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Global existence and blow-up solutions for doubly degenerate parabolic system with nonlocal source

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ABSTRACT

This paper deals with the following nonlocal doubly degenerate parabolic system

$$u_t - \operatorname{div}\big(\big|\nabla u^m\big|^{p-2}\nabla u^m\big) = a\int\limits_{\Omega} u^{\alpha_1}(x,t)v^{\beta_1}(x,t)\,dx,$$

$$v_t - \operatorname{div}\big(\big|\nabla v^n\big|^{q-2}\nabla v^n\big) = b\int\limits_{\Omega} u^{\alpha_2}(x,t)v^{\beta_2}(x,t)\,dx$$

with null Dirichlet boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^N$, where $m,n \geqslant 1$, $p,q > 2$, $\alpha_i,\beta_i \geqslant 0$, $i = 1,2$ and $a,b > 0$ are positive constants. We first get the non-existence result for a related elliptic systems of non-increasing positive solutions. Secondly by using this non-existence result, blow-up estimates for above non-Newton polytropic filtration systems with the homogeneous Dirichlet boundary value conditions are obtained. Then under appropriate hypotheses, we establish local theory of the solutions and prove that the solution either exists globally or blows up in finite time. In the special case, $\beta_1 = n(q-1) - \beta_2$, $\alpha_2 = m(p-1) - \alpha_1$, we also give a criterion for the solution to exist globally or blow up in finite time, which depends on a, b and $\zeta(x)$, $\vartheta(x)$ as defined in the main results.

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1. Introduction

In this paper, we consider the following nonlocal doubly degenerate parabolic system,

$$\begin{aligned}
&u_t - \Delta_{m,p}u = a\int\limits_{\Omega} u^{\alpha_1}(x,t)v^{\beta_1}(x,t)\,dx, \quad (x,t)\in\Omega_T,\\
&v_t - \Delta_{n,q}v = b\int\limits_{\Omega} u^{\alpha_2}(x,t)v^{\beta_2}(x,t)\,dx, \quad (x,t)\in\Omega_T,\\
&u(x,t) = v(x,t) = 0, \quad (x,t)\in\partial\Omega\times(0,T],\\
&u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \quad x\in\Omega,
\end{aligned} \tag{1.1}$$

where for $k > 0$, $\gamma > 2$ and $N \geqslant 1$,

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$$\Delta_{k,\gamma}\Theta = \nabla \cdot \left(\left|\nabla\Theta^k\right|^{\gamma-2} \cdot \nabla\Theta^k\right), \qquad \nabla\Theta^k = k\Theta^{k-1}(\Theta_{x_1}, \dots, \Theta_{x_N}),$$

$\Omega \subset \mathbb{R}^N$ $(N \geqslant 1)$ is a bounded domain with appropriately smooth boundary $\partial\Omega$; $m, n \geqslant 1$, $p, q > 2$, $\alpha_i, \beta_i \geqslant 0$, $i = 1, 2$, $\Omega_T = \Omega \times (0, T]$ and a, b are positive constants and u_0, v_0 satisfy compatibility and the following conditions:

$$(H) \quad u_0^m \in C(\overline{\Omega}) \cap W_0^{1,p}(\Omega), \qquad v_0^n \in C(\overline{\Omega}) \cap W_0^{1,q}(\Omega) \quad \text{and} \quad \nabla u_0^m \cdot \nu < 0,$$
$$\nabla v_0^n \cdot \nu < 0 \quad \text{on } \partial\Omega, \text{ where } \nu \text{ is unit outer normal vector on } \partial\Omega.$$

Parabolic systems like (1.1) arise in many applications in the fields of mechanics, physics and biology like, for instance, the description of turbulent filtration in porous media, the theory of non-Newtonian fluids perturbed by nonlinear terms and forced by rather irregular period in time excitations, the flow of a gas through a porous medium in a turbulent regime or the spread of biological (see [1–4] and the references given therein); in general, doubly nonlinear parabolic equations are used to model processes obeying a nonlinear Darcy law (see [5,6] and the references given therein). In the non-Newtonian fluids theory, the pair (p, q) is a characteristic quantity of medium. Media with $(p, q) > (2, 2)$ are called dilatant fluids and those with $(p, q) < (2, 2)$ are called pseudo-plastics. If $(p, q) = (2, 2)$, they are called Newtonian fluids. When $(p, q) = (2, 2)$ and $(m, n) > (1, 1)$ the connection with the flow in porous media is by now classical. When $(m, n) \geqslant (1, 1)$ and $(p, q) > (2, 2)$, the system models the non-stationary, polytropic flow of a fluid in a porous medium whose tangential stress has a power dependence on the velocity of the displacement under polytropic conditions (non-Newtonian elastic filtration); it has been intensively studied (see [7–9] and references therein). We refer to [10] for further information on these phenomena. Recently a connection has been revealed with soil science; specifically with flows in reservoirs exhibiting fractured media (see [11]).

The problems with nonlinear reaction term and nonlinear diffusion include blow-up and global existence conditions of solutions, blow-up rates and blow-up sets, etc. (see the surveys [12,13]). Here, we say solution blows up in finite time if the solution becomes unbounded (in the sense of maximum norm) at that time. System (1.1) has been studied by many authors. For $p = q = 2$, $m = n = 1$, it is a classical reaction–diffusion system of Fujita type (see [14,15] for nonlinear boundary conditions, see [16] for local nonlinear reaction terms, see [9,17] for nonlocal nonlinear reaction terms).

In the last three decades, many authors have studied the following degenerate parabolic problem:

$$\begin{aligned} &u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f(u), \quad (x,t) \in \Omega \times (0,T], \\ &u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T], \\ &u(x,0) = u_0(x), \quad x \in \Omega \end{aligned} \tag{1.2}$$

under different conditions. In [1,17–23], the existence, uniqueness, extinction phenomenon and regularity of solutions were obtained. If $f(u) = u^q$, $q > 1$, the results in [14,24–27] read: (1) the solution u exists globally if $q < p - 1$; (2) u blows up in finite time if $q > p - 1$ and $u_0(x)$ is sufficiently large. Li and Xie [17] studied the following Eq. (1.2) with $f(u) = \int_\Omega u^q(x,t)\,dx$ under null Dirichlet conditions and obtained that the solution either exists globally or blows up in finite time. Under appropriate hypotheses, they had local theory of the solution and obtained that the solution either exists globally or blows up in finite time.

Li et al. in [28] deal with the following reaction–diffusion system:

$$\begin{aligned} &u_t - \Delta u = \int_\Omega f\left(v(y,t)\right)dy, \quad x \in \Omega,\ t > 0, \\ &v_t - \Delta v = \int_\Omega g\left(u(y,t)\right)dy, \quad x \in \Omega,\ t > 0 \end{aligned}$$

with initial and boundary conditions. They proved that there exists a unique classical solution and the solution either exists globally or blows up in finite time. Furthermore, they obtained the blow-up set and asymptotic behavior provided that the solution blows up in finite time.

For p-Laplacian systems, Cui and Yang [29] and Li [30] studied the following equations:

$$\begin{aligned} &u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \int_\Omega v^\alpha\,dx, \quad (x,t) \in \Omega \times (0,T], \\ &v_t - \operatorname{div}\left(|\nabla v|^{q-2}\nabla v\right) = \int_\Omega u^\beta\,dx, \quad (x,t) \in \Omega \times (0,T], \\ &u(x,t) = v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T], \\ &u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \quad x \in \Omega, \end{aligned} \tag{1.3}$$

which derive some estimates near the blow-up point for positive solutions and non-existence of positive solutions of the relate elliptic systems, with global existence and blow-up of solutions for (1.3).

Very recently, Zhang and Yang [31] further studied the solutions for system (1.1) with $m = n = 1$. They first got the non-existence result for a related elliptic systems of non-increasing positive solutions and by using this result, blow-up estimates for above p-Laplacian systems with the homogeneous Dirichlet boundary value conditions were obtained. Then under appropriate hypotheses, they established local theory of the solutions and obtained that the solutions either exist globally or blow up in finite time. Zhou and Mu [32] and Yang and Lu [33] dealt with the global existence and blow-up properties of the system (1.1) coupled with nonlocal source with $\alpha_1 = \beta_2 = 0$.

This paper extends their results of the references cited above essentially to non-Newton polytropic filtration system (1.1). Therefore, this paper is also an extension of the above results. Due to the nonlinear diffusion terms and doubly degeneration for $u = 0$, $|\nabla u| = 0$ or $v = 0$, $|\nabla v| = 0$, we have some new difficulties to be overcome. Noticing that the system (1.1) includes the Newtonian filtration system ($p = 2$) and the non-Newtonian filtration system ($m = 1$) formally, so the method for it should be synthetic. In fact, we can use the methods for the above two systems to deal with it. Since we know that the blow-up rate is the key element in studying the blow-up properties, such as blow-up set, asymptotic behaviors, and similar solution, see Refs. [34,35]. To understand the blow-up behavior, the first step usually consists in deriving a bound for blow-up rate. To our knowledge there are no results on the blow-up rate estimates for system (1.1). Motivated by the results of the above cited papers, we use the non-existence result of the related elliptic system to establish the blow-up estimates for the doubly degenerate parabolic system (1.1) when $a = b = 1$. Then under appropriate hypotheses, we established local theory of the solutions. The method we used is the so-called 'test function method' and some modifications and adaptations of ideas from [31] and [9]. Our proof is based on argument by the different method of regularization, which involves considering the regularized problem firstly and making a priori estimates for the non-negative approximate solutions by carefully choosing special test functions and a scaling argument, then obtaining the results based on the estimates by a standard limiting process.

At last, we investigate the influence of nonlinear power exponents on the existence and non-existence of global solutions of the system (1.1). By supposing the initial data $(u_0(x), v_0(x))$ satisfies the conditions (H) and using upper- and lower-solution methods, when we allow the nonlinear diffusion terms, we obtain the solution of problem (1.1) blows up in finite time if one of the following conditions holds:

(i) $\beta_1\alpha_2 > [m(p-1) - \alpha_1][n(q-1) - \beta_2]$ and the initial data is sufficiently large;
(ii) $\beta_1\alpha_2 = [m(p-1) - \alpha_1][n(q-1) - \beta_2]$ and Ω contains a sufficiently large ball.

And the solution of problem (1.1) exists globally if one of the following conditions holds:

(i) $\beta_1\alpha_2 < [m(p-1) - \alpha_1][n(q-1) - \beta_2]$;
(ii) $\beta_1\alpha_2 = [m(p-1) - \alpha_1][n(q-1) - \beta_2]$ and the measure of the domain ($\|\Omega\|$) is small;
(iii) $\beta_1\alpha_2 > [m(p-1) - \alpha_1][n(q-1) - \beta_2]$ and the initial values are small.

Because equations in (1.1) are doubly degenerate, we will use the method of generalized regularization to establish the existence of weak solutions to the initial boundary value problem. In order to apply monotonicity, we establish the comparison principle for system (1.1) by choosing suitable test function and Gronwall's inequality. Then by the first eigenvalue and its corresponding eigenfunction to the nonlinear operator on some domain, we construct a positive supersolution. By the first eigenvalue and its corresponding eigenfunctions to the eigenvalue problem for the non-Newtonian filtration system, we construct a positive subsolution. By choosing suitable domains and positive constants for supersolution and subsolution, we can obtain a pair of well-ordered positive supersolution and subsolution. Using comparison principle, we achieve our purpose and obtain the global existence and blow-up of solutions to the problem (1.1).

The main results of the present paper is to extend the results of [31–33] to more generalized cases and this paper is organized as follows. In Section 2, we investigate the global non-existence for the related elliptic system. Section 3 is devoted to blow-up estimate for system (1.1). In Section 4, we give the local existence and uniqueness of system (1.1). In Sections 5 and 6, we give the global existence and blow-up property of solutions to (1.1). Finally, in Section 7, we consider a special case of problem (1.1) and also give a criterion for the solution to exist globally or blow up in finite time, which depends on a, b and $\zeta(x)$, $\vartheta(x)$ as defined in our main results.

2. Non-existence for positive radial solutions of the elliptic system

In order to establish the blow-up estimates for the doubly degenerate parabolic system (1.1) when $a = b = 1$, we first consider radially symmetric solutions of the related elliptic system

$$
\begin{aligned}
-\Delta_{m,p} u &= a \int_\Omega u^{\alpha_1}(x,t) v^{\beta_1}(x,t)\,dx, \quad x \in R^N,\\
-\Delta_{n,q} v &= b \int_\Omega u^{\alpha_2}(x,t) v^{\beta_2}(x,t)\,dx, \quad x \in R^N,
\end{aligned}
\tag{2.4}
$$

where $N \geqslant 3$. In this section, we derive some sufficient conditions under which the system (2.4) has no positive radially symmetric solution. Suppose that $u(x) = u(|x|) = u(r)$, $v(x) = v(|x|) = v(r)$, and $a = b = 1$. We have the following theorems. And the proof of the theorems is based upon a little modification of methods of Zhang and Yang [31] used to prove Theorems 2.1 and 2.2 (for brevity, we will omit the details of the proof).

