i) If $b = -1$, then from (2.9), we obtain that

$$g^{(k)}f^{(k)} \equiv 1.$$

(ii) If $b \neq -1$, then from (2.9), we obtain that

$$\frac{1}{f^{(k)}-1} = \frac{bg^{(k)}}{g^{(k)}-1}.$$

Since

$$\frac{1}{f^{(k)}} = \frac{bg^{(k)}}{(1+b)g^{(k)}-1}, \tag{2.10}$$

we can write

$$\overline{N}\left[r,\frac{1}{g^{(k)}-\frac{1}{1+b}}\right] \leqslant \overline{N}\left[r,\frac{g^{(k)}}{g^{(k)}-\frac{1}{1+b}}\right]. \tag{2.11}$$

From (2.10) and (2.11), we have

$$\overline{N}\left(r, \frac{1}{g^{(k)} - \frac{1}{1+b}}\right) \leqslant \overline{N}\left(r, \frac{1}{f^{(k)}}\right). \tag{2.12}$$

By the first fundamental theorem, we obtain the following inequality

$$\begin{aligned}\overline{N}\left(r, \frac{1}{f^{(k)}}\right) &\leqslant \overline{N}\left(r, \frac{f}{f^{(k)}}\right) + \overline{N}\left(r, \frac{1}{f}\right)\\ &\leqslant T\left(r, \frac{f}{f^{(k)}}\right) + \overline{N}\left(r, \frac{1}{f}\right)\\ &\leqslant N\left(r, \frac{f^{(k)}}{f}\right) + \overline{N}\left(r, \frac{1}{f}\right) + S(r, f).\end{aligned}$$

Clearly, any zero or pole of f of order m is a pole of $\frac{f^{(k)}}{f}$ of order at most k. Hence,

$$N\left(r, \frac{f^{(k)}}{f}\right) \leqslant k\left[\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right)\right].$$

Therefore

$$\begin{aligned}\overline{N}\left(r, \frac{1}{f^{(k)}}\right) &\leqslant k\left[\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right)\right] + \overline{N}\left(r, \frac{1}{f}\right) + S(r, f)\\ &\leqslant k\overline{N}(r, f) + (k+1)\overline{N}\left(r, \frac{1}{f}\right) + S(r, f).\end{aligned} \tag{2.13}$$

Therefore, from (2.12) and (2.13), we have

$$\overline{N}\left(r, \frac{1}{g^{(k)} - \frac{1}{1+b}}\right) \leqslant k\overline{N}(r, f) + (k+1)\overline{N}\left(r, \frac{1}{f}\right) + S(r, f). \tag{2.14}$$

From (2.14) and by Lemma 2.1, we have

$$\begin{aligned}T(r, g) &\leqslant \overline{N}(r, g) + (k+1)\overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g^{(k)} - \frac{1}{1+b}}\right) - N_0\left(r, \frac{1}{g^{(k+1)}}\right) + S(r, g)\\ &\leqslant \overline{N}(r, g) + (k+1)\overline{N}\left(r, \frac{1}{g}\right) + k\overline{N}(r, f) + (k+1)\overline{N}\left(r, \frac{1}{f}\right) + S(r, f) + S(r, g)\\ &\leqslant 2\overline{N}(r, g) + (k+2)\overline{N}(r, f) + (k+2)\left[\overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right)\right] + S(r, f) + S(r, g).\end{aligned}$$

That is,

$$T(r, g) \leqslant \left[(3k+8) - \Delta\right]T(r, g) + S(r, g)$$

for $r \in I$ and r is sufficiently large. That is, $\Delta \leqslant 3k+7$, which is a contradiction to our hypothesis $\Delta > 3k+7$.

Case 2. $b \neq 0$ and $a \neq b$.

Then from (2.9), we obtain

$$f^{(k)} - \left(1 + \frac{1}{b}\right) = \frac{-a}{b^2[g^{(k)} + \frac{a-b}{b}]}.$$

This implies

$$\overline{N}\left[r, \frac{1}{g^{(k)} + (\frac{a-b}{b})}\right] = \overline{N}\left[r, f^{(k)} - \left(1 + \frac{1}{b}\right)\right] = \overline{N}\left(r, f^{(k)}\right) = \overline{N}(r, f). \tag{2.15}$$

From Lemma 2.1 and from (2.15), we have

$$\begin{aligned}T(r,g) &\leqslant \overline{N}(r,g)+(k+1)\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}+\frac{a-b}{b}}\right)+S(r,g)\\ &\leqslant \overline{N}(r,g)+(k+1)\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}(r,f)+S(r,g)\\ &\leqslant 2\overline{N}(r,g)+(k+2)\overline{N}(r,f)+(k+2)\left[\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{f}\right)\right]+S(r,g).\end{aligned}$$

Using the argument as in Case 1, we get a contradiction.