Theorem 2.1. *Assume that*

(i) $\alpha_1 > m(p-1)$ (*or* $\beta_2 > n(q-1)$) *with* $p, q > 1$, $m, n \geqslant 1$, $\alpha_2, \beta_1 \geqslant 0$;
(ii) $\max\{p, q\} < N < \max\{A, B\}$, *where*

$$A = \frac{mn(p+1)(p-1)(q-1) + np(\alpha_1 - p + 1)(q-1) + qm\beta_1(p-1)}{m\beta_1(p-1) + n(\alpha_1 - p + 1)(q-1)},$$
$$B = \frac{mn(q+1)(p-1)(q-1) + mq(\beta_2 - q + 1)(q-1) + pn\alpha_2(q-1)}{n\alpha_2(q-1) + m(\beta_2 - q + 1)(p-1)}.$$

Then system (2.4) *has no positive radially symmetric solution.*

Theorem 2.2. *Suppose that*

(i) $\alpha_1, \alpha_2, \beta_1, \beta_2 \geqslant 0$, $m, n \geqslant 1$, $p > 1$, $q > 1$;
(ii) $\alpha_2\beta_1 > 0$, $mn\alpha_2\beta_1 > (m(p-1) - \alpha_1)(n(q-1) - \beta_2)$.

If one of the following conditions is satisfied:

(g1) $\alpha_1 < m(p-1)$ *and* $\beta_2 < n(q-1)$, $\max\{p, q\} < N \leqslant \max\{E, F\}$,
(g2) $\alpha_1 \leqslant m(p-1)$ *and* $\beta_2 \leqslant n(q-1)$, $\max\{p, q\} < N < \max\{E, F\}$,

where

$$E = \frac{m(p+1)[(p-1)(n(q-1) - \beta_2) + \beta_1(q+1)]}{\alpha_2\beta_1 - [(m(p-1) - \alpha_1)(n(q-1) - \beta_2)]} + p,$$
$$F = \frac{n(q+1)[(q-1)(m(p-1) - \alpha_1) + \alpha_2(p+1)]}{\alpha_2\beta_1 - [(m(p-1) - \alpha_1)(n(q-1) - \beta_2)]} + q,$$

then system (2.4) *has no radially symmetric positive solution.*

3. Blow-up estimate of doubly degenerate system (1.1)

Since we know that the blow-up rate is the key element in studying the blow-up properties, such as blow-up set, asymptotic behavior, and similar solution, to understand the blow-up behavior, the first step usually consists in deriving a bound for blow-up rate. The blow-up rate for positive solutions of the variational system (1.1) with $m = n = 1$ was obtained by Yang in [31]. To our knowledge there are no results on the blow-up rate estimates for system (1.1). Motivated by the results of the above cited papers, we use the non-existence result of the elliptic system (2.4) obtained in Section 2 to establish the blow-up estimates for the doubly degenerate parabolic system (1.1) when $a = b = 1$. The main result of the present section is to generalize the results in Ref. [31].

Theorem 3.1. *Let* (u, v) *be a solution of* (1.1). *Assume that*

(i) $u(\cdot, t)$, $v(\cdot, t)$ *are non-negative, radially symmetric, and radially decreasing functions of* $r = |x|$;
(ii) $u_t(x, t)$, $v_t(x, t)$ *attain the maxima at* $x = 0$ *for every* $t \in (0, T)$;
(iii) $u_t(x, t) \geqslant 0$, $v_t(x, t) \geqslant 0$ *for* $(x, t) \in Q_T = B_R \times (0, T)$;
(iv) u, v *have a blow-up time* $T < +\infty$;
(v) $\alpha_i + \beta_i > \max\{m(p-1), n(q-1)\}$ *with* $\alpha_i, \beta_i \geqslant 0$, $i = 1, 2$, $m, n \geqslant 1$, $p > 1$, $q > 1$;
(vi) $\min\{m(p\beta_2 - q\beta_1) - mnp(q-2), n(q\alpha_1 - p\alpha_2)\} > \max\{mnq(p-1), mnp\}$ *or* $\max\{m(p\beta_2 - q\beta_1), n(q\alpha_1 - p\alpha_2) - mnq(p-2)\} < \min\{mnq, mnp(q-1)\}$;
(vii) *there are positive constants* k_1, k_2 *and* $\eta < T$ *such that for* $t \in (\eta, T)$,

$$k_2\big(u(0,t)\big)^{\frac{\delta_2(m+m\delta_1-\delta_1)}{\delta_1(n+n\delta_2-\delta_2)}} \leqslant v(0,t) \leqslant k_1\big(u(0,t)\big)^{\frac{\delta_2(m+m\delta_1-\delta_1)}{\delta_1(n+n\delta_2-\delta_2)}}, \tag{3.5}$$

where

$$\delta_1=\frac{m\beta_1 q+(n(q-1)-\beta_2)mp}{\beta_1(p\alpha_2+mq(p-2))+p(\alpha_1-m)(n(q-1)-\beta_2)},\tag{3.6}$$

$$\delta_2=\frac{n\alpha_2 p+(m(p-1)-\alpha_1)nq}{\alpha_2(q\beta_1+np(q-2))+q(\beta_2-n)(m(p-1)-\alpha_1)}.\tag{3.7}$$

If one of the following conditions is satisfied:

(g1) $N=2$, $m,n\geqslant 1$, $p,q>1$, $\alpha_i,\beta_i\geqslant 0$, $i=1,2$;
(g2) $\alpha_1>m(p-1)$ *or* $\beta_2>n(q-1)$, $2\leqslant \max\{p,q\}<N<\max\{A,B\}$;
(g3) $\alpha_1<m(p-1)$ *and* $\beta_2<n(q-1)$, *with* $2\leqslant\max\{p,q\}<N\leqslant\max\{E,F\}$;
(g4) $\alpha_1\leqslant m(p-1)$ *and* $\beta_2\leqslant n(q-1)$, *with* $2\leqslant\max\{p,q\}<N<\max\{E,F\}$,

where A, B, E, F *are defined in Theorems* 2.1 *and* 2.2, *then there are positive constants* c_1, c_2 *and* $t_1\in(0,T)$ *such that*

$$u(x,t)\leqslant u(0,t)\leqslant c_1(T-t)^{-\frac{\delta_1}{m+m\delta_1-\delta_1}},\qquad v(x,t)\leqslant v(0,t)\leqslant c_2(T-t)^{-\frac{\delta_2}{n+n\delta_2-\delta_2}},$$

for $(x,t)\in Q_T\setminus Q_{t_1}$.

Remark 3.1. From the definitions of δ_1 and δ_2, we see that the conditions $\alpha_i+\beta_i>\max\{m(p-1),n(q-1)\}$ with $\alpha_i,\beta_i\geqslant 0$, $i=1,2$ and $\min\{m(p\beta_2-q\beta_1)-mnp(q-2),n(q\alpha_1-p\alpha_2)\}>\max\{mnq(p-1),mnp\}$ or $\max\{m(p\beta_2-q\beta_1),n(q\alpha_1-p\alpha_2)-mnq(p-2)\}<\min\{mnq,mnp(q-1)\}$ are natural for the discussion of the blow-up rate estimate.

Remark 3.2. Conditions (i)–(iii) in Theorem 3.1 are reasonable if we impose appropriate assumptions on the initial data $u_0(x)$ and $v_0(x)$, e.g., positivity, radial symmetry, and a suitable decreasing property with

$$\Delta_{m,p}u+a\int_\Omega u^{\alpha_1}(x,t)v^{\beta_1}(x,t)\,dx\geqslant 0,\qquad \Delta_{n,q}v+b\int_\Omega u^{\alpha_2}(x,t)v^{\beta_2}(x,t)\,dx\geqslant 0.$$

Remark 3.3. Clearly, condition (vii) seems too strong. If $p=q=2$, $m=n=1$, $\alpha_1+\beta_1=\alpha_2+\beta_2$, from Lemma 3.2 in Ref. [36], we know that $k_2u(x,t)^{\delta_2/\delta_1}\leqslant v(x,t)\leqslant k_1u(x,t)^{\delta_1/\delta_2}$ for $(x,t)\in Q_T\setminus Q_{t_0}$ with some $t_0\in(0,T)$. If $p\neq 2$, $q\neq 2$ or $m\neq 1$, $n\neq 1$ or $\alpha_1+\beta_1\neq\alpha_2+\beta_2$, we do not know whether or not condition (vii) holds. We hope this condition can be substantially improved in the future. This is an open problem.

From [33], we give the following Lemma 3.1.

Lemma 3.1. *Assume that conditions* (v) *and* (vi) *in Theorem* 3.1 *hold. Then we have*

$$\min(\delta_1,\delta_2)>0.$$

Proof of Theorem 3.1. Define $g(t)=[u^m(0,t)]^{1/\tau_1}$, $h(t)=[v^n(0,t)]^{1/\tau_2}$ for $t\in(0,T)$, where

$$\tau_1=\frac{m\beta_1 q+(n(q-1)-\beta_2)mp}{\alpha_2\beta_1-(m(p-1)-\alpha_1)(n(q-1)-\beta_2)},$$

$$\tau_2=\frac{n\alpha_2 p+(m(p-1)-\alpha_1)nq}{\alpha_2\beta_1-(m(p-1)-\alpha_1)(n(q-1)-\beta_2)}.$$

By putting $\omega_1(t)=(u^m(r/\gamma(t),t))/\gamma(t)^{\tau_1}$, $\omega_2(t)=(v^n(r/\gamma(t),t))/\gamma(t)^{\tau_2}$, $\gamma(t)=g(t)+h(t)$, $r=|x|$, using the symmetry and assumptions (ii)–(iii) in Theorem 3.1, it follows that

$$0\leqslant\big(\Phi_p(\omega_1')\big)'+\frac{N-1}{r}\Phi_p(\omega_1')+\int_0^r\omega_1^{\frac{\alpha_1}{m}}\omega_2^{\frac{\beta_1}{n}}\leqslant\frac{u_t(0,t)}{\gamma(t)^{p+(p-1)\tau_1}}+\frac{v_t(0,t)}{\gamma(t)^{q+(q-1)\tau_2}},\tag{3.8}$$

$$0\leqslant\big(\Phi_q(\omega_2')\big)'+\frac{N-1}{r}\Phi_q(\omega_2')+\int_0^r\omega_1^{\frac{\alpha_2}{m}}\omega_2^{\frac{\beta_2}{n}}\leqslant\frac{u_t(0,t)}{\gamma(t)^{p+(p-1)\tau_1}}+\frac{v_t(0,t)}{\gamma(t)^{q+(q-1)\tau_2}},\tag{3.9}$$

for any $t\in(0,T)$ and $r\in[0,R_\gamma(t))$.

Since $u(x,t)$, $v(x,t)$ achieve their maxima at $x=0$, we easily see that ω_1 and ω_2 are bounded. Indeed,

$$0\leqslant\omega_1(r,t)\leqslant\frac{u^m(0,t)}{\gamma(t)^{\tau_1}}\leqslant 1,\qquad 0\leqslant\omega_2(r,t)\leqslant\frac{v^n(0,t)}{\gamma(t)^{\tau_2}}\leqslant 1.\tag{3.10}$$

Multiplying (3.8) by $w_{1,r}$ (where $w_{1,r}$ express partial derivation of ω_1 for r), and then integrating with respect to r on $(0,r)$, we have

$$\frac{p-1}{p}|\omega_{1,r}|^p+\omega_1\int_0^r \omega_1^{\frac{\alpha_1}{m}}(s)\omega_2^{\frac{\beta_1}{n}}(s)\,ds-\int_0^r \omega_{1,r}\omega_1^{\frac{\alpha_1}{m}}(s)\omega_2^{\frac{\beta_1}{n}}(s)\,ds\leqslant 0. \tag{3.11}$$

From (3.11) and $\omega_{1,r}\leqslant 0$, it follows that

$$|\omega_{1,r}|\leqslant\left(\frac{mK_1p}{(p-1)(\alpha_1+m)}\right)^{1/p} \tag{3.12}$$

for $t\in(0,T)$ and $r\in[0,R_\gamma(t))$. Similarly, we get

$$|\omega_{2,r}|\leqslant\left(\frac{nK_2q}{(q-1)(\beta_2+n)}\right)^{1/q} \tag{3.13}$$

for $t\in(0,T)$ and $r\in[0,R_\gamma(t))$, where K_1, K_2 are positive constants.

Now we proceed by contradiction to claim that

$$\liminf_{t\to T}\frac{u_t(0,t)}{\gamma(t)^{p+(p-1)\tau_1}}+\frac{v_t(0,t)}{\gamma(t)^{q+(q-1)\tau_2}}=C>0. \tag{3.14}$$

Otherwise, suppose that there exists a sequence $\{t_n\}\subset(0,T)$ with $t_n\to T$ such that

$$\liminf_{t\to T}\frac{u_t(0,t)}{\gamma(t)^{p+(p-1)\tau_1}}+\frac{v_t(0,t)}{\gamma(t)^{q+(q-1)\tau_2}}=0.$$

By using Ascoli–Arzelà theorem, there exists a sequence (still denote by $\{t_n\}$) such that

$$\omega_1(\cdot,t_n)\to\overline{\omega}_1(\cdot),\qquad \omega_2(\cdot,t_n)\to\overline{\omega}_2(\cdot),\quad \text{as } n\to\infty, \tag{3.15}$$

hold uniformly on a compact subset of $[0,\infty)$. Now in the sense of distributions

$$0\leqslant\left(\Phi_p\left(\overline{\omega}_1'\right)\right)'+\frac{N-1}{r}\Phi_p\left(\overline{\omega}_1'\right)+\int_0^r\overline{\omega}_1^{\frac{\alpha_1}{m}}\overline{\omega}_2^{\frac{\beta_1}{n}}=0,$$

$$0\leqslant\left(\Phi_p\left(\overline{\omega}_1'\right)\right)'+\frac{N-1}{r}\Phi_p\left(\overline{\omega}_1'\right)+\int_0^r\overline{\omega}_1^{\frac{\alpha_1}{m}}\overline{\omega}_2^{\frac{\beta_1}{n}}=0. \tag{3.16}$$

The absolute continuity of ω_1, ω_2 implies $\overline{\omega}_1$, $\overline{\omega}_2$ are $C^1(0,\infty)$. By the local existence and uniqueness of initial value problem for (3.16) and using the argument in [37], we conclude that $\overline{\omega}_1,\overline{\omega}_2>0$ on $(0,\infty)$ with $\overline{\omega}_1'(0)=\overline{\omega}_2'(0)=0$.

If $N=2$, $p>2$, we proceed as follow. From (3.16), we infer that $r\Phi_p(\overline{\omega}_1')$, $r\Phi_q(\overline{\omega}_2')$ are decreasing and that there exist $M>0$ and $r_0>0$ such that

$$r\Phi_p\left(\overline{\omega}_1'\right)\leqslant M,\quad \text{for } r\in(r_0,\infty).$$

The last inequality implies that

$$\overline{\omega}_1(s)\geqslant\overline{\omega}_1(s)-\overline{\omega}_1(t)=(-M)^{\frac{1}{p-1}}\int_s^t r^{-\frac{1}{p-1}}\,dr=(-M)^{\frac{1}{p-1}}\left(t^{\frac{p-2}{p-1}}-s^{\frac{p-2}{p-1}}\right) \tag{3.17}$$

for $r_0\leqslant s\leqslant t$. Letting $t\to+\infty$ in (3.17), we obtain a contraction.

If $N=2$, $p=2$, proceeding similarly as above implies that

$$\overline{\omega}_1(s)>\overline{\omega}_1(s)-\overline{\omega}_1(t)>(-M)[\ln t-\ln s]$$

for $r_0\leqslant s\leqslant t$. Letting $t\to+\infty$ in the equality, we obtain a contraction.

Finally, if $N>\max\{p,q\}\geqslant 2$ holds, we know from Theorems 2.1 and 2.2 that system (3.16) has no positive solutions. We conclude that (3.14) is true. It follows from (3.14) that there exists $t_1\in(0,T)$ such that for any $t\in(t_1,T)$ we have

$$0\leqslant\frac{u_t(0,t)}{\gamma(t)^{p+(p-1)\tau_1}}+\frac{v_t(0,t)}{\gamma(t)^{q+(q-1)\tau_2}}\leqslant\frac{u_t(0,t)}{u(0,t)^{\frac{m(1+\delta_1)}{\delta_1}}}+\frac{v_t(0,t)}{v(0,t)^{\frac{n(1+\delta_2)}{\delta_2}}}. \tag{3.18}$$

Integrating (3.18) on $(t,s)\subset(t_1,T)$ and then letting $s\to T$, we obtain

$$c(T-t)\leqslant\frac{\delta_1}{m+m\delta_1-\delta_1}u(0,t)^{-\frac{m+m\delta_1-\delta_1}{\delta_1}}+\frac{\delta_2}{n+n\delta_2-\delta_2}v(0,t)^{-\frac{n+n\delta_2-\delta_2}{\delta_2}}. \tag{3.19}$$

By using condition (vi) and (3.19), we have

$$u(x,t)\leqslant u(0,t)\leqslant c_1(T-t)^{-\frac{\delta_1}{m+m\delta_1-\delta_1}},\quad \text{for any } (x,t)\in Q_T\setminus Q_{t_1}.$$

In the same way we have the blow-up estimate for v. The proof is complete. □

4. Local existence and uniqueness of system (1.1)

In order to study the globally existing and blowing-up solutions to problem (1.1), we need to firstly study the existence of local-in-time weak solutions of (1.1) under appropriate hypotheses in this section. For the Newtonian filtration system ($p=2$) and the evolution p-Laplacian equation ($m=1$), the analog problem was studied in [28,31]. Noticing that the system (1.1) includes the Newtonian filtration system ($p=2$) and the non-Newtonian filtration system ($m=1$) formally, so the method for it should be synthetic. Whereas the equations in (1.1) are degenerate for $u=0$, $|\nabla u|=0$ or $v=0$, $|\nabla v|=0$, there exist some new difficulties to be overcome. Our proof is based on argument by the different method of regularization, which involves considering the regularized problem firstly and making a priori estimates for the non-negative approximate solutions by carefully choosing special test functions and a scaling argument, then obtaining the results based on the estimates by a standard limiting process. From a physical point of view, we need only to consider the non-negative solutions. Moreover, if we assume that $u_0(x), v_0(x)\geqslant 0$ in Ω, by Lemma 4.1 (see it below), we can obtain that $(u(x,t),v(x,t))\geqslant(0,0)$ a.e. in $(\Omega\times(0,T))\times(\Omega\times(0,T))$. Thus we only consider the non-negative solutions in later sections.

As it is well known that doubly degenerate equations need not have classical solutions, we give a precise definition of a weak solution for problem (1.1). Let $\Omega_T=\Omega\times(0,T]$, $S_T=\partial\Omega\times[0,T]$, $T>0$.

Definition 4.1. A pair of functions (u,v) is called a solution of the problem (1.1) on $\overline{\Omega}_T\times\overline{\Omega}_T$ if and only if

$$\begin{aligned}
&u^m(x,t)\in C\big(0,T;L^\infty(\Omega)\big)\cap L^p\big(0,T;W_0^{1,p}(\Omega)\big),\\
&v^n(x,t)\in C\big(0,T;L^\infty(\Omega)\big)\cap L^q\big(0,T;W_0^{1,q}(\Omega)\big),\\
&\big(u^m\big)_t\in L^2\big(0,T;L^2(\Omega)\big),\qquad \big(v^n\big)_t\in L^2\big(0,T;L^2(\Omega)\big),\\
&u(x,0)=u_0(x),\qquad v(x,0)=v_0(x)
\end{aligned}$$

and the equalities

$$\begin{aligned}
&\int_\Omega u(x,t_2)\psi_1(x,t_2)\,dx-\int_\Omega u(x,t_1)\psi_1(x,t_1)\,dx\\
&\quad=\int_{t_1}^{t_2}\int_\Omega u\psi_{1t}\,dx\,dt-\int_{t_1}^{t_2}\int_\Omega\big|\nabla u^m\big|^{p-2}\nabla u^m\cdot\nabla\psi_1\,dx\,dt+a\int_{t_1}^{t_2}\int_\Omega\psi_1(x,t)\bigg(\int_\Omega u^{\alpha_1}v^{\beta_1}\,dx\bigg)dx\,dt,
\end{aligned}\tag{4.20}$$

$$\begin{aligned}
&\int_\Omega v(x,t_2)\psi_2(x,t_2)\,dx-\int_\Omega v(x,t_1)\psi_2(x,t_1)\,dx\\
&\quad=\int_{t_1}^{t_2}\int_\Omega v\psi_{2t}\,dx\,dt-\int_{t_1}^{t_2}\int_\Omega\big|\nabla v^n\big|^{q-2}\nabla v^n\cdot\nabla\psi_2\,dx\,dt+b\int_{t_1}^{t_2}\int_\Omega\psi_2(x,t)\bigg(\int_\Omega u^{\alpha_2}v^{\beta_2}\,dx\bigg)dx\,dt
\end{aligned}\tag{4.21}$$

hold for all $0<t_1<t_2<T$, where $\psi_1(x,t),\psi_2(x,t)\in C^{1,1}(\overline{Q}_T)$ such that $\psi_1(x,T)=\psi_2(x,T)=0$ and $\psi_1(x,t)=\psi_2(x,t)=0$ on S_T.

Similarly, to define a subsolution $(\underline{u}(x,t),\underline{v}(x,t))$ we need only to require that $\psi_1(x,t)\geqslant 0$, $\psi_2(x,t)\geqslant 0$, $(\underline{u}(x,0),\underline{v}(x,0))\leqslant(u_0(x),v_0(x))$ on $\Omega\times\Omega$, $(\underline{u}(x,t),\underline{v}(x,t))\leqslant(0,0)$ on $S_T\times S_T$ and the equalities in (4.20) and (4.21) are replaced by $\leqslant$. A supersolution can be defined similarly.

Definition 4.2. We say the solution (u,v) of the problem (1.1) blows up in finite time if there exists a positive constant $T^\star<\infty$, such that

$$\lim_{t\to T^{\star-}}\big(\big|u(\cdot,t)\big|_{L^\infty(\Omega)}+\big|v(\cdot,t)\big|_{L^\infty(\Omega)}\big)=+\infty.$$

We say the solution (u, v) exists globally if

$$\sup_{t\in(0,+\infty)}\big(\big|u(\cdot,t)\big|_{L^\infty(\Omega)}+\big|v(\cdot,t)\big|_{L^\infty(\Omega)}\big)<+\infty.$$

By a modification of the method given in [7,31], we obtain the following results.

Theorem 4.1. *Suppose that $(u_0, v_0) \geqslant (0,0)$ and satisfies the conditions (H), then there exists a constant $T_0 > 0$ such that the problem (1.1) admits a unique solution $(u, v) \in Q_{T_0} \times Q_{T_0}$, $u^m \in C(0,T; L^\infty(\Omega)) \cap L^p(0,T; W_0^{1,p}(\Omega))$, $v^n \in C(0,T; L^\infty(\Omega)) \cap L^q(0,T; W_0^{1,q}(\Omega))$.*

We first give a comparison lemma for the non-degenerate parabolic system, which plays a crucial role in the proof of our results.

Lemma 4.1 *(Comparison principle). Suppose that $(\underline{u}(x,t), \underline{v}(x,t))$ and $(\overline{u}(x,t), \overline{v}(x,t))$ are the lower and upper solution of problem (1.1) on $\overline{\Omega}_T \times \overline{\Omega}_T$, respectively. Then $(\underline{u}(x,t), \underline{v}(x,t)) \leqslant (\overline{u}(x,t), \overline{v}(x,t))$ a.e. on $\overline{\Omega}_T \times \overline{\Omega}_T$.*

Proof of this lemma is similar to that given in [38,1,7], we omit it here.

Proof of Theorem 4.1. Consider the following approximate problems for the problem (1.1):

$$\begin{aligned}
&u_{it}-\operatorname{div}\Big(\big(\big|\nabla u_i^m\big|^2+\varepsilon_i\big)^{\frac{p-2}{2}}\nabla u_i^m\Big)=a\int_\Omega u_i^{\alpha_1}v_i^{\beta_1}\,dx,\quad (x,t)\in\Omega_T,\\
&v_{it}-\operatorname{div}\Big(\big(\big|\nabla v_i^n\big|^2+\sigma_i\big)^{\frac{q-2}{2}}\nabla v_i^n\Big)=b\int_\Omega u_i^{\alpha_2}v_i^{\beta_2}\,dx,\quad (x,t)\in\Omega_T,\\
&u_i(x,t)=\varepsilon_i,\qquad v_i(x,t)=\sigma_i,\quad (x,t)\in S_T,\\
&u_i(x,0)=u_{0\varepsilon_i}(x)+\varepsilon_i,\qquad v_i(x,0)=v_{0\sigma_i}(x)+\sigma_i,\quad x\in\Omega.
\end{aligned}\tag{4.22}$$

Here ε_i, σ_i are strictly decreasing sequences, $0<\varepsilon_i,\sigma_i<1$, and $\varepsilon_i\to 0^+,\sigma_i\to 0^+$ as $i\to+\infty$. $u_{0\varepsilon_i}, v_{0\sigma_i}\in C_0^\infty(\Omega)$ are approximation functions for the initial data $u_0(x)$ and $v_0(x)$, respectively. $|u_{0\varepsilon_i}+\varepsilon_i|_{L^\infty(\Omega)}\leqslant|u_0+1|_{L^\infty(\Omega)}$, $|\nabla u_{0\varepsilon_i}^m|_{L^\infty(\Omega)}\leqslant|\nabla u_0^m|_{L^\infty(\Omega)}$, for all ε_i, and $(u_{0\varepsilon_i}+\varepsilon_i)^m\to u_0^m$ strongly in $W_0^{1,p}(\Omega)$; $|v_{0\sigma_i}+\sigma_i|_{L^\infty(\Omega)}\leqslant|v_0+1|_{L^\infty(\Omega)}$, $|\nabla v_{0\sigma_i}^n|_{L^\infty(\Omega)}\leqslant|\nabla v_0^n|_{L^\infty(\Omega)}$, for all σ_i, and $(v_{0\sigma_i}+\sigma_i)^n\to v_0^n$ strongly in $W_0^{1,q}(\Omega)$.

(4.22) is a non-degenerate problem for each fixed ε_i and σ_i; it is easy to prove that it admits a unique classic solution (u_i, v_i) by using the Schauder's fixed point theorem and $(u_i,v_i)\geqslant(\varepsilon_i,\sigma_i)>(0,0)$ by the classical theory for parabolic equations (see [39]).