Case 3. $b=0$.

From (2.9), we obtain

$$f=\frac{1}{a}g+p(z), \tag{2.16}$$

where $p(z)$ is a polynomial. If $p(z)\not\equiv 0$, then by second fundamental theorem for small functions we have

$$\begin{aligned}T(r,g) &\leqslant \overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g+ap(z)}\right)+S(r,f)\\ &\leqslant \overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{af}\right)+S(r,f)\\ &\leqslant \overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{f}\right)+S(r,f).\end{aligned} \tag{2.17}$$

Using the argument as in Case 2, we get a contradiction. Therefore, we get $p(z)\equiv 0$, that is,

$$f=\frac{1}{a}g. \tag{2.18}$$

If $a\neq 1$, then $f^{(k)}$ and $g^{(k)}$ sharing the value 1 CM, we deduce from (2.18) that $g^{(k)}\neq 1$, that is,

$$\overline{N}\left(r,\frac{1}{g^{(k)}-1}\right)=0.$$

We can deduce a contradiction as in Case 2. Thus we get that $a=1$, that is, $f\equiv g$.

Thus the proof of Lemma 2.2 is completed. □

Lemma 2.3. *Let $f(z)$ and $g(z)$ be two non-constant transcendental entire functions, k be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share the value 1 CM and if $\Delta=(k+2)[\Theta(0,f)+\Theta(0,g)]>2k+3$, then either $f^{(k)}g^{(k)}\equiv 1$ or $f\equiv g$.*

Proof. Since f and g are entire functions, we have $\overline{N}(r,f)=0$ and $\overline{N}(r,g)=0$.

Proceeding as in the proof of Lemma 2.2, we shall obtain conclusion of Lemma 2.3. □

Lemma 2.4. *(See [6].) Let f and g be two non-constant entire functions, $n\geqslant 1$. If $f^nf'g^ng'=1$, then $f(z)=c_1e^{-cz}$ and $g(z)=c_2e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1c_2)^{n+2}c^2=-1$.*

Lemma 2.5. *(See [3].) Let $Q(w)=(n-1)^2(w^n-1)(w^{n-2}-1)-n(n-2)(w^{n-1}-1)^2$, then $Q(w)=(w-1)^4(w-\beta_1)(w-\beta_2)\cdots(w-\beta_{2n}-6)$, where $\beta_j\in\mathbb{C}-\{0,1\}$ $(j=1,2,\dots,2n-6)$, which are distinct respectively.*

3. Proof of theorems

3.1. Proof of Theorem 1.1

Let $F=\frac{f^{n+1}}{n+1}$ and $G=\frac{g^{n+1}}{n+1}$. Then $F'=f^nf'$ and $G'=g^ng'$.
Consider

$$\overline{N}\left(r,\frac{1}{F}\right)=\overline{N}\left(r,\frac{1}{f^{n+1}}\right)\leqslant\frac{1}{s(n+1)}N\left(r,\frac{1}{F}\right)\leqslant\frac{1}{s(n+1)}\big[T(r,F)+O(1)\big].$$

Therefore

$$\Theta(0,F)=1-\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,\frac{1}{F})}{T(r,F)}\geqslant 1-\frac{1}{s(n+1)}.$$

Similarly

$$\Theta(0,G)\geqslant 1-\frac{1}{s(n+1)},\qquad \Theta(\infty,F)\geqslant 1-\frac{1}{s(n+1)},\qquad \Theta(\infty,G)\geqslant 1-\frac{1}{s(n+1)}.$$

Therefore

$$\Delta=(k+2)\Theta(\infty,F)+2\Theta(\infty,G)+(k+2)\big[\Theta(0,F)+\Theta(0,G)\big]\geqslant(3k+8)-\frac{3k+8}{s(n+1)}. \tag{3.1}$$

For $k=1$, we obtain $\Delta>10$.