To find limit functions $(u(x,t), v(x,t))$ of the sequence (u_i, v_i), we divide our proof into four steps:

Step 1: There exist a small constant $T_0>0$ and a positive constant M_1, independent of i, such that

$$|u_i|_{L^\infty(\Omega_{T_0})},|v_i|_{L^\infty(\Omega_{T_0})}\leqslant M_1.\tag{4.23}$$

To this end, we consider the following Cauchy problem:

$$\begin{aligned}
&\frac{dU_1}{dt}=a|\Omega|U_1^{\alpha_1}V_1^{\beta_1},\qquad \frac{dV_1}{dt}=b|\Omega|U_1^{\alpha_2}V_1^{\beta_2},\\
&U_1(0)=|u_0+1|_{L^\infty(\Omega)},\qquad V_1(0)=|v_0+1|_{L^\infty(\Omega)},
\end{aligned}\tag{4.24}$$

$$\begin{aligned}
&\frac{dU_2}{dt}=-a|\Omega|U_2^{\alpha_1}V_2^{\beta_1},\qquad \frac{dV_2}{dt}=-b|\Omega|U_2^{\alpha_2}V_2^{\beta_2},\\
&U_2(0)=-|u_0+1|_{L^\infty(\Omega)},\qquad V_2(0)=-|v_0+1|_{L^\infty(\Omega)}.
\end{aligned}\tag{4.25}$$

It is easy to verify that there exists a constant $t_0\in(0,T)$ such that (4.24) and (4.25) admit a solution $(U_1(t),V_1(t))$ and $(U_2(t),V_2(t))$ on $[0,t_0]$, respectively, moreover, the t_0 depends only on $|u_0+1|_{L^\infty(\Omega)},|v_0+1|_{L^\infty(\Omega)}$. By a comparison principle for the approximate problem (see [21,39]), we get

$$\big|u_i(x,t)\big|,\big|v_i(x,t)\big|\leqslant\max\big\{U_1(t),V_1(t),-U_2(t),-V_2(t)\big\}.$$

Setting $T_0=t_0/2$ and $M_1=\max\{U_1(t),V_1(t),-U_2(t),-V_2(t)\}$ we draw our conclusion.

Step 2: There exists a constant $M_2 > 0$, independent of i, such that

$$\left|\nabla u_i^m\right|_{L^p(\Omega_{T_0})}, \left|\nabla v_i^n\right|_{L^q(\Omega_{T_0})} \leqslant M_2. \tag{4.26}$$

In fact, multiplying the first equation in (4.22) by u_i^m, the second equation in (4.22) by v_i^n and integrating the results over Ω_{T_0} we have

$$\begin{aligned}
&\frac{1}{m+1}\int_\Omega u_i^{m+1}(x,T_0)\,dx + \int_0^{T_0}\int_\Omega \left(\left|\nabla u_i^m\right|^2+\varepsilon_i\right)^{\frac{p-2}{2}}\left|\nabla u_i^m\right|^2 dx\,dt \\
&\quad = \iint_{S_T} \left(\left|\nabla u_i^m\right|^2+\varepsilon_i\right)^{\frac{p-2}{2}} \nabla u_i^m \cdot \vec{n}\, u_i^m\,dx\,dt + \frac{1}{m+1}\int_\Omega \left(u_{0\varepsilon_i}(x)+\varepsilon_i\right)^{m+1} dx \\
&\qquad + a\int_0^{T_0}\left(\int_\Omega u_i^m(x,t)\,dx\right)\left(\int_\Omega u^{\alpha_1}v^{\beta_1}\,dx\right)dt, \\
&\frac{1}{n+1}\int_\Omega v_i^{n+1}(x,T_0)\,dx + \int_0^{T_0}\int_\Omega \left(\left|\nabla v_i^n\right|^2+\sigma_i\right)^{\frac{q-2}{2}}\left|\nabla v_i^n\right|^2 dx\,dt \\
&\quad = \iint_{S_T} \left(\left|\nabla v_i^n\right|^2+\sigma_i\right)^{\frac{q-2}{2}} \nabla v_i^n \cdot \vec{n}\, v_i^n\,dx\,dt + \frac{1}{n+1}\int_\Omega \left(v_{0\sigma_i}(x)+\sigma_i\right)^{n+1} dx \\
&\qquad + b\int_0^{T_0}\left(\int_\Omega v_i^n(x,t)\,dx\right)\left(\int_\Omega u^{\alpha_2}v^{\beta_2}\,dx\right)dt.
\end{aligned}$$

By $|u_{0\varepsilon_i}+\varepsilon_i|_{L^\infty(\Omega)} \leqslant |u_0+1|_{L^\infty(\Omega)}$, $|v_{0\sigma_i}+\sigma_i|_{L^\infty(\Omega)} \leqslant |v_0+1|_{L^\infty(\Omega)}$, and (4.23) we get

$$\int_0^{T_0}\int_\Omega \left|\nabla u_i^m\right|^p dx\,dt \leqslant \frac{|\Omega|}{m+1}|u_0+1|_{L^\infty(\Omega)}^{m+1} + aT_0|\Omega|^2 M_1^{m+\alpha_1+\beta_1} \triangleq M_{2u},$$

$$\int_0^{T_0}\int_\Omega \left|\nabla v_i^n\right|^q dx\,dt \leqslant \frac{|\Omega|}{n+1}|v_0+1|_{L^\infty(\Omega)}^{n+1} + bT_0|\Omega|^2 M_1^{n+\alpha_2+\beta_2} \triangleq M_{2v}.$$

Setting

$$M_2 = \max\{M_{2u}, M_{2v}\},$$

we draw our conclusion.

Step 3: There exists a constant $M_3 > 0$, independent of i, such that

$$\left|\left(u_i^m\right)_t\right|_{L^2(\Omega_{T_0})}, \left|\left(v_i^n\right)_t\right|_{L^2(\Omega_{T_0})} \leqslant M_3. \tag{4.27}$$

To do so, multiplying the first equation in (4.22) by $(u_i^m)_t$, the second equation in (4.22) by $(v_i^n)_t$ and integrating the results over Ω_{T_0} we get

$$\begin{aligned}
\int_0^{T_0}\int_\Omega m u_i^{m-1}(u_{it})^2(x,t)\,dx\,dt = &-\int_0^{T_0}\int_\Omega \left(\left|\nabla u_i^m\right|^2+\varepsilon_i\right)^{\frac{p-2}{2}}\nabla u_i^m\cdot\nabla\left(u_i^m\right)_t\,dx\,dt \\
&+ a\int_0^{T_0}\int_\Omega \left(u_i^m\right)_t(x,t)\left(\int_\Omega u^{\alpha_1}v^{\beta_1}\,dx\right)dx\,dt,
\end{aligned}$$

$$\int_0^{T_0}\int_\Omega n v_i^{n-1}(v_{it})^2(x,t)\,dx\,dt = -\int_0^{T_0}\int_\Omega \left(\left|\nabla v_i^n\right|^2+\sigma_i\right)^{\frac{q-2}{2}}\nabla v_i^n\cdot\nabla\left(v_i^n\right)_t\,dx\,dt$$
$$+\,b\int_0^{T_0}\int_\Omega \left(v_i^n\right)_t(x,t)\left(\int_\Omega v^{\alpha_2}v^{\beta_2}\,dx\right)dx\,dt.$$

By Hölder's inequality, $|u_{0\varepsilon_i}+\varepsilon_i|_{L^\infty(\Omega)}\leqslant|u_0+1|_{L^\infty(\Omega)}$, $|v_{0\sigma_i}+\sigma_i|_{L^\infty(\Omega)}\leqslant|v_0+1|_{L^\infty(\Omega)}$, inequality (4.23) and the equalities

$$\int_0^{T_0}\int_\Omega \left(\left|\nabla u_i^m\right|^2+\varepsilon_i\right)^{\frac{p-2}{2}}\nabla u_i^m\cdot\nabla\left(u_i^m\right)_t\,dx\,dt = \frac{1}{p}\int_\Omega\left(\left|\nabla u_{\varepsilon_i}^m(x,T_0)\right|^2+\varepsilon_i\right)^{\frac{p}{2}}dx-\frac{1}{p}\int_\Omega\left(\left|\nabla u_{0\varepsilon_i}^m\right|^2+\varepsilon_i\right)^{\frac{p}{2}}dx,$$
$$\int_0^{T_0}\int_\Omega \left(\left|\nabla v_i^n\right|^2+\sigma_i\right)^{\frac{q-2}{2}}\nabla v_i^n\cdot\nabla\left(v_i^n\right)_t\,dx\,dt = \frac{1}{q}\int_\Omega\left(\left|\nabla v_{\sigma_i}^n(x,T_0)\right|^2+\sigma_i\right)^{\frac{q}{2}}dx-\frac{1}{q}\int_\Omega\left(\left|\nabla v_{0\sigma_i}^n\right|^2+\sigma_i\right)^{\frac{q}{2}}dx,$$

we obtain

$$\int_0^{T_0}\int_\Omega m u_i^{m-1}(u_{it})^2(x,t)\,dx\,dt$$
$$\leqslant-\frac{1}{p}\int_\Omega\left(\left|\nabla u_{\varepsilon_i}^m(x,T_0)\right|^2+\varepsilon_i\right)^{\frac{p}{2}}dx+\frac{1}{p}\int_\Omega\left(\left|\nabla u_{0\varepsilon_i}^m\right|^2+\varepsilon_i\right)^{\frac{p}{2}}dx+\frac{1}{2}\int_0^{T_0}\int_\Omega m u_i^{m-1}(u_{it})^2(x,t)\,dx\,dt$$
$$+\,2a^2\int_0^{T_0}\int_\Omega m u_i^{m-1}\left(\int_\Omega u_i^{\alpha_1}v_i^{\beta_1}\,dx\right)^2dx\,dt,$$
$$\int_0^{T_0}\int_\Omega n v_i^{n-1}(v_{it})^2(x,t)\,dx\,dt$$
$$\leqslant-\frac{1}{q}\int_\Omega\left(\left|\nabla v_{\sigma_i}^n(x,T_0)\right|^2+\sigma_i\right)^{\frac{q}{2}}dx+\frac{1}{q}\int_\Omega\left(\left|\nabla v_{0\sigma_i}^n\right|^2+\sigma_i\right)^{\frac{q}{2}}dx+\frac{1}{2}\int_0^{T_0}\int_\Omega n v_i^{n-1}(v_{it})^2(x,t)\,dx\,dt$$
$$+\,2b^2\int_0^{T_0}\int_\Omega n v_i^{n-1}\left(\int_\Omega u_i^{\alpha_2}v_i^{\beta_2}\,dx\right)^2dx\,dt.$$

So

$$\int_0^{T_0}\int_\Omega m u_i^{m-1}(u_{it})^2(x,t)\,dx\,dt\leqslant C,$$
$$\int_0^{T_0}\int_\Omega n v_i^{n-1}(v_{it})^2(x,t)\,dx\,dt\leqslant C.$$

Using (4.23) and Young's inequality, we get

$$\left|\left(u_i^m\right)_t\right|_{L^2(\Omega_{T_0})}=\int_0^{T_0}\int_\Omega m u_i^{m-1}\left[m u_i^{m-1}(u_{it})^2(x,t)\right]dx\,dt\leqslant M_3,$$
$$\left|\left(v_i^n\right)_t\right|_{L^2(\Omega_{T_0})}=\int_0^{T_0}\int_\Omega n v_i^{n-1}\left[n v_i^{n-1}(v_{it})^2(x,t)\right]dx\,dt\leqslant M_3$$

for some $M_3>0$.

Therefore, by virtue of (4.23), (4.26) and (4.27) and the Ascoli–Arzelà theorem, we can choose a subsequence, still denoted by $\{(u_i, v_i)\}$ for convenience, such that

$$u_i \to u, \qquad v_i \to v \quad \text{a.e. for } (x,t) \in \Omega_{T_0}, \tag{4.28}$$

$$\nabla u_i^m \to \nabla u^m \quad \text{weakly in } L^p\big(0, T_0; L^p(\Omega)\big), \tag{4.29}$$

$$\nabla v_i^n \to \nabla v^n \quad \text{weakly in } L^q\big(0, T_0; L^q(\Omega)\big), \tag{4.30}$$

$$\big(u_i^m\big)_t \to u_t^m, \qquad \big(v_i^n\big)_t \to v_t^n \quad \text{weakly in } L^2\big(0, T_0; L^2(\Omega)\big), \tag{4.31}$$

$$\big|\nabla u_i^m\big|^{p-2}\big(u_i^m\big)_{x_i} \to w_i \quad \text{weakly in } L^{\frac{p}{p-1}}\big(0, T_0; L^{\frac{p}{p-1}}(\Omega)\big), \tag{4.32}$$

$$\big|\nabla v_i^n\big|^{q-2}\big(v_i^n\big)_{x_i} \to z_i \quad \text{weakly in } L^{\frac{q}{q-1}}\big(0, T_0; L^{\frac{q}{q-1}}(\Omega)\big). \tag{4.33}$$

Step 4: We show that $w_i = |\nabla u^m|^{p-2}(u^m)_{x_i}$ and $z_i = |\nabla v^n|^{q-2}(v^n)_{x_i}$.

Multiplying the first equation in (4.22) by $\psi_1(u_i^m - u^m)$, the second equation in (4.22) by $\psi_2(v_i^n - v^n)$ and integrating the results over Q_{T_0}, we have

$$\begin{aligned}
&\int_0^{T_0}\int_\Omega \psi_1\big(u_i^m - u^m\big)u_{it}\,dx\,dt + \int_0^{T_0}\int_\Omega \psi_1\big(\big|\nabla u_i^m\big|^2 + \varepsilon_i\big)^{\frac{p-2}{2}}\nabla u_i^m \cdot \nabla\big(u_i^m - u^m\big)\,dx\,dt \\
&\qquad + \int_0^{T_0}\int_\Omega \big(u_i^m - u^m\big)\big(\big|\nabla u_i^m\big|^2 + \varepsilon_i\big)^{\frac{p-2}{2}}\nabla u_i^m \cdot \nabla\psi_1\,dx\,dt \\
&\quad = a\int_0^{T_0}\int_\Omega \psi_1\big(u_i^m - u^m\big)\bigg(\int_\Omega u_i^{\alpha_1} v_i^{\beta_1}\,dx\bigg)\,dx\,dt, \\
&\int_0^{T_0}\int_\Omega \psi_2\big(v_i^n - v^n\big)v_{it}\,dx\,dt + \int_0^{T_0}\int_\Omega \psi_2\big(\big|\nabla v_i^n\big|^2 + \sigma_i\big)^{\frac{q-2}{2}}\nabla v_i^n \cdot \nabla\big(v_i^n - v^n\big)\,dx\,dt \\
&\qquad + \int_0^{T_0}\int_\Omega \big(v_i^n - v^n\big)\big(\big|\nabla v_i^n\big|^2 + \sigma_i\big)^{\frac{q-2}{2}}\nabla v_i^n \cdot \nabla\psi_2\,dx\,dt \\
&\quad = b\int_0^{T_0}\int_\Omega \psi_2\big(v_i^n - v^n\big)\bigg(\int_\Omega u_i^{\alpha_2} v_i^{\beta_2}\,dx\bigg)\,dx\,dt.
\end{aligned}$$

Using (4.23), (4.28) and (4.31) we get

$$\lim_{n\to\infty}\int_0^{T_0}\int_\Omega \psi_1\big|\nabla u_i^m\big|^{p-2}\nabla u_i^m \cdot \nabla\big(u_i^m - u^m\big)\,dx\,dt = 0,$$

$$\lim_{n\to\infty}\int_0^{T_0}\int_\Omega \psi_2\big|\nabla v_i^n\big|^{q-2}\nabla v_i^n \cdot \nabla\big(v_i^n - v^n\big)\,dx\,dt = 0,$$

where $\psi_1, \psi_2 \in C^{1,1}(\Omega_{T_0})$, $\psi_1, \psi_2 \geqslant 0$.

The left arguments are as same as those of Theorem 1 in [9], so we omit them.

We complete the existence part by a standard limiting process.

The uniqueness of the solution is obvious. In fact, assume that (u_1, v_1), (u_2, v_2) are two non-negative solutions of (1.1); using Lemma 4.1 repeatedly, we can get $u_1 = u_2$, $v_1 = v_2$ a.e. in $\overline{\Omega} \times [0, T_0]$. $\square$

5. Global existence of a solution

In this section, we investigate the global existence property of the solutions to problem (1.1) and prove Theorem 5.1. The main method is constructing a globally upper solution and using comparison principle to achieve our purpose.

Theorem 5.1. *Suppose that the initial data $(u_0(x), v_0(x))$ satisfies the conditions (H), then the solution of problem (1.1) exists globally if one of the following conditions holds*:

(i) $\beta_1\alpha_2 < [m(p-1)-\alpha_1][n(q-1)-\beta_2]$;
(ii) $\beta_1\alpha_2 = [m(p-1)-\alpha_1][n(q-1)-\beta_2]$ *and the measure of the domain* $(\|\Omega\|)$ *is small*;
(iii) $\beta_1\alpha_2 > [m(p-1)-\alpha_1][n(q-1)-\beta_2]$ *and the initial values are small.*

In order to study the globally existing solutions to problem (1.1), we need to study the following elliptic system

$$-\Delta_{k,\gamma}\Theta = 1, \quad x\in\Omega,$$
$$\Theta = 1, \quad x\in\partial\Omega, \tag{5.34}$$

where $\Delta_{k,\gamma}\Theta$ is defined in (1.1), and we obtain the following lemma.

Lemma 5.1. *Problem* (5.34) *has a unique solution $\Theta(x)$, and satisfies the following relations,*

$$\Theta(x) > 1 \quad \textit{in } \Omega, \qquad \nabla\Theta\cdot\nu < 0 \quad \textit{on } \partial\Omega, \qquad \sup_{x\in\Omega}\Theta = M < +\infty,$$

where M is a positive constant.

Proof of this lemma is similar to that given in [32], we omit it here.

Proof of Theorem 5.1. Let $\varphi(x)$ and $\psi(x)$ be the unique solutions of the following elliptic problem

$$\begin{cases} -\Delta_{m,p}\varphi = 1, & x\in\Omega,\\ \varphi = 1, & x\in\partial\Omega,\end{cases} \qquad \begin{cases} -\Delta_{n,q}\psi = 1, & x\in\Omega,\\ \psi = 1, & x\in\partial\Omega.\end{cases}$$

Then from Lemma 5.1, we obtain the following relations

$$\varphi(x), \psi(x) > 1 \quad \text{in } \Omega, \qquad \nabla\varphi\cdot\nu, \nabla\psi\cdot\nu < 0 \quad \text{on } \partial\Omega, \tag{5.35}$$
$$M = \max\Big\{\sup_{x\in\Omega}\varphi, \sup_{x\in\Omega}\psi\Big\} < +\infty, \tag{5.36}$$

where $M > 0$ is a positive constant.

Let $\overline{u}(x,t) = \Lambda_1\varphi(x)$, $\overline{v}(x,t) = \Lambda_2\psi(x)$, where $\Lambda_1, \Lambda_2 > 0$ will be determined later. Then with a direct computation we obtain

$$\overline{u}_t - \Delta_{m,p}\overline{u} = \Lambda_1^{m(p-1)}, \qquad \overline{v}_t - \Delta_{n,q}\overline{v} = \Lambda_2^{n(q-1)},$$

and

$$a\int_\Omega \overline{u}^{\alpha_1}\overline{v}^{\beta_1}\,dx \leqslant a\|\Omega\|\Lambda_1^{\alpha_1}\Lambda_2^{\beta_1}M^{\alpha_1+\beta_1}, \qquad b\int_\Omega \overline{u}^{\alpha_2}\overline{v}^{\beta_2}\,dx \leqslant b\|\Omega\|\Lambda_1^{\alpha_2}\Lambda_2^{\beta_2}M^{\alpha_2+\beta_2}.$$

So, $(\overline{u}(x,t), \overline{v}(x,t))$ is an upper solution of problem (1.1), if

$$\Lambda_1^{m(p-1)} \geqslant a\|\Omega\|\Lambda_1^{\alpha_1}\Lambda_2^{\beta_1}M^{\alpha_1+\beta_1}, \qquad \Lambda_2^{n(q-1)} \geqslant b\|\Omega\|\Lambda_1^{\alpha_2}\Lambda_2^{\beta_2}M^{\alpha_2+\beta_2},$$
$$\overline{u}(x,t)|_{\partial\Omega} \geqslant 0, \qquad \overline{v}(x,t)|_{\partial\Omega} \geqslant 0,$$
$$\overline{u}(x,0) = u_0(x), \qquad \overline{v}(x,0) = v_0(x). \tag{5.37}$$

Next we prove (5.37) in three cases.

(i) When $\beta_1\alpha_2 < [m(p-1)-\alpha_1][n(q-1)-\beta_2]$, if we choose Λ_1, Λ_2 large enough such that

$$\Lambda_1 > \max\Big\{\max_{x\in\overline{\Omega}}u_0(x), \big(ab^{\frac{\beta_1}{n(q-1)-\beta_2}}\|\Omega\|^{1+\frac{\beta_1}{n(q-1)-\beta_2}}M^{\alpha_1+\beta_1+\frac{(\alpha_2+\beta_2)\beta_1}{n(q-1)-\beta_2}}\big)^{\frac{1}{m(p-1)-\alpha_1-\frac{\alpha_2\beta_1}{n(q-1)-\beta_2}}}\Big\},$$

$$\Lambda_2 > \max\Big\{\max_{x\in\overline{\Omega}}v_0(x), \big(a^{\frac{\alpha_2}{m(p-1)-\alpha_1}}b\|\Omega\|^{1+\frac{\alpha_2}{m(p-1)-\alpha_1}}M^{\alpha_2+\beta_2+\frac{(\alpha_1+\beta_1)\alpha_2}{m(p-1)-\alpha_1}}\big)^{\frac{1}{n(q-1)-\beta_2-\frac{\alpha_2\beta_1}{m(p-1)-\alpha_1}}}\Big\},$$

then (5.37) holds.

(ii) When $\beta_1\alpha_2 = [m(p-1)-\alpha_1][n(q-1)-\beta_2]$, we can choose Λ_1, Λ_2 large enough such that

$$\Lambda_1 > \max_{x\in\overline{\Omega}} u_0(x), \qquad \Lambda_2 > \max_{x\in\overline{\Omega}} v_0(x),$$

and $\|\Omega\|$ small enough such that

$$\|\Omega\| \leqslant \min\Big\{\big(ab^{\frac{\beta_1}{n(q-1)-\beta_2}} M^{\alpha_1+\beta_1+\frac{(\alpha_2+\beta_2)\beta_1}{n(q-1)-\beta_2}}\big)^{-1/(1+\frac{\beta_1}{n(q-1)-\beta_2})}, \big(a^{\frac{\alpha_2}{m(p-1)-\alpha_1}} bM^{\alpha_2+\beta_2+\frac{(\alpha_1+\beta_1)\alpha_2}{m(p-1)-\alpha_1}}\big)^{-1/(1+\frac{\alpha_2}{m(p-1)-\alpha_1})}\Big\},$$

then (5.37) holds.

(iii) When $\beta_1\alpha_2 > [m(p-1)-\alpha_1][n(q-1)-\beta_2]$, we can take Λ_1, Λ_2 small enough such that

$$\Lambda_1 \leqslant \big(ab^{\frac{\beta_1}{n(q-1)-\beta_2}} \|\Omega\|^{1+\frac{\beta_1}{n(q-1)-\beta_2}} M^{\alpha_1+\beta_1+\frac{(\alpha_2+\beta_2)\beta_1}{n(q-1)-\beta_2}}\big)^{\frac{-1}{\frac{\alpha_2\beta_1}{n(q-1)-\beta_2}-[m(p-1)-\alpha_1]}},$$

$$\Lambda_2 \leqslant \big(a^{\frac{\alpha_2}{m(p-1)-\alpha_1}} b\|\Omega\|^{1+\frac{\alpha_2}{m(p-1)-\alpha_1}} M^{\alpha_2+\beta_2+\frac{(\alpha_1+\beta_1)\alpha_2}{m(p-1)-\alpha_1}}\big)^{\frac{-1}{\frac{\alpha_2\beta_1}{m(p-1)-\alpha_1}-[n(q-1)-\beta_2]}}.$$

Furthermore, if the initial data is sufficiently small such that $u_0(x) < \Lambda_1$ and $v_0(x) < \Lambda_2$, then (5.37) holds. The proof of Theorem 5.1 is complete. □

6. Blow-up of a solution

In this section, we investigate the blow-up property of the solutions to problem (1.1) and prove Theorem 6.1. The main method is constructing a blowing-up lower solution and using the comparison principle to achieve our purpose.

Theorem 6.1. *Suppose the initial data $(u_0(x), v_0(x))$ satisfies the conditions (H), then the solution of problem* (1.1) *blows up in finite time if one of the following conditions holds*:

(i) $\beta_1\alpha_2 > [m(p-1)-\alpha_1][n(q-1)-\beta_2]$ *and the initial data is sufficiently large*;
(ii) $\beta_1\alpha_2 = [m(p-1)-\alpha_1][n(q-1)-\beta_2]$ *and Ω contains a sufficiently large ball.*

Proof. (i) When $\beta_1\alpha_2 > [m(p-1)-\alpha_1][n(q-1)-\beta_2]$, and the initial data is large enough, set

$$\underline{u}(x,t) = (\tau-t)^{-\gamma_1} V_1(\xi), \qquad \xi = |x|(\tau-t)^{-\sigma_1}, \qquad V_1(\xi) = \left(1+\frac{A}{2}-\frac{\xi^2}{2A}\right)_+^{1/m},$$

$$\underline{v}(x,t) = (\tau-t)^{-\gamma_2} V_2(\eta), \qquad \eta = |x|(\tau-t)^{-\sigma_2}, \qquad V_2(\eta) = \left(1+\frac{A}{2}-\frac{\eta^2}{2A}\right)_+^{1/n},$$

where $\gamma_i, \sigma_i > 0$ $(i=1,2)$, $A > 1$ and $0 < \tau < 1$ are parameters to be determined. It is easy to see that $\underline{u}(x,t)$, $\underline{v}(x,t)$ blow up at time τ, so it is enough to prove that $(\underline{u}(x,t), \underline{v}(x,t))$ is a lower solution of problem (1.1). If we choose τ small enough such that

$$\operatorname{supp}\underline{u}(\cdot,t) = \overline{B\big(0, R(\tau-t)^{\sigma_1}\big)} \subset \overline{B\big(0, R\tau^{\sigma_1}\big)} \subset \Omega,$$
$$\operatorname{supp}\underline{v}(\cdot,t) = \overline{B\big(0, R(\tau-t)^{\sigma_2}\big)} \subset \overline{B\big(0, R\tau^{\sigma_2}\big)} \subset \Omega,$$

where $R = (A(2+A))^{1/2}$, then $\underline{u}(x,t)|_{\partial\Omega} = 0$, $\underline{v}(x,t)|_{\partial\Omega} = 0$. Next if we choose the initial data large enough such that

$$u_0(x) \geqslant \frac{1}{\tau^{\gamma_1}} V_1\left(\frac{|x|}{\tau^{\sigma_1}}\right), \qquad v_0(x) \geqslant \frac{1}{\tau^{\gamma_2}} V_2\left(\frac{|x|}{\tau^{\sigma_2}}\right),$$

then $(\underline{u}(x,t), \underline{v}(x,t))$ is a lower solution of problem (1.1) if

$$\underline{u}_t - \Delta_{m,p}\underline{u} \leqslant a\int_\Omega \underline{u}^{\alpha_1}(x,t)\underline{v}^{\beta_1}(x,t)\,dx, \quad (x,t)\in\Omega\times(0,\tau], \tag{6.38}$$

$$\underline{v}_t - \Delta_{n,q}\underline{v} \leqslant b\int_\Omega \underline{u}^{\alpha_2}(x,t)\underline{v}^{\beta_2}(x,t)\,dx, \quad (x,t)\in\Omega\times(0,\tau]. \tag{6.39}$$