Here $F'=f^nf'$ and $G'=g^ng'$ share the value 1 and $\Delta>10$. Then by Lemma 2.2, we get either

$$F'G'\equiv 1\quad\text{or}\quad F\equiv G. \tag{3.2}$$

Consider the case $F'G'\equiv 1$, that is,

$$f^nf'g^ng'\equiv 1. \tag{3.3}$$

Suppose that f has a pole z_0 (with order $p\geqslant s$ say). Then z_0 is a zero of g (with order $m\geqslant s$ say). By (3.3), we get

$$nm+m-1=np+p+1.$$

That is, $(m-p)(n+1)=2$, which is impossible since $n\geqslant 2$ and m, p are positive integers. Therefore, we conclude that f and g are entire functions. From Lemma 2.6, we get $f(z)=c_1e^{cz}$ and $g(z)=c_2e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1c_2)^{n+1}c^2=-1$.

Next we consider another case $F\equiv G$. This gives

$$\frac{f^{n+1}}{n+1}\equiv\frac{g^{n+1}}{n+1},\quad\text{i.e.,}\quad f^{n+1}=g^{n+1}.$$

Hence $f=dg$ for some $(n+1)$-th root of unity d.

3.2. Proof of Theorem 1.2

Let

$$F=\frac{1}{n+2}f^{n+2}-\frac{1}{n+1}f^{n+1},\qquad G=\frac{1}{n+2}g^{n+2}-\frac{1}{n+1}g^{n+1}.$$

Then

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F}\right)&=\overline{N}\left[r,\frac{1}{f^{n+1}(f-\frac{n+2}{n+1})}\right]\\&\leqslant\frac{1}{s(n+1)}N\left[r,\frac{1}{f^{n+1}(f-\frac{n+2}{n+1})}\right]\\&\leqslant\frac{1}{s(n+1)}N\left[r,\frac{1}{F}\right]\\&\leqslant\frac{1}{s(n+1)}\big[T(r,F)+0(1)\big].\end{aligned}$$

Therefore

$$\Theta(0,F)\geqslant 1-\frac{1}{s(n+1)}.$$

Similarly

$$\Theta(0,G)\geqslant 1-\frac{1}{s(n+1)}.$$

Now,

$$\overline{N}(r,F) \leqslant \frac{1}{s(n+2)} N(r,F) \leqslant \frac{1}{s(n+1)} N(r,F).$$

That is,

$$\overline{N}(r,F) \leqslant \frac{1}{s(n+1)} \big[T(r,F) + O(1)\big].$$

Therefore

$$\Theta(\infty,F) \geqslant 1 - \frac{1}{s(n+1)}.$$

Similarly

$$\Theta(\infty,G) \geqslant 1 - \frac{1}{s(n+1)}.$$

In the same manner as in the proof of Theorem 1.1, we obtain $\Delta > 10$ for $k = 1$. Here $F' = f^n(f-1)f'$ and $G' = g^n(g-1)g'$ share the value 1 and $\Delta > 10$. Then by Lemma 2.2, we get either

$$F'G' \equiv 1 \quad \text{or} \quad F \equiv G. \tag{3.4}$$

Consider the case $F'G' \equiv 1$, that is,

$$f^n(f-1)f'\,g^n(g-1)g' \equiv 1. \tag{3.5}$$

Let z_0 be a zero of f of order p_0. From (3.5) we know that z_0 is a pole of g. Suppose that z_0 is a pole of g of order q_0. Again by (3.5), we obtain

$$np_0 + p_0 - 1 = nq_0 + 2q_0 + 1,$$

that is, $(n+1)(p_0 - q_0) = q_0 + 2$, which implies

$$p_0 \geqslant q_0 + 1, \quad \text{that is,} \quad p_0 \geqslant s + 1. \tag{3.6}$$

Let z_1 be a zero of $f-1$ of order p_1, then from (3.5) z_1 is a pole of g of order q_1. Again by (3.5), we get

$$p_1 + p_1 - 1 = nq_1 + 2q_1 + 1,$$

i.e.,

$$2p_1 \geqslant ns + 2s + 2 \quad \text{or} \quad p_1 \geqslant \frac{ns + 2s + 2}{2}. \tag{3.7}$$

Let z_2 be a zero of f' of order p_2, that is not zero of $f(f-1)$, then from (3.5) z_2 is a pole of g of order q_2. Again by (3.5), we get

$$p_2 = nq_2 + 2q_1 + 1 \quad \text{or} \quad p_2 \geqslant ns + 2s + 1. \tag{3.8}$$