After a direct computation, we obtain

$$\underline{u}_t = \frac{\gamma_1 V_1(\xi) + \sigma_1 \xi V_1'(\xi)}{(\tau - t)^{\gamma_1 + 1}}, \qquad \underline{v}_t = \frac{\gamma_2 V_2(\eta) + \sigma_2 \eta V_2'(\eta)}{(\tau - t)^{\gamma_2 + 1}},$$
$$\nabla \underline{u}^m = \frac{x}{A(\tau - t)^{m\gamma_1 + 2\sigma_1}}, \qquad -\Delta \underline{u}^m = \frac{N}{A(\tau - t)^{m\gamma_1 + 2\sigma_1}},$$
$$\nabla \underline{v}^n = \frac{x}{A(\tau - t)^{m\gamma_2 + 2\sigma_2}}, \qquad -\Delta \underline{v}^n = \frac{N}{A(\tau - t)^{m\gamma_2 + 2\sigma_2}}, \tag{6.40}$$

and

$$-\Delta_{m,p} \underline{u} = \left|\nabla \underline{u}^m\right|^{p-2} \Delta \underline{u}^m + (p-2)\left|\nabla \underline{u}^m\right|^{p-4} \left(\nabla \underline{u}^m\right)^\tau \cdot \left(H_x\left(\underline{u}^m\right)\right) \cdot \nabla \underline{u}^m$$
$$= \left|\nabla \underline{u}^m\right|^{p-2} \Delta \underline{u}^m + (p-2)\left|\nabla \underline{u}^m\right|^{p-4} \sum_{j=1}^N \sum_{i=1}^N \frac{\partial \underline{u}^m}{\partial x_i} \frac{\partial^2 \underline{u}^m}{\partial x_i \partial x_j} \frac{\partial \underline{u}^m}{\partial x_j}, \tag{6.41}$$
$$-\Delta_{n,q} \underline{v} = \left|\nabla \underline{v}^n\right|^{q-2} \Delta \underline{v}^n + (q-2)\left|\nabla \underline{v}^n\right|^{q-4} \left(\nabla \underline{v}^n\right)^\tau \cdot \left(H_x\left(\underline{v}^n\right)\right) \cdot \nabla \underline{v}^n$$
$$= \left|\nabla \underline{v}^n\right|^{q-2} \Delta \underline{v}^n + (q-2)\left|\nabla \underline{v}^n\right|^{q-4} \sum_{j=1}^N \sum_{i=1}^N \frac{\partial \underline{v}^n}{\partial x_i} \frac{\partial^2 \underline{v}^n}{\partial x_i \partial x_j} \frac{\partial \underline{v}^n}{\partial x_j}, \tag{6.42}$$

where $H_x(\underline{u}^m)$, $H_x(\underline{v}^n)$ denote the Hessian matrix of $\underline{u}^m(x,t)$, $\underline{v}^n(x,t)$, respectively.

Use the notation $d(\Omega) = \operatorname{diam}(\Omega)$, then from (6.41) and (6.42), we obtain

$$|\Delta_{m,p} \underline{u}| \leqslant \frac{N}{A(\tau - t)^{m\gamma_1 + 2\sigma_1}} \left(\frac{d(\Omega)}{(\tau - t)^{m\gamma_1 + 2\sigma_1}}\right)^{p-2}$$
$$+ (p-2)\left(\frac{d(\Omega)}{(\tau - t)^{m\gamma_1 + 2\sigma_1}}\right)^{p-4} \left(\frac{d(\Omega)}{(\tau - t)^{m\gamma_1 + 2\sigma_1}}\right)^2 \frac{N}{A(\tau - t)^{m\gamma_1 + 2\sigma_1}}$$
$$= \frac{N(p-1)(d(\Omega))^{p-2}}{A(\tau - t)^{(m\gamma_1 + 2\sigma_1)(p-1)}}. \tag{6.43}$$

Similarly, from (6.41) and (6.42) we obtain

$$|\Delta_{n,q} \underline{v}| \leqslant \frac{N}{A(\tau - t)^{n\gamma_2 + 2\sigma_2}} \left(\frac{d(\Omega)}{(\tau - t)^{n\gamma_2 + 2\sigma_2}}\right)^{q-2}$$
$$+ (q-2)\left(\frac{d(\Omega)}{(\tau - t)^{n\gamma_2 + 2\sigma_2}}\right)^{q-4} \left(\frac{d(\Omega)}{(\tau - t)^{n\gamma_2 + 2\sigma_2}}\right)^2 \frac{N}{A(\tau - t)^{n\gamma_2 + 2\sigma_2}}$$
$$= \frac{N(q-1)(d(\Omega))^{q-2}}{A(\tau - t)^{(n\gamma_2 + 2\sigma_2)(q-1)}}. \tag{6.44}$$

Next, we compute the nonlocal term of (6.38) and (6.39)

$$a \int_\Omega \underline{u}^{\alpha_1}(x,t) \underline{v}^{\beta_1}(x,t)\, dx$$
$$= \frac{a}{(\tau - t)^{\gamma_1 \alpha_1 + \gamma_2 \beta_1}} \int_{B(0, R(\tau - t)^{\sigma_1}) \cap B(0, R(\tau - t)^{\sigma_2})} V_1^{\alpha_1}\left(\frac{|x|}{(\tau - t)^{\sigma_1}}\right) V_2^{\beta_1}\left(\frac{|x|}{(\tau - t)^{\sigma_2}}\right) dx$$
$$\geqslant \frac{aM_1}{(\tau - t)^{\gamma_1 \alpha_1 + \gamma_2 \beta_1 - N(\sigma_1 + \sigma_2)}},$$
$$b \int_\Omega \underline{u}^{\alpha_2}(x,t) \underline{v}^{\beta_2}(x,t)\, dx$$
$$= \frac{b}{(\tau - t)^{\gamma_1 \alpha_2 + \gamma_2 \beta_2}} \int_{B(0, R(\tau - t)^{\sigma_1}) \cap B(0, R(\tau - t)^{\sigma_2})} V_1^{\alpha_2}\left(\frac{|x|}{(\tau - t)^{\sigma_1}}\right) V_2^{\beta_2}\left(\frac{|x|}{(\tau - t)^{\sigma_2}}\right) dx$$
$$\geqslant \frac{bM_2}{(\tau - t)^{\gamma_1 \alpha_2 + \gamma_2 \beta_2 - N(\sigma_1 + \sigma_2)}}, \tag{6.45}$$

where

$$M_1 = \int_{B(0,R)} V_1^{\alpha_1} V_2^{\beta_1}\big(|\xi|\big)\,d\xi, \qquad M_2 = \int_{B(0,R)} V_1^{\alpha_2} V_2^{\beta_2}\big(|\eta|\big)\,d\eta.$$

If $0 \leqslant \xi, \eta \leqslant A$, then $1 \leqslant V_1(\xi) \leqslant (1 + A/2)^{1/m}$, $1 \leqslant V_2(\eta) \leqslant (1 + A/2)^{1/n}$ and $V_1'(\xi) \leqslant 0$, $V_2'(\eta) \leqslant 0$. Combining the above inequalities and the definition of M_1 and M_2, we obtain

$$\begin{aligned} M_1 &= \int_{B(0,R)} V_1^{\alpha_1} V_2^{\beta_1}\big(|\xi|\big)\,d\xi \geqslant \int_{B(0,A)} V_1^{\alpha_1} V_2^{\beta_1}\big(|\xi|\big)\,d\xi \geqslant \big\|B(0,A)\big\|, \\ M_2 &= \int_{B(0,R)} V_1^{\alpha_2} V_2^{\beta_2}\big(|\eta|\big)\,d\eta \geqslant \int_{B(0,A)} V_1^{\alpha_2} V_2^{\beta_2}\big(|\eta|\big)\,d\eta \geqslant \big\|B(0,A)\big\|. \end{aligned} \tag{6.46}$$

Then from (6.40)–(6.46) we obtain

$$\begin{aligned} &\underline{u}_t - \Delta_{m,p}\underline{u} - a\int_\Omega \underline{u}^{\alpha_1}(x,t)\underline{v}^{\beta_1}(x,t)\,dx \\ &\quad \leqslant \frac{\gamma_1(1+\frac{A}{2})^{1/m}}{(\tau-t)^{\gamma_1+1}} + \frac{N(p-1)(d(\Omega))^{p-2}}{A(\tau-t)^{(m\gamma_1+2\sigma_1)(p-1)}} - \frac{a\|B(0,A)\|}{(\tau-t)^{\gamma_1\alpha_1+\gamma_2\beta_1-N(\sigma_1+\sigma_2)}}, \end{aligned} \tag{6.47}$$

$$\begin{aligned} &\underline{v}_t - \Delta_{n,q}\underline{v} - b\int_\Omega \underline{u}^{\alpha_2}(x,t)\underline{v}^{\beta_2}(x,t)\,dx \\ &\quad \leqslant \frac{\gamma_2(1+\frac{A}{2})^{1/n}}{(\tau-t)^{\gamma_2+1}} + \frac{N(q-1)(d(\Omega))^{q-2}}{A(\tau-t)^{(n\gamma_2+2\sigma_2)(q-1)}} - \frac{b\|B(0,A)\|}{(\tau-t)^{\gamma_1\alpha_2+\gamma_2\beta_2-N(\sigma_1+\sigma_2)}}. \end{aligned} \tag{6.48}$$

If $\xi, \eta \geqslant A$, since $m, n \geqslant 1$, we obtain $V_1(\xi) \leqslant 1$, $V_2(\eta) \leqslant 1$ and $V_1'(\xi) \leqslant -1/m$, $V_2'(\eta) \leqslant -1/n$. Combining the above inequalities (6.40)–(6.46), and $M_1 \geqslant 0$, $M_2 \geqslant 0$, we obtain

$$\underline{u}_t - \Delta_{m,p}\underline{u} - a\int_\Omega \underline{u}^{\alpha_1}(x,t)\underline{v}^{\beta_1}(x,t)\,dx \leqslant \frac{\gamma_1 - \frac{1}{m}\sigma_1 A}{(\tau-t)^{\gamma_1+1}} + \frac{N(p-1)(d(\Omega))^{p-2}}{A(\tau-t)^{(m\gamma_1+2\sigma_1)(p-1)}}, \tag{6.49}$$

$$\underline{v}_t - \Delta_{n,q}\underline{v} - b\int_\Omega \underline{u}^{\alpha_2}(x,t)\underline{v}^{\beta_2}(x,t)\,dx \leqslant \frac{\gamma_2 - \frac{1}{n}\sigma_2 A}{(\tau-t)^{\gamma_2+1}} + \frac{N(q-1)(d(\Omega))^{q-2}}{A(\tau-t)^{(n\gamma_2+2\sigma_2)(q-1)}}. \tag{6.50}$$

If $0 \leqslant \xi \leqslant A$ and $\eta \geqslant A$, we have that (6.47) and (6.50) hold. If $\xi \geqslant A$ and $0 \leqslant \eta \leqslant A$, we have that (6.48) and (6.49) hold.

So, from the above discussions, (6.38) and (6.39) hold if the right-hand sides of (6.47)–(6.50) are non-positive.

Since $p, q > 2$, $m, n \geqslant 1$ and $\beta_1\alpha_2 > [m(p-1) - \alpha_1][n(q-1) - \beta_2] > (\alpha_1 - 1)(\beta_2 - 1)$, we can choose two constants $\sigma_1, \sigma_2 > 0$ small enough such that

$$\begin{aligned} \frac{(\beta_1-\beta_2+1)[N(\sigma_1+\sigma_2)+1]}{\beta_1\alpha_2-(\alpha_1-1)(\beta_2-1)} &< \frac{1-2\sigma_1(p-1)}{m(p-1)-1}, \\ \frac{(\alpha_2-\alpha_1+1)[N(\sigma_1+\sigma_2)+1]}{\beta_1\alpha_2-(\alpha_1-1)(\beta_2-1)} &< \frac{1-2\sigma_2(q-1)}{n(q-1)-1}. \end{aligned}$$

Then we can choose two constants γ_1, γ_2 such that

$$\begin{aligned} \frac{(\beta_1-\beta_2+1)[N(\sigma_1+\sigma_2)+1]}{\beta_1\alpha_2-(\alpha_1-1)(\beta_2-1)} &< \gamma_1 < \frac{1-2\sigma_1(p-1)}{m(p-1)-1}, \\ \frac{(\alpha_2-\alpha_1+1)[N(\sigma_1+\sigma_2)+1]}{\beta_1\alpha_2-(\alpha_1-1)(\beta_2-1)} &< \gamma_2 < \frac{1-2\sigma_2(q-1)}{n(q-1)-1}, \end{aligned}$$

that is

$$(m\gamma_1 + 2\sigma_1)(p-1) < \gamma_1 + 1 < \gamma_1\alpha_1 + \gamma_2\beta_1 - N(\sigma_1+\sigma_2),$$

$$(m\gamma_2 + 2\sigma_2)(q-1) < \gamma_2 + 1 < \gamma_1\alpha_2 + \gamma_2\beta_2 - N(\sigma_1+\sigma_2).$$

Furthermore, if we choose $A > \max\{1, m\gamma_1/\sigma_1, n\gamma_2/\sigma_2\}$, then for $\tau > 0$ sufficiently small, the right-hand sides of (6.47)–(6.50) are non-positive, so (6.38) and (6.39) hold, and we obtain Theorem 6.1(i).