In the same manner as above, we have the similar results for the zeros of $g(g-1)g'$. From (3.5), we can write

$$\overline{N}\big(r, f^n(f-1)f'\big) = \overline{N}\left(r, \frac{1}{g^n(g-1)g'}\right),$$

i.e.,

$$\overline{N}(r,f) = \overline{N}\left(r, \frac{1}{g^n(g-1)g'}\right).$$

From (3.6) to (3.8) and $n \geqslant 4$ satisfying $(n+1)s \geqslant 12$, we obtain

$$\begin{aligned}
\overline{N}(r,f) &\leqslant \overline{N}\left(r, \frac{1}{g^n}\right) + \overline{N}\left(r, \frac{1}{g-1}\right) + \overline{N}\left(r, \frac{1}{g'}\right) \\
&\leqslant \frac{1}{n(s+1)} N\left(r, \frac{1}{g}\right) + \frac{2}{ns+2s+2} N\left(r, \frac{1}{g-1}\right) + \frac{1}{ns+2s+1} N\left(r, \frac{1}{g'}\right) \\
&\leqslant \frac{1}{4(s+1)} N\left(r, \frac{1}{g}\right) + \frac{2}{(n+1)s+s+2} N\left(r, \frac{1}{g-1}\right) + \frac{1}{(n+1)s+s+1} N\left(r, \frac{1}{g'}\right)
\end{aligned}$$

$$\leqslant \frac{1}{8}N\left(r,\frac{1}{g}\right)+\frac{2}{12+1+2}N\left(r,\frac{1}{g-1}\right)+\frac{1}{12+1+1}N\left(r,\frac{1}{g'}\right)$$
$$\leqslant \left(\frac{1}{8}+\frac{2}{15}+\frac{1}{14}\right)T(r,g)+S(r,g). \tag{3.9}$$

By the second fundamental theorem and (3.9), we have

$$\begin{aligned} T(r,f) &\leqslant \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)+\overline{N}(r,f)+S(r,f)\\ &\leqslant \frac{1}{s+1}N\left(r,\frac{1}{f}\right)+\frac{2}{15}N\left(r,\frac{1}{f-1}\right)+\overline{N}(r,f)\\ &\leqslant \left(\frac{1}{2}+\frac{2}{15}\right)T(r,f)+\left(\frac{1}{8}+\frac{2}{15}+\frac{1}{14}\right)T(r,g)+S(r,f)+S(r,g)\\ &\leqslant (0.63333)T(r,f)+(0.329733)T(r,g)+S(r,f)+S(r,g). \end{aligned} \tag{3.10}$$

Similarly, we have

$$T(r,g)\leqslant (0.63333)T(r,g)+(0.329733)T(r,f)+S(r,f)+S(r,g). \tag{3.11}$$

From (3.10) and (3.11), we obtain

$$T(r,f)+T(r,g)\leqslant (0.963066)\big[T(r,f)+T(r,g)\big]+S(r,f)+S(r,g),$$

which is a contradiction.

Thus from (3.4), we get $F\equiv G$, that is,

$$f^{n+1}\left(f-\frac{n+2}{n+1}\right)=g^{n+1}\left(g-\frac{n+2}{n+1}\right). \tag{3.12}$$

Let $h=\frac{f}{g}$. If $h\not\equiv 1$, then by (3.12), we have

$$f=\frac{(n+2)(1-h^{n+1})h}{(n+1)(1-h^{n+2})}=\frac{n+2}{n+1}\left(\frac{h^{n+1}}{1+h+\cdots+h^{n+1}}-1\right), \tag{3.13}$$

where h is a non-constant meromorphic function. It follows that $T(r,f)=(n+1)T(r,h)+S(r,f)$. From (3.13), we note that a pole of h is not a pole of $(\frac{h^{n+1}}{1+h+h^2+\cdots+h^{n+1}}-1)$. Hence $\overline{N}(r,f)=\overline{N}(r,\frac{1}{h^{n+2}-1})$. That is,

$$\overline{N}(r,f)=\sum_{j=1}^{n+1}\overline{N}\left(r,\frac{1}{h-\alpha_j}\right),$$

where α_j $(\neq 1)$ $(j=1,2,\ldots,n+1)$ are distinct roots of the algebraic equation $h^{n+2}=1$.

So by Nevanlinna's second fundamental theorem, we obtain

$$\begin{aligned} (n-1)T(r,h) &\leqslant \sum_{j=1}^{n+1}\overline{N}\left(r,\frac{1}{h-\alpha_j}\right)+S(r,h)\\ &\leqslant \overline{N}(r,f)+S(r,h)\\ &\leqslant \big[1-\Theta(\infty,f)+\epsilon\big]T(r,f)+S(r,h)\\ &\leqslant (n+1)\big[1-\Theta(\infty,f)+\epsilon\big]T(r,h)+S(r,h), \end{aligned}$$

where $\epsilon>0$. Therefore, we have $\Theta(\infty,f)\leqslant\frac{2}{n+1}$, which contradicts the assumption. Thus $f\equiv g$. This completes the proof of Theorem 1.2. □

3.3. Proof of Theorem 1.3

Let

$$F=\frac{1}{n+3}f^{n+3}-\frac{2}{n+2}f^{n+2}+\frac{1}{n+1}f^{n+1}\quad\text{and}\quad G=\frac{1}{n+3}g^{n+3}-\frac{2}{n+2}g^{n+2}+\frac{1}{n+1}g^{n+1}.$$

Then

$$F' = f^n(f-1)^2 f' \quad \text{and} \quad G' = g^n(g-1)^2 g'.$$

Proceeding as in the proof of Theorem 1.2, we obtain either

$$F \equiv G \quad \text{or} \quad F'G' \equiv 1. \tag{3.14}$$

Suppose $F'G' \equiv 1$, that is, $f^n(f-1)^2 f' g^n(g-1)^2 g' \equiv 1$. In the same manner as in the proof of Theorem 1.2, we deduce a contradiction. From (3.14), we get $F \equiv G$, that is,

$$\frac{1}{n+3} f^{n+3} - \frac{2}{n+2} f^{n+2} + \frac{1}{n+1} f^{n+1} = \frac{1}{n+3} g^{n+3} - \frac{2}{n+2} g^{n+2} + \frac{1}{n+1} g^{n+1}. \tag{3.15}$$

Setting $h = \frac{f}{g}$, we substitute $f = hg$ in (3.15). It follows that

$$(n+2)(n+1)g^2\big(h^{n+3}-1\big) - 2(n+3)(n+1)g\big(h^{n+2}-1\big) + (n+2)(n+3)\big(h^{n+1}-1\big) = 0. \tag{3.16}$$

If h is not constant, using Lemma 2.5 and (3.16), we obtain

$$\big[(n+1)(n+2)\big(h^{n+3}-1\big)g - (n+3)(n+1)\big(h^{n+2}-1\big)\big]^2 = -(n+3)(n+1)Q(h),$$

where $Q(h) = (h-1)^4(h-\beta_1)(h-\beta_2)\cdots(h-\beta_{2n})$, $\beta_j \in \mathbb{C} - \{0,1\}$ $(j = 1, 2, \ldots, 2n)$, which are pairwise distinct.

This implies that every zero of $h - \beta_j$ $(j = 1, 2, \ldots, 2n)$ has a multiplicity of at least 2. By the second fundamental theorem we obtain that $n \leqslant 2$, which is again a contradiction. Therefore, h is a constant. From (3.16), we have $h^{n+1} - 1 = 0$ and $h^{n+2} - 1 = 0$, which imply $h = 1$ and hence $f \equiv g$.

3.4. *Proof of Theorem 1.4*

Proceeding as in the proof of Theorem 1.1 and applying Lemma 2.3 we shall obtain that Theorem 1.4 holds.

3.5. *Proof of Theorem 1.5*

Proceedings as in the proof of Theorem 1.2 and applying Lemma 2.3, we can easily prove Theorem 1.5.

4. Open problems

Question 4.1. Can n in Theorems 1.1–1.3 be still reduced?

Question 4.2. Is the condition $(n+1)s \geqslant 12$ sharp in Theorems 1.1–1.5?

Question 4.3. Can the condition $\Theta(\infty, f) > \frac{2}{n+1}$ in Theorem 1.2 be deleted?

Question 4.4. Can CM shared value be replaced by an IM shared value in Theorems 1.1–1.5?

Question 4.5. Can the differential polynomials in Theorems 1.1–1.5 be replaced by differential polynomials of the form $(f^n)^{(k)}$ and $[f^n(f-1)]^{(k)}$?

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