(ii) When $\beta_1\alpha_2 = [m(p-1)-\alpha_1][n(q-1)-\beta_2]$ and Ω contains a sufficiently large ball, we assume that $0 \in \Omega$ and a ball $B(0,R) \subset\subset \Omega$. Then we only need to show that the radial solution of problem (1.1) on $(\overline{B(0,R)} \times [0,T]) \times (\overline{B(0,R)} \times [0,T])$ blows up in finite time.

Since $p, q > 2$, $m, n \geqslant 1$ and $\beta_1\alpha_2 = [m(p-1)-\alpha_1][n(q-1)-\beta_2]$, we can choose two constants $\sigma_1, \sigma_2 > 0$ such that

$$\frac{\beta_1}{m(p-1)-\alpha_1} = \frac{\sigma_1}{\sigma_2} = \frac{n(q-1)-\beta_2}{\alpha_2},$$

that is

$$\sigma_1\alpha_1 + \sigma_2\beta_1 = m(p-1)\sigma_1, \qquad \sigma_1\alpha_2 + \sigma_2\beta_2 = n(q-1)\sigma_2.$$

Firstly, let us consider the following elliptic problem on $(0,R)$,

$$\begin{cases} -\dfrac{d}{dr}\left(r^{N-1}\left|\dfrac{d\varphi^m}{dr}\right|^{p-2}\dfrac{d\varphi^m}{dr}\right) = r^{N-1}, \\ \varphi'(0) = 0, \quad \varphi(R) = 0, \end{cases} \qquad \begin{cases} -\dfrac{d}{dr}\left(r^{N-1}\left|\dfrac{d\psi^n}{dr}\right|^{q-2}\dfrac{d\psi^n}{dr}\right) = r^{N-1}, \\ \psi'(0) = 0, \quad \psi(R) = 0. \end{cases}$$

Then it is easy to show

$$\varphi(r) = \left(\frac{p-1}{p}\right)^{1/m}\left(\frac{1}{N}\right)^{1/m(p-1)}\left(R^{p/(p-1)} - r^{p/(p-1)}\right)^{1/m},$$
$$\psi(r) = \left(\frac{q-1}{q}\right)^{1/n}\left(\frac{1}{N}\right)^{1/n(q-1)}\left(R^{q/(q-1)} - r^{q/(q-1)}\right)^{1/n}.$$

By assumption (H) on initial data, we can choose $s_0 > 0$ small enough that

$$u_0(r) \geqslant s_0^{\sigma_1}\varphi(r), \qquad v_0(r) \geqslant s_0^{\sigma_2}\psi(r), \quad \forall r \in [0,R).$$

Next, let us consider the following Cauchy problem with $s(0) = s_0$,

$$s'(t) = \min\left\{\frac{ac_1-1}{\sigma_1 M_1}, \frac{bc_2-1}{\sigma_2 M_2}\right\}s^{\Upsilon}(t),$$
$$\Upsilon = \min\bigl\{m(p-1)\sigma_1 - \sigma_1 + 1, n(q-1)\sigma_2 - \sigma_2 + 1\bigr\},$$

where for R large enough and $\omega(N)$ as the volume of the unit ball in N-dimensional space,

$$\begin{aligned} c_1 &= \int_{B(0,R)} \varphi^{\alpha_1}\psi^{\beta_1}\bigl(|x|\bigr)\,dx \\ &= \int_0^R dr \int_{\partial B(0,r)} \varphi^{\alpha_1}\psi^{\beta_1}(r)\,d\sigma \\ &= \int_0^R N\omega(N)\varphi^{\alpha_1}(r)\psi^{\beta_1}(r)r^{N-1}\,dr \\ &= N\omega(N)\left(\frac{p-1}{p}\right)^{\frac{\alpha_1}{m}}\left(\frac{1}{N}\right)^{\frac{\alpha_1}{m(p-1)}}\left(\frac{q-1}{q}\right)^{\frac{\beta_1}{n}}\left(\frac{1}{N}\right)^{\frac{\beta_1}{n(q-1)}} \\ &\quad \cdot \int_0^R \bigl(R^{\frac{p}{p-1}} - r^{\frac{p}{p-1}}\bigr)^{\frac{\alpha_1}{m}}\bigl(R^{\frac{q}{q-1}} - r^{\frac{q}{q-1}}\bigr)^{\frac{\beta_1}{n}} r^{N-1}\,dr > \frac{1}{a}, \\ c_2 &= \int_{B(0,R)} \varphi^{\alpha_2}\psi^{\beta_2}\bigl(|x|\bigr)\,dx \\ &= \int_0^R dr \int_{\partial B(0,r)} \varphi^{\alpha_2}\psi^{\beta_2}(r)\,d\sigma \end{aligned}$$

$$
\begin{aligned}
&= \int_0^R N\omega(N)\varphi^{\alpha_2}(r)\psi^{\beta_2}(r)r^{N-1}\,dr \\
&= N\omega(N)\left(\frac{p-1}{p}\right)^{\frac{\alpha_2}{m}}\left(\frac{1}{N}\right)^{\frac{\alpha_2}{m(p-1)}}\left(\frac{q-1}{q}\right)^{\frac{\beta_2}{n}}\left(\frac{1}{N}\right)^{\frac{\beta_2}{n(q-1)}} \\
&\quad\cdot \int_0^R \left(R^{\frac{p}{p-1}} - r^{\frac{p}{p-1}}\right)^{\frac{\alpha_2}{m}}\left(R^{\frac{q}{q-1}} - r^{\frac{q}{q-1}}\right)^{\frac{\beta_2}{n}} r^{N-1}\,dr > \frac{1}{b},
\end{aligned}
$$

$$
M_1 = \left(\frac{p-1}{p}\right)^{\frac{1}{m}}\left(\frac{1}{N}\right)^{\frac{1}{m(p-1)}} R^{\frac{p}{m(p-1)}}, \qquad M_2 = \left(\frac{q-1}{q}\right)^{\frac{1}{n}}\left(\frac{1}{N}\right)^{\frac{1}{n(q-1)}} R^{\frac{q}{n(q-1)}}.
$$

Since $\Upsilon > 1$, then there exists a constant $\tau^\star$ such that $\lim_{t\to\tau^\star} s(t) = +\infty$.

Finally, we construct $\underline{u}(r,t) = s^{\sigma_1}\varphi(r)$ and $\underline{v}(r,t) = s^{\sigma_2}\psi(r)$, then $(\underline{u}(r,t), \underline{v}(r,t))$ blows up in finite time. So it is enough to prove that $(\underline{u}(r,t), \underline{v}(r,t))$ is a lower solution of problem (1.1) on $(\overline{B(0,R)} \times [0,T]) \times (\overline{B(0,R)} \times [0,T])$. Let us make some simple computations:

$$
\begin{aligned}
\Delta_{m,p}\underline{u} &= \nabla\cdot\left(\left|(\underline{u}^m)_r\right|^{p-2}(\underline{u}^m)_r \frac{x}{r}\right) \\
&= \sum_{i=1}^N \left(\left|(\underline{u}^m)_r\right|^{p-2}(\underline{u}^m)_r \frac{x_i}{r}\right)_{x_i} \\
&= \left(\left|(\underline{u}^m)_r\right|^{p-2}(\underline{u}^m)_r\right)_r + \left|(\underline{u}^m)_r\right|^{p-2}(\underline{u}^m)_r \frac{N-1}{r} \\
&= r^{1-N}\left(r^{N-1}\left|(\underline{u}^m)_r\right|^{p-2}(\underline{u}^m)_r\right)_r := \Im(\underline{u}), \\
\Delta_{n,q}\underline{v} &= r^{1-N}\left(r^{N-1}\left|(\underline{v}^n)_r\right|^{q-2}(\underline{v}^n)_r\right)_r := \Im(\underline{v}).
\end{aligned}
$$

Then problem (6.41) becomes the following equations,

$$
\begin{aligned}
&\underline{u}_t - \Im(\underline{u}) - a\int_\Omega \underline{u}^{\alpha_1}(x,t)\underline{v}^{\beta_1}(x,t)\,dx \\
&\quad = \sigma_1\varphi s^{\sigma_1-1}s'(t) + s^{m(p-1)\sigma_1}(t) - ac_1 s^{\sigma_1\alpha_1+\sigma_2\beta_1} \\
&\quad = \varphi s^{\sigma_1-1}\left(\sigma_1 s'(t) + \varphi^{-1}s^{m(p-1)\sigma_1-\sigma_1+1}(t) - ac_1\varphi^{-1}s^{\sigma_1\alpha_1+\sigma_2\beta_1-\sigma_1+1}\right) \\
&\quad = \varphi s^{\sigma_1-1}\left(\sigma_1 s'(t) - (ac_1-1)\varphi^{-1}s^{m(p-1)\sigma_1-\sigma_1+1}(t)\right) \\
&\quad \leqslant \varphi s^{\sigma_1-1}\left(\sigma_1 s'(t) - (ac_1-1)M_1^{-1}s^{m(p-1)\sigma_1-\sigma_1+1}(t)\right) \\
&\quad < 0, \quad \forall (r,t) \in B(0,R)\times(0,\tau), \\
&\underline{v}_t - \Im(\underline{v}) - b\int_\Omega \underline{u}^{\alpha_2}(x,t)\underline{v}^{\beta_2}(x,t)\,dx \\
&\quad = \sigma_2\psi s^{\sigma_2-1}s'(t) + s^{n(q-1)\sigma_2}(t) - bc_2 s^{\sigma_1\alpha_2+\sigma_2\beta_2} \\
&\quad = \psi s^{\sigma_2-1}\left(\sigma_2 s'(t) + \psi^{-1}s^{n(q-1)\sigma_2-\sigma_2+1}(t) - bc_2\psi^{-1}s^{\sigma_1\alpha_2+\sigma_2\beta_2-\sigma_2+1}\right) \\
&\quad = \psi s^{\sigma_2-1}\left(\sigma_2 s'(t) - (bc_2-1)\psi^{-1}s^{n(q-1)\sigma_2-\sigma_2+1}(t)\right) \\
&\quad \leqslant \psi s^{\sigma_2-1}\left(\sigma_2 s'(t) - (bc_2-1)M_2^{-1}s^{n(q-1)\sigma_2-\sigma_2+1}(t)\right) \\
&\quad < 0, \quad \forall (r,t) \in B(0,R)\times(0,\tau), \\
&r^{N-1}\left|(\underline{u}^m)_r\right|^{p-2}(\underline{u}^m)_r\big|_{r=0} = 0, \qquad r^{N-1}\left|(\underline{v}^n)_r\right|^{q-2}(\underline{v}^n)_r\big|_{r=0} = 0, \quad \forall t \in [0,\tau], \\
&\underline{u}(R,t) = s^{\sigma_1}\varphi(R) = 0, \qquad \underline{v}(R,t) = s^{\sigma_2}\psi(R) = 0, \quad \forall t \in [0,\tau], \\
&\underline{u}(r,0) = s_0^{\sigma_1}\varphi(r) \leqslant u_0(r), \qquad \underline{v}(r,0) = s_0^{\sigma_2}\psi(r) \leqslant v_0(r), \quad \forall r \in [0,R].
\end{aligned}
$$

So, $(\underline{u}(r,t), \underline{v}(r,t))$ is a lower solution of problem (6.41) on $(\overline{B(0,R)} \times [0,T]) \times (\overline{B(0,R)} \times [0,T])$, we obtain Theorem 6.1(ii). The proof of Theorem 6.1 is complete. □

7. The special case $\beta_1 = n(q-1) - \beta_2$, $\alpha_2 = m(p-1) - \alpha_1$

In this section we consider problem (1.1) for a special case $\beta_1 = n(q-1) - \beta_2$, $\alpha_2 = m(p-1) - \alpha_1$; similar to Sections 5 and 6, we prove Theorem 7.1 by constructing special upper and lower solutions.

Theorem 7.1. *Suppose the initial data $(u_0(x), v_0(x))$ satisfies the conditions (H) and that $\alpha_2 = m(p-1) - \alpha_1$, $\beta_1 = n(q-1) - \beta_2$.*

(i) *If $\lambda\mu \leqslant (ab)^{-1}$, then the solution of problem* (1.1) *exists globally*;
(ii) *If $\lambda\mu > (ab)^{-1}$, $\alpha_1 < 1$, $\beta_2 < 1$, then the solution of problem* (1.1) *blows up in finite time, where*

$$\lambda = \int\limits_{\Omega} \zeta^{\alpha_1}(x)\vartheta^{\beta_1}(x)\,dx, \qquad \mu = \int\limits_{\Omega} \zeta^{\alpha_2}(x)\vartheta^{\beta_2}(x)\,dx,$$

and $\zeta(x)$, $\vartheta(x)$ are the unique solutions of the following elliptic equation (*see* [14,32]),

$$\begin{cases} -\Delta_{m,p}\zeta = 1, & x \in \Omega, \\ \zeta = 0, & x \in \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta_{n,q}\vartheta = 1, & x \in \Omega, \\ \vartheta = 0, & x \in \partial\Omega. \end{cases}$$

7.1. Global existence

In this section we prove the conclusion (i) of Theorem 7.1 Since $\lambda\mu \leqslant (ab)^{-1}$, we can take two positive constants Λ_1, Λ_2 large enough such that

$$a\mu \leqslant \frac{\Lambda_1^{m(p-1)-\alpha_1}}{\Lambda_2^{n(q-1)-\beta_2}} \leqslant (b\lambda)^{-1}, \qquad \Lambda_1\zeta(x) \geqslant u_0(x), \qquad \Lambda_2\vartheta \geqslant v_0(x).$$

Set $\overline{u}(x,t) = \Lambda_1\zeta(x)$, $\overline{v}(x,t) = \Lambda_2\vartheta(x)$, then we show that $(\overline{u}(x,t), \overline{v}(x,t))$ is an upper solution of problem (1.1), which exists globally. After a simple computation, we obtain

$$\begin{aligned}
&\overline{u}_t - \Delta_{m,p}\overline{u} - a\int\limits_{\Omega} \overline{u}^{\alpha_1}(x,t)\overline{v}^{\beta_1}(x,t)\,dx \\
&\quad = \Lambda_1^{m(p-1)} - a\mu\Lambda_1^{\alpha_1}\Lambda_2^{n(q-1)-\beta_2} = \Lambda_1^{\alpha_1}\big[\Lambda_1^{m(p-1)-\alpha_1} - a\mu\Lambda_2^{n(q-1)-\beta_2}\big] \geqslant 0, \\
&\overline{v}_t - \Delta_{n,q}\overline{v} - b\int\limits_{\Omega} \overline{u}^{\alpha_2}(x,t)\overline{v}^{\beta_2}(x,t)\,dx \\
&\quad = \Lambda_2^{n(q-1)} - b\lambda\Lambda_1^{m(p-1)-\alpha_1}\Lambda_2^{\beta_2} = \Lambda_2^{\beta_2}\big[\Lambda_2^{n(q-1)-\beta_2} - b\lambda\Lambda_1^{m(p-1)-\alpha_1}\big] \geqslant 0.
\end{aligned}$$

Noting that $\overline{u}(x,t) = \overline{v}(x,t) = 0$ on $\partial\Omega \times [0,\infty)$, we obtain that $(\overline{u}(x,t), \overline{v}(x,t))$ is an upper solution of problem (1.1). Then conclusion (i) of Theorem 7.1 holds.

7.2. Blow-up

In this section we prove conclusion (ii) of Theorem 7.1. First, we introduce the following useful lemma.

Lemma 7.1. *Suppose that the initial data $(u_0(x), v_0(x))$ satisfies the assumption (H) and $\lambda\mu > (ab)^{-1}$, then there exist two positive constants σ_1, σ_2 such that*

$$u(x,t) \geqslant \sigma_1\zeta(x), \qquad v(x,t) \geqslant \sigma_2\vartheta(x), \quad \forall (x,t) \in \overline{Q}_T.$$

Proof of Theorem 7.1(ii). Since $\lambda\mu > (ab)^{-1}$, we can take two appropriate σ_1, σ_2 positive constants such that

$$a\mu \geqslant \frac{\sigma_1^{m(p-1)-\alpha_1}}{\sigma_2^{n(q-1)-\beta_2}} \geqslant (b\lambda)^{-1}, \qquad \sigma_1\zeta(x) \leqslant u_0(x), \qquad \sigma_2\vartheta(x) \leqslant v_0(x).$$

Let $\underline{u}(x,t) = \sigma_1\zeta(x)$, $\underline{v}(x,t) = \sigma_2\vartheta(x)$, then we will show $(\underline{u}(x,t), \underline{v}(x,t))$ is a lower solution of problem (1.1). After a simple computation, we obtain

$$\begin{aligned}
&\underline{u}_t - \Delta_{m,p}\underline{u} - a\int_\Omega \underline{u}^{\alpha_1}(x,t)\underline{v}^{\beta_1}(x,t)\,dx \\
&\quad = \sigma_1^{m(p-1)} - a\mu\sigma_1^{\alpha_1}\sigma_2^{n(q-1)-\beta_2} = \sigma_1^{\alpha_1}\big[\sigma_1^{m(p-1)-\alpha_1} - a\mu\sigma_2^{n(q-1)-\beta_2}\big] \leqslant 0, \\
&\underline{v}_t - \Delta_{n,q}\underline{v} - b\int_\Omega \underline{u}^{\alpha_2}(x,t)\underline{v}^{\beta_2}(x,t)\,dx \\
&\quad = \sigma_2^{n(q-1)} - b\lambda\sigma_1^{m(p-1)-\alpha_1}\sigma_2^{\beta_2} = \sigma_2^{\beta_2}\big[\sigma_2^{n(q-1)-\beta_2} - b\lambda\sigma_1^{m(p-1)-\alpha_1}\big] \leqslant 0.
\end{aligned}$$

Noting that $\underline{u}(x,t) = \underline{v}(x,t) = 0$, we obtain that $(\underline{u}(x,t), \underline{v}(x,t))$ is a lower solution of problem (1.1). The proof of Lemma 7.1 is complete.

Now we can prove Theorem 7.1(ii). For $\Omega_1 \subset\subset \Omega$, let us consider the following elliptic equation,

$$\begin{cases} -\Delta_{m,p}\zeta_1 = 1, & x \in \Omega_1, \\ \zeta_1 = 0, & x \in \partial\Omega_1, \end{cases} \qquad \begin{cases} -\Delta_{n,q}\vartheta_1 = 1, & x \in \Omega_1, \\ \vartheta_1 = 0, & x \in \partial\Omega_1. \end{cases}$$

Then the comparison principle asserts that $\zeta(x)|_{\Omega_1} \geqslant \zeta_1(x)$, $\vartheta(x)|_{\Omega_1} \geqslant \vartheta_1(x)$. Take

$$\mu_1 = \int_\Omega \zeta_1^{\alpha_1}\vartheta_1^{\beta_1}(x)\,dx, \qquad \lambda_1 = \int_\Omega \zeta_1^{\alpha_2}\vartheta_1^{\beta_2}(x)\,dx.$$

Since $\lambda\mu > (ab)^{-1}$ and $\zeta(x)|_{\partial\Omega} = 0$, $\vartheta(x)|_{\partial\Omega} = 0$, we can choose some Ω_1 such that $\lambda_1\mu_1 > (ab)^{-1}$. From Lemma 7.1, we can see $u(x,t)|_{\Omega_1} \geqslant \sigma_1\zeta_1(x)$, $v(x,t)|_{\Omega_1} \geqslant \sigma_2\vartheta_1(x)$.

Next let us take a domain $\Omega_2 \subset\subset \Omega_1$ and use the notation

$$\varepsilon = \min\Big\{\inf_{x\in\overline{\Omega}_2} \sigma_1\zeta_1(x), \inf_{x\in\overline{\Omega}_2} \sigma_2\vartheta_1(x)\Big\} > 0.$$

Then,

$$u(x,t)|_{\overline{\Omega}_2} \geqslant \sigma_1\zeta_1(x)|_{\overline{\Omega}_2} \geqslant \varepsilon, \qquad v(x,t)|_{\overline{\Omega}_2} \geqslant \sigma_2\vartheta_1(x)|_{\overline{\Omega}_2} \geqslant \varepsilon.$$

So, the above discussion ensures that the solution $(u(x,t), v(x,t))$ of problem (1.1) is an upper solution of the following problem in $(\overline{\Omega}_2 \times [0,T]) \times (\overline{\Omega}_2 \times [0,T])$,

$$\begin{aligned}
&\bar{u}_t - \Delta_{m,p}\bar{u} = a\int_\Omega \bar{u}^{\alpha_1}(x,t)\bar{v}^{\beta_1}(x,t)\,dx, \quad (x,t) \in \Omega_2 \times (0,T], \\
&\bar{v}_t - \Delta_{n,q}\bar{v} = b\int_\Omega \bar{u}^{\alpha_2}(x,t)\bar{v}^{\beta_2}(x,t)\,dx, \quad (x,t) \in \Omega_2 \times (0,T], \\
&\bar{u}(x,t) = \bar{v}(x,t) = \varepsilon, \quad (x,t) \in \partial\Omega_2 \times (0,T], \\
&\bar{u}(x,0) = \varepsilon, \qquad \bar{v}(x,0) = \varepsilon, \quad x \in \Omega_2.
\end{aligned} \tag{7.51}$$

Denote $\wp = \max\{\sup_{x\in\overline{\Omega}_2}\zeta_1(x), \sup_{x\in\overline{\Omega}_2}\vartheta_1(x)\}$, and consider the following Cauchy problem,

$$\begin{aligned}
&\wp s_1'(t) + s_1^{\alpha_1}\big[s_1^{m(p-1)-\alpha_1} - a\mu_1 s_2^{n(q-1)-\beta_2}\big] = 0, \qquad s_1(0) = \varepsilon/\wp, \\
&\wp s_2'(t) + s_2^{\beta_2}\big[s_2^{n(q-1)-\beta_2} - b\lambda_1 s_1^{m(p-1)-\alpha_1}\big] = 0, \qquad s_2(0) = \varepsilon/\wp.
\end{aligned} \tag{7.52}$$

Multiplying the first equation of (7.52) by $b\lambda_1 + 1$, the second equation of (7.52) by $a\mu_1 + 1$ and combining them together, we obtain

$$\wp\frac{b\lambda_1+1}{1-\alpha_1}\big(s_1^{1-\alpha_1}\big)'(t) + \wp\frac{a\mu_1+1}{1-\beta_2}\big(s_2^{1-\beta_2}\big)'(t) = (ab\lambda_1\mu_1 - 1)\big(s_1^{m(p-1)-\alpha_1} + s_2^{n(q-1)-\beta_2}\big).$$

Since $\frac{m(p-1)-\alpha_1}{1-\alpha_1} > 1$, $\frac{n(q-1)-\beta_2}{1-\beta_2} > 1$ and $ab\lambda_1\mu_1 > 1$, there exists a constant $T' < +\infty$ such that

$$\lim_{t\to T'}\big(s_1^{1-\alpha_1}(t) + s_2^{1-\beta_2}(t)\big) = +\infty.$$

Noticing $\alpha_1 < 1$, $\beta_2 < 1$, we obtain that

$$\lim_{t\to T'}\big(s_1(t) + s_2(t)\big) = +\infty.$$

Set $\tilde{u}(x,t) = s_1(t)\zeta_1(x)$, $\tilde{v}(x,t) = s_2(t)\vartheta_1(x)$, then $(\tilde{u}(x,t), \tilde{v}(x,t))$ blows up in finite time. So, the solution of problem (7.52) blows up in finite time if $(\tilde{u}(x,t), \tilde{v}(x,t))$ is a lower solution of problem (7.52). After a simple computation, we obtain

$$\begin{aligned}
&\tilde{u}_t - \Delta_{m,p}\tilde{u} - a\int_\Omega \tilde{u}^{\alpha_1}(x,t)\tilde{v}^{\beta_1}(x,t)\,dx \\
&\quad = \zeta_1(x)s_1'(t) + s_1^{m(p-1)} - a\mu_1 s_1^{\alpha_1} s_2^{n(q-1)-\beta_2} \\
&\quad \leqslant \wp s_1'(t) + s_1^{\alpha_1}\big[s_1^{m(p-1)-\alpha_1} - a\mu_1 s_2^{n(q-1)-\beta_2}\big] = 0, \quad \forall (x,t) \in \Omega_2 \times (0,T], \\
&\tilde{v}_t - \Delta_{n,q}\tilde{v} - b\int_\Omega \tilde{u}^{\alpha_2}(x,t)\tilde{v}^{\beta_2}(x,t)\,dx \\
&\quad = \vartheta_1(x)s_2'(t) + s_2^{n(q-1)} - b\lambda_1 s_1^{m(p-1)-\alpha_1} s_2^{\beta_2} \\
&\quad \leqslant \wp s_2'(t) + s_2^{\beta_2}\big[s_2^{n(q-1)-\beta_2} - b\lambda_1 s_1^{m(p-1)-\alpha_1}\big] = 0, \quad \forall (x,t) \in \Omega_2 \times (0,T], \\
&\tilde{u}(x,t) = s_1(t)\zeta_1(x) = 0, \qquad \tilde{v}(x,t) = s_2(t)\vartheta_2(x) = 0, \quad \forall (x,t) \in \partial\Omega_2 \times (0,T], \\
&\tilde{u}(x,0) = s_1(0)\zeta_1(x) \leqslant \varepsilon, \qquad \tilde{v}(x,0) = s_2(0)\vartheta_2(x) \leqslant \varepsilon, \quad \forall x \in \Omega_2.
\end{aligned}$$

So, $(\tilde{u}(x,t), \tilde{v}(x,t))$ is a lower solution of problem (7.52). Then the conclusion of Theorem 7.1(ii) holds. The proof of Theorem 7.1 is complete. □

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